Técnicas de primitivação (T4 e T5)

<u>Observação</u>:

$$\frac{\operatorname{sen}^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow \boxed{\operatorname{tg}^2 x + 1 = \sec^2 x}$$

$$\frac{\operatorname{sen}^2 x}{\operatorname{sen}^2 x} + \frac{\cos^2 x}{\operatorname{sen}^2 x} = \frac{1}{\operatorname{sen}^2 x} \Rightarrow \boxed{1 + \operatorname{cotg}^2 x = \operatorname{cosec}^2 x}$$

$$\cos(a+b) = \cos a \cos b - \operatorname{sen} a \operatorname{sen} b$$

$$(a=b=x) \Rightarrow \cos(2x) = \cos^2 x - \cancel{\operatorname{sen}^2 x}$$

$$+ \quad 1 = \cos^2 x + \cancel{\operatorname{sen}^2 x}$$

$$1 + \cos(2x) = 2\cos^2 x$$

$$\Rightarrow \boxed{\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)}$$

$$- \quad 1 = \cancel{\cos^2 x} + \operatorname{sen}^2 x$$

$$\cos(2x) = \cancel{\cos^2 x} - \operatorname{sen}^2 x$$

$$1 - \cos(2x) = 2\operatorname{sen}^2 x$$

$$\boxed{\operatorname{sen}^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)}$$

<u>Exercícios</u>: Calcule as seguintes integrais

a) $\int \cos^2 x \, dx = \int \left[\frac{1}{2} + \frac{1}{2}\cos(2x)\right] dx$

$$= \frac{x}{2} + \frac{1}{2}\left(\frac{1}{2}\operatorname{sen}(2x)\right) + k$$

$$\parallel$$

$$\operatorname{sen} x . \cos x$$

$$\operatorname{sen}(2x) = 2\operatorname{sen} x \cos x$$

a) $\underset{I}{\int \cos^2 x \, dx} = \int \cos x \cos x \, dx = \operatorname{sen} x \cos x - \int (-\operatorname{sen} x)\operatorname{sen} x \, dx$

$$\boxed{\begin{array}{ll} u = \cos x & du = -\operatorname{sen} x \, dx \\ dv = \cos x \, dx & v = \operatorname{sen} x \end{array}}$$

$$= \operatorname{sen} x \cos x + \int (1 - \cos^2 x) \, dx = \operatorname{sen} x \cos x + x - \underbrace{\int \cos^2 x \, dx}_{I}$$

$$\Rightarrow I = \frac{1}{2}(\operatorname{sen} x \cos x + x) + k$$

b) $\int \operatorname{sen}^2 x \; dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)dx = \frac{x}{2} - \frac{\operatorname{sen}(2x)}{4} + k$

b) $\underbrace{\int \operatorname{sen}^2 x \; dx}_{= I} = \int \operatorname{sen} x \operatorname{sen} x \, dx = -\operatorname{sen} x \cos x + \int \cos^2 x \, dx =$

$\boxed{\begin{array}{ll} u = \operatorname{sen} x & du = \cos x \, dx \\ dv = \operatorname{sen} x \, dx & v = -\cos x \end{array}}$

$= -\operatorname{sen} x \cos x + \int (1 - \operatorname{sen}^2 x) \, dx = -\operatorname{sen} x \cos x + x - I$

$\Rightarrow I = \frac{1}{2}\left(-\operatorname{sen} x \cos x + x\right) + C$

III- Substituição "trigonométrica":

$\int f(x) \, dx = \int f(\varphi(u)) \varphi'(u) \, du$

$\boxed{x = \varphi(u)}$, φ derivável

φ inversível e a inversa derivável

$dx = \varphi'(u) \, du$

Seja f definida num intervalo I. Suponhamos que $x = \varphi(u)$ seja inversível, com inversa $u = \theta(x), x \in I$, sendo φ e θ deriváveis. Então

$$\int f(x) \, dx = \int f(\varphi(u)) \, \varphi'(u) \, du \text{, onde } u = \theta(x)$$

Exemplos:

a) $\int \sqrt{1 - x^2} \; dx = \int \sqrt{1 - \operatorname{sen}^2 u} \cos u \, du = \int \sqrt{\cos^2 u} \cdot \cos u \, du$

$u = \operatorname{arcsen} x$ $\quad$ $x = \operatorname{sen} u$ $\quad$ $dx = \cos u \, du$ $\quad$ $-\frac{\pi}{2} < u < \frac{\pi}{2}$

$= \int |\cos u| \cos u \, du = \int \cos u \cdot \cos u \, du = \int \cos^2 u \, du = \frac{u}{2} + \frac{1}{4}\operatorname{sen}(2u) + k$

$= \frac{\operatorname{arcsen} x}{2} + \frac{1}{4}\operatorname{sen}(\underbrace{2 \operatorname{arcsen} x}_{y}) + k = \frac{\operatorname{arcsen} x}{2} + \frac{1}{4} 2x\sqrt{1 - x^2} + k$

$y = \operatorname{sen}(2 \arcsin x)$

$y = 2 \operatorname{sen}(\arcsin x) \cdot \cos(\arcsin x)$?

$y = 2x \cdot \sqrt{1-x^2}$

$z = \cos(\arcsin x)$

$z^2 = \cos^2(\arcsin x) = 1 - \operatorname{sen}^2(\arcsin x)$

$= 1 - x^2$

$\cos(w) \geq 0$

$\Rightarrow z = \pm\sqrt{1-x^2}$

b) $\displaystyle\int \sqrt{1+x^2}\, dx = \int \sqrt{1+\operatorname{tg}^2 u}\ \sec^2 u\, du = \int \sqrt{\sec^2 u} \cdot \sec^2 u\, du$

$x = \operatorname{tg} u$

$dx = \sec^2 u$

$= \displaystyle\int |\sec u| \cdot \sec^2 u\, du = \int \sec^3 u\, du$

$= \frac{1}{2}\left[\sec u \operatorname{tg} u + \ln|\sec u + \operatorname{tg} u|\right] + C$ onde $u = \operatorname{arctg} x$

(exercício)

$I = \frac{1}{2}\left[\sec x \operatorname{tg} x + \ln|\sec x + \operatorname{tg} x|\right] + C$

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c) $\displaystyle\int \sqrt{2x - x^2}\, dx = \int \sqrt{1 - 1 + 2x - x^2}\, dx = \int \sqrt{1 - (x^2 - 2x + 1)}\, dx$

$x(2-x) = 0$

$= \displaystyle\int \sqrt{1 - (x-1)^2}\, dx =$ (ex.

$\operatorname{sen} u$

$x - 1 = \operatorname{sen} u$

$dx = \cos u\, du$

$u = \arcsin(x-1)$

d) $\displaystyle\int \sqrt{x^2+2x+2}\, dx = \int \sqrt{x^2+2x+1+1}\, dx = \int \sqrt{(x+1)^2+1}\, dx =$ (ex)

$x^2+2x+2 = (x+1)^2 + 1 \geq 0$

$-\frac{\pi}{2} < u < \frac{\pi}{2}$

$x + 1 = \operatorname{tg} u$

$dx = \sec u\, du$

IV- Frações parciais:

IV-a- Integrais do tipo $\int \frac{p(x)}{(x-\alpha)(x-\beta)}dx$

Teorema: Sejam α, β, m e n reais dados, com $\alpha \neq \beta$, então existem constantes A e B tais que

1. $$\frac{mx+n}{(x-\alpha)(x-\beta)} = \frac{A}{(x-\alpha)} + \frac{B}{(x-\beta)}$$

2. $$\frac{mx+n}{(x-\alpha)^2} = \frac{A}{(x-\alpha)} + \frac{B}{(x-\alpha)^2}$$

Dem:

a) $$\frac{mx+n}{(x-\alpha)(x-\beta)} = \frac{\overset{?}{A}}{x-\alpha} + \frac{\overset{?}{B}}{x-\beta} = \frac{A(x-\beta)+B(x-\alpha)}{(x-\alpha).(x-\beta)} = \frac{(A+B)x - A\beta - B\alpha}{(x-\alpha)(x-\beta)}$$

$$\Rightarrow mx+n = (A+B)x - A\beta - B\alpha$$

$$\begin{cases} A+B = m \\ A\beta + B\beta = m\beta \\ -A\beta - B\alpha = n \end{cases}$$

$$0 + B\beta - B\alpha = m\beta + n \Rightarrow \boxed{B = \frac{m\beta+n}{\beta-\alpha}} \; (\alpha \neq \beta)$$

$A = m - B$

$A = m - \left(\frac{m\beta+n}{\beta-\alpha}\right)$

$A = m + \frac{m\beta+n}{\alpha-\beta}$

$A = \frac{m\alpha - \cancel{m\beta} + \cancel{m\beta} + n}{\alpha-\beta}$

$\boxed{A = \underline{m\alpha + n}} \; (\alpha \neq \beta)$