W[1]-Hardness of the k-Center Problem Parameterized by the Skeleton Dimension

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1 Introduction

In the k-Center problem, we are given a graph $G = (V, E)$ with positive edge weights and an integer k and the goal is to select k center vertices $C \subseteq V$ such that the maximum distance from any vertex to the closest center vertex is minimized. On general graphs, this problem is NP-hard. Typical applications of the k-Center problem can be found in logistics or urban planning and hence, it is natural to study the problem on transportation networks. Such networks are often characterized as graphs that are (almost) planar or have low doubling dimension d, highway dimension hd, skeleton dimension κ or treewidth tw. Intuitively a graph has low highway dimension if there are locally sparse shortest path hitting sets, and low skeleton dimension, if taking any shortest path tree and pruning the last portion of every branch gives a tree with only a few branches.

It was shown by Feldmann and Marx that k-Center is W[1]-hard on planar graphs of constant doubling dimension when parameterizing by k, hd and tw [2]. We extend their result and show the following theorem:

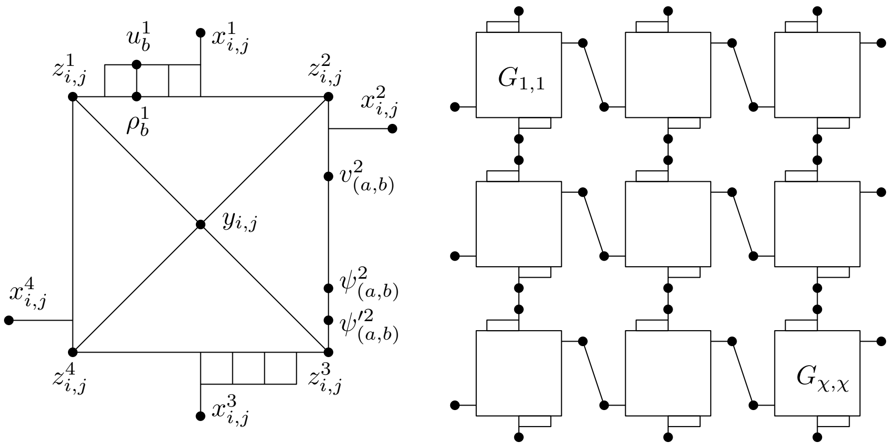
Theorem 1. *On planar graphs of constant doubling dimension, the k-Center problem is $W[1]$-hard for the combined parameter (k, tw, hd, κ) where tw is the treewidth, hd the highway dimension and κ the skeleton dimension of the input graph. Assuming ETH there is no $f(k, tw, hd, \kappa) \cdot |V|^{o(tw+\kappa+\sqrt{k+hd})}$ time algorithm for any computable function f.*

2 The Reduction

Following the idea of Feldmann and Marx [2], we present a reduction from the Grid Tiling with Inequality ($GT_{\leq}$) problem. This problem asks the following question: Given χ^2 sets $S_{i,j} \subseteq [n]^2$ of pairs of integers, where $(i, j) \in [\chi]^2$, is it possible to choose one pair $s_{i,j} \in S_{i,j}$ from every set, such that

- if $s_{i,j} = (a, b)$ and $s_{i+1,j} = (a', b')$ we have $a \leq a'$, and
- if $s_{i,j} = (a, b)$ and $s_{i,j+1} = (a', b')$ we have $b \leq b'$.

It is known that the $GT_{\leq}$ problem is $W[1]$-hard for parameter χ and, unless the Exponential Time Hypothesis (ETH) fails, it has no $f(\chi) \cdot n^{o(\chi)}$ time algorithm for any computable function f [1].



(a) A gadget $G_{i,j}$.

(b) The graph $G_{\mathcal{I}}$.

Fig. 1: A single gadget $G_{i,j}$ and the complete graph $G_{\mathcal{I}}$.

Given an instance $\mathcal{I}$ of $(\mathrm{GT}_{\leq})$, we construct a graph $G_{\mathcal{I}}$. For any of the χ^2 sets $S_{i,j}$, there is a gadget $G_{i,j}$ that contains a cycle $O_{i,j}$. For any pair $(a,b) \in S_{i,j}$ the cycle $O_{i,j}$ contains four vertices $v^1_{(a,b)}, \ldots, v^4_{(a,b)}$. We connect neighboring gadgets and choose edge weights such that the distance between a vertex $v^3_{(a,b)}$ from $G_{i,j}$ and a vertex $v^1_{(a',b')}$ from $G_{i+1,j}$ is $2^{n+2} + \frac{a-a'}{n}$. Moreover the distance between a vertex $v^3_{(a,b)}$ from $G_{i,j}$ and a vertex $v^1_{(a',b')}$ from $G_{i,j+1}$ is $2^{n+2} - 1 + 2^b + 2^{b'} + \frac{a-a'}{n}$.

We can show that for $k = 5\chi^2$, the k-Center instance $(G_{\mathcal{I}}, k)$ has a solution of cost of most 2^{n+1} if and only if the $\mathrm{GT}_{\leq}$-instance $\mathcal{I}$ is a yes-instance.

The construction from [2] yields a graph where the maximum degree Δ might be as large as $\Omega(n)$ and as we have $\Delta \leq \kappa$, one cannot bound the skeleton dimension κ in terms of χ. In contrast, our graph $G_{\mathcal{I}}$ has maximum degree $\Delta = 4$ and we can show that $\kappa \in \mathcal{O}(\chi)$. Moreover, $G_{\mathcal{I}}$ is planar and has doubling dimension $d \in \mathcal{O}(1)$, highway dimension $hd \in \mathcal{O}(\chi^2)$ and treewidth $tw \in \mathcal{O}(\chi^2)$. These properties of the graph $G_{\mathcal{I}}$ imply Theorem 1.

References

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