A modified ROM-based parareal method for the simulation of urban floods

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Introduction

- Numerical simulation of urban floods using the shallow water equations (SWE);

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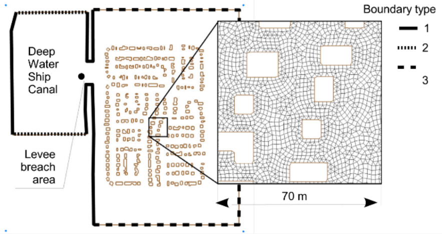
- Accurate results: **high computational cost**;

[Guinot et al. (2017)]



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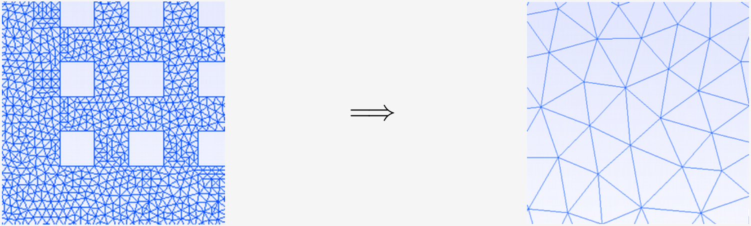

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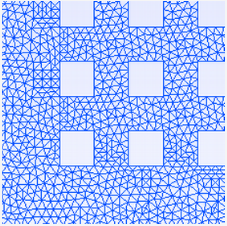
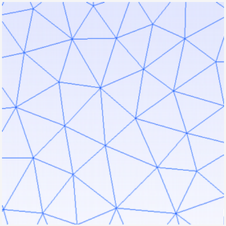
  - Porosity-based SWE [Defina (2000); Guinot and Soares-Frazão (2006), ...]
- Coarser mesh, larger time step $\Longrightarrow$ smaller computational cost (2-3 orders of magnitude);
- Good global approximations, but less accurate inside the urban zone (zones of interest)



$\Longrightarrow$



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  - Adaptations for solving hyperbolic problems: use of **reduced-order models (ROMs)**;
  - Further improvements

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The parareal method: definitions

[Lions et al. (2001)]

- A simple problem:

$$\begin{cases} \frac{d}{dt}\boldsymbol{y}(t) + A\boldsymbol{y}(t) = 0, \quad \text{in } [0, T] \\ \boldsymbol{y}(0) = \boldsymbol{y}_0 \end{cases} \tag{1}$$

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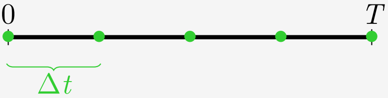
- $\mathcal{G}_{\Delta t}$: A **coarser discretization (propagator)** of (1)
  - Time step $\Delta t > \delta t$
  - Much cheaper but inaccurate



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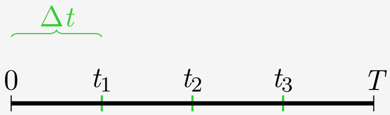

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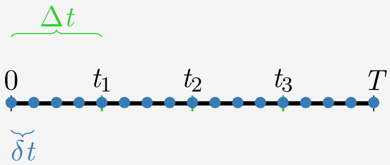

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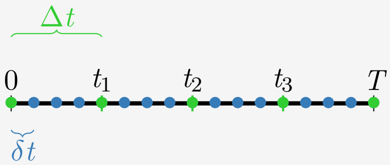
- $\boldsymbol{y}_n^k$: solution at instant t_n and iteration k.



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- Predictor-corrector iterative method;
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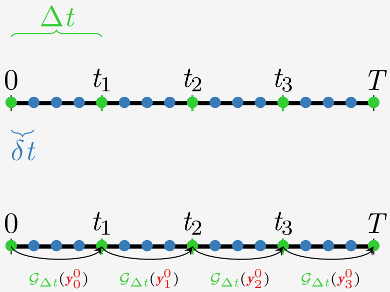
$$\boldsymbol{y}_{n+1}^0 = \mathcal{G}_{\Delta t}(\boldsymbol{y}_n^0) \qquad \text{(seq.)}$$



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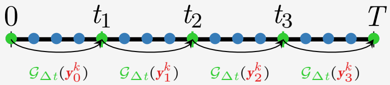

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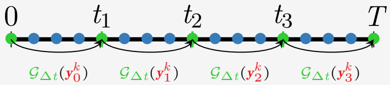

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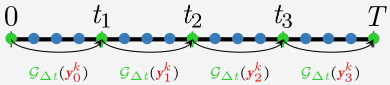

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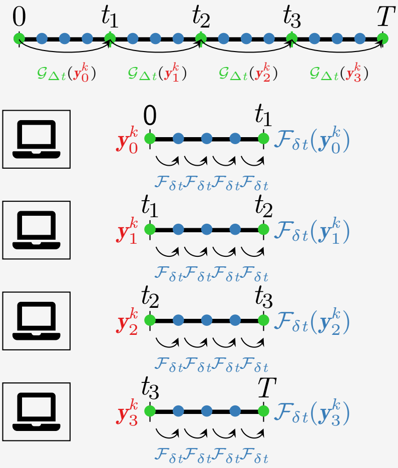
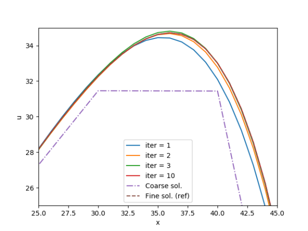
  - $\boldsymbol{y}_n^k$ available for all $n = 0, \ldots, N_{\Delta t}$
  - Coarse prediction (sequentially):
  - Fine correction (in parallel):



Parareal method: performance

- Fast convergence for many problems;
  - Parabolic, diffusive problems

$$u_t + u_{xx} = f$$



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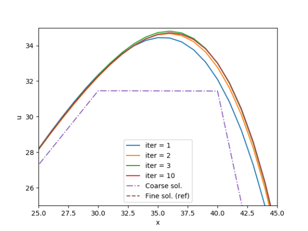

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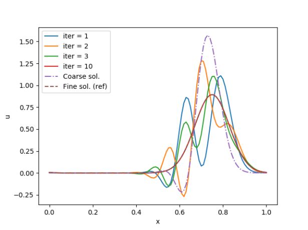

Parareal method: performance

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- Causes [Ruprecht (2018)]:
  - Mismatch of discrete phase speeds between $\mathcal{F}_{\delta t}$ and $\mathcal{G}_{\Delta t}$
  - Mainly on high wave numbers (damped in parabolic problems);

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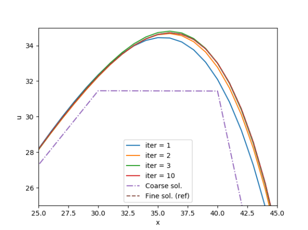

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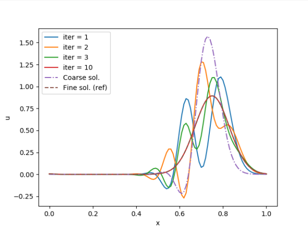

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  - **POD**: proper orthogonal decomposition (reduction of linear term)
    - Approximation in a space $\mathcal{S}_q$ with dimension $q \ll N$ (SVD);
  - **EIM/POD-DEIM:** (discrete) empirical interpolation method (reduction of nonlinear term)
    - Approximation in a space $\widehat{\mathcal{S}}_m$ with dimension $m \ll N$;
    - Interpolation from m points in space.

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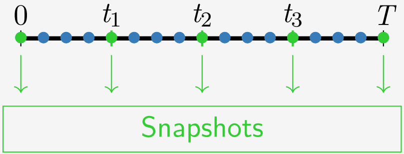
  - The ROM is reformulated **on-the-fly** at each iteration, using the solutions from all coarse time steps of all previous iterations.



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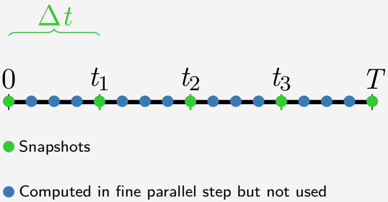

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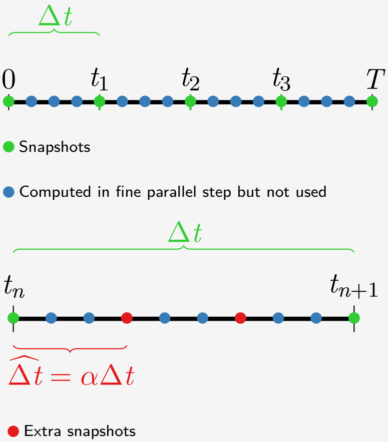

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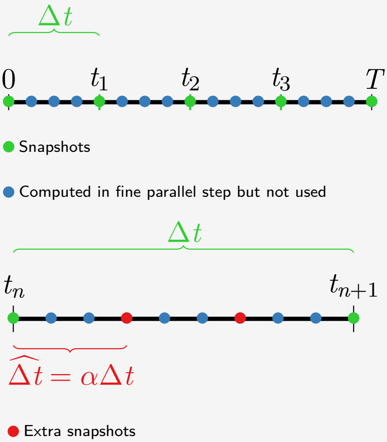

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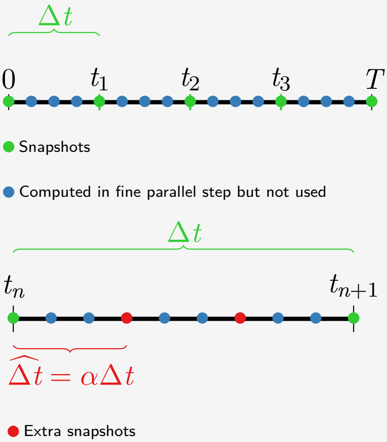

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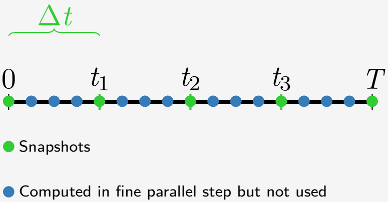
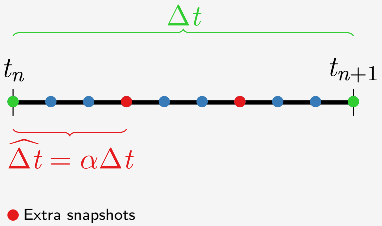
  - Keep α large (*e.g.* $\alpha = 1/2$).





Numerical tests - SWE

$$\frac{\partial}{\partial t}\boldsymbol{U}(t) + \frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{U}(t)) + \frac{\partial}{\partial y}\boldsymbol{G}(\boldsymbol{U}(t)) = \boldsymbol{S}(\boldsymbol{U}(t))$$

$$\boldsymbol{U} = \begin{pmatrix} h \\ hu_x \\ hu_y \end{pmatrix}, \ \boldsymbol{F} = \begin{pmatrix} hu_x \\ hu_x^2 + gh^2/2 \\ hu_x u_y \end{pmatrix}, \ \boldsymbol{G} = \begin{pmatrix} hu_y \\ hu_x u_y \\ hu_y^2 + gh^2/2 \end{pmatrix}, \ \boldsymbol{S} = \begin{pmatrix} 0 \\ S_{0,x} + S_{f,x} \\ S_{0,y} + S_{f,y} \end{pmatrix}$$

Numerical tests - SWE

$$\frac{\partial}{\partial t}\boldsymbol{U}(t) + \frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{U}(t)) + \frac{\partial}{\partial y}\boldsymbol{G}(\boldsymbol{U}(t)) = \boldsymbol{S}(\boldsymbol{U}(t))$$

$$\boldsymbol{U} = \begin{pmatrix} h \\ hu_x \\ hu_y \end{pmatrix}, \quad \boldsymbol{F} = \begin{pmatrix} hu_x \\ hu_x^2 + gh^2/2 \\ hu_x u_y \end{pmatrix}, \quad \boldsymbol{G} = \begin{pmatrix} hu_y \\ hu_x u_y \\ hu_y^2 + gh^2/2 \end{pmatrix}, \quad \boldsymbol{S} = \begin{pmatrix} 0 \\ S_{0,x} + S_{f,x} \\ S_{0,y} + S_{f,y} \end{pmatrix}$$

- Explicit FV scheme;

Numerical tests - SWE

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- Explicit FV scheme;
- Referential solution:

$$\begin{cases} \boldsymbol{y}_{\text{ref},0} = \boldsymbol{y}_0 \\ \boldsymbol{y}_{\text{ref},n+1} = \mathcal{F}_{\delta t}(\boldsymbol{y}_{\text{ref},n}) \qquad n = 0, \ldots, N_{\Delta t} - 1 \end{cases}$$

Numerical tests - SWE

$$\frac{\partial}{\partial t}\boldsymbol{U}(t) + \frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{U}(t)) + \frac{\partial}{\partial y}\boldsymbol{G}(\boldsymbol{U}(t)) = \boldsymbol{S}(\boldsymbol{U}(t))$$

$$\boldsymbol{U} = \begin{pmatrix} h \\ hu_x \\ hu_y \end{pmatrix}, \ \boldsymbol{F} = \begin{pmatrix} hu_x \\ hu_x^2 + gh^2/2 \\ hu_x u_y \end{pmatrix}, \ \boldsymbol{G} = \begin{pmatrix} hu_y \\ hu_x u_y \\ hu_y^2 + gh^2/2 \end{pmatrix}, \ \boldsymbol{S} = \begin{pmatrix} 0 \\ S_{0,x} + S_{f,x} \\ S_{0,y} + S_{f,y} \end{pmatrix}$$

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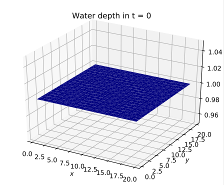
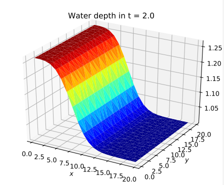
- Error per instant and iteration:

$$e_n^k := \frac{\sum_{i=1}^{3N} |[\boldsymbol{y}_n^k]_i - [\boldsymbol{y}_{\text{ref, n}}]_i|}{\sum_{i=1}^{3N} |[\boldsymbol{y}_{\text{ref},n}]_i|}$$

First test case (pseudo-2D)



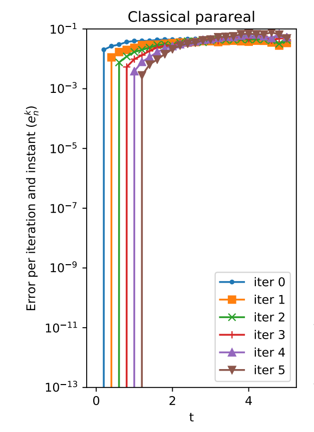


- $\Omega = [0, 20]^2$;
- Initial solution: lake-in-rest, $h(t = 0) \equiv 1$, flat bottom;
- Boundary conditions: inward unitary mass flux at $x = 0$.
- Propagators:

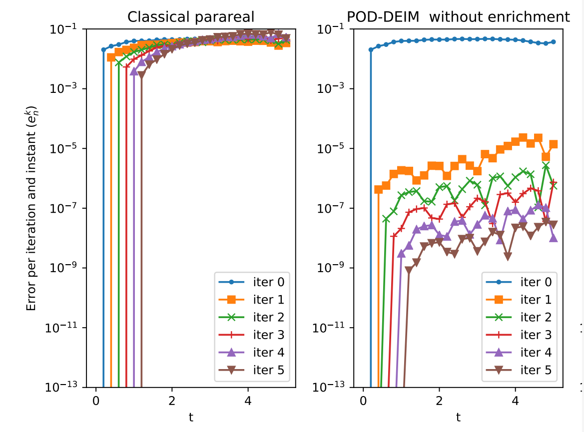
$\mathcal{F}_{\delta t}$	SWE	$\delta t = 0.001$	$\delta x = 1$	$\delta y = 1$
$\mathcal{G}_{\Delta t}$	SWE	$\Delta t = 0.2$	$\Delta x = 1$	$\Delta y = 1$

- $T = 5$, $N_{\Delta t} = 25$, $P = 20$ processors;
- Enriched snapshots: $\widehat{\Delta t} = 0.1$ $(\alpha = 1/2)$;

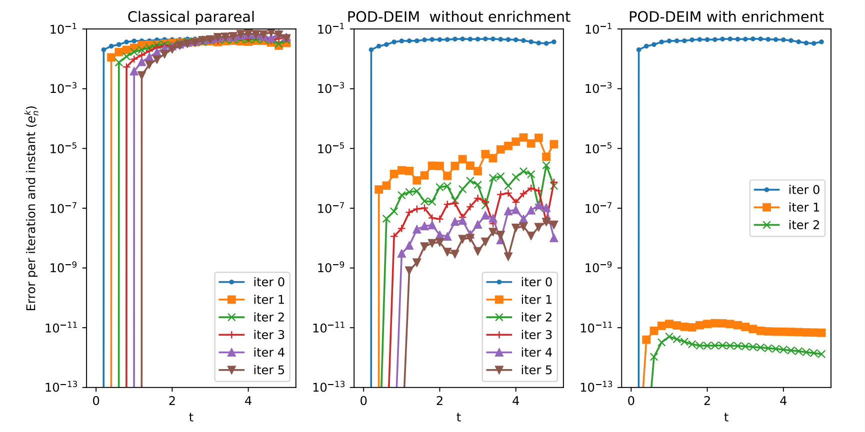
First test case (pseudo-2D): errors



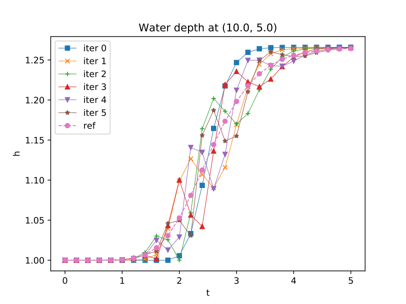
First test case (pseudo-2D): errors

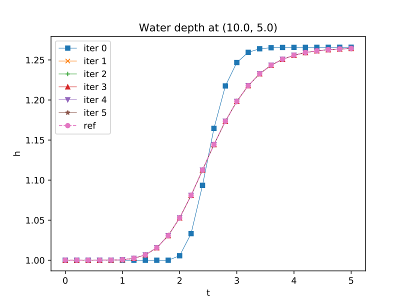


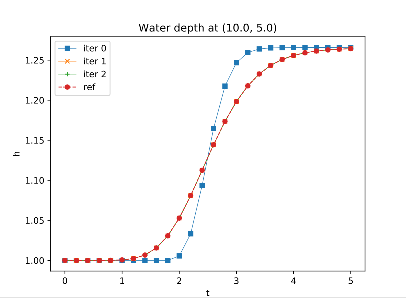
First test case (pseudo-2D): errors



First test case (pseudo-2D): h at $(x, y) = (10, 5)$

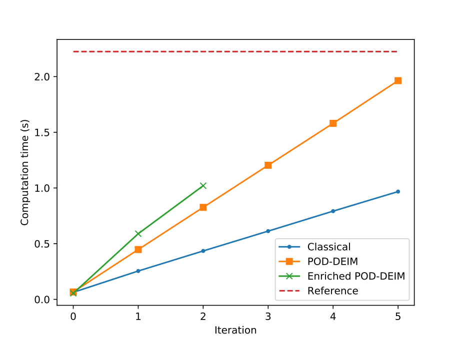


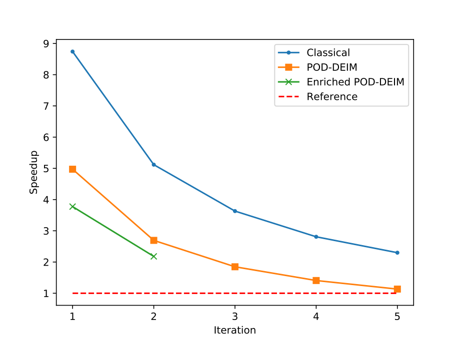




First test case (pseudo-2D): speedup

$$\text{speedup} = \frac{\tau_{\text{ref}}}{\tau_{\text{parareal}}}$$



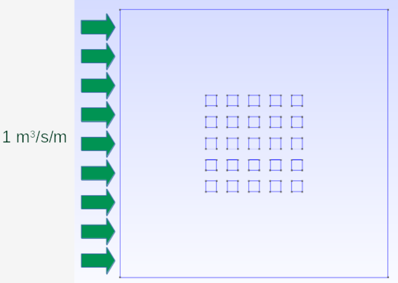


Second test case: a fictional flood simulation

- $\Omega = [0, 100]^2$;
- Initial solution:

$$\begin{cases} h(t=0, x, y) = 0.1 \\ u_x(t=0, x, y) = 0 \qquad (x, y) \in \Omega \\ u_y(t=0, x, y) = 0 \end{cases}$$

- Inward unitary flux on the western boundary.

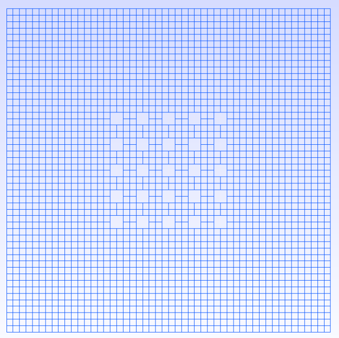


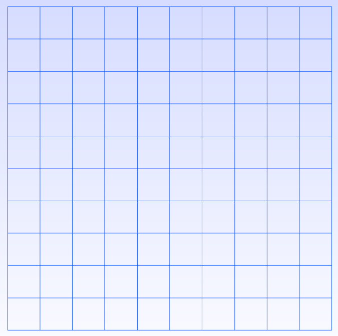
Second test case: a fictional flood simulation

- Propagators:

$\mathcal{F}_{\delta t}$	SWE	$\delta t = 0.001$	$\delta x = 2$	$\delta y = 2$
$\mathcal{G}_{\Delta t}$	por-SWE	$\Delta t = 1$	$\Delta x = 10$	$\Delta y = 10$

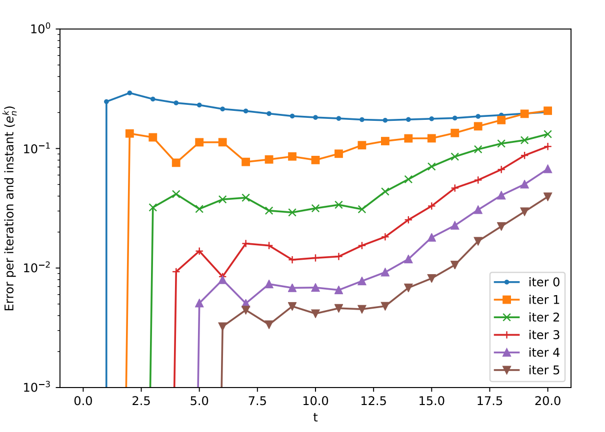
- $T = 20,\ N_{\Delta t} = 20,\ P = 20$ processors
- ROM-based parareal with 4 extra snapshots per Δt ($\alpha = 1/5$);



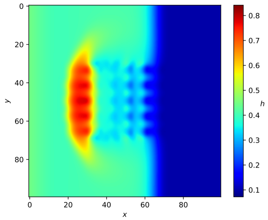


Second test case: a fictional flood simulation

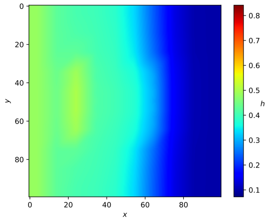
Error per instant and iteration



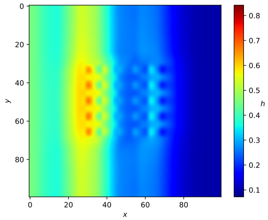
Test case: water depth at $t = 20$



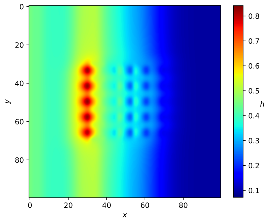
(a) Ref. (SWE)



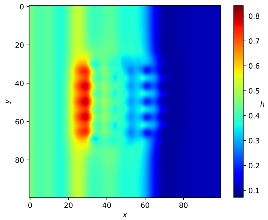
(b) Iter. 0 (por-SWE)



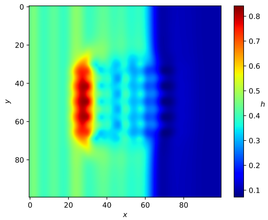
(c) Iteration 1



(d) Iteration 2

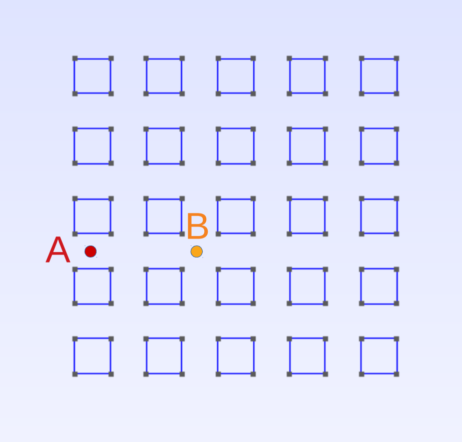


(e) Iteration 3

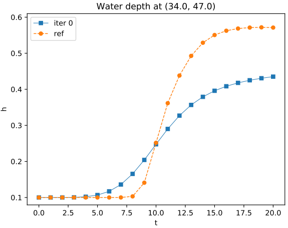


(f) Iteration 4

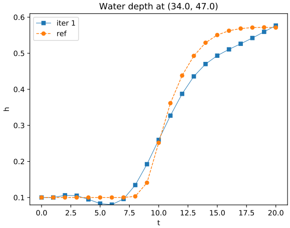
Test case: probes locations



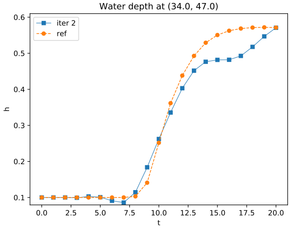
Second test case: water depth at $A(34, 47)$



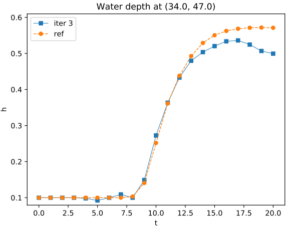
(a) Iteration 0



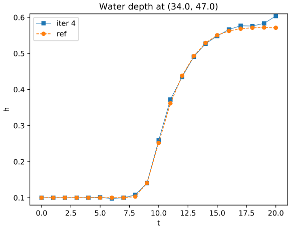
(b) Iteration 1



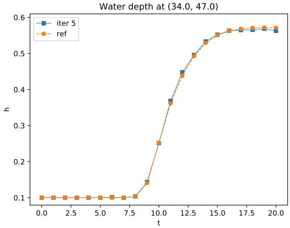
(c) Iteration 2



(d) Iteration 3

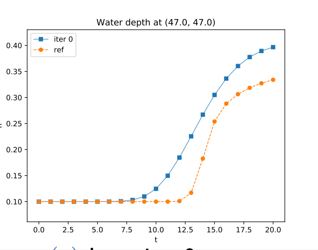


(e) Iteration 4

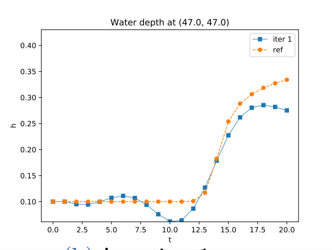


(f) Iteration 5

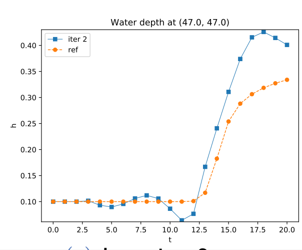
Test case: water depth at $B(47, 47)$



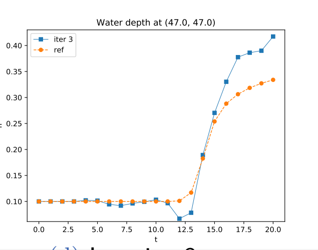
(a) Iteration 0



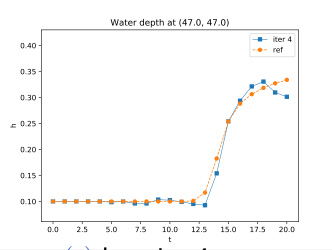
(b) Iteration 1



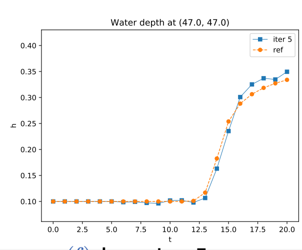
(c) Iteration 2



(d) Iteration 3



(e) Iteration 4

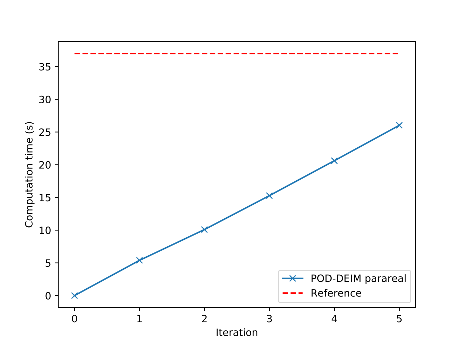


(f) Iteration 5

Test case: speedup

$$\text{speedup} = \frac{\tau_{\text{ref}}}{\tau_{\text{parareal}}}$$

- Coarse model (iteration 0) $\approx$ 0.11 s (speedup ≈ 3500)



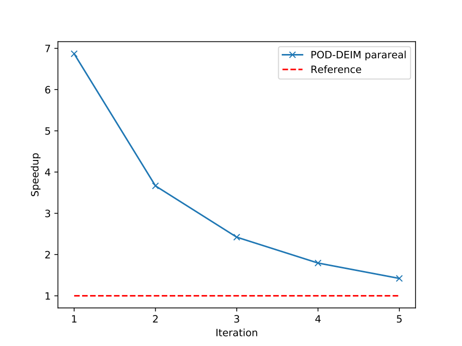


Table of Contents

Adaptative parareal

[Maday and Mula (2020)]

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  - Ideally: same precision is attained.

Adaptative ROM-based parareal

- Define $\widehat{\mathcal{F}}^1_{\delta t}, \widehat{\mathcal{F}}^2_{\delta t}, \ldots, \widehat{\mathcal{F}}^{\overline{n}}_{\delta t} = \mathcal{F}_{\delta t}$ with increasing accuracy.

Adaptative ROM-based parareal

- Define $\widehat{\mathcal{F}}^1_{\delta t}, \widehat{\mathcal{F}}^2_{\delta t}, \ldots, \widehat{\mathcal{F}}^{\overline{n}}_{\delta t} = \mathcal{F}_{\delta t}$ with increasing accuracy.
- ROM-based parareal iteration:

$$\boldsymbol{y}^{k+1}_{n+1} = \mathcal{F}^{k}_{\delta t,r}(\boldsymbol{P}^k \boldsymbol{y}^{k+1}_n) + \mathcal{F}_{\delta t}(\boldsymbol{y}^k_n) - \mathcal{F}^{k}_{\delta t,r}(\boldsymbol{P}^k \boldsymbol{y}^k_n)$$

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- Reduced cost of the fine propagator;

Adaptative ROM-based parareal

- Define $\widehat{\mathcal{F}}_{\delta t}^{1}, \widehat{\mathcal{F}}_{\delta t}^{2}, \ldots, \widehat{\mathcal{F}}_{\delta t}^{\overline{n}} = \mathcal{F}_{\delta t}$ with increasing accuracy.
- ROM-based parareal iteration:

$$\boldsymbol{y}_{n+1}^{k+1} = \widehat{\mathcal{F}}_{\delta t,r}^{k}(\boldsymbol{P}^{k}\boldsymbol{y}_{n}^{k+1}) + \widehat{\mathcal{F}}_{\delta t}^{k}(\boldsymbol{y}_{n}^{k}) - \widehat{\mathcal{F}}_{\delta t,r}^{k}(\boldsymbol{P}^{k}\boldsymbol{y}_{n}^{k})$$

- Reduced cost of the fine propagator;
- Reduced cost for formulating the ROMs;

Adaptative ROM-based parareal

- Define $\widehat{\mathcal{F}}_{\delta t}^{1}, \widehat{\mathcal{F}}_{\delta t}^{2}, \ldots, \widehat{\mathcal{F}}_{\delta t}^{\overline{n}} = \mathcal{F}_{\delta t}$ with increasing accuracy.
- ROM-based parareal iteration:

$$\boldsymbol{y}_{n+1}^{k+1} = \widehat{\mathcal{F}}_{\delta t,r}^{k}(\boldsymbol{P}^{k}\boldsymbol{y}_{n}^{k+1}) + \widehat{\mathcal{F}}_{\delta t}^{k}(\boldsymbol{y}_{n}^{k}) - \widehat{\mathcal{F}}_{\delta t,r}^{k}(\boldsymbol{P}^{k}\boldsymbol{y}_{n}^{k})$$

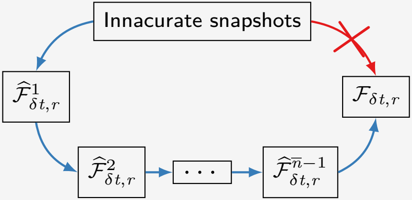
- Reduced cost of the fine propagator;
- Reduced cost for formulating the ROMs;
- Reduced cost for solving the ROMs;

Adaptative ROM-based parareal

- Define $\widehat{\mathcal{F}}^1_{\delta t}, \widehat{\mathcal{F}}^2_{\delta t}, \dots, \widehat{\mathcal{F}}^{\overline{n}}_{\delta t} = \mathcal{F}_{\delta t}$ with increasing accuracy.
- ROM-based parareal iteration:

$$\boldsymbol{y}^{k+1}_{n+1} = \widehat{\mathcal{F}}^k_{\delta t,r}(\boldsymbol{P}^k \boldsymbol{y}^{k+1}_n) + \widehat{\mathcal{F}}^k_{\delta t}(\boldsymbol{y}^k_n) - \widehat{\mathcal{F}}^k_{\delta t,r}(\boldsymbol{P}^k \boldsymbol{y}^k_n)$$

- Reduced cost of the fine propagator;
- Reduced cost for formulating the ROMs;
- Reduced cost for solving the ROMs;
- More stable ROMs.



Second test case with the adaptative ROM-parareal

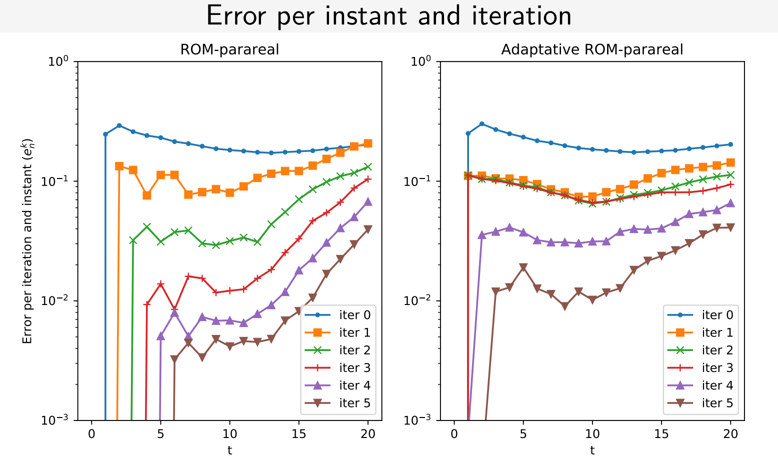
- **Coarse model (iteration 0):** $\Delta t = 1$; $\Delta x = \Delta y = 10$

Second test case with the adaptative ROM-parareal

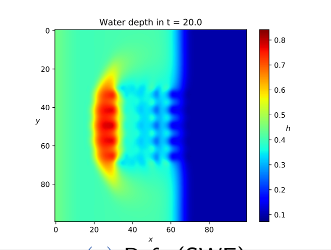
- **Coarse model (iteration 0):** $\Delta t = 1$; $\Delta x = \Delta y = 10$
- **Fine models:**

Iteration	**Fine model**	δt	$\delta x = \delta y$
1	$\widehat{\mathcal{F}}^1_{\delta t}$	0.05	4
2	$\widehat{\mathcal{F}}^2_{\delta t}$	0.01	4
3	$\widehat{\mathcal{F}}^3_{\delta t}$	0.005	4
4	$\widehat{\mathcal{F}}^4_{\delta t}$	0.005	2
5	$\widehat{\mathcal{F}}^5_{\delta t} = \mathcal{F}_{\delta t}$	0.001	2

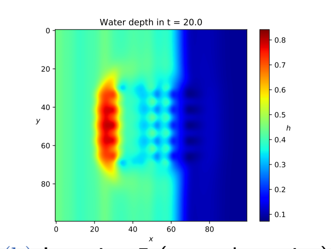
Adaptative test case: errors



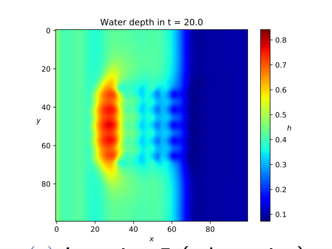
Adaptative test case: water depth at $t = 20$



(a) Ref. (SWE)

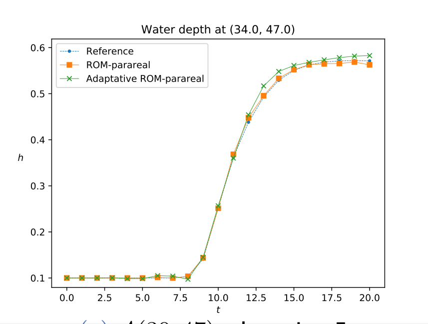


(b) Iteration 5 (non-adaptative)

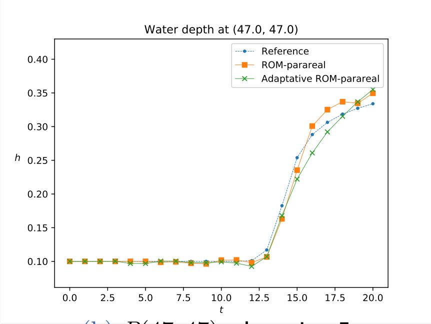


(c) Iteration 5 (adaptative)

Adaptative test case: water depth at probes

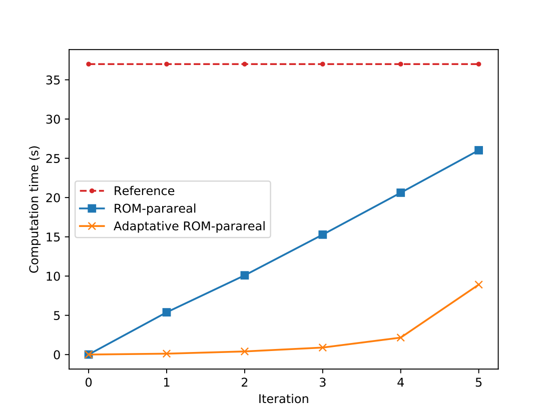


(a) $A(30, 47)$ - Iteration 5



(b) $B(47, 47)$ - Iteration 5

Test case: speedup



- **Speedup (iteration 5):**
  - Non-adaptative: 1.4
  - Adaptative: 4.2

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Conclusions

- ROM-based parareal methods are efficient for speeding up nonlinear hyperbolic problems;
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  - Quadratic dependency of the POD on the number of snapshots
- Adaptative approach
  - Progressive refinement of $\mathcal{F}_{\delta t}$;
  - Improved speedup and stability;

Perspectives

- Adaptative approach: study configurations, propose objective criteria for defining the fine models;
- Application to more realistic test cases:
  - Long simulations;
  - Solutions with strong variation;
  - Interpolation between spatial meshes

Thank you for your attention

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