Aula de exercícios

1. Aplicações de Integral Definida - Cálculo de Áreas

1. Calcule a área da região compreendida entre os gráficos das funções $f(x) = \frac{1}{1+x^2}$ e $g(x) = 1 - \frac{x^2}{5}$ e as retas $x = 0$ e $x = 3$.

$$f(x) = g(x) \Rightarrow \frac{1}{1+x^2} = 1 - \frac{x^2}{5} \Rightarrow \frac{1}{1+x^2} = \frac{5-x^2}{5} \Rightarrow 5 = (5-x^2)(1+x^2)$$

$$\Rightarrow x^4 - 4x^2 - 5 + 5 = 0 \Rightarrow x^2(x^2-4) = 0$$

$$\Rightarrow x^2(x+2)(x-2) = 0 \Leftrightarrow x = 0,\ x = 2 \text{ ou } x = -2$$

Considere um ponto qualquer entre 0 e 2 e outro acima de 2

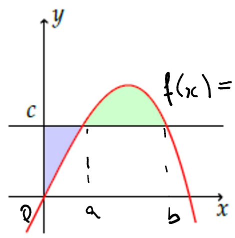
$$\left.\begin{array}{l} f(1) = \frac{1}{2} = \frac{5}{10} \\ g(1) = 1 - \frac{1}{5} = \frac{4}{5} = \frac{8}{5} \end{array}\right\} \Rightarrow g(x) > f(x),\ \forall\ 0 < x < 2 \qquad \left.\begin{array}{l} f(3) = \frac{1}{10} \\ g(3) = 1 - \frac{9}{5} = -\frac{4}{5} \end{array}\right\} \Rightarrow f(x) > g(x),\ \forall x > 2$$

$$A = \int_0^2 (g(x) - f(x))\,dx + \int_2^3 (f(x) - g(x))\,dx = \int_0^2 \left(1 - \frac{x^2}{5} - \frac{1}{1+x^2}\right) dx + \int_2^3 \left(\frac{1}{1+x^2} - 1 + \frac{x^2}{5}\right) dx$$

$$= \left[x - \frac{x^3}{15} - \operatorname{arctg} x\right]_0^2 + \left[\operatorname{arctg} x - x + \frac{x^3}{15}\right]_2^3 = 2 - \frac{8}{15} - \operatorname{arctg} 2 - 0 + \operatorname{arctg} 3 - 3 + \frac{27}{15} \operatorname{arctg} 2 + 2 - \frac{8}{15}$$

$$= \operatorname{arctg} 3 - 2\operatorname{arctg} 2 + \frac{26}{15}$$

5. A reta horizontal $y = c$ intercepta a curva $y = 2x - 3x^3$ no primeiro quadrante como mostra a figura. Determine c para que as áreas das duas regiões sombreadas sejam iguais.



$$f(x) = 2x - 3x^3 \Rightarrow f(b) = c \Rightarrow \boxed{c = 2b - 3b^3}\ (*)$$

$$A = B \Rightarrow \int_0^a (c - f(x))\,dx = \int_a^b (f(x) - c)\,dx \Rightarrow \int_0^a (c - f(x))\,dx - \int_a^b (f(x) - c)\,dx$$

$$\Rightarrow \int_0^a (c - f(x))\,dx + \int_a^b (c - f(x))\,dx = 0 \Rightarrow \int_0^b (c - f(x))\,dx = 0$$

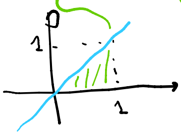
$$\Rightarrow \int_0^b (c - 2x + 3x^3)\,dx = 0 \Rightarrow \left(cx - x^2 + \frac{3x^4}{4}\right)\Big|_0^b = 0$$

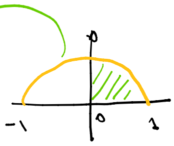
$$\Rightarrow cb - b^2 + \frac{3b^4}{4} = 0 \overset{(*)}{\Rightarrow} (2b - 3b^3)b - b^2 + \frac{3b^4}{4} = 0$$

$$\Rightarrow b^2\left(1 - \frac{9}{4}b^2\right) = 0 \Rightarrow \cancel{b = 0},\ \cancel{b = -\frac{2}{3}} \text{ ou } \boxed{b = \frac{2}{3}}$$

$$\Rightarrow c = 2\left(\frac{2}{3}\right) - 3\left(\frac{2}{3}\right)^3 = \frac{4}{3} - \frac{8}{9} = \frac{4}{9} \Rightarrow \boxed{c = \frac{4}{9}}$$

6. Calcule $\int_0^1 x + \sqrt{1-x^2}\,dx$, interpretando-a como uma área.





$$\int_0^1 \left(x+\sqrt{1-x^2}\right)dx = \frac{1}{2}+\frac{\pi}{4}$$

2. Integrais Indefinidas

6. $\int \tan^3 x \sec^2 x\,dx \overset{P}{=} \int u^3\,du = \frac{u^4}{4}+k = \frac{tg^4 x}{4}+k$

$u = tg\,x$

$du = sec^2 x\,dx$

26. $\int 2x(x+1)^{2010}\,dx \overset{P}{=} \int 2(u-1)u^{2010}\,du = \int \left(2u^{2011}-2u^{2010}\right)du$

$u = x+1$

$x = u-1$

$dx = du$

$$= \frac{2u^{2012}}{2012} - \frac{2u^{2011}}{2011}+k = \frac{2(x+1)^{2012}}{2012} - \frac{2(x+1)^{2011}}{2011}+k$$

33. $\int \arcsin x\,dx \overset{P}{=} x\,arcsen\,x - \int \frac{x}{\sqrt{1-x^2}}\,dx \overset{P}{=} x\,arcsen\,x + \int \frac{1}{2\sqrt{w}}\,dw$

$u = arcsen\,x \qquad du = \frac{1}{\sqrt{1-x^2}}\,dx$

$dv = dx \qquad v = x$

$w = 1-x^2$

$dw = -2x\,dx$

$$= x\,arcsen\,x + \sqrt{w}+k = x\,arcsen\,x + \sqrt{1-x^2}+k$$

33. $\int \arcsin x\,dx \overset{P}{=} \int arcsen\,(sen\,u)\cdot\cos u\,du = \int u\cos u\,du \overset{P}{=} u\,sen\,u - \int sen\,u\,du =$

$x = sen\,u$

$dx = \cos u\,du$

$u = arcsen\,x$

$v = u \qquad dv = du$

$dw = \cos u\,du \qquad w = sen\,u$

$$= x\,arcsen\,x + \cos(arcsen\,x)+k = x\,arcsen\,x + \sqrt{1-x^2}+k$$

36. $\int \sqrt{a^2+b^2x^2}\,dx$

$(a=0) \Rightarrow \int\sqrt{a^2+b^2x^2}\,dx = \int\sqrt{b^2x^2}\,dx = \int|bx|\,dx = \begin{cases} \frac{bx^2}{2}, & se\ bx\geq 0 \\ -\frac{bx^2}{2}, & se\ bx<0 \end{cases}$

$(b=0) \Rightarrow \int\sqrt{a^2+b^2x^2}\,dx = \int\sqrt{a^2}\,dx = |a|x+k$

$$(b=0) \Rightarrow \int \sqrt{a^2+b^2x^2}\,dx = \int \sqrt{a^2}\,dx = |a|x + k$$

$$(a\neq 0 \text{ e } b\neq 0) \Rightarrow \int \sqrt{a^2+b^2x^2}\,dx = \int \sqrt{a^2\left(1+\left(\frac{bx}{a}\right)^2\right)}\,dx = |a| \int \sqrt{1+\left(\frac{bx}{a}\right)^2}\,dx$$

$$\underset{P}{=} |a| \int \sqrt{1+\operatorname{tg}^2 u}\cdot\frac{a}{b}\sec^2 u\,du = \frac{|a|a}{b}\int \sec^3 u\,dx =$$

$\frac{bx}{a} = \operatorname{tg} u$

$x = \frac{a}{b}\operatorname{tg} u$

$dx = \frac{a}{b}\sec^2 u\,du$

$$= \frac{|a|a}{b}\cdot\frac{1}{2}\left[\sec u \operatorname{tg} u + \ln|\sec u + \operatorname{tg} u|\right] + k$$

$$= \frac{|a|a}{2b}\left[\frac{bx}{a}\,\frac{\sqrt{a^2+b^2x^2}}{|a|} + \ln\left|\frac{\sqrt{a^2+b^2x^2}}{|a|} + \frac{bx}{a}\right|\right] + k$$

$$= \frac{x}{2}\sqrt{a^2+b^2x^2} + \frac{|a|a}{2b}\ln\left|\frac{\sqrt{a^2+b^2x^2}}{|a|} + \frac{bx}{a}\right| + k$$

41. $\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx = \int (1-\operatorname{sen}^2 x)\cos x\,dx \underset{P}{=} \int (1-u^2)\,du = u + \frac{u^3}{3} + k =$

$u = \operatorname{sen} x$

$du = \cos x\,dx$

$$= \operatorname{sen} x + \frac{\operatorname{sen}^3 x}{3} + k$$

3. Testes

Questão 1. O valor de $\int_{-\pi/4}^{\pi/4} (\cos t + \sqrt{1+t^2}\sin^3 t\cos^3 t)\,dt$ é:

(a) $\sqrt{2}$; (b) $\frac{\sqrt{2}}{2}$; (c) $-\sqrt{2}$; (d) $-\frac{\sqrt{2}}{2}$; (e) 0.

$$\int_{-\pi/4}^{\pi/4} (\cos t + \sqrt{1+t^2}\operatorname{sen}^3 t\cos^3 t)\,dt = \int_{-\pi/4}^{\pi/4} \underbrace{\cos t}_{\text{f. par}}\,dt + \int_{-\pi/4}^{\pi/4} \underbrace{\sqrt{1+t^2}\operatorname{sen}^3 t\cos^3 t}_{\text{função ímpar}}\,dt$$

$$= 2\int_0^{\pi/4} \cos t\,dt + 0 = 2\operatorname{sen} t\Big|_0^{\pi/4} = 2\cdot\frac{\sqrt{2}}{2} = \sqrt{2}$$

Questão 4. Seja $F : \mathbb{R} \to \mathbb{R}$ uma função derivável. Sabendo que $F(3) = -1$ e que $\int_0^3 F(x)dx = 0$, o valor de $\int_0^3 xF'(x)dx$ é: (a) -9; (b) 0; (c) 9; (d) 3; (e) -3.

$$\int_0^3 xF'(x)\,dx = (xF(x))\Big|_0^3 - \int_0^3 F(x)\,dx = 3F(3) - 0F(0) - \int_0^3 F(x)\,dx$$

$u = x$ $\quad du = dx$

$dv = F'(x)\,dx$ $\quad v = F(x)$

$$= 3(-1) - 0 - 0 = -3$$