Towards Exponential Semi-Lagrangian Parallel-in-Time Methods for the Shallow Water Equations on the Rotating Sphere

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Full Outline

- Introduction
- Coarse time stepping methods
- MGRIT
- Numerical tests

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PinT in atmospheric modelling

- Challenges:
  - Hyperbolic nature, small-scale spatial features;
  - Improve stability and convergence
- Choice of coarse time stepping method?

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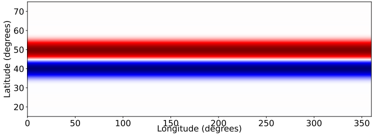

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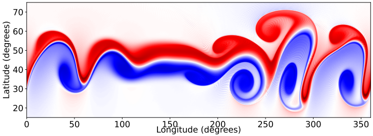

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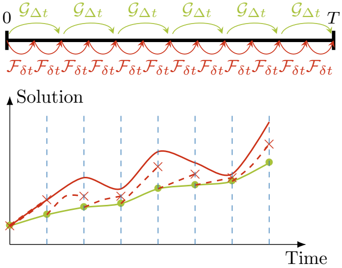

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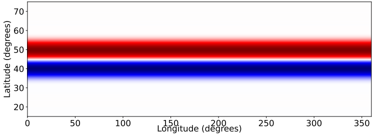

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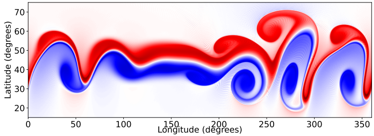

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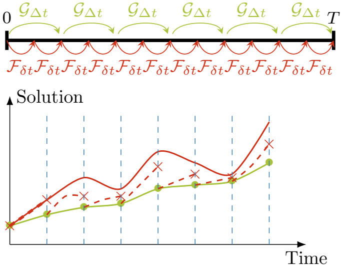

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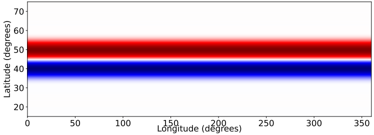

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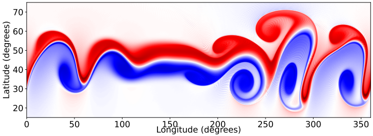

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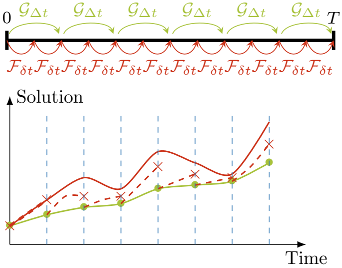

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Some possibilites of coarse time stepping method

- Implicit-explicit (IMEX)

$$\boldsymbol{U}^{n+1} = \boldsymbol{F}_I^{\Delta t/2}\left(\boldsymbol{F}_E^{\Delta t}\left(\boldsymbol{F}_I^{\Delta t/2}\left(\boldsymbol{U}^n\right)\right)\right)$$

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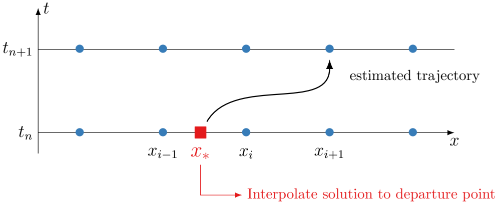
- Implicit-explicit (IMEX)
- Semi-Lagrangian methods
  - Large time steps
  - Operational application (*e.g.* SL-SI-SETTLS in IFS-ECMWF[Hortal (2002)])
  - Application in PinT [Schmitt et al. (2018), De Sterck et al. (2021)]



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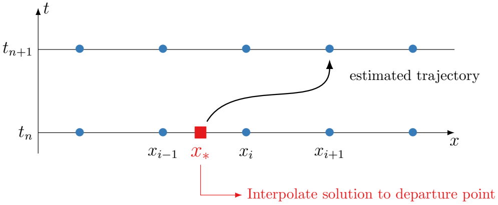

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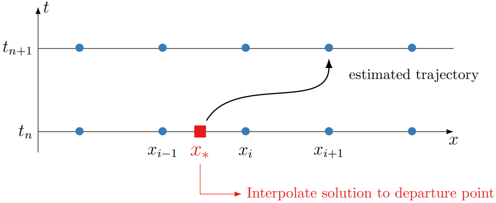

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  - Exact integration of linear terms
  - *e.g.* ETD2RK [Cox and Matthews (2002)]

$$\frac{\partial \boldsymbol{U}}{\partial t} = \boldsymbol{L}\boldsymbol{U} + \boldsymbol{N}(\boldsymbol{U}) \implies \boldsymbol{U}^{n+1} = \underbrace{e^{\Delta t \boldsymbol{L}}\boldsymbol{U}^n}_{\text{exact}} + \underbrace{e^{\Delta t \boldsymbol{L}} \int_{t_n}^{t_{n+1}} e^{-(s-t_n)\boldsymbol{L}} \boldsymbol{N}(\boldsymbol{U}(s))ds}_{\text{approx.}}$$

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- **Semi-Lagrangian exponential integration methods** [Peixoto and Schreiber (2019)]

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S. Friedhoff et al., "A Multigrid-in-Time Algorithm for Solving Evolution Equations in Parallel", *Presented at: Sixteenth Copper Mountain Conference on Multigrid Methods* (2013)

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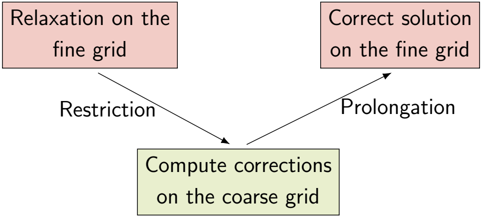

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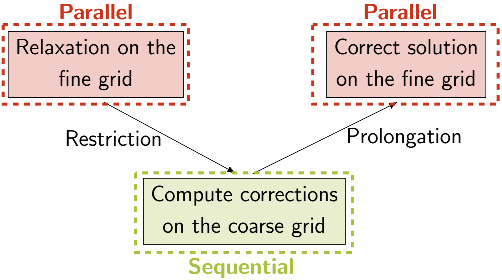

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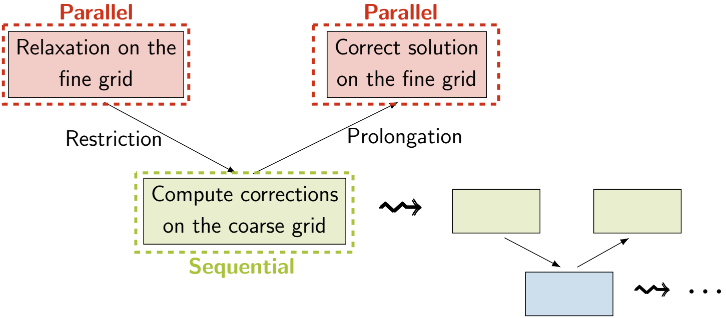

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  - Number of levels;
  - Coarsening factor between levels: $\Delta t_{l+1} = m\Delta t_l$
  - Relaxation strategy: F, FCF, FCFCF, ..., F(CF)$^{N_{\text{relax}}}$

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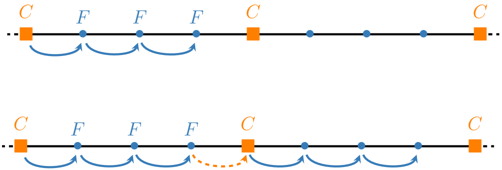

Stability analysis of MGRIT

- Extension of stability analysis of Parareal [Staff and Rønquist (2005)]
- Linearized ODE: $\frac{\partial u}{\partial t} = \lambda_{\boldsymbol{L}} u + \boldsymbol{N}(u) \implies \frac{\partial u}{\partial t} = \lambda_{\boldsymbol{L}} u + \lambda_{\boldsymbol{N}} u$
- Stability in function of $\xi_{\boldsymbol{N}} := \lambda_{\boldsymbol{N}} \Delta t \in \mathbb{C}$ for fixed $\xi_{\boldsymbol{L}} = \lambda_{\boldsymbol{L}} \Delta t \in i\mathbb{R}^*$:

$$|A_{\text{scheme}}(Re(\xi_{\boldsymbol{N}}), Im(\xi_{\boldsymbol{N}}))| \leq 1, \qquad u^{n+1} = A_{\text{scheme}} u^n$$

- Stability function of MGRIT (two levels with $\text{F(CF)}^{N_{\text{relax}}}$):

$$u_n^k = \underbrace{\left(\sum_{i=0}^{\lfloor k/(N_{\text{relax}}+1) \rfloor} \binom{n - iN_{\text{relax}}}{i} \boldsymbol{A_f}^{mN_{\text{relax}}} \left(\boldsymbol{A_f}^m - \boldsymbol{A_c}\right)^i \boldsymbol{A_c}^{n-i(N_{\text{relax}}+1)} \right)}_{A_{\text{MGRIT}}} u_0$$

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  - IMEX
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- Stability in function of the coarse time stepping method and N_{relax}
  - Fine scheme: IMEX
  - Coarse scheme: IMEX , SL-SI-SETTLS , ETD2RK , SL-ETD2RK
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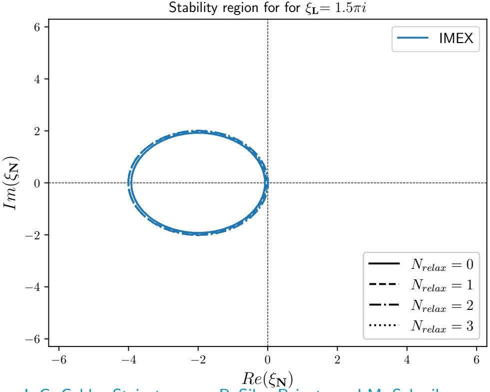

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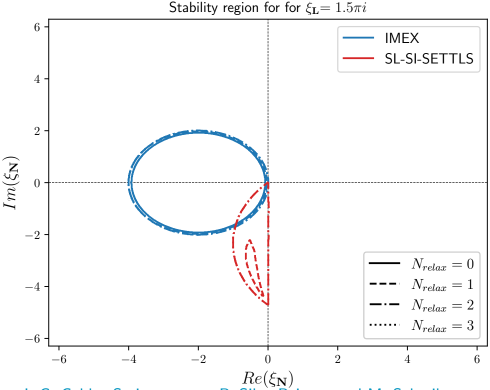

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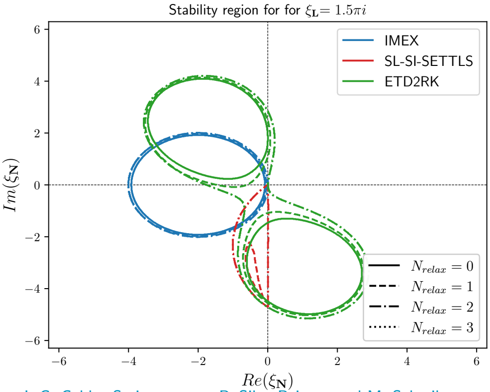

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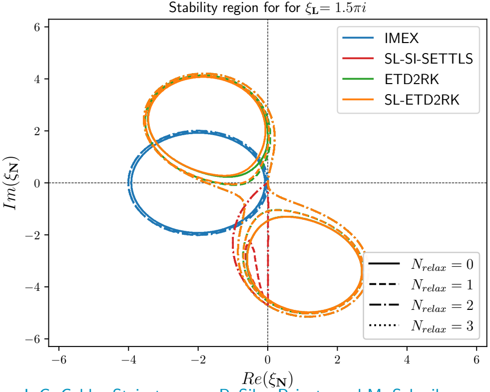

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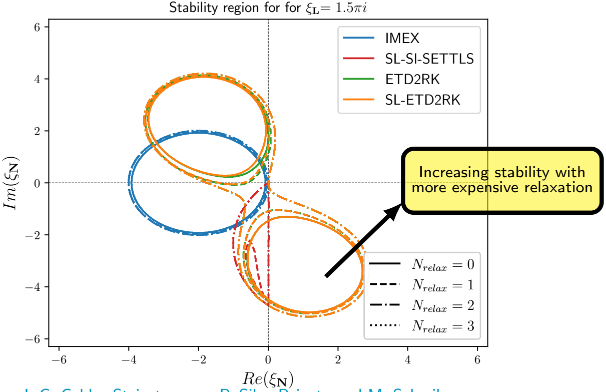

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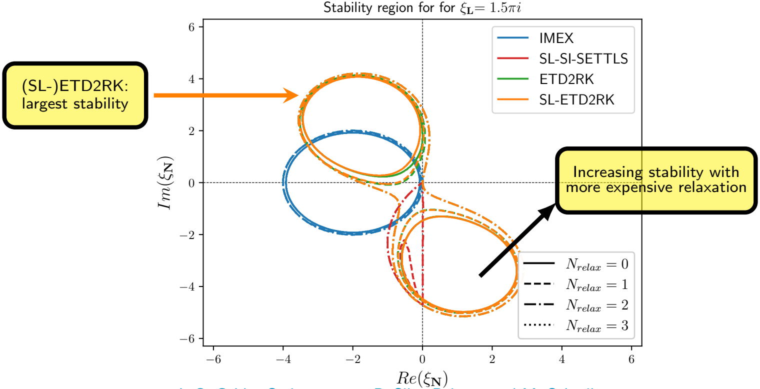

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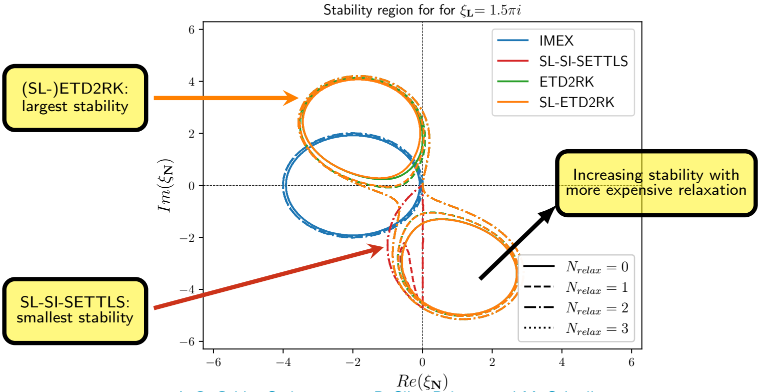

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Test case

- Shallow water equations on the rotating sphere
- Gaussian bump perturbation in geopotential field Φ
- T $= 102400$
- Reference solution: $\delta t = 60$
- Coarse levels: coarsening factor m
- Artificial diffusion
- Reference solution (geopotential field Φ):

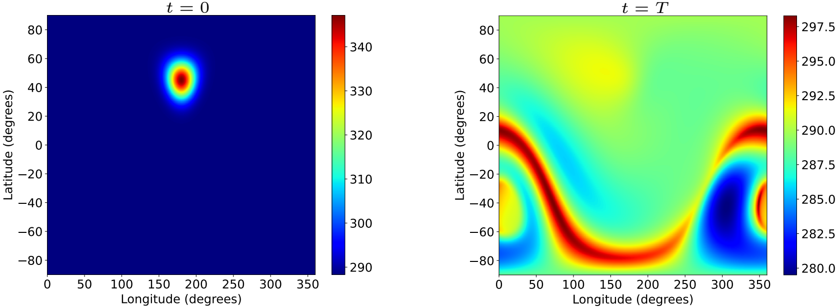
$$\frac{\partial}{\partial t}\boldsymbol{U} = \boldsymbol{L_G U} + \boldsymbol{L_C U} + \boldsymbol{N_A}(\boldsymbol{U}) + \boldsymbol{N_R}(\boldsymbol{U})$$

$$\boldsymbol{U} = \begin{pmatrix} \Phi \\ \xi \\ \delta \end{pmatrix}, \qquad \boldsymbol{L_G}(\boldsymbol{U}) = \begin{pmatrix} \overline{\Phi}\delta \\ 0 \\ -\nabla^2\Phi \end{pmatrix}, \qquad \boldsymbol{L_C}(\boldsymbol{U}) = \begin{pmatrix} 0 \\ -\nabla\cdot(f\boldsymbol{V}) \\ -\boldsymbol{k}\cdot\nabla\times(f\boldsymbol{V}) \end{pmatrix}$$

$$\boldsymbol{N_A}(\boldsymbol{U}) = \begin{pmatrix} -\boldsymbol{V}\cdot\nabla\Phi \\ -\nabla\cdot(\xi\boldsymbol{V}) \\ \nabla^2\left(\frac{\boldsymbol{V}\cdot\boldsymbol{V}}{2}\right) + \boldsymbol{k}\cdot\nabla\times(\xi\boldsymbol{V}) \end{pmatrix}, \qquad \boldsymbol{N_R}(\boldsymbol{U}) = \begin{pmatrix} -\Phi'\delta \\ 0 \\ 0 \end{pmatrix}$$

Test case

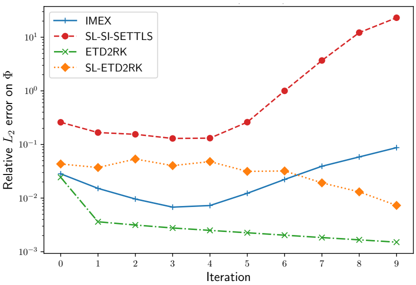
- Shallow water equations on the rotating sphere
- Gaussian bump perturbation in geopotential field Φ
- T $= 102400$
- Reference solution: $\delta t = 60$
- Coarse levels: coarsening factor m
- Artificial diffusion
- Reference solution (geopotential field Φ):



Test case

Errors along iterations in function of the coarse time stepping method:

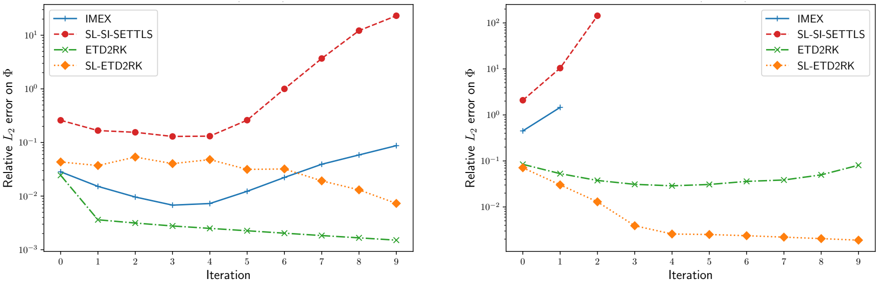
$$N_{\text{levels}} = 2,\ m = 2,\ N_{\text{relax}} = 4$$



Test case

Errors along iterations in function of the coarse time stepping method:

$N_{\text{levels}} = 2,\ m = 2,\ N_{\text{relax}} = 4$ $\qquad$ $N_{\text{levels}} = 3,\ m = 4,\ N_{\text{relax}} = 1$

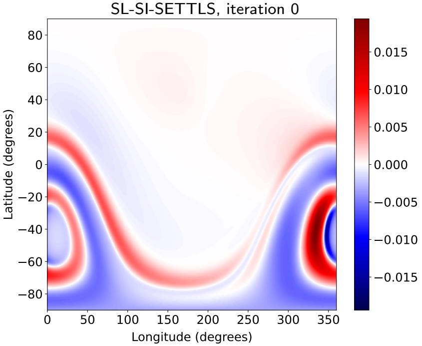


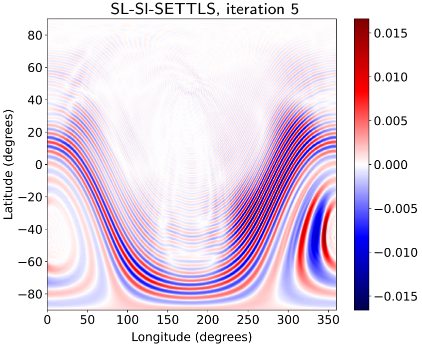
- More levels $\implies$ larger timestep in coarsest level $\implies$ stability issues
- Faster convergence and more stability with exponential methods

Test case

Relative error of geopotential Φ at $t = T$:

- Coarse method: SL-SI-SETTLS

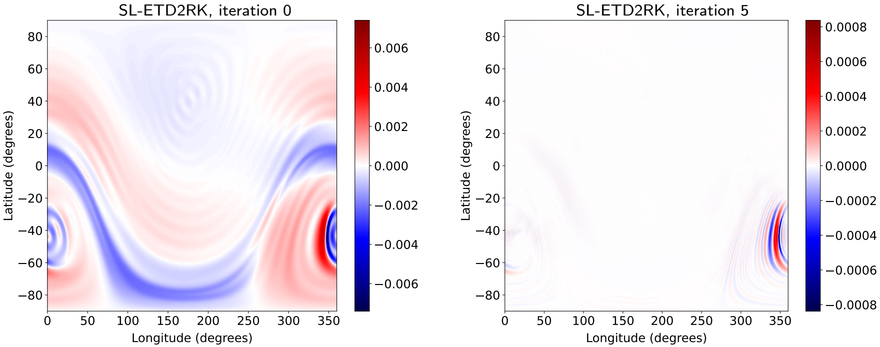




Test case

Relative error of geopotential Φ at $t = T$:

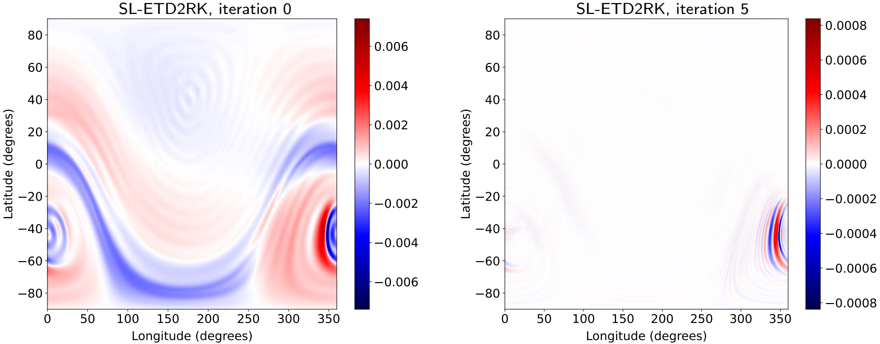
- Coarse method: SL-ETD2RK



Test case

Relative error of geopotential Φ at $t = T$:

- Coarse method: SL-ETD2RK



- Errors on small-scale spatial features

Conclusions and perspectives

- PinT for the SWE on the rotating sphere, choice of coarse time stepping method
- Poor stability properties using SL-SI-SETTLS in MGRIT
- Improved stability and convergence with ETD2RK and SL-ETD2RK
- Small-scale spatial features are still a bottleneck
- Parallel implementation and study of numerical speedup
- Evaluation of influence of artificial diffusion

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Thank you!

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FAPESP SÃO PAULO RESEARCH FOUNDATION

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Outline

- Introduction
- Coarse time stepping methods
- MGRIT
- Numerical tests

Cox, S.M. and P.C. Matthews (2002). "Exponential Time Differencing for Stiff Systems". In: *Journal of Computational Physics* 176.2, pp. 430–455. ISSN: 0021-9991. DOI: https://doi.org/10.1006/jcph.2002.6995. URL: https://www.sciencedirect.com/science/article/pii/S0021999102969950.

De Sterck, Hans et al. (2021). "Optimizing multigrid reduction-in-time and Parareal coarse-grid operators for linear advection". In: *Numerical Linear Algebra with Applications*. ISSN: 1099-1506. DOI: 10.1002/nla.2367. URL: http://dx.doi.org/10.1002/nla.2367.

Friedhoff, S. et al. (2013). "A Multigrid-in-Time Algorithm for Solving Evolution Equations in Parallel". In: *Presented at: Sixteenth Copper Mountain Conference on Multigrid Methods*. Copper Mountain, CO, United States, Mar 17 - Mar 22, 2013. URL: http://www.osti.gov/scitech/servlets/purl/1073108.

Galewsky, Joseph, Richard K. Scott, and Lorenzo M. Polvani (Jan. 2004). "An initial-value problem for testing numerical models of the global shallow-water equations". In: *Tellus A: Dynamic Meteorology and Oceanography* 56.5, pp. 429–440. DOI: 10.3402/tellusa.v56i5.14436. URL: https://doi.org/10.3402/tellusa.v56i5.14436.

Hortal, Mariano (July 2002). "The development and testing of a new two-time-level semi-Lagrangian scheme (SETTLS) in the ECMWF forecast model". In: *Quarterly Journal of the Royal Meteorological Society* 128.583, pp. 1671–1687. DOI: 10.1002/qj.200212858314. URL: https://doi.org/10.1002/qj.200212858314.

Peixoto, Pedro S. and Martin Schreiber (2019). "Semi-Lagrangian Exponential Integration with Application to the Rotating Shallow Water Equations". In: *SIAM Journal on Scientific Computing* 41.5, B903–B928. DOI: 10.1137/18M1206497. eprint: https://doi.org/10.1137/18M1206497. URL: https://doi.org/10.1137/18M1206497.

Schmitt, A. et al. (2018). "A numerical study of a semi-Lagrangian Parareal method applied to the viscous Burgers equation". In: *Computing and Visualization in Science* 19, pp. 45–57.

Staff, Gunnar Andreas and Einar M. Rønquist (2005). "Stability of the Parareal Algorithm". In: *Lecture Notes in Computational Science and Engineering*. Springer-Verlag, pp. 449–456. DOI: 10.1007/3-540-26825-1_46. URL: https://doi.org/10.1007/3-540-26825-1_46.