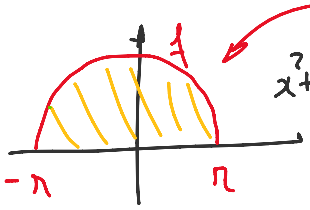
Aplicações da integral definida - parte 2 (T4)

Exercício: Calcule o volume de uma esfera de raio $r$.

$\frac{4}{3}\pi r^3$



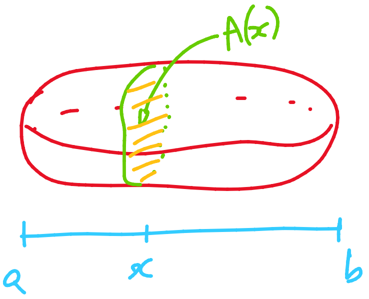

$x^2+y^2=r^2 \Rightarrow y=\pm\sqrt{r^2-x^2}$

$f(x)=\sqrt{r^2-x^2}$

$$V=\pi\int_{-r}^{r}\left(\sqrt{r^2-x^2}\right)^2 dx = 2\pi\int_0^r (r^2-x^2)\,dx = 2\pi\left[r^2x-\frac{x^3}{3}\right]_0^r$$

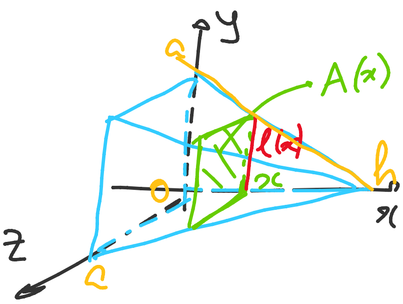
$$=2\pi\left[\left(r^3-\frac{r^3}{3}\right)-(0)\right]=2\pi.\frac{2r^3}{3}=\frac{4}{3}\pi r^3$$

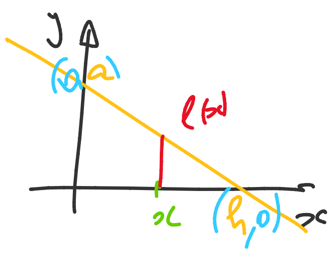
## Volume de um sólido "qualquer"




$$V=\int_a^b A(x)\,dx$$

Exercício: Calcule o volume de uma pirâmide de base quadrada de lado $a$ e altura $h$.




$y=mx+n$

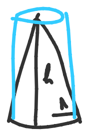
$$\begin{cases} a=m0+n \\ 0=mh+n \end{cases} \Rightarrow \begin{cases} n=a \\ m=-\frac{a}{h} \end{cases} \Rightarrow y=-\frac{a}{h}x+a$$

$$\| \\ l(x)$$

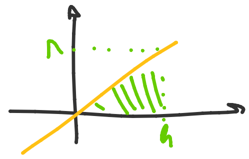
$$V=\int_0^h A(x)\,dx=\int_0^h (\ell(x))^2\,dx=\int_0^h\left(-\frac{a}{h}x+a\right)^2dx = a^2\int_0^h\left(-\frac{x}{h}+1\right)^2dx$$

$$=a^2\int_0^h\left(\frac{x^2}{h^2}-\frac{2x}{h}+1\right)dx = a^2\left(\frac{x^3}{3h^2}-\frac{x^2}{h}+x\right)\Big|_0^h = a^2\left[\left(\frac{h}{3}-\cancel{h}+\cancel{h}\right)-0\right]=\frac{a^2h}{3}$$

Exercício: Calcule o volume de um cone de base circular de raio $r$ e altura $h$.




$V=\frac{1}{3}\pi r^2 h$

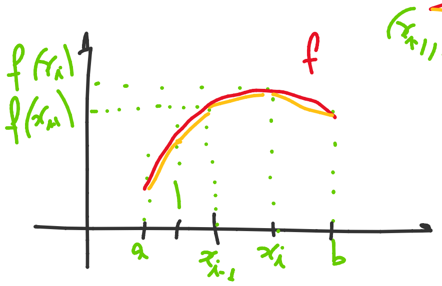


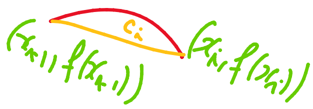

$y=mx$

$r=mh \Rightarrow m=\frac{r}{h}$

$$V=\pi\int_0^h\left(\frac{r}{h}x\right)^2dx=\pi\frac{r^2}{h^2}\frac{x^3}{3}\Big|_0^h=\frac{\pi r^2 h}{3}$$

## Comprimento de gráfico de função



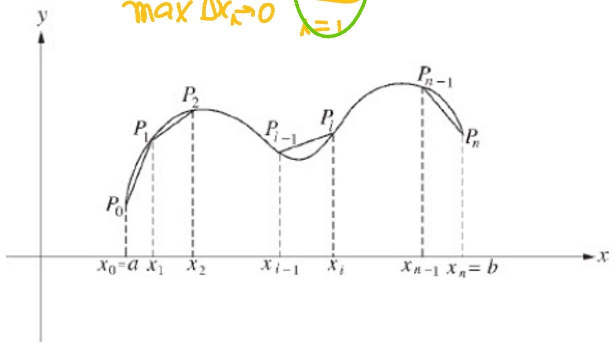




$$C_i=\sqrt{(f(x_i)-f(x_{i-1}))^2+(x_i-x_{i-1})^2}=(*)$$

$(T.V.M) \Rightarrow \exists\, m_i \in\, ]x_{i-1}, x_i[$ t.q. $f(x_i)-f(x_{i-1})=f'(m_i)(x_i-x_{i-1})$

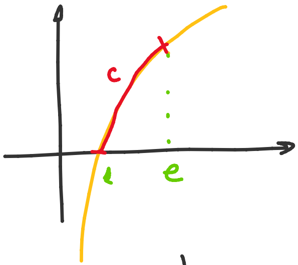
$$(*)=\sqrt{(f'(m_i)\,\Delta x_i)^2+\Delta x_i^2}=\sqrt{(f'(m_i))^2+1}\;\Delta x_i$$

$$C=\lim_{\max \Delta x_i\to 0}\sum_{i=1}^{n}\sqrt{1+(f'(m_i))^2}\,\Delta x_i=\int_a^b\sqrt{1+(f'(x))^2}\,dx$$




$$\boxed{C=\int_a^b\sqrt{1+(f'(x))^2}\,dx}$$

Exercício: Calcule o comprimento do gráfico de $y = \ln(x), 1 \le x \le e$.




$$C = \int_1^e \sqrt{1 + (f'(x))^2}\, dx$$

$$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x} \Rightarrow (f'(x))^2 = \frac{1}{x^2} \Rightarrow 1 + (f'(x))^2 = 1 + \frac{1}{x^2} \Rightarrow \sqrt{1 + (f'(x))^2} = \sqrt{1 + \frac{1}{x^2}}$$

$$C = \int_1^e \sqrt{1 + \frac{1}{x^2}}\, dx = \int_1^e \sqrt{\frac{1 + x^2}{x^2}}\, dx = \int_1^e \frac{\sqrt{1+x^2}}{x}\, dx$$

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$$\int \frac{\sqrt{1+x^2}}{x}\, dx = \int \frac{\sqrt{1 + tg^2 u}}{tg\, u} \sec^2 u\, du = \int \frac{\sec^3 u}{tg\, u}\, du = \int \frac{\frac{1}{\cos^3 u}}{\frac{\operatorname{sen} u}{\cos u}}\, du$$

$x = tg\, u$

$dx = \sec^2 u\, du$

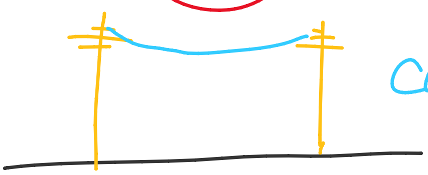
$v = \cos u$

$dv = -\operatorname{sen} u\, du$

$$= \int \frac{1}{\operatorname{sen} u \cos^2 u} \frac{\operatorname{sen} u}{\operatorname{sen} u}\, du = \int \frac{\operatorname{sen} u\, du}{(1 - \cos^2 u)\cos^2 u} = \int \frac{-dv}{(1 - v^2)v^2} = \text{etc.}$$

Exercício: Calcule o comprimento da curva $y = \cosh(x), -3 \le x \le 4$.

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{senh} x = \frac{e^x - e^{-x}}{2}$$



catenária

$$(\cosh x)' = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{e^x - e^{-x}}{2} = \operatorname{senh} x$$

$$(\operatorname{senh} x)' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \cosh x$$

$= e^{x-x} = e^0$

$$((\cosh x)')^2 = \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} = \frac{e^{2x} - 2 \cdot 1 + e^{-2x}}{4}$$

$$= \frac{e^{2x} + 2 + e^{-2x} - 4}{4} = \frac{e^{2x} + 2 \cdot e^x \cdot e^{-x} + e^{-2x}}{4} - 1$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 - 1 = (\cosh x)^2 - 1$$

$$\left((\cosh x)'\right)^2 + 1 = (\cosh x)^2 - 1 + 1$$

$$\sqrt{\left((\cosh x)'\right)^2 + 1} = \sqrt{\cosh^2 x} = \cosh x$$

$$C = \int_{-3}^{4} \cosh x \, dx = [\operatorname{senh} x]_{-3}^{4} = \operatorname{senh} 4 - \operatorname{senh}(-3) = \underline{\operatorname{senh} 4 + \operatorname{senh} 3}$$

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Ex.: Mostre que

a) $\cosh x$ é par

b) $\operatorname{senh} x$ é ímpar