Coefficients of the solid angle and Ehrhart quasi-polynomials*

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1 Introduction

Given a rational polytope $P \subseteq \mathbb{R}^d$, we consider $L_P(t) := |tP \cap \mathbb{Z}^d|$, the number of integer points in the real dilates $tP := \{tx : x \in P\}$, and also the *solid angle sum* $A_P(t) := \sum_{x\in\mathbb{Z}^d} \omega_{tP}(x)$, which counts the integer points weighted by the proportion of the space around that point which the polytope occupies.

Ehrhart and Macdonald showed that these quantities can be written as *quasi-polynomials* functions of t (see Linke [3] for the extension from integer to real values of t), that is, as expressions of the form

$$\begin{aligned} L_P(t) &:= |tP \cap \mathbb{Z}^d| = \mathrm{vol}(P)t^d + e_{d-1}(t)t^{d-1} + \cdots + e_0(t), \\ A_P(t) &:= \sum_{x\in\mathbb{Z}^d} \omega_{tP}(x) = \mathrm{vol}(P)t^d + a_{d-1}(t)t^{d-1} + \cdots + a_0(t), \end{aligned}$$

where each *quasi-coefficient* $e_k(t), a_k(t)$ is a periodic function with period dividing the *denominator* of P, defined to be the smallest integer m such that mP in an integer polytope (see e.g., Beck and Robins [1]).

One of the motivations for studying these coefficients is that they capture geometric information about the polytope. McMullen [4] proved the existence of functions μ, not unique, such that for a rational P,

$$e_k(t) = \sum_{F\subseteq P,\ \dim(F)=k} \mathrm{vol}^*(F)\mu(tP, tF),$$

where the sum is taken over all k-dimensional faces of P, $\mathrm{vol}^*(F)$ is the k-dimensional volume of the face F, normalized so that the fundamental domain of the lattice of integer points in the linear space parallel to F has volume 1 and μ depends only on “local” geometric data associated to the face F. Such formulas are called *local formulas* for the quasi-coefficients.

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2 Main results

Our methods offer a further development for the initial approach of Diaz, Le, and Robins [2]. Our main result is an explicit, local formula for the codimension two quasi-coefficient $a_{d-2}(t)$ of the solid angle sum $A_P(t)$ of a rational polytope P. This formula simplifies when we only consider integer polytopes and integer dilations and, in particular, it gives an explicit formula for the solid angle sum of an integer polytope in dimensions 3 and 4, extending Pick's formula valid for dimension 2.

Theorem 1. *Let $P \subseteq \mathbb{R}^d$ be a full-dimensional rational polytope. Then for positive real values of t, the codimension two quasi-coefficient of the solid angle sum $A_P(t)$ has the following finite form:*

$$a_{d-2}(t) = \sum_{\substack{G \subseteq P, \\ \dim G = d-2}} \mathrm{vol}^*(G) \Bigg[\frac{c_G}{2k} \left(\frac{\|v_{F_2}\|}{\|v_{F_1}\|} \overline{B}_2\big(\langle v_{F_1}, \bar{x}_G \rangle \, t \big) + \frac{\|v_{F_1}\|}{\|v_{F_2}\|} \overline{B}_2\big(\langle v_{F_2}, \bar{x}_G \rangle \, t \big) \right)$$
$$+ \left(\omega_P(G) - \frac{1}{4} \right) \mathbf{1}_{\Lambda_G^*} (t \bar{x}_G) - s\big(h, k; (x_1 + h x_2)t, -k x_2 t\big) \Bigg].$$

Similar formulas are also obtained for the Ehrhart quasi-coefficients $e_{d-2}(t)$ and $e_{d-1}(t)$. The formula for $e_{d-1}(t)$ extends the classical interpretation of half the sum of the relative volumes of the facets, valid for integer dilations, to a formula valid for real dilations.

Theorem 2. *Let P be a full-dimensional rational polytope in $\mathbb{R}^d$. Then for all positive real values of t, the codimension one quasi-coefficient of the Ehrhart function $L_P(t)$ has the following finite form:*

$$e_{d-1}(t) = - \sum_{\substack{F \subseteq P, \\ \dim(F) = d-1}} \mathrm{vol}^*(F) \overline{B}_1^{+}\big(\langle v_F, x_F \rangle \, t \big).$$

In the formulas above, $\overline{B}_k$ are the periodized Bernoulli polynomials of degree k and $\overline{B}_1^{+}$ is the right-sided limit of $\overline{B}_1$. We denote by s the Dedekind-Rademacher sum and the other parameters are the "local" information associated to the position of the affine span of a face and the polytope P.

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