# Técnicas de primitivação (T4 e T5)

Ex: Calcule $\int \frac{6x-15}{x^2-5x+6}\,dx$ de duas formas diferentes.

IV- <u>Frações parciais</u>:

IV-a- Integrais do tipo $\int \frac{p(x)}{(x-\alpha)(x-\beta)}\,dx$

<u>Teorema</u>: Sejam $\alpha, \beta, m$ e $n$ reais dados, com $\alpha \neq \beta$, então existem constantes $A$ e $B$ tais que

1. $\frac{mx+n}{(x-\alpha)(x-\beta)} = \frac{A}{(x-\alpha)} + \frac{B}{(x-\beta)}$

2. $\frac{mx+n}{(x-\alpha)^2} = \frac{A}{(x-\alpha)} + \frac{B}{(x-\alpha)^2}$

$$\int \frac{x+3}{x^2+2x+1}\,dx = \int \frac{x+3}{(x+1)^2}\,dx = \int \left[\frac{A}{x+1} + \frac{B}{(x+1)^2}\right] dx = (*)$$

$$① \quad \frac{x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{A(x+1)+B}{(x+1)^2} = \frac{Ax+(A+B)}{(x+1)^2}$$

$$\begin{cases} A = 1 \\ A+B=3 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=2 \end{cases}$$

$x+1=u$
$dx=du$

$$(*) = \int \left[\frac{1}{x+1} + \frac{2}{(x+1)^2}\right] dx = \ln|x+1| + \int \frac{2}{(x+1)^2}\,dx = (**)$$

$$= \ln|x+1| + \int \frac{2}{u^2}\,du = \ln|x+1| + \int 2u^{-2}\,du = \ln|x+1| + 2\,\frac{u^{-2+1}}{-2+1} + K$$

$$= \ln|x+1| - \frac{2}{x+1} + k$$

$$\int \frac{x+3}{x^2+2x+1}\,dx = \int \frac{x+3}{(x+1)^2}\,dx = \int \frac{w+2}{w^2}\,dw = \int \frac{1}{w}\,dw + 2\int \frac{1}{w^2}\,dw$$

$x+1=w$
$dx=dw$
$x+3=w+2$

$$= \ln|w| - \frac{2}{w} + K = \ln|x+1| - \frac{2}{x+1} + k$$

Exemplos:

a) $\displaystyle\int \frac{x+3}{x^2-3x+2}\,dx = \int \frac{x+3}{(x-1)(x-2)}\,dx = \int\left[\frac{A}{x-1}+\frac{B}{x-2}\right]dx$

$$= \int \frac{-4}{x-1}\,dx + \int \frac{5}{x-2}\,dx$$

$A=-4$
$B=5$

$$= -4\ln|x-1| + 5\ln|x-2| + k$$

Obs: Se $\alpha$ e $\beta$ são raízes de $p(x) = x^2+bx+c$,
então $p(x) = (x-\alpha)(x-\beta)$

b) $\displaystyle\int \frac{x^2+2}{x^2-3x+2}\,dx = \int\left[1+\frac{3x}{(x-1)(x-2)}\right]dx = x + \int \frac{3x}{(x-1)(x-2)}\,dx$

$$= x + \int\left[\frac{A}{x-1}+\frac{B}{x-2}\right]dx = x + \int\left[\frac{1}{x-1}+\frac{2}{x-2}\right]dx$$

$$= x + \ln|x-1| + 2\ln|x-2| + k$$

c) $\displaystyle\int \sec x\,dx = \int \frac{1}{\cos x}\,dx = \int \frac{\cos x\,dx}{\cos x\cos x} = \int \frac{\cos x\,dx}{\cos^2 x}$

$$= \int \frac{\cos x\,dx}{1-\operatorname{sen}^2 x} = \int \frac{du}{1-u^2} = \int\left[\frac{J}{1-u}+\frac{M}{1+u}\right]du = (*)$$

$u=\operatorname{sen} x$
$du=\cos x\,dx$

$$\frac{0\mu+1}{1-\mu^2} = \frac{J}{1-\mu}+\frac{M}{1+\mu} = \frac{(J-M)\mu+(J+M)}{(1-\mu)(1+\mu)} \qquad \begin{cases} J-M=0 \\ J+M=1 \end{cases} \Rightarrow J=M=\tfrac{1}{2}$$

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$$(*) = \int\left[\frac{1/2}{1-\mu}+\frac{1/2}{1+\mu}\right]d\mu = (**)$$

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$$\frac{1}{2}\int\frac{1}{1-\mu}\,d\mu = \frac{1}{2}\int\frac{1}{v}(-dv) = -\frac{1}{2}\int\frac{1}{v}\,dv = -\frac{1}{2}\ln|v|+k_1 = -\frac{1}{2}\ln|1-\mu|+k_1$$

$v = 1-\mu$
$dv = -d\mu$

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$$(**) = -\frac{1}{2}\ln|1-\mu|+\frac{1}{2}\ln|1+\mu|+k = \frac{1}{2}\ln\frac{|1+\mu|}{|1-\mu|}+k = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|+k$$

$$= \ln\left|\frac{1+\sin x}{1-\sin x}\right|^{1/2}+k = \ln\left|\frac{(1+\sin x)(1+\sin x)}{(1-\sin x)(1+\sin x)}\right|^{1/2}+k =$$

$$= \ln\left|\frac{(1+\sin x)^2}{\cos^2 x}\right|^{1/2}+k = \ln\left|\frac{1+\sin x}{\cos x}\right|+k = \ln|\sec x+\operatorname{tg} x|+k$$

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$$\int \sec x\,dx = \ln|\sec x+\operatorname{tg} x|+C \qquad \ln(a)-\ln(b) = \ln\left(\frac{a}{b}\right)$$

IV-b- Integrais do tipo $\displaystyle\int\frac{p(x)}{(x-\alpha)(x-\beta)(x-\gamma)}dx$

<u>Teorema</u>: Sejam $\alpha,\beta,\gamma,m,n$ e $p$ reais dados, com $\alpha,\beta,\gamma$ distintos entre si, então existem constantes $A, B, C$ tais que

a) $$\frac{mx^2+nx+p}{(x-\alpha)(x-\beta)(x-\gamma)} = \frac{A}{x-\alpha}+\frac{B}{x-\beta}+\frac{C}{x-\gamma}$$

b) $$\frac{mx^2+nx+p}{(x-\alpha)(x-\beta)^2} = \frac{A}{x-\alpha}+\frac{B}{x-\beta}+\frac{C}{(x-\beta)^2}$$

Exemplos:

a) $\displaystyle\int \frac{x^4+2x+1}{x^3-x^2-2x}\,dx = \int\left[x+1+\frac{3x^2+4x+1}{x(x^2-x-2)}\right]dx$

$\displaystyle = \frac{x^2}{2}+x+\int\frac{3x^2+4x+1}{x(x+1)(x-2)}\,dx = \frac{x^2}{2}+x+\int\left[\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-2}\right]dx$

$\displaystyle = \frac{x^2}{2}+x+\int\left[\frac{-1/2}{x}+\frac{0}{x+1}+\frac{21/6}{x-2}\right]dx = \frac{x^2}{2}+x-\frac{1}{2}\ln|x|+\frac{21}{6}\ln|x-2|+k$

$$\begin{array}{l|l}
x^4 \quad + 2x+1 & x^3-x^2-2x \\
\hline
-x^4-x^3-2x^2 & x+1 \\
0+x^3+x^2+2x & \\
-x^3-x^2-2x & \\
\hline
0+3x^2+4x+1 &
\end{array}$$

b) $\displaystyle\int \frac{2x+1}{x^3-x^2-x+1}\,dx = \int\frac{2x+1}{(x+1)(x-1)^2}\,dx =$

$$\begin{array}{l|l}
x^3-x^2-x+1 & x+1 \\
\hline
x^3+x^2 & x^2-2x+1 \\
-2x^2-x & \\
-2x^2-2x & \\
\hline
x+1 & \\
x+1 & \\
\hline
0 &
\end{array}
\qquad
\begin{array}{l|l}
x^2-2x+1 & x-1 \\
\hline
x^2-x & x-1 \\
-x+1 & \\
-x+1 & \\
\hline
0 &
\end{array}$$

$\displaystyle = \int\left[\frac{-1/4}{x+1}+\frac{1/4}{x-1}+\frac{3/2}{(x-1)^2}\right]dx$

$\displaystyle = -\frac{1}{4}\ln|x+1|+\frac{1}{4}\ln|x-1|-\frac{3}{2(x-1)}+h$

IV-b- Integrais do tipo $\displaystyle\int\frac{p(x)}{ax^2+bx+c}dx$, com $\Delta = b^2-4ac<0$

Teorema: Sejam $m, n, p, a, b, c$ e $\alpha$ números reais dados tais que $\Delta = b^2-4ac<0$, então existem constantes $A, B, D$ tais que

a) $\displaystyle\frac{mx^2+nx+p}{(x-\alpha)(ax^2+bx+c)} = \frac{A}{x-\alpha}+\frac{Bx+D}{ax^2+bx+c}$

<u>Obseração</u>: Nos exemplos abaixo, se existe o termo de grau 1 no denominador, então completa-se o quadrado, no outro caso, não.

a) $\displaystyle\int \frac{x+1}{x^2+2x+2}\,dx = \int \frac{x+1}{x^2+2x+1+1}\,dx = \int \frac{x+1}{(x+1)^2+1}\,dx = \int \frac{u}{u^2+1}\,du$

$\Delta < 0$

$x+1=u$
$dx=du$

$\displaystyle = \frac{1}{2}\int \frac{dv}{v} = \frac{1}{2}\ln|v| + k = \frac{1}{2}\ln|u^2+1| + k = \frac{1}{2}\ln(x^2+2x+2)+k$

$v=u^2+1$
$dv=u\,du$

b) $\displaystyle\int \frac{x+1}{x^2+2}\,dx = \int \frac{x+1}{2\left(\frac{x^2}{2}+1\right)}\,dx = \int \frac{x+1}{2\left[\left(\frac{x}{\sqrt{2}}\right)^2+1\right]}\,dx = \frac{1}{2}\int \frac{x+1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\,dx = 1\int \frac{y}{y^2+1}\,dy + \frac{\sqrt{2}}{2}\int \frac{1}{y^2+1}\,dy$

$\displaystyle = \frac{1}{2}\int \frac{(\sqrt{2}y+1)\sqrt{2}\,dy}{y^2+1}$

$\frac{x}{\sqrt{2}}=y$
$x=\sqrt{2}y \Rightarrow x+1=\sqrt{2}y+1$
$dx=\sqrt{2}\,dy$

$\displaystyle = \frac{1}{2}\ln(y^2+1) + \sqrt{2}\,\mathrm{arctg}(y) + k$

$\displaystyle = \frac{1}{2}\ln\left(\frac{x^2}{2}+1\right) + \frac{\sqrt{2}}{2}\,\mathrm{arctg}\left(\frac{x}{\sqrt{2}}\right) + k$