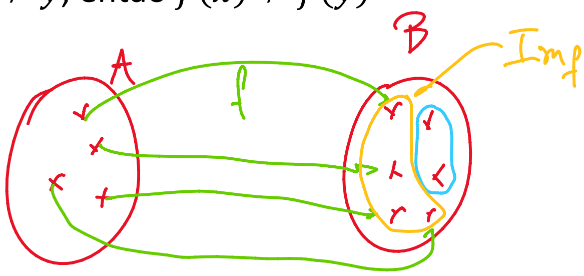
Funções inversas

Definição: Seja $f: A \longrightarrow \mathrm{B}$ uma função. Dizemos que f é:

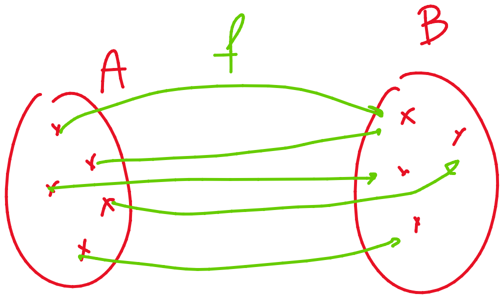
a) **injetora** se $x, y \in A$, com $x \neq y$, então $f(x) \neq f(y)$

Exemplo: $f(x) = \sqrt{x}$



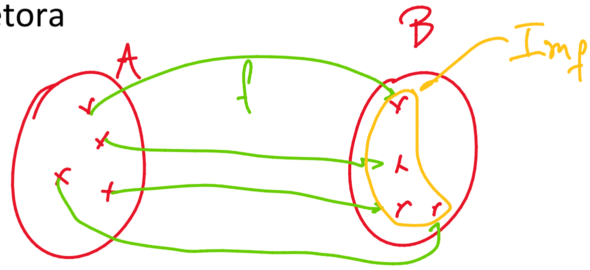
b) **sobrejetora** se dado $y \in B$, então $\exists x \in A$ tal que $y = f(x)$

Exemplo: $f(x) = \mathrm{tg}(x)$



c) **bijetora** se f for injetora e sobrejetora

Exemplo: $f(x) = x^3$



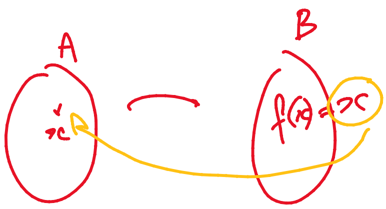
Definição: Seja $f: A \subset \mathbb{R} \longrightarrow B \subset \mathbb{R}$ uma função bijetora. Dizemos que f é **inversível** se existe uma função $g: B \longrightarrow A$ tal que $f(x) = y$ e $g(y) = x$ para $\forall x \in A$ e $\forall y \in B$.
Chamamos g de função inversa de f e denotamos g por f^{-1}.

$$\begin{cases} f(x) = y \\ g(y) = x \end{cases} \Rightarrow \begin{cases} f(g(y)) = y \\ g(f(x)) = x \end{cases}$$

Exemplos:

identidade

a) $f(x) = x \Rightarrow f^{-1}(x) = x$



$f^{-1}(x) = x$

$f: \mathbb{R} \longrightarrow \mathbb{R}$ = contradomínio
$x \longmapsto x$

$D_f = \mathbb{R}$, $\mathrm{Im}_f = \mathbb{R}$

$x \neq y \Rightarrow x = f(x) \neq f(y) = y$

b) $f(x) = x^3 \Rightarrow f^{-1}(x) = \sqrt[3]{x}$

$D_f = \mathbb{R}$

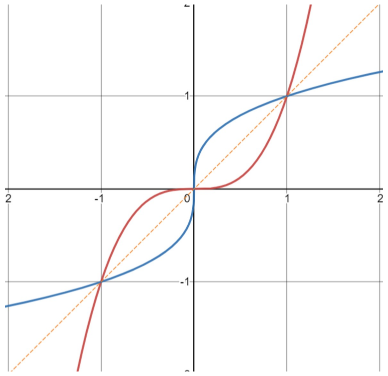
$Im_f = \mathbb{R}$

$y = f(x) = x^3$

$f^{-1}(y) = \boxed{g(y) = \sqrt[3]{y}}$

$f(2) = 2^3 = 8$

$g(8) = \sqrt[3]{8} = 2$



c) $f(x) = x^2 \Rightarrow f^{-1}(x) =$

$f: \mathbb{R} \to \mathbb{R}$

$D_f = \mathbb{R}$ $\neq$ não é sobrejetora

$Im_f = \mathbb{R}_+$

$f(-2) = f(2)$ não é injetora

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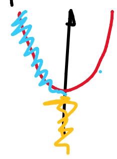


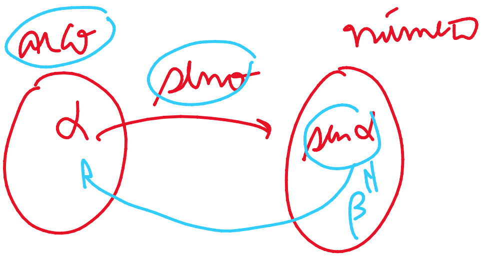
Considere $f: \mathbb{R}_+ \to \mathbb{R}_+$ bijetora $\Rightarrow g = f^{-1}: \mathbb{R}_+ \to \mathbb{R}_+$

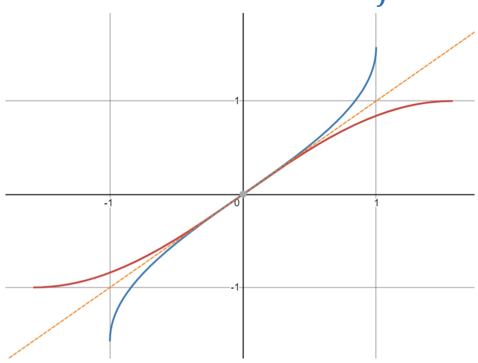
$x \mapsto x^2$ $\qquad\qquad\qquad\qquad x \mapsto \sqrt{x}$

d) $f(x) = \text{sen}(x) \Rightarrow f^{-1}(x) = \text{arcsen}(x)$

$f: [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$

$f^{-1}: [-1, 1] \to [-\pi/2, \pi/2]$

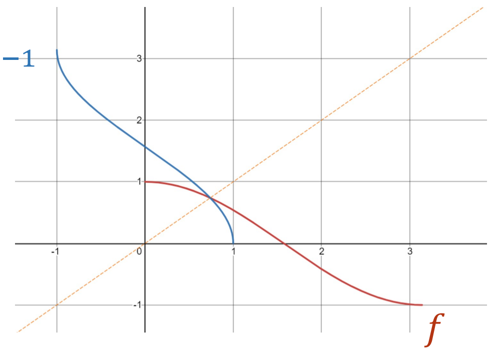




e) $f(x) = \cos(x) \Rightarrow f^{-1}(x) = \arccos(x)$

$f: [0, \pi] \to [-1, 1]$

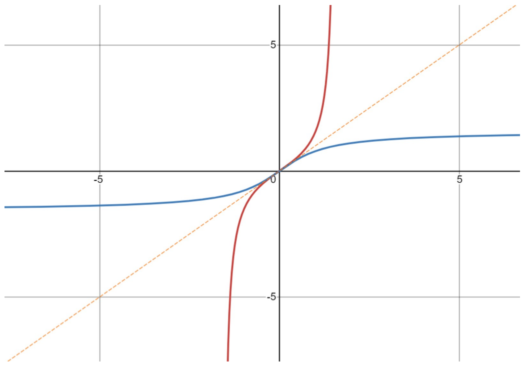
$f^{-1}: [-1, 1] \to [0, \pi]$



f) $f(x) = \text{tg}(x) \Rightarrow f^{-1}(x) = \text{arctg}(x)$

$f: [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$

$f^{-1}: \mathbb{R} \to [-\pi/2, \pi/2]$



Ob: $f^{-1} \neq \frac{1}{f}$

<u>Derivada das funções inversas</u>

Sejam $f: A \longrightarrow B$ uma função derivável e $g: B \longrightarrow A$ a inversa de f também derivável. Portanto, temos que:

$$\begin{cases} f(y) = x, \forall x \\ g(x) = y, \forall y \end{cases} \Rightarrow \begin{cases} f(g(x)) = x, \forall x \\ g(f(y)) = y, \forall y \end{cases}$$

$g = f^{-1}$

$$f(g(x)) = x \Rightarrow (f(g(x)))' = (x)' = 1$$

$$\| \quad f'(g(x))\, g'(x)$$

$$\boxed{g'(x) = \frac{1}{f'(g(x))}}, \quad f'(g(x)) \neq 0$$

$y = f(x) \quad \frac{dy}{dx} = f'(x)$

$x = g(y) \quad \frac{dx}{dy} = g'(y)$

$$\boxed{\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}}$$

<u>Exercícios</u>: Calcule as derivadas

a) $f(x) = \text{arcsen}(x) \Rightarrow \begin{cases} y = \text{arcsen}\, x \\ x = \text{sen}\, y \end{cases} \Rightarrow \begin{cases} \text{sen}(\text{arcsen}\, x) = x \\ \text{arcsen}(\text{sen}\, y) = y \end{cases}$

$D_f = [-1, 1] \quad]-1, 1[$

$$\cos(\text{arcsen}\, x) \cdot (\text{arcsen}\, x)' = 1$$

$$\boxed{(\text{arcsen}\, x)' = \frac{1}{\cos(\text{arcsen}\, x)}} \overset{(*)}{=} \frac{1}{\sqrt{1-x^2}}$$

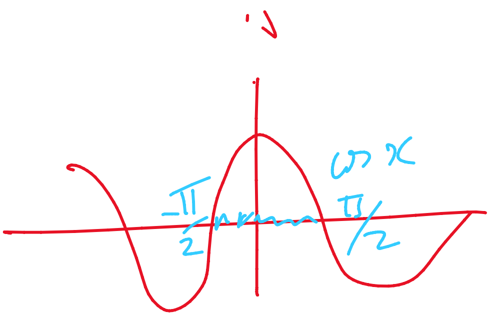
$z^2 = \cos^2(\text{arcsen}\, x) = 1 - \text{sen}^2(\text{arcsen}\, x)$
$= 1 - x^2$

$z = \pm\sqrt{1-x^2} \overset{(*)}{\Rightarrow} z = +\sqrt{1-x^2}$

$[-\pi/2, \pi/2]$

$\boxed{z = \cos(\text{arcsen}\, x)} \Rightarrow z \geq 0$

b) $f(x) = \arccos(x)$



$$(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$$

c) $f(x) = \operatorname{arctg}(x)$

$$\begin{cases} \operatorname{arctg}(\operatorname{tg} y) = y \\ \operatorname{tg}(\operatorname{arctg} x) = x \end{cases}$$

$$(\operatorname{tg}(\operatorname{arctg}(x)))' = (x)' = 1$$

$$\sec^2(\operatorname{arctg} x)(\operatorname{arctg} x)' = 1 = \frac{1}{1+x^2}$$

$$(\operatorname{arctg} x)' \underset{(*)}{=} \frac{1}{\sec^2(\operatorname{arctg} x)}$$

$$w = \sec^2(\operatorname{arctg} x) = 1 + \operatorname{tg}^2(\operatorname{arctg} x) = 1 + x^2$$

$$(*) \quad \frac{\cos^2 z}{\cos^2 z} + \frac{\sin^2 z}{\cos^2 z} = \frac{1}{\cos^2 z} \Rightarrow \boxed{1 + \operatorname{tg}^2 z = \sec^2 z}$$

$$(\operatorname{arctg} z)' = \frac{1}{1+z^2}$$

$$(\operatorname{arctg}(f(x)))' = \frac{1}{1+(f(x))^2} \cdot f'(x)$$

$$\boxed{(\operatorname{arctg} \sqrt{x})'} = \frac{1}{1+(\sqrt{x})^2} \cdot (\sqrt{x})' \qquad (\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

$x \geq 0$

$$= \frac{\frac{1}{2\sqrt{x}}}{1+x} = \frac{1}{2\sqrt{x} + x\sqrt{x}}$$