Maximum acyclic subgraph under conflict constraints on bounded degree graphs*

Leonardo de Abreu, Manoel Campêlo, Ana Karolinna Maia
Universidade Federal do Ceará, Fortaleza, Brazil
{leonardo.abreu,mcampelo}@lia.ufc.br, karolmaia@ufc.br

Universidade Federal do Ceará
Fortaleza, Brazil

1. Introduction

A directed graph D is acyclic if it does not contain any directed circuit.

Maximum degree, $\Delta(D)$, of a Directed Graph $D = (V, A)$ is defined as $\max_{v \in V}\{\delta^+(v)+\delta^-(v)\}$ where δ^- and δ^+ denote the indegree and outdegree, respectively.

Conflict constraint states that a given pair of elements (arcs) cannot be simultaneously in any feasible solution.

A **Conflict graph** G is a simple graph whose edges define conflict constraints. A feasible solution is always an independent set in G.

We call a directed circuit whose arcs do not have any conflict constraint between them a *proper cycle*. Otherwise, we call it an *unproper cycle*.

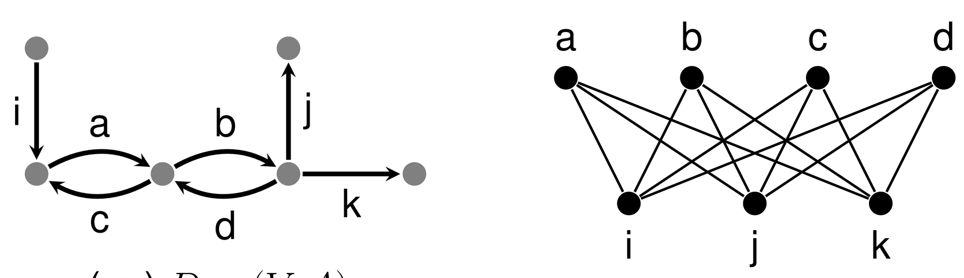
In [1] and [2], conflict constrained versions of minimum spanning tree and other classical problems were introduced.

In [3] a conflict constrained version of the maximum acyclic subgraph problem were introduced. The authors also developed approximation algorithms for it.

The Maximum Acyclic Subgraph under Conflict Constraints problem (MASCC) is defined as follows,

Given (D, G) where $D = (V, A)$ is a digraph and $G = (A, E)$ is a conflict graph, find a set $S \subseteq A$ with maximum cardinality such that $D[S]$ is acyclic and S is independent in G.

The figure below is an example of an instance for MASCC.



(a) $D = (V, A)$ (b) $G = (A, E)$

Figure 1: *Instance of MASCC. G is the conflict graph over the arcs in D.*

2. Complexity

The following table summarizes our complexity results for MASCC on bounded degree graphs.

$\Delta(D)$	$\Delta(G)$	Class	Description
≤ 1	≤ 2	*P*	D is acyclic, G is union of disjunct cycles and paths.
≤ 1	≥ 3	*NP-H*	At least as hard as maximum independent set problem.
$= 2$	≤ 1	*P*	Theorem 2.1*
$= 2$	$= 2$	*NP-H*	Theorem 2.2*
≥ 2	≥ 3	*NP-H*	At least as hard as maximum independent set problem.

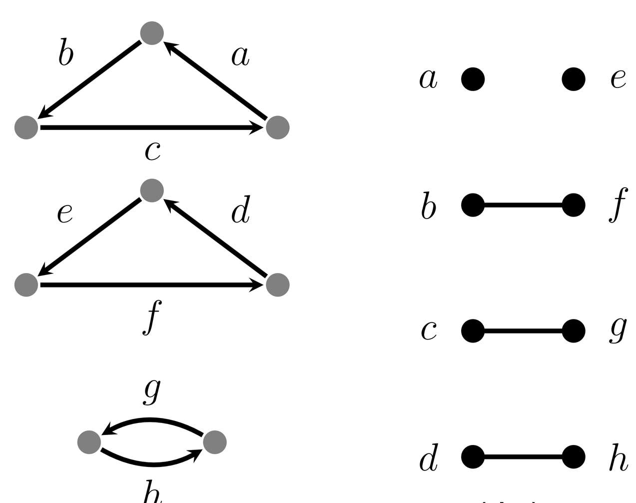
Table 1: *Overview of MASCC complexity on bounded degree graphs*

Results marked with an asterisk (*) are the main contribution of this work.

Theorem 2.1. *MASCC on (D, G) is polynomial-time solvable if $\Delta(D) = 2$ and $\Delta(G) \leq 1$.*

In order to prove this theorem, we present a method to obtain an equivalent instance where D is union of proper cycles, each one containing at least 2 arcs with degree 1 in G (see figure 2).

We developed a greedy algorithm that solves such instances.



(a) D (b) G

Figure 2: *Instance of MASCC. Notice that each cycle has 2 arcs with degree 1 in G.*

Theorem 2.2. *MASCC is NP-HARD, even if $\Delta(D) = \Delta(G) = 2$.*

We show that if we could decide in polynomial time if there is a set $A' \subseteq A$ so that $|A'| \geq |A|/2$, $D[A']$ is acyclic and A' is independent in G, then we could verify if any 3-CNF formula is satisfiable in polynomial time.

When we reduce a 3-SAT instance to this problem, the graphs we obtain are union of disjoint cycles.

Corollary 2.1. *From results summarized in table 1 we get a perfect dichotomy: MASCC on (D, G) is NP-Hard when $\min\{\Delta(D), \Delta(G)\} \geq 2$ or $\max\{\Delta(D), \Delta(G)\} \geq 3$ and is polynomial-time solvable otherwise.*

3. Approximability

We extend approximability results to MASCC.

Let $\mathcal{A}$ be an algorithm for maximum acyclic subgraph problem that returns a solution $\mathcal{A}(D) \subseteq A$ of size $|\mathcal{A}(D)| \geq |A|/2$. Let $\mathcal{B}$ be an β-approximation algorithm for maximum independent set problem.

Lemma 3.1. *The algorithm $(\mathcal{A} \circ \mathcal{B})(D, G)$, defined as $\mathcal{A}(D')$ with $D' = (V, \mathcal{B}(G))$, is $(\beta/2)$-approximation algorithm for MASCC.*

Let $\mathcal{A}$ be any $(1/2)$-approximation algorithm for the maximum acyclic subgraph problem cited in [3] and let $\mathcal{B}_k$ be the $(\frac{5}{3+\Delta} - \epsilon)$-approximation algorithm for the maximum independent set problem in [4], and improved in [5], for a fixed integer k.

Theorem 3.2. *$\mathcal{A} \circ \mathcal{B}_k$ is $\left(\frac{2.5}{3+\Delta(G)} - \epsilon\right)$-approximation algorithm for MASCC.*

When $\Delta(G) = 2$, the algorithm provides an approximation ratio of $1/2$ just like the algorithms in [3], which run in polynomial time only if the maximum independent set of the conflict graph is known.

We observe that under the Unique Games Conjecture, we cannot expect a factor better than $1/2$, even if G is empty [6].

On the other hand, unless $P = NP$, there is no constant approximation factor (for general G) even if D is already acyclic [7].

Although this algorithm has a good approximation ratio for small degree G, it is too expensive if $\Delta(G)$ is large.

To circumvent this issue, we present another algorithm.

Theorem 3.3. *Let $\mathcal{B}'$ be the AEKS-SR algorithm in [5]. $\mathcal{A} \circ \mathcal{B}'$ is $\left(\frac{\log\log\Delta(G)}{O(\Delta(G))}\right)$-approximation algorithm for MASCC.*

This later algorithm has better performance for large degree graphs.
Theorems 3.2 and 3.3 together give good theoretical algorithms for any instance of MASCC.

4. Fixed Parameter Tractability

A closely related problem to the MASCC is finding a set $F \subset A$ with minimal cardinality so that $D[A \setminus F]$ is acyclic and F is vertex cover of G. This problem is called feedback arc set under positive disjunctive constraints (FASPDC).

Note that if $A' \subseteq A$ is a solution to MASCC, $A \setminus A'$ is solution to FASPDC.

Theorem 4.1. *FASPDC is FPT on the size of solution.*

We show that by a reduction to the classical feedback vertex set problem.
The idea is to realize that constraints in G are equivalent to 2-cycles in D.

5. Future Work

- Extending the approximation algorithm of [8] may garantee a better approximation factor, at least for digraphs with small maximum degree.
- One may question how the girth of D and G affect the compexity of MASCC. It is known that some W-Hard problems become FPT for graphs with no short cycles.

References

[1] A. Darmann, U. Pferschy, and J. Schauer, “Determining a minimum spanning tree with disjunctive constraints,” in *Algorithmic Decision Theory*, Berlin, Heidelberg: Springer Berlin Heidelberg, 2009, pp. 414–423.

[2] A. Darmann, U. Pferschy, J. Schauer, and G. J. Woeginger, “Paths, trees and matchings under disjunctive constraints,” *Discrete Applied Mathematics*, vol. 159, no. 16, pp. 1726–1735, 2011, CTW 2009.

[3] S. M. S. Mapa and S. Urrutia, “On the maximum acyclic subgraph problem under disjunctive constraints,” *Information Processing Letters*, vol. 115, no. 2, pp. 119–124, 2015.

[4] P. Berman and M. Fürer, “Approximating maximum independent set in bounded degree graphs,” in *Proceedings of SODA '94*, 1994, pp. 365–371.

[5] M. M. Halldórsson and J. Radhakrishnan, “Improved approximations of independent sets in bounded-degree graphs,” in *Algorithm Theory — SWAT '95*, 1994, pp. 195–206.

[6] V. Guruswami, R. Manokaran, and P. Raghavendra, “Beating the random ordering is hard: Inapproximability of maximum acyclic subgraph,” in *49th Annual IEEE Symposium on Foundations of Computer Science*, 2008, pp. 573–582.

[7] C. Bazgan, B. Escoffier, and V. T. Paschos, “Completeness in standard and differential approximation classes: Poly-(D)APX- and (D)PTAS-completeness,” *Theoretical Computer Science*, vol. 339, no. 2, pp. 272–292, 2005.

[8] A. Newman, “Approximating the maximum acyclic subgraph,” Ph.D. dissertation, Massachusetts Institute of Technology, 2000.

[9] P. Berman and T. Fujito, “On approximation properties of the independent set problem for low degree graphs,” *Theory of Computing Systems*, vol. 32, no. 2, pp. 115–132, 1999.

*Partially supported by CNPq and FUNCAP