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# Knowledge, Uncertainty and Ignorance in Logic: Bilattices and beyond

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ABSTRACT. *In the paper we present a survey of some approaches to the semantics of many-valued propositional systems. These approaches are inspired on one hand by classical problems in the investigations of logical aspects of epistemic activity: knowledge and truth, contradictions, beliefs, reliability of data, etc. On the other hand they reflect contemporary concerns of researchers in Artificial Intelligence (and Cognitive Science in general) with inferences drawn from imperfect information, even from total ignorance. We treat the mathematical apparatus that has emerged recently: algebraic structures related to the new logical systems in the same way Boolean algebras correspond to classical logic.*

*Keywords*: *bilattices, info-algebras, logical practices, knowledge, many-valued logics, set expansions, truth values, uncertainty.*

## Introduction

Judging from the vast numbers of papers published, it could be said that the interest in the study of reasoning has never been keener. Although there is a millennia old tradition in this field, only recently logicians have been joined by cognitive scientists, specialists in artificial intelligence and information transfer, knowledge engineers, linguists, etc. in the pursuit of the ultimate goal: to find out how humans reason in order to distill some universal principles of effective reasoning and thus be able to design intelligent artefacts (that possess or at least simulate reasoning capabilities apparently characteristic of humans).

* Editorial note. This is the last paper of George Gargov. It has been found by his wife in Gargov's computer. Probably this is an initial step for a book on bilattices, which George planned to write conjointly with Prof. Melvin Fitting.

Let us begin our very short survey of the basic intuitions behind truth values, truth-table methods and semantical inference with an outline of the classical logical doctrine concerning reasoning. According to it logic deals with *correct* reasoning, this notion being explicated as referring to transformations of statements which, if applied to true ones, lead to true statements, hence the importance of *truth values.* The basic thesis of classical formal logic concerning truth values seems to be that *in every epistemic situation a well-formed statement A is always assumed either true or false,* but not both, although sometimes the exact truth value is (temporarily!) unknown. Moreover, the truth value of a compound statement is recoverable from its syntactic structure and the truth values of the components (although this might lead to enquiries about other epistemic situations).

Thus (1) the problem of how the truth values are obtained is radically separated from the ontological problem of their existence; (2) the definiteness of truth values regardless of any difficulties in their actual establishing is assumed; and (3) in a sense perfect information about every conceivable (even remote) situation is postulated, independent from the state of the *observer* (the intelligent agent).

For the formal implementation of the above doctrine one associates with an epistemic situation a *truth assignment* (a semantical evaluation function) v which assigns each statement A a *definite* truth value from the set {**true, false**}(v: Fml→{**true,false**}). For typographical reasons we use below 1 instead of **true** and 0 instead of **false**. Assuming the usual interpretation of the classical connectives, i.e., assuming that all connectives are *truth-funcional,* this set (the smallest possible logical matrix) is the Boolean algebra $\mathbf{2} = \langle\{0,1\},\wedge,\vee,\sim,0,1\rangle$. Thus classical *semantics* is represented by some set H of homomorphisms into **2**. The definition of semantic consequence relation: $\Gamma \models A$ (where $\Gamma$ is a set of statements, A – a statement) if $\forall v \in H\ (\forall B \in \Gamma (v(B) = 1) \Rightarrow v(A)=1)$, captures the basic intuition about sound inference: that it should transmit the truth forward, i.e. if all hypotheses of an inference are true (in a situation) then the conclusion should also be true (in the same situation).

For reasoning involving intensional connectives (not truth-functional in **2**) like modalities, tense operators, etc., a more sophisticated version is needed:

**Example 1** Here we allow many epistemic situations, or *possible worlds,* with several accessibility relations between them (but keeping them all binary): thus we can accommodate most of the unary intensional connectives (modal, temporal, deontic, etc.) and some of the binary ones such as conditionals, data connectives, etc. In this approach a *frame* F is a tuple $\langle W,\{R_i\}_{i\in I}\rangle$, of which

(1) W is a non-empty set of *possible worlds*;
(2) $R_i$ are binary relations in W, i.e., $R_i \subseteq W\times W$.

A *model* M (on a frame F) is a pair $\langle F,\phi\rangle$ where $\phi$ is a truth assignment (*valuation function*), i.e.

$$\phi: W\times Var(\mathcal{L}) \to \mathbf{2}.$$

In a model M the function $\phi$ can be extended to a mapping $\phi_M$: $W \times \mathcal{L} \rightarrow \mathbf{2}$ by the well-known truth conditions for different connectives, for example (writing M,w |= A instead of $\phi_M(w,A) = 1$)

$$M,w \models A \wedge B \quad \text{iff} \quad M,w \models A \text{ and } M,w \models B, \text{ or}$$

$$M,w \models \Box_i A \quad \text{iff} \quad \forall w'(wR_i w' \Rightarrow M,w' \models A), \text{ etc.}$$

Writing $\|A\|_M$ for $\{w: M,w \models A\}$ we get a mapping of $\mathcal{L}$ into the intensional algebra **A**(F) of the frame F (i.e. the algebra of all subsets of W $<\wp(W),\cap,\cup,\ldots,\{\Box_i\}_{i\in I}>$, where the intensional (e.g. modal, temporal, etc.) operations are defined as, e.g., $\Box_i Z = \{w: \forall w'(wR_i w' \Rightarrow w' \in Z)\}$). $\|.\|_M$ is a homomorphism: $\|A \wedge B\| = \|A\| \cap \|B\|, \ldots, \|\Box_i A\| = \Box_i \|A\|$. We denote $\|A\|_M = W$ by M |= A and the fact that for all models M based on F, M |= A , by F |= A.

The important point for our exposition is that a possible worlds frame F is synonymous with an intensional (modal) algebra **A**(F), while a model M corresponds to a homomorphism of $\mathcal{L}$ into **A**(F), i.e. a member of Hom($\mathcal{L}$,**A**(F)). In this way all connectives become in fact truth-functional, though in respect to another (more complex) logical matrix, in which the truth values are sets of possible worlds, traditionally called *propositions*.

Given a class of such models there are at least three possibilities for defining the notion of *semantic consequence*.

1. $\Gamma \models_0 A$ iff $\forall M \forall w \in W(\forall B \in \Gamma(w \models B) \Rightarrow w \models A)$;
2. $\Gamma \models_1 A$ iff $\forall M(\forall B \in \Gamma(M \models B) \Rightarrow M \models A)$;
3. $\Gamma \models_2 A$ iff $\forall F(\forall B \in \Gamma(F \models B) \Rightarrow F \models A)$.

Expressed in algebraic terms these conditions become:

1'. $\Gamma \models_0 A$ iff $\forall \mathbf{A}(F) \forall h \in H \subseteq \text{Hom}(\mathcal{L},\mathbf{A}(F))(h(\bigwedge\{B: B \in \Gamma\}) \leq h(A))$

2'. $\Gamma \models_1 A$ iff $\forall \mathbf{A}(F) \forall h \in H \subseteq \text{Hom}(\mathcal{L},\mathbf{A}(F))(\forall B \in \Gamma(h(B) = 1) \Rightarrow h(A)=1)$.

3'. $\Gamma \models_2 A$ iff $\forall \mathbf{A}(F)(\forall B \in \Gamma$ (B is an **A**(F)tautology) $\Rightarrow$ A is an **A**(F) tautology).

As is well known, the first of these consequence operations is the one suitable for reasoning in relational models, while the last is inherently second-order with all the ensuing difficulties (incompleteness, lack of compactness, etc.). In the present paper we concentrate on the second possibility, which is familiar mainly from the so-called *matrix approach* in the study of many-valued logics [8, 61].

The above notions of the truth of a statement and semantic consequence can be questioned on several points.

**1.** The first goes back in time to the intuitionistic criticism of the classical approach to mathematical truth. It questions the rationality of assuming that one can *always* assign a truth value to a particular statement (and hold this as a methodological principle when dealing with still unsettled mathematical problems). Such a criticism leads to admitting statements which are *undefined*. Analyzing the notion of algorithm, in particular the statements one can make concerning their behavior, Kleene came up in [36] with the "strong Kleene truth tables" that included *undefined* as a third possibility, but even earlier Lukasiewicz had introduced the third value when investigating the status of statements about contingent future events (there is an obvious connection between these two concerns). This opened the door to considering the truth values as partial objects, to influence from the denotational semantics of programming languages, and to applications of fix point techniques (cf. e.g. [55]). For example in the theory of truth developed by Kripke [37] and others [15, 60] the fixed points of certain monotone operators on the family of all truth assignments were studied. The importance of the relation of "being more defined" and its connection with the "being more true" relation began gradually to emerge.

**2.** Another point on which the classical view has been questioned is the contention, having its origin even before Aristotle, that no statement is both true and false (in one and the same epistemic situation). Arguments put forward by the like of Hegel, Wittgenstein, etc., seem to show that this is open to a discussion. Some recent publications give expositions of what can be done abandoning the view that "everything is consistent" and have spoken of the "consistency of the world" problem, cf., e.g. [42,44,47]. Nevertheless the assumption of such a consistency, equivalent to the well–known *law of non–contradiction*, is considered by the majority of logicians as the final and indisputable principle of logic beyond which there is absolutely no ground for a rational epistemic activity, cf. Lewis [39].

Philosophically speaking the consistency and completeness of knowledge are determined by its "correspondence" to the "outside world". Thus contradictions may be the result of:

- defects in the correspondence,
- defects in the knowledge,
- defects in the world.

Concentrating on defects in knowledge, it is an interesting problem what reasoning procedures can be developed in order to accommodate the possibility of contradictory statements. The simplest option is to permit statements to be both true and false and keep this as the *only* possibility beyond the classical assumptions. This leads to a picture where for a statement A and an epistemic situation we have just three ways with the truth value: *A is only true; A is only false; A is both true and false.* Formally this approach can be described by truth assignments into the set $\{\{0\},\{1\},\{0,1\}\}$, as done by Priest in [43,44]. The corresponding consequence relation tolerates inconsistencies in the sense that there is no general way to infer logically *all* statements from a contradiction.

Some features of the recent AI approaches to reasoning, where one studies *default inferences* (if information on certain parameters of the situation cannot be obtained in reasonable bounds of resources – time, computational space, etc., they are replaced by default values), the *closed world assumption* (only items that are explicitly mentioned exist) and other *non-monotonic* schemata of deriving conclusions, e.g. reasoning by truth in preferred models (the so-called *pragmatic logics* [5] in which a statement pragmatically follows from a set of premises, if it is true in all preferred models of the premises), prompted a renewed interest in the problems of dealing with inconsistencies.

3. A further step is to combine the assumptions of partiality and contradictoriness. By this step we arrive at a class of assignments that have values in the set $\{\perp,\{1\},\{0\},\{1,0\}\}$. In this case the corresponding consequence relation is also contradiction tolerant. The arising logic with two *designated* truth values – $\{0,1\}$ = **Both** and $\{1\}$ = **True**) is also well-known and has been extensively studied, e.g. by N. Belnap [6,7], etc. Recently this logic has found numerous applications in computer science – as a suitable basis for studying the semantics of the programming languages [16,20].

***Truth-value spaces*** Along the path indicated by the above lines of criticism of the classical semantical schema we arrive at the notion of *truth value space*. The classical spaces (spaces for classical logic) were in general Boolean algebras with additional operators representing the intensional connectives occurring in the language. Early examples of non-classical spaces were the pseudo Boolean algebras, Post algebras, the unit interval [0,1] in fuzzy logic, etc.

From the very beginning deviations from the classical scheme were justified by appealing to *uncertainty* of information (on the basis of which the decision to declare something true is taken), *indefiniteness* of data, *vagueness* (fuzziness) of notions, i.e. all kinds of imperfections in the available knowledge, or lack of suitable knowledge due to difficulties in understanding (subjective non-significance), and even objective non-significance (as for example in Bochvar, cf. [14], who studied propositions in the foundations of mathematics that destroyed any theory they appeared in).

The truth–value spaces that were used and are in use at present reflect in their internal structure different views and assumptions (philosophical, mathematical, logical, pragmatic, etc.) concerning truth and inference. But there seem to be some general features common to all known examples of truth spaces: they represent methods of evaluation of information, i.e., truth values of statements are determined on the basis of the available information. We can even in general identify them with the *available relevant information* (about the state of affairs described, or referred to by the statement). This information can be characterized in two ways:

**truth degree** – reflecting the truth content of a statement. No doubt here we need a theory of truth (e.g., correspondence theory, or any other coherent view on how information is to be considered true, on the necessity of an external world, etc.)

but clearly truth degrees generate a partial order among truth values. Moreover it is rational to assume that the order is a *lattice* order.

**degree of knowledge** – reflecting the definiteness of information, or the completeness of the knowledge about the truth value (this could involve an estimation how reliable the information is, indications whether we find it plausible, etc.). Again it is reason-able to assume a lattice ordering.

An obvious way to make the truth value spaces from Example 1 more "realistic" is to admit either partial or contradictory models, or both. This has been done by many (see the historical survey in the Conclusion) and from different viewpoints, e.g. [1,6,12,13,35,42,47,57,59]. Recently Ginsberg [32,33] promoted a notion of a truth value space incorporating most the ideas discussed above. His *bilattices* (algebras with two complete lattice orders) were intended to combine model theoretic and computational advantages in treating reasoning with imperfect information: they could be used either as conventional logical matrices of as in denotational semantics – as a background for fixed point calculations (in the latter case truth value assignments do not presuppose the truth functionality of any logical connective – an important point for non-monotonic inference).

Another way to account for the uncertainty of knowledge is to consider *sets* of truth values, e.g., set of propositions, as representatives of the "temporarily unknown" truth–value of a statement. We find analogous ideas in fields like fuzzy set theory and logic [2], probabilistic logic [9, 10,24], AI [49], many-valued logic [26,27], etc. Here we propose a codification of such uses in the notion of *set expansion* of a given truth value space.

In our paper we treat in the spirit of Rasiowa and Sikorski [46] the mathematics (part 1) and logic (part 2 – for simplicity of presentation we restrict it to propositional languages) of two broad classes of truth value spaces: the bilattices and the set expansions. The many-valued logics determined by different subclasses of these depend on a number of parameters. One of the goals of the paper is to present a classification of the corresponding logics.

*Part I*

# ALGEBRAIC ASPECTS

## *1 The general theory of bilattices*

The notion of *bilattice* (and the term itself) was first introduced by Ginsberg [32]. In general an algebra of the similarity type <2,2,2,2,0,0,0,0)> is called a *bilattice*, if the two pairs of operations $\wedge$, $\vee$ and $\otimes$, $\oplus$ define two lattice structures which, together with the pairs of constants 0,1 and $\bot$, $\top$, constitute two bounded lattices with respective orders $\leq_t$ and $\leq_k$. Ginsberg in [32], as well as Fitting in [15,16,18] require the two lattices to be complete, i.e., supX and infX to exist for any subset X (which implies their boundedness).

We would not in general impose the condition of completeness in this paper, but results which hold only for such complete bilattices will be specially noted. Since there are several different definitions of basic notions available, we adhere to the approach of Fitting and first define the most general case:

**Definition 1.1** A *pre-bilattice* is a structure $\mathbf{B} = (B,\leq_t,\leq_k)$ where B is a non-empty set (of truth-values) and $\leq_t$ , $\leq_k$ are partial orders on B each generating on it the structure of a bounded lattice. The greatest and least elements of B with respect to $\leq_k$ are denoted by $\top$, $\bot$ while the greatest and least elements w.r.t. $\leq_t$ are 1 and 0. A pre-bilattice is *non-degenerate* (*non-trivial*) if all these four elements are different. The finitary lattice operations corresponding to $\leq_t$ are denoted by $\wedge$, $\vee$, the operations corresponding to $\leq_k$ – by $\otimes$ and $\oplus$. The respective infinitary operations are denoted by: $\bigwedge$, $\bigvee$ , $\prod$, and $\sum$.

In order to formally reflect the interplay between the truth-degree and the degree of knowledge we need more restricted classes of pre-bilattice structures. Fitting considered in [15] the class of *interlaced* bilattices.

**Definition 1.2** A pre-bilattice **B** is an interlaced bilattice, if $\wedge$ and $\vee$ are k-monotone, while $\oplus$ and $\otimes$ are t-monotone.

The meaning of these requirements is easily deciphered: we insist that e.g. the conjunction of two better known (or more defined) statements is better known (more defined) than the original conjunction, etc. or that the truth content of a union of informations about two statements does not decrease with the increase of knowledge.

In any interlaced bilattice the four constants 0, 1, $\bot$, and $\top$ are related as follows:

$0 \oplus 1 = \top$, $0 \otimes 1 = \bot$,
$\top \vee \bot = 1$, $\top \wedge \bot = 0$.

**Example 2** The simplest non–degenerate billatice **4** is given in Fig. 1 (on the left).

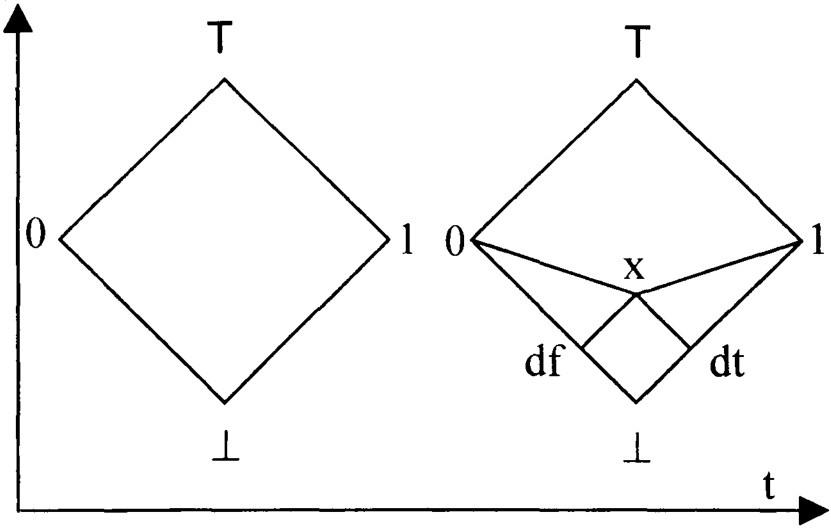


Figure 1
The simplest bilattice **4** and the bilattice of default logic **D**

The bilattice **4** is an interlaced bilattice and moreover it is a sub-bilattice of every non-trivial bilattice. The bilattice on the right (the bilattice of simple default logic, cf. [32]) is an example of a useful bilattice which lacks the property of interlacedness, e.g. the k-operations are not t-monotone since in **D:** $x = 0 \otimes 1$, instead of $\bot = 0 \otimes 1$.

**Example 3** The possible worlds example continued (Ginsberg [32], but before him many others, cf. the Conclusion). Here we deal with a generalization of Example 1 in that valuation functions $\phi$: W×Var($\mathcal{L}$)→**2** defining models M on a frame F are replaced with mappings into **4**.

Let us first assume valuations to be *partial* functions into **2**. This change leads to two notions of forcing:

(1) *positive*
M,w |=A when $\phi_M(w,A) = 1$, and

(2) *negative*
M,w=|A when $\phi_M(w,A) = 0$.

Denoting by $||A||_1$ the set $\{w:M,w|=A\}$ and by $||A||_0$ the set $\{w:M,w=|A\}$, we have for each partial model M an evaluation of formulae A by pairs $<||A||_1,||A||_0>$. Note that total models are characterized by $||A||_1= W \setminus ||A||_0$, so for them $||A||_0$ is in fact redundant. Not so in the general case where we have only the restriction $||A||_1 \cap ||A||_0 = \emptyset$ and thus the necessity to formulate separate truth conditions for =|, e.g.:

$$M,w=| A \wedge B \text{ iff } M,w=| A \text{ or } M,w=| B,$$
$$M,w=| A \vee B \text{ iff } M,w=| A \text{ and } M,w=| B,$$
$$M,w=| \Box A \text{ iff } \exists w'(wR_i w' \text{ and } M,w'=| A).$$

Recalling the trick of viewing partial functions into **2** as total functions into $\{0,1,\bot\}$ we can say that partial models are defined by a special class of mappings into **4**, namely those omitting T.

Now the next generalization step would be to interpret an *arbitrary* $\phi$: $W \times \mathcal{L} \to \mathbf{4}$ as defining a model. Fortunately one possibility is almost obvious:

$$M,w|=A \text{ if } 1 \leq_k \phi_M(w,A), \text{ i.e., if } \phi_M(w,A) = 1 \text{ or } \phi_M(w,A) = \mathrm{T};$$
$$M,w=|A \text{ if } 0 \leq_k \phi_M(w,A), \text{ i.e., if } \phi_M(w,A) = 0 \text{ or } \phi_M(w,A) = \mathrm{T}.$$

The corresponding pair of subsets of W - $<||A||_1,||A||_0>$ would be no more a disjoint pair in general, but the requirements for $\phi$ being a homomorphism determine the following combination laws for such pairs <U,V>:

$$<U,V> \wedge <U',V'> = <U \cap U', V \cup V'>;$$
$$<U,V> \vee <U',V'> = <U \cup U', V \cap V'>;$$
$$<U,V> \otimes <U',V'> = <U \cap U', V \cap V'>;$$
$$<U,V> \oplus <U',V'> = <U \cup U', V \cup V'>.$$

The first components carry the positive information (about worlds in which a statement has to be accepted as true or *forced*), while the second components codify negative information (about worlds where the statement has to be accepted as false, or *rejected*).

It is not difficult to check that with respect to the above operations the set of all pairs of subsets of W is a bilattice with constants $1 = <W,\emptyset>$, $0 = <\emptyset,W>$, $\bot = <\emptyset,\emptyset>$ and $\mathrm{T} = <W,W>$, and with partial orders

$$<U,V> \leq_t <U',V'> \text{ if } U \subseteq U' \text{ and } V' \subseteq V;$$
$$<U,V> \leq_k <U',V'> \text{ if } U \subseteq U' \text{ and } V \subseteq V'.$$

We call this bilattice the *frame bilattice* of F and denote it by **B**(F) The parallel drawn in Example 1 can be now extended: generalized models on a possible worlds frame F correspond on one hand to mappings into **4**, but on the other hand - to homomorphisms into **B**(F). Such models are uniquely determined by two forcing relations |= and =| (these being completely independent from each other).

***The pairs construction***This construction is a further generalization of the above possible worlds example (cf. again Ginsberg [32]). Consider two bounded lattices $\mathbf{L_1}$=<$L_1$,∧,∨,0,1> and $\mathbf{L_2}$=<$L_2$,∧,∨,0,1> – we use the same notation for operations in both lattices since it will be clear from the context which lattice is referred to.

**Definition 1.3** On the Cartesian product $L_1 \times L_2$ one can introduce the following operations:

$$\langle a,b\rangle \wedge \langle c,d\rangle = \langle a\wedge c, b\vee d\rangle;$$
$$\langle a,b\rangle \vee \langle c,d\rangle = \langle a\vee c, b\wedge d\rangle;$$
$$\langle a,b\rangle \oplus \langle c,d\rangle = \langle a\vee c, b\vee d\rangle;$$
$$\langle a,b\rangle \otimes \langle c,d\rangle = \langle a\wedge c, b\wedge d\rangle.$$

Denote the resulting algebra by $\mathbf{L_1}\times\mathbf{L_2}$.

**Lemma 1.4** *$L_1 \times L_2$ is an (interlaced) bilattice with 0 = <0,1>, 1 = <1,0>, ⊥ = <0,0> and T = <1,1> and orders*

$$\langle a,b\rangle \leq_t \langle c,d\rangle \text{ iff } a\leq c \text{ and } d\leq b,$$
$$\langle a,b\rangle \leq_k \langle c,d\rangle \text{ iff } a\leq c \text{ and } b\leq d.$$

*Proof*: Conditions are easily checked.■

**Remarks** 1. If both lattices $\mathbf{L_1}$ and $\mathbf{L_2}$ are complete, then $\mathbf{L_1}\times\mathbf{L_2}$ is a complete bilattice with infinitary operations defined as follows:

$\bigwedge\{\langle a,b\rangle: a\in X, b\in Y\} = \langle \inf X, \sup Y\}$, $\bigvee\{\langle a,b\rangle: a\in X, b\in Y\} = \langle \sup X, \inf Y\rangle$, $\prod\{\langle a,b\rangle: a\in X, b\in Y\} = \langle \inf X, \inf Y\}$, and $\sum\{\langle a,b\rangle: a\in X, b\in Y\} = \langle \sup X, \sup Y\rangle$.

2. The motivation of this construction is clear from the above example: the product of the two lattices codifies judgements concerning the status of a statement – an element *<a,b>* represents both positive information (by the component *a* – a degree of belief in the truth of the statement) and negative information (with *b* which represents a degree of belief against the truth of the statement, or a belief in the falsity of this statement). It is important to mention that since $\mathbf{L_1}$ and $\mathbf{L_2}$ can be in general quite different this construction supports the option that the beliefs *for* and *against* can be incommensurable.

3. Note that **4** ≅ **2**×**2** and **B**(F) ≅ **A**(F)×**A**(F).

**Example 4** Besides the above examples, there is another relatively well known example - **[0,1]×[0,1]**, where **[0,1]** is the unit interval viewed as a lattice with respect to the operations corresponding to the usual (linear) ordering of real numbers - min(a,b) and max(a,b). The bilattice operations are as follows:

$$\langle a,b\rangle \wedge \langle c,d\rangle = \langle \min(a,c),\max(b,d)\rangle;$$
$$\langle a,b\rangle \vee \langle c,d\rangle = \langle \max(a,c),\min(b,d)\rangle;$$
$$\langle a,b\rangle \oplus \langle c,d\rangle = \langle \max(a,c),\max(b,d)\rangle;$$
$$\langle a,b\rangle \otimes \langle c,d\rangle = \langle \min(a,c),\min(b,d)\rangle.$$

Elements of **[0,1]×[0,1]** generalize real numbers as degrees of membership to fuzzy sets (cf. [1,2,3]). Again the first component gives a (positive) degree of membership, while the second number measures a degree of (belief in) non-membership.

Since in the present paper the emphasis is on bilattices as *logical matrices* defining some logics and since in the majority of cases such logics are distributive we concentrate mainly on *distributive* bilattices.

**Definition 1.5** A pre–bilattice **B** is called a *distributive* bilattice if all 12 possible distributive laws (involving $\wedge$, $\vee$, $\oplus$ and $\otimes$) hold.

Of course, the distributivity of a bilattice is a much stronger property than the mere distributivity of the two underlying lattices since it involves also their interaction. Thus distributivity is a consequence of specific hypotheses about the combinations of pieces of information contributing to the truth values, implying in particular that it is always possible to decompose contributions of the involved data according to the structure of the statement. In the bilattice **D** shown on Fig. 1 such a decomposition is impossible.

**Fact 1.6** (cf. [15]) *If **B** is distributive, then it is interlaced.*

*Proof*: In order to check, e.g., the monotonicity of $\wedge$ with respect to $\leq_k$ we proceed as follows: assume $x\leq_k x'$ and $y\leq_k y'$ (i.e., $x\oplus x' = x'$ and $y\oplus y' = y'$), then $x'\wedge y' = (x\oplus x')\wedge(y\oplus y') = (x\wedge y)\oplus(x\wedge y')\oplus(x'\wedge y)\oplus(y\wedge y')$, therefore $x\wedge y\leq_k x'\wedge y'$. In the rest of the cases we argue similarly. ■

**Lemma 1.7** *If $L_1$ and $L_0$ are two distributive lattices, then $B = L_1 \times L_0$ is a distributive bilattice.*

The easy proof is left to the reader.

For distributive bilattices there is a nice representation theorem asserting the converse of the above lemma:

**Theorem 1.8** (cf. Ginsberg [32])*If **B** is a distributive bilattice, then there exist two bounded distributive lattices* $\mathbf{L}_1$ *and* $\mathbf{L}_0$ *such that* $\mathbf{L}_1 \times \mathbf{L}_0 \cong \mathbf{B}$.

The proof is of some interest with the notions introduced in its course, so we give it in full.

**Definition 1.9** Let **B** be a pre–bilattice. For $x \in B$ set

$$(x)_0 = x \wedge \bot,$$
$$(x)_1 = x \vee \bot.$$

**Facts 1.10** *In an interlaced bilattice **B**:*

*1. the operations* $(x)_0$ *and* $(x)_1$ *are monotone;*

*2.* $0 \leq_t (x)_0 \leq_t \bot$ *and* $\bot \leq_k (x)_0 \leq_k 0$;

$\bot \leq_t (x)_1 \leq_t 1$ *and* $\bot \leq_k (x)_1 \leq_k 1$.

**Lemma 1.11** *In a distributive bilattice **B** the following hold:*

*1.* $(x)_0 \oplus (x)_1 = x (x)_0 \otimes (x)_1 = \bot$

*2.* $(x \wedge y)_0 = (x)_0 \wedge (y)_0 (x \wedge y)_1 = (x)_1 \wedge (y)_1$

$(x \vee y)_0 = (x)_0 \vee (y)_0 (x \vee y)_1 = (x)_1 \vee (y)_1$

$(x \oplus y)_0 = (x)_0 \oplus (y)_0 (x \oplus y)_1 = (x)_1 \oplus (y)_1$

$(x \otimes y)_0 = (x)_0 \otimes (y)_0 (x \otimes y)_1 = (x)_1 \otimes (y)_1$

*Proof*: 1. $(x)_0 \oplus (x)_1 = (x \wedge \bot) \oplus (x \vee \bot) = ((x \wedge \bot) \oplus x) \vee ((x \wedge \bot) \oplus \bot) =$

$= ((x \oplus x) \wedge (\bot \oplus x)) \vee (x \wedge \bot) = (x \wedge x) \vee (x \wedge \bot) = x \vee (x \wedge \bot) = x.$

$(x)_0 \otimes (x)_1 = (x \wedge \bot) \otimes (x \vee \bot) = ((x \wedge \bot) \otimes x) \vee ((x \wedge \bot) \otimes \bot) =$

$= (x \otimes x) \wedge (\bot \otimes x)) \vee \bot = (x \wedge \bot) \vee \bot = \bot.$

2. We check only a few samples, e.g.: $(x \oplus y)_0 = (x \oplus y) \wedge \bot = (x \wedge \bot) \oplus (y \wedge \bot) =$

$= (x)_0 \oplus (y)_0$ or $(x \vee y)_1 = (x \vee y) \vee \bot = (x \vee \bot) \vee (y \vee \bot) = (x)_1 \vee (y)_1$. ■

**Lemma 1.12** *In a distributive bilattice **B**, for all x and y:*

*1.* $y \leq_t (x)_0$ *implies* $(x)_0 \leq_k y$;

*2.* $(x)_1 \leq_t y$ *implies* $(x)_1 \leq_k y$.

*Proof*: 1. Assume $y \leq_t (x)_0$, i.e. $y \leq_t x\wedge\bot$, which implies $y \leq_t x$ and consequently $x\wedge y=y$. Consider $(x)_0\oplus y = (x\wedge\bot)\oplus y = (x\oplus y)\wedge(\bot\oplus y) = (x\oplus y)\wedge y = (x\wedge y)\oplus y = y\oplus y = y$, that is $(x)_0 \leq_k y$.

The same for (2).■

**Lemma 1.13** $(x\oplus y)_0=(x)_0\wedge(y)_0 (x\oplus y)_1=(x)_1\vee(y)_1$

$(x\otimes y)_0=(x)_0\vee(y)_0 (x\otimes y)_1=(x)_1\wedge(y)_1$

*Proof*: Note that $(x)_0\wedge(y)_0 \leq_t (x)_0$ and by the previous lemma $(x)_0 \leq_k (x)_0\wedge(y)_0$, analogously $(y)_0 \leq_k (x)_0\wedge(y)_0$, so $(x)_0\oplus(y)_0 \leq_k (x)_0\wedge(y)_0$.

On the other hand $(x)_0 \leq_k (x)_0\oplus(y)_0$ and $(y)_0 \leq_k (x)_0\oplus(y)_0$, so by the monotonicity of $\wedge$ one has $(x)_0\wedge(y)_0 \leq_k (x)_0\oplus(y)_0$. Thus we get the equality we need.

Similar arguments work for the rest of the cases. ■

**Remark** Note that $(x)_0 = \prod\{y:y \leq_t x\}$ and $(x)_1= \prod\{y:x \leq_t y\}$, if the bilattice is complete and completely distributive. Thus $(x)_0$ represents the essential negative information encoded in x, while $(x)_1$ represents the positive content of x.

To continue the proof of the representation theorem we define two lattices $\mathbf{L_1}$ and $\mathbf{L_0}$ and establish that $\mathbf{B}$ is isomorphic to $\mathbf{L_1}\times\mathbf{L_0}$.

Let $\mathbf{L_1}=\langle L_1,\wedge_1,\vee_1,0_1,1_1\rangle$, where

$L_1=\{(x)_1: x\in B\}=\{x:x=(x)_1\}$

$\wedge_1$ is $\wedge$ restricted to $L_1$

$\vee_1$ is $\vee$ restricted to $L_1$

$0_1 = \bot$

$1_1 = 1$.

Now, by Lemma 1.11 this definition is correct, $\mathbf{L_1}$ is indeed a bounded distributive lattice, and its partial order $\leq$ coincides with $\leq_t$ restricted to $L_1$.

Let, in the same spirit, $\mathbf{L_0} = \langle L_0, \wedge_0, \vee_0, 0_0, 1_0 \rangle$, where

$$L_0 = \{(x)_0 : x \in B\} = \{x : x = (x)_0\}$$

$\wedge_0$ is $\vee$ restricted to $L_0$

$\vee_0$ is $\wedge$ restricted to $L_0$

$0_0 = \bot$

$1_0 = 0.$

In the latter case note the reversal of the partial order and hence the interchange of the corresponding operations.

Consider now $\mathbf{L_1} \times \mathbf{L_0}$. Define a map $f : \mathbf{B} \to \mathbf{L_1} \times \mathbf{L_0}$ by

$$f(x) = \langle (x)_1, (x)_0 \rangle.$$

**Lemma 1.14** *The map f is an isomorphism between* $\boldsymbol{B}$ *and* $\boldsymbol{L_1} \times \boldsymbol{L_0}$.

*Proof*: Let us first check that f is a homomorphism.

$f(1) = \langle (1)_1, (1)_0 \rangle = \langle 1 \vee \bot, 1 \wedge \bot \rangle = \langle 1, \bot \rangle = \langle 1_1, 0_0 \rangle = 1$ in $\mathbf{L_1} \times \mathbf{L_0}$;

$f(0) = \langle (0)_1, (0)_0 \rangle = \langle 0 \vee \bot, 0 \wedge \bot \rangle = \langle \bot, 0 \rangle = \langle 0_1, 1_0 \rangle = 0$ in $\mathbf{L_1} \times \mathbf{L_0}$;

$f(\bot) = \langle (\bot)_1, (\bot)_0 \rangle = \langle \bot \vee \bot, \bot \wedge \bot \rangle = \langle \bot, \bot \rangle = \langle 0_1, 0_0 \rangle = \bot$ in $\mathbf{L_1} \times \mathbf{L_0}$;

$f(\top) = \langle (\top)_1, (\top)_0 \rangle = \langle \top \vee \bot, \top \wedge \bot \rangle = \langle 1, 0 \rangle = \langle 1_1, 1_0 \rangle = \top$ in $\mathbf{L_1} \times \mathbf{L_0}$;

$f(x \wedge y) = \langle (x \wedge y)_1, (x \wedge y)_0 \rangle = \langle (x)_1 \wedge (y)_1, (x)_0 \wedge (y)_0 \rangle =$

$= \langle (x)_1 \wedge_1 (y)_1, (x)_0 \vee_0 (y)_0 \rangle = \langle (x)_1, (x)_0 \rangle \wedge \langle (y)_1, (y)_0 \rangle = f(x) \wedge f(y);$

$f(x \vee y) = \langle (x \vee y)_1, (x \vee y)_0 \rangle = \langle (x)_1 \vee (y)_1, (x)_0 \vee (y)_0 \rangle =$

$= \langle (x)_1 \vee_1 (y)_1, (x)_0 \wedge_0 (y)_0 \rangle = \langle (x)_1, (x)_0 \rangle \vee \langle (y)_1, (y)_0 \rangle = f(x) \vee f(y);$

$f(x \oplus y) = \langle (x \oplus y)_1, (x \oplus y)_0 \rangle = \langle (x)_1 \oplus (y)_1, (x)_0 \oplus (y)_0 \rangle =$

$= \langle (x)_1 \vee (y)_1, (x)_0 \wedge y)_0 \rangle = \langle (x)_1 \vee_1 (y)_1, (x)_0 \vee_0 (y)_0 \rangle =$

$= \langle (x)_1, (x)_0 \rangle \oplus \langle (y)_1, (y)_0 \rangle = f(x) \oplus f(y);$

$f(x \otimes y) = \langle (x \otimes y)_1, (x \otimes y)_0 \rangle = \langle (x)_1 \otimes (y)_1, (x)_0 \otimes (y)_0 \rangle =$

$= \langle (x)_1 \wedge (y)_1, (x)_0 \vee (y)_0 \rangle = \langle (x)_1 \wedge_1 (y)_1, (x)_0 \wedge_0 (y)_0 \rangle =$

$= \langle (x)_1, (x)_0 \rangle \otimes \langle (y)_1, (y)_0 \rangle = f(x) \otimes f(y).$

The map f is a bijection: $f(x) = f(y)$ implies that $(x)_1 = (y)_1$ and $(x)_0 = (y)_0$; therefore, by the above lemma, $x = (x)_1 \oplus (x)_0 = (y)_1 \oplus (y)_0 = y$. On the other hand f is a surjective mapping, since $<x,y>$ is the value of f at $x \oplus y$. ■

***A theory of homomorphisms***Let us concentrate for a moment on homomorphisms of bilattices. In this sub-section we shall be concerned exclusively with distributive bilattices, so the term *bilattice* will mean a distributive bilattice, if it is not explicitly stated otherwise.

If f: **B**→**4** is a homomorphism, it generates four sets which are the pre-images of the four elements of **4** under f:

$$B_1 = f^{-1}(1); B_0 = f^{-1}(0);$$

$$B_\top = f^{-1}(\top); B_\perp = f^{-1}(\perp).$$

These are pairwise disjoint and their union is B. Define $X_1 = B_1 \cup B_\top$ and $X_0 = B_0 \cup B_\top$. Clearly given the two sets $X_1$ and $X_2$ one can reconstruct the four pre-images:

$$B_1 = X_1 \setminus X_0; B_0 = X_0 \setminus X_1;$$

$$B_\top = X_1 \cap X_0; B_\perp = B \setminus (X_1 \cup X_0);$$

**Definition 1.15** Let **B** be a bilattice. A non-empty subset F of B is called a *bi-filter,* if the following conditions are met:

F1. F is upward closed with respect to $\leq_t$, i.e., if $x \in F$ and $x \leq_t y$, then $y \in F$;

F2. F is upward closed with respect to $\leq_k$,i.e., if $x \in F$ and $x \leq_k y$, then $y \in F$;

F3. $x \wedge y \in F$ iff $x \in F$ and $y \in F$;F4. $x \otimes y \in F$ iff $x \in F$ and $y \in F$.

A bi-filter is *proper,* if it is different from the whole B.

**Definition 1.16** A bi-filter is *prime,* if it is proper and has the properties:F5. $x \vee y \in F$ iff $x \in F$ or $y \in F$;F6. $x \oplus y \in F$ iff $x \in F$ or $y \in F$.

Now, it is easy to check that $X_1$ is a prime bi-filter for any homomorphism into **4**. As for the situation with $X_0$ - it is captured by the next definition.

**Definition 1.17** A non-empty subset I of B is called a *bi-ideal* if the following hold:

I1. I is downward closed with respect to $\leq_t$, i.e., if $x \in I$ and $y \leq_t x$, then $y \in I$;

I2. I is upward closed with respect to $\leq_k$,i.e., if $x \in I$ and $x \leq_k y$, then $y \in I$;

I3. $x \vee y \in I$ iff $x \in I$ and $y \in I$;

I4. $x \otimes y \in I$ iff $x \in I$ and $y \in I$.

Again we have the notion of a *prime* bi-ideal - a proper bi-ideal with the additional properties:

I5. $x \wedge y \in I$ iff $x \in I$ or $y \in I$;

I6. $x \oplus y \in I$ iff $x \in I$ or $y \in I$.

For any homomorphism $f$:**B**→**4**, the corresponding $X_0$ is a prime bi-ideal (easily checked). Let us note also the following obvious facts:

1. If F is a proper bi-filter, then $1, T \in F$ but $0, \bot$ do not;
2. If I is a proper bi-ideal, then $0, T \in I$ but $1, \bot$ do not.

**Lemma 1.18** *Let $X_1$, $X_0$ be a prime bi-filter and a prime bi-ideal in **B** respectively. Such a pair determines a homomorphism f: **B**→**4** such that $F = X_1$ , $I = X_0$ .*

*Proof:* First we define in the way shown above four subsets of the bilattice: $B_1$, $B_0$, $B_T$, and $B_\bot$ – they form a partition of B. Then we set for an $x \in B$:

$$f(x) = i, \text{ if } x \in B_i \ (i = 0, 1, T, \bot).$$

To show that f respects the operations is a quite straightforward (but tedious) task – one has to check a lot of cases, e.g., $f(\bot) = \bot$: since $\bot$ belongs neither to $X_1$ nor to $X_0$, it is a member of $B_\bot$. Let us consider one more case in detail:

$$f(x \vee y) = f(x) \vee f(y). \ (*)$$

If $f(x \vee y) = \bot$, then $x \vee y \in B_\bot$ and neither x nor y can be in $X_1$ (because so would be $x \vee y$), or both be in $X_0$ (then so would be $x \vee y$); thus we have either $x \in B_0$ and $y \in B_\bot$ or $y \in B_0$ and $x \in B_\bot$. In both cases (*) holds.

If $f(x \vee y) = 1$, i.e., when $x \vee y \in B_1$, then x or y belong to $X_1$, but it is not the case that they both belong to $X_0$, so if one of them is in $B_T$ the other has to be in $B_\bot$, or if one is in $B_1$, the other can be in $B_\bot$ or in $B_0$, etc. In all cases (*) holds.

In the same way one checks the remaining two possibilities for $f(x \vee y)$. Other cases are treated similarly. ■

**Lemma 1.19** *Let X be a non-empty subset of B. $[X]_f = \{x : \exists x_1,...,x_n, y(x_1,...,x_n \in X$ and $x_1 \wedge ... \wedge x_n \leq_t y \leq_k x)\}$ is he smallest bi-filter containing X. In case X has the following multiplicative property:*

$$\forall x_1,...,x_n \in X \ (x_1 \wedge ... \wedge x_n \neq 0 \text{ and } x_1 \otimes ... \otimes x_n \neq \bot),$$

*$[X]_f$ is a proper bi-filter:*

*Proof:* Note first that $[X]_f = \{x: \exists x_1,...,x_n, y(x_1,...,x_n \in X \text{ and } x_1\otimes...\otimes x_n \leq_k y \leq_t x)\}$. Indeed, if $x_1\wedge....\wedge x_n \leq_t y \leq_k x$, then $y\oplus x = x$ and also $(x_1\wedge....\wedge x_n)\oplus x \leq_t y\oplus x = x$, thus $(x_1\wedge....\wedge x_n)\oplus x \leq_t x$. On the other hand $(x_1\otimes...\otimes x_n) \leq_k (x_1\wedge...\wedge x_n)$, so $(x_1\otimes...\otimes x_n) \leq_k (x_1\wedge...\wedge x_n)\oplus x \leq_t x$. In the opposite direction: if $x_1\otimes...\otimes x_n \leq_k y \leq_t x$, then $y\vee x = x$ and also $(x_1\otimes...\otimes x_n)\vee x \leq_k y\vee x = x$. On the other hand $x_1\wedge...\wedge x_n \leq_t x_1\otimes...\otimes x_n$, thus $x_1\wedge...\wedge x_n \leq_t (x_1\otimes...\otimes x_n)\vee x \leq_k x$.

Having the above alternative description, to check F1 - F4 is fairly easy, e.g., $[X]_f$ is clearly closed upward with respect to both orders, it has property F3 since $x_1\otimes...\otimes x_n \leq_k y \leq_t x$ and $x_1'\otimes...\otimes x_m' \leq_k y' \leq_t z$ implies $(x_1\otimes...\otimes x_n)\wedge(x_1'\otimes...\otimes x_m') \leq_k y\wedge y' \leq_t x\wedge z$, but $(x_1\otimes...\otimes x_n\otimes x_1'\otimes...\otimes x_m') \leq_k (x_1\otimes ... \otimes x_n)\wedge(x_1'\otimes...\otimes x_m')$; property F4 concerning $\otimes$ is treated similarly.

Thus $[X]_f$ is a bi-filter that contains X, moreover the multiplicativity property of X implies that $[X]_f$ is proper since obviously then $0\notin[X]_f$. If a bi-filter F contains X, then it must include also all finite conjunction and meets of elements of X and then by the upward closure it must contain X. Therefore $[X]_f$ is the minimal bi-filter extending X. ■

**Lemma 1.20** *If for an element x and a subset X of a bilattice **B** $x\notin[X]_f$, then there exists a prime bi-filter F such that $[X]_f \subseteq F$ and $x\notin F$.*

*Proof:* The proof is standard: consider the family of proper bi-filters G extending $[X]_f$ and such that $x\notin G$, it possesses the Zorn property, i.e. each chain of elements is majorized by an element of the family (the union of that chain is a suitable majorant), so there are maximal elements. Let F be one of them. F is prime: if $y\vee z\in F$, then either y or z belong to F, otherwise we would have $x\in[F\cup\{y\}]_f$ and $x\in[F\cup\{z\}]_f$, i.e. $x_1\otimes y \leq_k y' \leq_t x$ and $x_2\otimes z \leq_k z' \leq_t x$, for some $x_1,x_2$ from F; combining these inequalities we would get $(x_1\otimes y)\vee(x_2\otimes z) \leq_k y'\vee z' \leq_t x$, and consequently $(x_1\vee x_2)\otimes(y\vee x_2)\otimes(x_1\vee z)\otimes(y\vee z) \leq_k y'\vee z' \leq_t x$, but the former meet being a member of F, we would finally have $x\in F$ - contrary to the assumption. The case of $\oplus$ is similar. ■

As an important corollary we obtain the existence of homomorphisms into **4** which map some prescribed subsets of B into {1,T} while for a given element their value is outside that set.

The existence of bi-ideals $[X]_i$ extending subsets of B and omitting certain elements can be established in the same way.

**Theorem 1.21** (representation theorem for distributive bilattices) *Any distributive bilattice **B** is embeddable in a product of frame lattices.*

*Proof:* Let $W_1$ be the set of all prime bi-filters in **B**, $W_0$ - the set of all prime bi-ideals. Consider the lattices $F_1^+ = \langle P(W_1), \cap, \cup, \emptyset, W_1 \rangle$ and $F_0^+ = \langle P(W_0), \cap, \cup, \emptyset, W_0 \rangle$. Define for $x \in B$:

$$|x|_1 = \{F: F \text{ is a bi-filter and } x \in F\};$$

$$|x|_0 = \{I: I \text{ is a bi-ideal and } x \in F\}.$$

Our claim is that the mapping f which assigns to x the pair $\langle |x|_1, |x|_0 \rangle$ is a monomorphism into $F_1^+ \times F_0^+$. This follows from a series of identities:

$$|x \wedge y|_1 = |x|_1 \cap |y|_1 \quad |x \wedge y|_0 = |x|_0 \cup |y|_0$$

$$|x \vee y|_1 = |x|_1 \cup |y|_1 \quad |x \vee y|_0 = |x|_0 \cap |y|_0$$

$$|x \otimes y|_1 = |x|_1 \cap |y|_1 \quad |x \otimes y|_0 = |x|_0 \cap |y|_0,$$

$$|x \oplus y|_1 = |x|_1 \cup |y|_1 \quad |x \oplus y|_0 = |x|_0 \cup |y|_0,$$

which are corollaries of the properties of prime bi-filters and bi-ideals. ■

**Remarks** 1. Now that we have two set-theoretical representations - one as in the above theorem, the other obtained by applying Theorem 1.8 and then the Stone representation theorem to the two lattices in the product - a question arises as to their relations. It turns out that the two approaches are exactly equivalent: the lattices $\mathbf{L}_1$ and $\mathbf{L}_0$ from Theorem 1.8 and the two projections of the image of the monomorphism from Theorem 1.21 are isomorphic.

2. Below we'll need sometimes another representation of the pre-images of a homomorphism $f: \mathbf{B} \to \mathbf{4}$. Putting $Y_1 = B_1 \cup B_\perp$ and $Y_1 = B_1 \cup B_\perp$, we have $B_T = Y_1 \cap Y_0$; $B_0 = Y_0 \setminus Y_1$; $B_1 = Y_1 \setminus Y_0$; $B_\perp = B \setminus (Y_1 \cup Y_0)$. $Y_1$ can be called *dual prime bi-filter* and $Y_0$ – a *dual prime bi-ideal*, where the two new dual notions have a common feature – they apply to *downward closed with respect*

*to* $\leq_k$ sets, otherwise a dual bi-filter has the properties F1 and F3, it satisfies a ⊕-version of F3, i.e., $x \oplus y \in D$ iff $x \in D$ and $y \in D$, and the prime dual bi-filters additionally satisfy: $x \vee y \in D$ iff $x \in D$ or $y \in D$, $x \otimes y \in D$ iff $x \in D$ or $y \in D$. The properties of dual bi-ideals similarly correspond to those of bi-ideals: they satisfy I1 and I3, they have a ⊕-version of I3, while the prime bi-ideals also $x \wedge y \in J$ iff $x \in J$ or $y \in J$, $x \otimes y \in J$ iff $x \in J$ or $y \in J$. Thus the theory of the dual notions turns out to be naturally dual to the theory of bi-filters and bi-ideals, leading in particular to a corollary concerning the possibility to homomorphically map a given subset of B into $D_0 = \{\bot, 1\}$ of **4**, while mapping another element outside $D_0$. Finally let us remark that the complement of a prime bi-filter is a prime dual bi-ideal and *vice versa*. The same relation holds between prime bi-ideals and dual prime bi-filters.



## 2 *Negation, conflation and other operations in a bilattice*

Besides the basic bilattice operations as a rule the truth value spaces actually in use contain additional operations – some of them are in fact indispensable, if one is to apply these spaces to problems of inference.

***Negation*** We start with the introduction of an operation which is usually present in a useful truth value space. In fact Ginsberg [32] included the existence of negation into the definition of bilattice.

**Definition 2.1** We say that the unary operation ¬ in a bilattice **B** is a *weak negation*, if the following holds:

1. $x \leq_t y$ implies $\neg y \leq_t \neg x$
2. $x \leq_k y$ implies $\neg x \leq_k \neg y$.

A bilattice has a *pseudo negation*, if in addition to the above two conditions a third one is satisfied:

3. $x \leq_t \neg\neg x$.

**Fact 2.2** *If ¬ is a pseudo negation in a bilatice **B**, then:*
$\neg 1 = 0$, $\neg 0 = 1$;
$\neg \bot = \bot$, $\neg T = T$.

**Remark** The following examples of negation like-operations in **[0,1]×[0,1]**: $\neg_{\mu,\bullet}\langle a,b\rangle = \langle \bullet b, \mu a\rangle$, which are weak negations for all $\mu, \bullet$ such that $0 < \mu, \bullet \leq 1$, but a pseudo-negation only for $\mu = 1$, $\bullet = 1$, and for which $\neg_{\mu,\bullet}\langle 1,0\rangle = \langle 0,\mu\rangle \bullet \langle 0,1\rangle$, show that conditions (1) and (2) do not suffice for the above facts.

**Definition 2.3** A bilattice has a *negation*, if (3) is strengthened to:

4. $\neg\neg x = x$ .

**Fact 2.4** *In a bilattice with negation ¬ the de Mogran laws hold:*

*5.* $\neg(x \wedge y) = \neg x \vee \neg y$

*6.* $\neg(x \vee y) = \neg x \wedge \neg y$

*and ¬ distributes over k–operations:*

*7.* $\neg(x \oplus y) = \neg x \oplus \neg y$

*8.* $\neg(x \otimes y) = \neg x \otimes \neg y$.

*In the case of complete bilattices the above identities have infinitary counterparts:*

*5'.* $\neg \bigwedge X = \bigvee \neg X$ *(where* $\neg X = \{\neg x : x \in X\}$*)*

*6'.* $\neg \bigvee X = \bigwedge \neg X$

*7'.* $\neg \Sigma X = \Sigma \neg X$

*8'.* $\neg \Pi X = \Pi \neg X$.

**Example 5** An important class of bilattices with negation is formed by the bilattices of the kind **L**×**L** (for *any* bounded lattice **L**), where ¬ is defined as follows:

$$\neg\langle a,b\rangle = \langle b,a\rangle.$$

It can be easily checked that the operation ¬ just defined is indeed a negation. Note that **L** need not necessarily have a negation itself, but it is possible to introduce a negation in its square due to the horizontal symmetry of **L**×**L**.

**Remark** The equations of 2.2 show that in **4** ¬ is not only a pseudo negation, but a "real" de Morgan one. In a generalized possible worlds models the truth conditions for ¬ are:

M,w|= ¬A iff M,w=| A;

M,w=| ¬A iff M,w|= A.

Again, for distributive bilattices with a negation a nice converse of the above holds – one can prove the following representation theorem:

**Theorem 2.5** *If **B** is a distributive bilattice with negation, then **B** ≅ **L**×**L** for a distributive lattice **L**.*

*Proof*: We need only to supplement the proof of Theorem 1.8 with an argument dealing with negation. Observe first that :

$$\neg(x)_0 = \neg(x \wedge \bot) = \neg x \vee \neg \bot = \neg x \vee \bot = (\neg x)_1$$

$$\neg(x)_1 = \neg(x \vee \bot) = \neg x \wedge \neg \bot = \neg x \wedge \bot = (\neg x)_0$$

Thus ¬ maps $L_0$ into $L_1$ and *vice versa*. Moreover the following holds:

**Lemma 2.6** ¬ *is an isomorphism of* $\mathbf{L}_0$ *and* $\mathbf{L}_1$.

*Proof.* Check using de Morgan laws. ■

Now take an isomorphic copy of $\mathbf{L}_0$ – **L** and consider a map f defined as in the proof of Theorem 1.8 but with values in **L** (by identifying elements of $L_0$ and $L_1$ with their counterparts in L). This f is an isomorphism of **B** and L×L with respect to the negation, too: if f(x) = <a,b>, then $f(\neg x) = <(\neg x)_1,(\neg x)_0> = <\neg(x)_0,\neg(x)_1> =$ <b,a> = ¬<a,b> = ¬f(x). ■

So the presence of a negation in a distributive bilattice **B** indicates a vertical symmetry in **B** which allows its representation as a square of some lattice and even as a sub-bilattice of a *frame* bilatice (as mentioned e.g. in Ginsberg's [32]). The same result can be obtained by applying the theory of homomorphisms to bilattices with negation, where the crucial observation is that for a bi-filter F the set ¬F = {¬x:x∈F} is a bi-ideal and *vice versa*.

***Conflation*** Fitting considered in [15] an operation, which seems to have appeared initially in Visser [60] inspired by his approach to the truth-values gaps and gluts theory of Kripke and others (cf. [37,60]). It relates to a possible *vertical* symmetry of a billatice just as the negation is associated with horizontal symmetry. In [19] Fitting used a term *convolution* for the operation, but we stick to the original name.

**Definition 2.7** A mapping – of a bilattice **B** into itself is called a *conflation*, if:

1. $x \leq_t y$ implies $-x \leq_t -y$;
2. $x \leq_k y$ implies $-y \leq_k -x$;
3. $--x = x$.

**Example 6** Bilattices of the kind **L**×**L** (for a lattice **L** having a negation ~), where – is defined as follows:

– <a,b> = <~b,~a>.

provide the most important class of examples. Note that in such bilattices ¬ and – commute:

– ¬ <a,b> = – <b,a> = <~a,~b> = ¬ <~b,~a> = ¬ – <a,b>.

**Fact 2.8** *In a bilattice* ***B*** *with conflation:*

$-1 = 1$, $-0 = 0$,

$-\bot = \top$, $-\top = \bot$.

The operation – is a dual counterpart of ¬ with respect to $\leq_k$ and $\leq_t$, as the following identities show:

4. $-(x \wedge y) = -x \wedge -y$

5. $-(x \vee y) = -x \vee -y$
6. $-(x \oplus y) = -x \otimes -y$
7. $-(x \otimes y) = -x \oplus -y$

When **B** is complete we have:

4'. $-\bigwedge X = \bigwedge -X$ (where $-X = \{-x : x \in X\}$)

5'. $-\bigvee X = \bigvee -X$
6'. $-\Sigma X = -\Pi X$
7'. $-\Pi X = \Sigma -X$.

**Theorem 2.9** (Fitting).*If* ***B*** *is a distributive bilattice with a negation and a conflation that commute, then* ***B*** $\cong$ ***L***×***L*** *for a distributive lattice* **L** *with a negation (a* de Morgan *lattice).*

*Proof*: Since **B** has a negation, $\mathbf{L_1}$ and $\mathbf{L_0}$ are isomorphic (and give us the required **L**). We need an operation ~ in **L** to play the role of a negation. To this end we define for an element $a \in L$ (*a* can be taken to belong to $\mathbf{L_1}$ without loss of generality):

$$\sim a = (-\neg a)_1.$$

We claim that ~ is a negation in **L**, i.e. $a \leq b$ implies $\sim b \leq \sim a$ and $\sim\sim a = a$. Indeed the anti–monotonicity w.r.t. ≤ is clear and the double negation law follows from the fact that – and ¬ commute. ■

In bilattices with conflation two notions become available which also figure most prominently in applications:

**Definition 2.10** Let **B** be a bilattice with conflation.

1. An element $x \in B$ is a *consistent* truth value, if $x \leq_k -x$;
2. $x \in B$ is *exact* , if $x = -x$.

The meaning of these definitions becomes clearer when we consider our favorite examples. Let us note first that in bilattices of the kind **L**×**L** <a,b> is consistent iff a ≤~b and an exact value has the form <a,~a>. In the possible worlds models truth conditions for – look as follows:

$$M,w \models -A \text{ iff } M,w \not\mathrel{=\!\!|} A;$$
$$M,w \mathrel{=\!\!|} -A \text{ iff } M,w \not\models A.$$

Thus an element <U,V> of **B**(F) is consistent, if $U \cap V = \varnothing$, i.e., when there are no worlds to simultaneously force and reject the given formula; <U,V> is exact if $U = W \setminus V$, i.e., the formula is (classically) forced exactly when it is not rejected. In **[0,1]**×**[0,1]** conflation is defined by $-\langle a,b \rangle = \langle 1-b, 1-a \rangle$, so <a,b> is consistent if $a+b \leq 1$, and <a,b> is exact, if $a = 1-b$.

The existence of conflation is not necessary for the notion of a consistent truth value to make sense in a bilattice: in **L**×**L**, where **L** is a pseudo Boolean algebra, an element <a,b> with $a \wedge b = 0$ can reasonably be called consistent, although <a,~a> lacks some of the properties of the exact values (due to the fact that ~ is only a pseudo negation).

Let us remark also that the familiar consistency condition $x \wedge \neg x = 0$ is rather strong in the bilattice setting. It entails not merely consistency in the above sense but also the fact that x is an *exact* value of a special kind: x = <a,b> with $a \wedge b = 0$ and $a \vee b = 1$. Only in Boolean algebras is this condition equivalent to exactness of x.

The proofs of the following two lemmata are omitted since they are relatively straightforward and moreover can be found in Fitting [15].

**Lemma 2.11** *In a bilattice with conflation:*

*a) 0, 1 and ⊥ are consistent, T is not consistent;*

*b) the set of consistent values is closed under* ∧, ∨, *and* ⊗ *(and their infinitary counterparts in the case of a complete bilattice), but not under* ⊕;

*c) the sum of a* k–directed *family of consistent values is also consistent (k–directed means that any two members are majorized in* $\leq_k$ *by a member of the family).*

*d) if f is a k–monotone operation which commutes with –, then the set of consistent values is closed under f.*

**Lemma 2.12** *In a bilattice with conflation:*

*a) 0, 1 are exact, ⊥, T are not;*

*b) the set of exact values is closed under* ∧, ∨ *(and their infinitary counterparts when the bilattice is complete), but not under* ⊕ *or* ⊗;

*c) all exact elements are consistent;*

*d) for any operation f which commutes with conflation the set of exact values is closed under f.*

Now it should be clear that bilattices with the two unary operations – and ¬ possess two way symmetry – horizontal and vertical – allowing their representation as sub-bilattices of possible world bilattices. In set based bilattices, which stem from Kripke models for languages with modalities □, ◊ and possibly other intensional connectives, there would be more bilattice operations - as a rule they are k-monotone and commute with conflation.

***External modalities*** Closely related to the notion of an exact truth value are the following two operations that can be defined in the presence of conflation. We call them *modalities* following a tradition in many-valued logic (cf. [27,54,56]).

**Definition 2.13** In a bilattice with conflation define:

$$\blacksquare x = (x)_1 \wedge -(x)_1;$$
$$\blacklozenge x = (x)_0 \vee -(x)_0.$$

Note that these two can be expressed equivalently as:

$$\blacksquare x = (x\vee\bot)\wedge(-x\vee T);$$
$$\blacklozenge x = (x\wedge\bot)\vee(-x\wedge T).$$

**Fact 2.14** *In a bilattice with* $\blacksquare$ *and* $\blacklozenge$*:*

$$\blacksquare 1 = \blacklozenge 1 = 1$$
$$\blacksquare 0 = \blacklozenge 0 = 0$$
$$\blacksquare T = 1,\ \blacklozenge T = 0$$
$$\blacksquare \bot = 0,\ \blacklozenge \bot = 1$$

**Lemma 2.15** *1.* $-\blacksquare x = \blacksquare x,\ -\blacklozenge x = \blacklozenge x;$

*2.* $\blacksquare\blacksquare x = \blacklozenge\blacksquare x = \blacksquare x$

$\blacklozenge\blacklozenge x = \blacksquare\blacklozenge x = \blacklozenge x;$

*3.* $\blacksquare(x\wedge y) = \blacksquare x\wedge\blacksquare y\ \ \blacklozenge(x\wedge y) = \blacklozenge x\wedge\blacklozenge y$

$\blacksquare(x\vee y) = \blacksquare x\vee\blacksquare y\ \ \blacklozenge(x\vee y) = \blacklozenge x\vee\blacklozenge y$

$\blacksquare(x\otimes y) = \blacksquare x\wedge\blacksquare y\ \ \blacklozenge(x\otimes y) = \blacklozenge x\vee\blacklozenge y$

$\blacksquare(x\oplus y) = \blacksquare x\vee\blacksquare y\ \ \blacklozenge(x\oplus y) = \blacklozenge x\wedge\blacklozenge y$

*If the billatice is with a negation commuting with –, then*

*4.* $\neg\blacksquare x = \blacklozenge\neg x.$

*Proof*: Easy check. For example (1) is established by observing that

$$-\blacksquare x = -((x\vee\bot)\wedge(-x\vee T)) = (-x\vee T)\wedge(--x\vee\bot) = \blacksquare x, \text{etc.}\ \blacksquare$$

**Lemma 2.16** *1.* $\blacksquare$ *and* $\blacklozenge$ *are t–monotone operations, but not k–monotone;*

*2. for consistent truth values x,* $\blacksquare x \leq_t \blacklozenge x;$

*3. exact truth values can be characterized by the condition* $\blacksquare x = x$ *(or, equivalently,* $\blacklozenge x = x$*);*

*4. in a distributive bilattice*

$$x = (\blacksquare x\wedge T)\vee(\blacklozenge x\wedge\bot),$$
$$-x = (\blacksquare x\wedge\bot)\vee(\blacklozenge x\wedge T).$$

*Proof*: We check only the first identity of (4): replacing in the right–hand side $\blacksquare x$ and $\blacklozenge x$ with their equivalent expressions from Definition 2.13 we get

$$((x\vee\bot)\wedge(-x\vee T)\wedge T)\vee(((x\wedge\bot)\vee(-x\wedge T))\wedge\bot) =$$
$$= ((x\vee\bot)\wedge T)\vee((x\wedge\bot)\vee(-x\wedge 0)) = (x\wedge T)\vee 0\vee(x\wedge\bot)\vee 0 = x\wedge(T\vee\bot) = x.$$

■

**Remarks** 1. Note that in bilattices of the kind L×L (where L is a de Morgan lattice) the operations have the following outlook:

$$\blacksquare\langle a,b\rangle = \langle a,\sim a\rangle$$

$\blacklozenge$<a,b> = <~b,b>.

Accordingly, in a possible worlds model the truth conditions for $\blacksquare$A and $\blacklozenge$A take the form:

$$M,w \models \blacksquare A \text{ iff } M,w \models A;$$
$$M,w \models \blacklozenge A \text{ iff } M,w \not=\!| A;$$
$$M,w =\!| \blacksquare A \text{ iff } M,w \not\models A;$$
$$M,w =\!| \blacklozenge A \text{ iff } M,w =\!| A.$$

In **[0,1]**×**[0,1]**: $\blacksquare$<a,b> = <a, 1-a>, $\blacklozenge$<a,b> = <1-b,b>.

2. Using the representations $\blacksquare x = (x\vee\bot)\wedge(-x\vee T)$ and $\blacklozenge x = (x\wedge\bot)\vee(-x\wedge T)$ one can show that if F is a bi-filter, $x\in F$ iff $\blacksquare x\in F$; if D is a dual bi-filter, $x\in D$ iff $\blacklozenge x\in D$.

### *3 Set expansions of lattices of truth values*

Let **L** be a bounded lattice **L** = <L, ∧, ∨, 0, 1>. Elements of **L** represent truth values, e.g., they can be elements of an abstract logical matrix, or sets of possible worlds, etc. We assume also any number of additional *finitary* operations $o(x_1, ..., x_n)$. For example, we could have unary operations like negation (in the case of de Morgan lattices and Boolean algebras), modalities, or a binary operation of implication when we consider Heyting algebras, etc.

The basic idea explored in this section is to view subsets X, Y, Z, ... of L as new, "expanded" truth values. A set X could be said to embody the knowledge an observer has about the "real" truth value of a statement A, so $||A|| \in X$, where $||A||$ is the "real" truth value. Consequently, it can be envisioned as an infinitary disjunction $\bigvee\{||A|| = x : x \in X\}$ (cf. the Conclusion for a discussion). In this section we develop the algebraic aspects of such an approach.

***Internal operations*** Set expansions of lattices will be introduced step-by-step. As a first step we define a class of operations which are *expanded* counterparts of the operations in the basic algebra.

**Definition 3.1** The set expansion $\mathbf{L}^{set}$ of **L** = $<L, \wedge, \vee, 0, 1, \{o_i\}_{i \in I}>$ is an algebra based on the set of *all* subsets of L – $\wp(L)$ and having the following *internal* operations:

$$o(X_1,...,X_n) = \{o(x_1, ..., x_n) : x_k \in X_k,\ k=1,...,n\}.$$

where $o(x_1, ..., x_n)$ is an operation of **L**, e.g.:

$$X \wedge Y = \{x \wedge y : x \in X, y \in Y\}$$
$$X \vee Y = \{x \vee y : x \in X, y \in Y\}$$
$$1 = \{1\}$$
$$0 = \{0\}.$$

and if, say, the lattice L has a negation ~ or a modality  then

$$\sim X = \{\sim x : x \in X\}$$
$$X = \{ x : x \in X\}$$

The set expansion contains another pair of remarkable elements:

$$\top = \varnothing, \bot = L .$$

These are called *external* constants of $\mathbf{L}^{set}$ in contrast to the internal constants 0,1.

**Remark** The internal expansions of the lattice operations reflect a certain view on the interaction of information about the truth values of components of compound sentences, namely on how the structure of the compound formula guides us as to the set of possibilities for its "real" truth value. Put briefly: individual possible truth values interact completely independently from each other (cf. also the Conclusion).

Below we list several elementary properties of the introduced operations.

**Proposition 3.2** *1.* $X \wedge Y = Y \wedge X$, $X \vee Y = Y \vee X$;
*2.* $X \wedge (Y \wedge Z) = (X \wedge Y) \wedge Z$,
$X \vee (Y \vee Z) = (X \vee Y) \vee Z$;
*3.* $X \wedge 1 = X$, $X \vee 0 = X$.

*In the case when* **L** *is a say a de Morgan lattice we would have also:*
*4.* $\sim\sim X = X$,
$\sim(X \wedge Y) = \sim X \vee \sim Y$,
$\sim(X \vee Y) = \sim X \wedge \sim Y$,

*etc.*

*Proof:* The proof is easy: investigate the form of a typical element of the left–hand side and show that it can be transformed into an element of the right–hand side and *vica versa.* ■

Unfortunately not all identities concerning internal operations and valid in **L** are preserved in $\mathbf{L}^{\mathbf{set}}$, most notably the lattice laws of of idempotence, absorption and (if **L** happens to be distributive) distributivity. Let us look into this problem more closely. We start with a partial list of identities that are not in general preserved in the set expansion of a lattice **L**.

**Proposition 3.3** $X \wedge 0 = 0$, $X \vee 1 = 1$ *do not hold in the set expansion of* **2**, *while identities like* $X \wedge X = X$, $X \vee X = X$, $X \wedge (X \vee Y) = X$, $X \vee (X \wedge Y) = X$, $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$, $X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z)$, *etc., can all be refuted in the set expansion of the four element Boolean algebra.*

Such observations lead to a question natural from the algebraic prospective: *which identities in the above operations are preserved under set expansion*?

In order to formulate a partial answer we need some definitions and basic facts. Consider *internal* terms s, t,..., i.e., terms built from variables $v_1$, $v_2$, ..., and symbols of internal operations and constants. Such terms can be evaluated both in the lattice **L** and the expansion $\mathbf{L}^{\mathbf{set}}$. If an evaluation function v assigns to $v_i$ an element $X_i$ of $\wp(L)$ we write the value of a term $s(v_1,v_2,...,v_n)$ under v as $s(X_1,X_2,...,X_n)$. An expression like $s(a_1,...,a_n)$, where $a_1,...,a_n \in L$, should be understood in the same spirit. Note that for the value of an internal term $s(X_1,...,X_n)$, if $X_i$ =T, then s = T, and *vica versa* – if all $X_i$ • T, then s • T. The variables occurring in a term t form a set Var(t). Call a term *lean*, if all its variables have single occurrences.

Let now $K$ be a class of lattices. An equation s = t is a $K$–identity if it is valid in all members of $K$. A characterization of all $K^{set}$ identities in terms of $K$-identities can be obtained for certain classes of lattices. Clearly a $K^{set}$–identity is also a $K$–identity: any valuation $v$ for a lattice **L** is transformed into a valuation $v^*$ for $\mathbf{L}^{set}$ by putting $v^*(X) = \{v(x)\}$, i.e. taking the corresponding singletons. This construction has the following property:

$$v^*(s)=\{v(s)\}.$$

Thus any refuted in $K$ equation is refuted also in $K^{set}$: $v(s) \bullet v(t) \Rightarrow v^*(s) \bullet v^*(t)$.

**Definition 3.4** An identity s = t is *K–lean* if there exist a substitution σ and an equation u = v such that $s = u^{\sigma}$, $t = v^{\sigma}$, u = v is a $K$–identity and both u and v are lean. An identity s = t is *balanced,* if Var(s) = Var(t).

**Definition 3.5** 1. An inequality s ≠ t is *satisfiable in a class K* of lattices if there is a lattice $\mathbf{L} \in K$ and a valuation ν into **L** such that ν(s) ≠ ν(t).

2. A family of inequalities $\{s_i \neq t_i\}_{i \in I}$ is *simultaneously satisfiable* in $K$ if there exist a lattice $\mathbf{L} \in K$ and a valuation ν into **L** such that $\nu(s_i) \neq \nu(t_i)$, for all $i \in I$.

**Theorem 3.6** *Let K be a class of lattices such that any finite family of satisfiable in K inequalities is simultaneously satisfiable in K. Then the following two conditions are equivalent:*

*1. s=t is a $K^{set}$ identity;*

*2. s=t is balanced and K–lean.*

*Proof:* Let s = t be a balanced and $K$-lean identity. Note that, due to the fact that s = t is balanced, when evaluated in an expansion $\mathbf{L}^{set}$ s and t get values T in exactly the same instances. Thus in order to check if s = t is an identity in $\mathbf{L}^{set}$, it is sufficient to consider only valuations which *do not have* T in their range.

With the above restriction we have the following representation of the values of terms, when $s_1(v_1,\ldots,v_n)$ and $t_1(v_1,\ldots,v_n)$ are lean:

$$s_1(X_1,\ldots, X_n) = \{s_1(a_1,\ldots,a_n):a_i \in X_i\},$$

$$t_1(X_1,\ldots, X_n) = \{t_1(a_1,\ldots,a_n):a_i \in X_i\}.$$

Let now $s_1 = t_1$ be that $K$-identity of which s = t is a substitutional instance. Since $s_1 = t_1$ is an identity in **L**, the above two sets are equal, so $s_1$ and $t_1$ get the same values in $\mathbf{L}^{\mathbf{set}}$, but then as substitutional instances s and t also get the same values. Thus s = t is indeed a $\boldsymbol{K}^{\mathbf{set}}$-identity.

In the opposite direction we reason by contraposition: if s = t is either not $\boldsymbol{K}$-lean or not balanced, then it is not a $\boldsymbol{K}^{\mathbf{set}}$-identity. It is immediately clear that a non-balanced equation cannot be an identity in $\mathbf{L}^{\mathbf{set}}$. Thus we are left with the case of s = t being not $\boldsymbol{K}$-lean, which means that any $s_1 = t_1$, which is lean and of which s = t is a substitutional instance, is not a $\boldsymbol{K}$-identity, i.e. $s_1 \neq t_1$ is satisfiable in $\boldsymbol{K}$. In particular, if we consider the term $s^*$ obtained from $s(v_1,...,v_k)$ by assigning to consecutive occurrences of a variable $v_i$ different new variables $v_{i1}, v_{i2},...,v_{im_i}$, we get a family of satisfiable in $\boldsymbol{K}$ inequalities:

$$\{s^*(v_{11},...,v_{km_k}) \neq t^*: t^* \in T\}$$

Here T is the family of all lean terms t* obtained from t by the procedure of replacing different occurrences of a variable $v_i$ by different variables $v_{ij}$ in all possible combinations. Clearly T is finite and the above family of inequalities has to be simultaneously satisfiable in $\boldsymbol{K}$, so in some lattice $\mathbf{L} \in \boldsymbol{K}$ we have for some elements $a_{ij}$:

$$s^*(a_{11},...,a_{km}) \neq t^*(..,a_{ij},...).$$

Let $X_1 = \{a_{11},...,a_{1m_1}\}$, ..., $X_i = \{a_{i1},...,a_{im_i}\}$, etc. The claim $s(X_1,...,X_k) \neq t(X_1, ...,X_k)$ follows from the observation that $s^*(a_{11},...,a_{km}) \in s(X_1,...,X_k)$, but $s^*(a_{11},...,a_{km_k}) \notin t(X_1,...,X_k)$. ■

**Remarks** 1. The property from Definition 3.5 is possessed by a variety of classes of lattices, e.g., any class of Boolean algebras containing arbitrary large finite Boolean algebras or some infinite ones, any class of pseudo Boolean algebras with the same restrictions, etc.

2. A useful counterexample to a more liberal formulation of the above theorem is the class of all linear orders ***Lin***, for which, e.g., $X \wedge X = X$ and $X \vee X = X$ are $\boldsymbol{Lin}^{\mathrm{set}}$- identities (since in a linear order $a \wedge b$ and $a \vee b$ equal either a or b).

***External operations*** Since the set expansion of a lattice is built up from *sets* of elements, it is only natural to introduce also the set–theoretical operations:

$$X \oplus Y = X \cap Y,$$
$$X \otimes Y = X \cup Y,$$

their infinitary versions $\Sigma$ and $\Pi$, as well as the relation

$$X \leq_k Y \text{ iff } X \supseteq Y,$$

In $L^{set}$ it is also possible to consider the complement of a set X:

$$X^c = L \setminus X.$$

With such additional operations (called *external* to contrast them with the previous class of operations) set expansions turn into truth value spaces similar to bilattices: $\leq_k$ represents ordering by degree of knowledge (the smaller the set, the more we know about the "real" truth value), $\oplus$ and $\otimes$ give us different combinations of information about this possible truth value – "accept anything" joining of the information in $\oplus$, and "consensus" reduction to the information common to both sets in $\otimes$.

The analogy is not perfect though, as will be seen below. Note that the constant T represents a kind of non-significance value capable of destroying any internal statement (thus the information leading to T is not simply contradictory, but rather *senseless*).

**Proposition 3.5** *The internal operations are k-monotone, e.g.:*

$$X \leq_k X', Y \leq_k Y' \text{ imply } X \vee Y \leq_k X' \vee Y', \text{ etc.}$$

**Proposition 3.6** *With respect to the introduced external operations $L^{set}$ is a co-atomic, complete and completely distributive Boolean algebra (the algebra of all subsets of L with inverse inclusion) with least element $\perp$ and greatest element T. In particular the following laws hold:*

$$(\Sigma\{X_i: i \in I\}) \otimes Y = \Sigma\{X_i \otimes Y: i \in I\},$$

$$(\Pi\{X_i: i \in I\}) \oplus Y = \Pi\{X_i \oplus Y: i \in I\},$$

*etc.*

Let us examine some of the relations between external and internal operations.

**Proposition 3.7** *The following hold only as inequalities with respect to the k–order:*

$$X \wedge X \leq_k X, \quad X \vee X \leq_k X,$$

$$X \wedge (X \vee Y) \leq_k X, \quad X \vee (X \wedge Y) \leq_k X,$$

$$(X \wedge Y) \vee (X \wedge Z) \leq_k X \wedge (Y \vee Z), \quad (X \vee Y) \wedge (X \vee Z) \leq_k X \vee (Y \wedge Z), \text{etc.}$$

**Lemma 3.8** *In the set expansion of any bounded lattice we have:*

$$X \wedge (Y \otimes Z) = (X \wedge Y) \otimes (X \wedge Z), \quad X \vee (Y \otimes Z) = (X \vee Y) \otimes (X \vee Z),$$

*and in general for any* lean *internal term s(...,X,...):*

*$s(\ldots, Y\otimes Z, \ldots) = s(\ldots, Y, \ldots)\otimes s(\ldots, Z, \ldots)$;*
*with an infinitary version: $s(\ldots, \Pi\{X_i : i\in I\}, \ldots) = \Pi\{s(\ldots, X_i, \ldots) : i\in I\}$.*

As for ⊕ – we can claim only that $(X\vee Y)\oplus(X\vee Z)\leq_k X\vee(Y\oplus Z)$, $(X\wedge Y)\oplus(X\wedge Z) \leq_k X\wedge(Y\oplus Z)$, $s(\ldots, Y, \ldots)\otimes s(\ldots, Z, \ldots)\leq_k s(\ldots, Y\otimes Z, \ldots)$. For example the triple X = 1, Y= 0, Z = 1 shows the failure of an inequality opposite to the first one: $Y\oplus Z = \top$, so $X\vee(Y\oplus Z) = \top$ while $(X\vee Y)\oplus(X\vee Z) = 1$.

**Example 8** The simplest set expansion is $\mathbf{2}^{\mathbf{set}}$: it has four elements, all of them signature constants – 0,1, ⊥ and ⊤. With respect to ⊕, ⊗, and the complementation $c$ $\mathbf{2}^{\mathbf{set}}$ is a Boolean algebra, though with respect to ∨, ∧ it is *not* even a lattice since ⊤ is a *non–significance* value. $\mathbf{2}^{\mathbf{set}}$ can be considered with its internal negation ~ (inherited from the Boolean algebra **2**), it also has a sort of *conflation* operation –, related to $c$ and the internal negation ~ by $-X = (\sim X)^c$ (by the way in this structure this equals $\sim(X^c)$). In this way $-1 = 1$, $-0 = 0$, $-\bot = \top$, $-\top = \bot$ (just as a conflation should act), moreover we have $X \leq_k Y$ implies $-Y \leq_k -X$.

***Singletons and other curiosities in $L^{set}$*** In this sub-section we present several examples of interesting defining possibilities in $\mathbf{L}^{\mathbf{set}}$. Let us start with the observation that some relevant properties of subsets of L can be guaranteed by simple equations in $\mathbf{L}^{\mathbf{set}}$, e.g., $X\vee\bot = X$ defines all subsets of L that are *upward closed*; $X\wedge\bot = X$ defines all subsets of L that are *downward closed*; $X\vee X = X$ defines all ∨-closed subsets of L, i.e. subsets with the property $a,b\in X \Rightarrow a\vee b\in X$; $X\wedge X = X$ defines all ∧-closed subsets, and in general $o(X, \ldots, X) = X$ defines the sets that are closed with respect to the operation *o*. The meaning of the "bilattice projection operations" $(x)_0$ and $(x)_1$ may also be of interest, so we mention that $(X)_0 = X\wedge\bot$ is the *downward cone* of X; $(X)_1 = X\vee\bot$ is the *upward cone* of X. The combinations $(X)_0\oplus(X)_1 = \{y: \exists x_0,x_1\in X(x_0\leq y\leq x_1)\}$ and $(X)_0\otimes(X)_1 = \{y: \exists x_0,x_1\in X(y\leq x_0 \text{ and } x_1\leq y)\}$ give us the so–called *convex hull* and *cylinder* of X, respectively.

Using these observations one can give a definition of filters in **L** as the solutions of a simple system of equations:

$X\vee\bot = X$ (i.e. $(X)_1 = X$)

$X\wedge X = X$.

For ideals there is a dual system:

$X\wedge\bot = X$ (i.e. (X)0= X)

$X \vee X = X$

In order to define prime filters and ideals one must use the operation of complementation and add to the above systems $X^c \vee X^c = X^c$ and $X^c \wedge X^c = X^c$ respectively. In the case of singletons $\{x\}$, $\{x\} \vee \bot$ is the principle filter defined by x. It is easy to check that $\{x\} \vee \bot$ satisfies the three equations characterizing a prime filter *iff* x is an atom in **L**.

We may define singletons in $\mathbf{L}^{\mathbf{set}}$ as atoms in the lattice of set-theoretic operations, or, equivalently, as *co–atoms* in the $\leq_k$ order, where

X is a co–atom iff $X \neq T$ and $\forall Y(X \leq_k Y \Rightarrow X{=}Y \text{ or } Y{=}T)$.

Singletons have the following characteristic properties:

$\prod\{X_i{:}i \in I\} \leq_k \{x\} \Rightarrow \exists i\, X_i \leq_k \{x\}; X = \prod\{\{x\}{:}X \leq_k \{x\}\}$.

***Homomorphisms of set expansions*** General algebraic considerations suggest the importance of studying homomorphisms of set expansions and in particular homomorphisms into the smallest such algebra $\mathbf{2}^{\mathbf{set}}$.

As turns out, if homomorphisms $f{:}\mathbf{L}^{\mathbf{set}} \to \mathbf{M}^{\mathbf{set}}$ should respect the infinitary k-operations, i.e., $f(\Sigma\{X_i{:}i \in I\}) = \Sigma\{f(X_i){:}i \in I\}$ and $f(\prod\{X_i{:}i \in I\}) = \prod\{f(X_i){:}i \in I\}$, then the rigid structure of the set expansions leaves no room for variety, as witnessed by the following facts:

1. *Homomorphisms isolate T.* For a homomorphism $f{:}\mathbf{L}^{\mathbf{set}} \to \mathbf{M}^{\mathbf{set}}$ and a singleton $\{x\}$ in $\mathbf{L}^{\mathbf{set}}$ one has $\{x\} \vee 1 = 1$, so $f(\{x\}) \vee 1 = 1$ which implies that $f(\{x\}) \neq T$. Now using the representation $X = \prod\{\{x\}{:}x \in X\}$ we get $f(X) = \prod\{f(\{x\}){:}x \in X\}$. Thus if $X \neq T$, then $f(X) \neq T$.

2. *Source singletons are mapped into special partitions of the target singletons.* If $x \neq y$, then in $\mathbf{L}^{\mathbf{set}}$ $\{x\} \oplus \{y\} = T$, so $f(\{x\}) \oplus f(\{y\}) = T$. Thus different singletons are mapped into disjoint elements of $\mathbf{M}^{\mathbf{set}}$, moreover from $\prod\{\{x\}{:}x \in L\} = \bot$, we get that $\prod\{f(\{x\}){:}x \in L\} = \bot$, so the restriction of f over the singletons generates a special partition of M: let $P_x = f(\{x\})$, then the family $\{P_x{:}x \in L\}$ is a partition of M of a special kind, i.e., besides $x \neq y \Rightarrow P_x \cap P_y = \emptyset$ and $\cup\{P_x{:}x \in L\} = M$, we have also $P_x \vee P_y = P_{x \vee y}$; $P_x \wedge P_y = P_{x \wedge y}$, and in general $P_{o(x,\ldots,y)} = o(P_x, \ldots, P_y)$.

Applied to homomorphisms of $\mathbf{2}^{\mathbf{set}}$ this has the following effect: *if **L** is different from **2**, then there are no homomorphisms $f{:}\mathbf{2}^{set} \to \mathbf{L}^{set}$.*

3. *Epimorphisms are isomorphisms.* Epimorphisms of set expansions map singleton onto singletons: if $f(X) = \{z\}$ for a $z \in M$, then having, for any $x \in X$,

$X \leq_k \{x\}$ and consequently $\{z\} \leq_k f(\{x\})$, by the properties of co-atoms in $\mathbf{M}^{set}$, $\{z\} = f(\{x\})$, but that, in view of (2) above, implies $X = \{x\}$. Therefore there exist no epimorphisms of set-expansions except isomorphisms. Applied to homomorphism into $\mathbf{2}^{set}$, which are all epimorphisms (since all elements of $\mathbf{2}^{set}$ are signature constants) the latter fact yields the following corollary: *if* $\mathbf{L}$ *is different from* $\mathbf{2}$*, then there are no homomorphisms* $f{:}\mathbf{L}^{set} \to \mathbf{2}^{set}$.

***The external modalities*** If it is assumed that **L** is complete, then one can introduce in $\mathbf{L}^{set}$ a pair of unary operations: $\blacksquare X = \{\inf X\}$; $\blacklozenge X = \{\sup X\}$, with the following elementary properties

$$\blacksquare\blacksquare X = \blacklozenge\blacksquare X = \blacksquare X \qquad \blacklozenge\blacklozenge X = \blacksquare\blacklozenge X = \blacklozenge X$$
$$\blacksquare 1 = 1 \quad \blacksquare 0 = 0 \qquad \blacklozenge 1 = 1 \quad \blacklozenge 0 = 0$$
$$\blacksquare T = 1 \quad \blacksquare\bot = 0 \qquad \blacklozenge T = 0 \quad \blacklozenge\bot = 1$$

These can be easily checked as well as the next proposition.

**Proposition 3.9** *In the set expansion of a complete lattice* **L** *the following identities hold:*

$$\blacksquare(X \otimes Y) = \blacksquare X \wedge \blacksquare Y, \qquad \blacklozenge(X \otimes Y) = \blacklozenge X \vee \blacklozenge Y,$$

*and their infinitary versions*

$$\blacksquare\Pi\{X_i : i \in I\} = \bigwedge\{\blacksquare X_i : i \in I\};$$

$$\blacklozenge\Pi\{X_i : i \in I\} = \bigvee\{\blacklozenge X_i : i \in I\}.$$

*For* $X, Y \neq T$ *we have also:*

$$\blacksquare(X \wedge Y) = \blacksquare X \wedge \blacksquare Y, \qquad \blacklozenge(X \vee Y) = \blacklozenge X \vee \blacklozenge Y,$$

*and the infinitary (for* $X_i \neq T$*)*

$$\blacksquare\bigwedge\{X_i : i \in I\} = \bigwedge\{\blacksquare X_i : i \in I\}; \quad \blacklozenge\bigvee\{X_i : i \in I\} = \bigvee\{\blacklozenge X_i : i \in I\}.$$

*If* $\mathbf{L}$ *is in addition completely distributive, then for* $X, Y \neq T$

$$\blacksquare(X \vee Y) = \blacksquare X \vee \blacksquare Y, \qquad \blacklozenge(X \wedge Y) = \blacklozenge X \wedge \blacklozenge Y,$$

*and the infinitary laws (for* $X_i \neq T$*):*

$$\blacksquare\bigvee\{X_i : i \in I\} = \bigvee\{\blacksquare X_i : i \in I\};$$

$$\blacklozenge\bigwedge\{X_i : i \in I\} = \bigwedge\{\blacklozenge X_i : i \in I\}.$$

*If there is a de Morgan negation* ~ *among the internal operations, then*

$$\sim\blacksquare X = \blacklozenge\sim X \text{ and } \sim\blacklozenge X = \blacksquare\sim X.$$

Unfortunately the complementation and ⊕ behave rather erratically with respect to the external modalities.Having ■ and ♦ the notion "X is a singleton" can be expressed as ■X = X (which is equivalent to ♦X = X).

**Proposition 3.10** *The set S of singletons in $L^{set}$ can be characterized as follows:*

*(1) S={X:■X = X} = {X:♦X = X}*

*(2) S is the set of co–atoms in $L^{set}$,*

**Corollary 3.11** 1. *By the above we have in $L^{set}$:*

*■X = X iff $X \neq T$ and $\forall Y (X \leq_k Y \Rightarrow X = Y$ or $Y = T)$.*

2. *With the internal operations ∧, ∨ restricted to it S becomes a lattice isomorphic to* **L**.

3. *If s=t is an* **L**-*identity, then s'=t' is an $L^{set}$–identity,where s' is $s^\sigma$,t' is $t^\sigma$ and the subsitution σ assigns to a variable X the term ■X.*

***Info–algebras*** Here we try to algebraically define the "useful" part of the set expansions of lattices of truth values following the analogy with bilattices. Perhaps it is already clear from the results cited above that the operation ⊕ (together with the constant T) is the cause of most of the discrepancies between set algebras and bilattices. One way out of this is to drop ⊕ and *c* from the signature (but keep T as some interesting applications involve such nonsensical values).

**Definition 3.12** An info-algebra **A** of a given *internal* signature (∧,∨,...,0,1) is a partially ordered set <A,$\leq_k$> which is a *complete lower semi-lattice.* The least and greatest elements of **A** are ⊥ and T respectively. The join operation is denoted by ∏ (its finitary version by ⊗). The algebra has the following properties:

A1. Internal structure:

1.1. internal operations are monotone with respect to $\leq_k$;

1.2. s(...,T,...) = T for internal terms;

1.3. $s(...,\prod\{X_i:i\in I\},...) = \prod\{s(...,X_i,...):i\in I\}$, for *lean* terms.

A2. Singletons:

2.1. the set of singletons a,b,c ...of **A** - $L_A$ is a distributive lattice with respect to the restrictions of ∧ and ∨ (and it is bounded by 0 and 1);

2.2. $\prod\{X_i:i\in I\}\leq_k a \Rightarrow \exists i\; X_i\leq_k a$;

2.3. $X = \prod\{a:X\leq_k a\}$.

**Lemma 3.13** *For an internal operation $o(X_1, ..., X_n)$ and a singleton a:*

$$o(X_1, ..., X_n) \leq_k a \Rightarrow \exists b_1,...,b_n \in L_A (X_i \leq_k b_i\ (i=1,...,n) \text{ and } o(b_1,...,b_n) = a).$$

*Proof:* By 2.3 $o(X_1, ..., X_n) = o(\prod\{a:X_1 \leq_k a\},...,\prod\{a:X_n \leq_k a\})$ . Applying 1.3 we get $o(X_1, ..., X_n) = \prod\{o(b_1,...,b_n):X_1 \leq_k b_1,...,X_n \leq_k b_n\}$. Now if $o(X_1, ..., X_n) \leq_k a$, then $\prod\{o(b_1,...,b_n):X_1 \leq_k b_1,...,X_n \leq_k b_n\} \leq_k a$ and using 2.2 we get the desired result. ■



It is not difficult to see that $\mathbf{L}^{\mathbf{set}}$ is an info-algebra. Conversely, any info-algebra **A** can be represented as a set expansion (with the restricted signature), namely as $\mathbf{L}_{\mathbf{A}}^{\mathbf{set}}$, as the next lemma points out.

**Lemma 3.14** *Let A be an info-algebra, then $A \cong L_A^{set}$ (as info-algebras).*

*Proof:* The map $f:A \to \wp(L_A)$ defined as $f(X) = \{a:X \leq_k a\}$ is an isomorphism of A and $\mathbf{L}_{\mathbf{A}}^{\mathbf{set}}$:

$$f(o(X_1, ..., X_n)) = \{a: o(X_1, ..., X_n) \leq_k a\} =$$
$$= \{a: \exists b_1,..., b_n (X_i \leq_k b_i\ (i = 1,...,n) \text{ and } o(b_1,...,b_n) = a)\} =$$
$$= \{a: \exists b_1,..., b_n (b_i \in f(X_i) \text{ and } o(b_1,...,b_n) = a\} = o(f(X_1),...,f(X_n));$$

$$f(\prod\{X_i:i\in I\}) = \{a: \prod\{X_i:i\in I\} \leq_k a\} = \{a: \exists i\ X_i \leq_k a\} = \prod\{\{a: X_i \leq_k a\}:i\in I\} =$$
$$= \prod\{f(X_i):i\in I\}.$$

The rest of conditions are checked as usual. ■

***Homomorphisms of info-algebras*** An interesting special case of the info-algebra homomorphims are the homomorphisms of info-algebras into $2^{\mathbf{set}}$ which exist in abundant numbers in contrast with the full set expansions.

**Lemma 3.15** *Let $f:A \to 2^{set}$ be a homomorphism. The subset F of $L_A$ defined by the following stipulation:*

$$F = \{a: f(a) \in D_1\}, \text{ where } D_1 = \{T, 1\},$$

*is a prime filter in $\boldsymbol{L_A}$.*

*Proof:* Checking that F possesses the properties of a prime filter:

1. If $a \in F$ and $a \leq b$, then $b \in F$: $a \in F$ means that $f(a) \in D_1$, $a \leq b$ implies that $a \vee b = b$. Consider $a \vee b = a \vee b = b$. Apply f and get $f(a) \vee f(b) = f(b)$. In $\mathbf{2}^{set}$ the equation $a \vee b = b$ together with $a \in D_1$ guarantees that $b \in D_1$, therefore $f(b) \in D_1$.

2. $a \wedge b \in F$ iff $a \in F$ and $b \in F$: if $a \wedge b \in F$, then clearly by 1. $a, b \in F$. Assume now that $a \in F$ and $b \in F$, which is equivalent to $f(a) \in D_1$ and $f(b) \in D_1$, that gives $f(a) \wedge f(b) \in D_1$ (by the laws of $\mathbf{2}^{set}$) and, since f is a homomorphism, $f(a \wedge b) \in D_1$, so $a \wedge b \in F$.

3. $a \vee b \in F$ iff $a \in F$ or $b \in F$: clearly, if $a \in F$ or $b \in F$, then $a \vee b \in F$. In the opposite direction: if $a \vee b \in D_1$, then $f(a) \vee f(a) \in D_1$, but in $\mathbf{2}^{set}$ this guarantees that either $f(a) \in D_1$ or $f(b) \in D_1$, i.e., $a \in F$ or $b \in F$. ■

**Lemma 3.16** *Let A be an info-algebra of the minimal internal signature $(\wedge, \vee, 0, 1)$. Then the mapping $f: A \to 2^{set}$ defined from a prime filter F in $L_A$ by first stipulating for the singletons:*

$$f(a) = \begin{cases} 1, \text{ if } a \in F \\ 0, \text{ otherwise} \end{cases}$$

*and then for arbitrary elements of $L^{set}$: $f(X) = \prod \{f(a) : X \leq_k a\}$, is a homomorphism.*

*Proof:* Clearly the constants $0, 1, \bot, \top$ are mapped correctly by f: $f(1) = 1$ and $f(0) = 0$ by the fact that F is a proper filter, $f(\bot) = \bot$, since $1 \otimes 0 = \bot$ and $f(\top) = \top$ trivially. Let us also note that for $a, b \in L$

$$f(a \wedge b) = f(a) \wedge f(b);$$
$$f(a \vee b) = f(a) \vee f(b),$$

since F is prime. To check if $f(X \wedge Y) = f(X) \wedge f(Y)$ reason as follows:

$$f(X \wedge Y) = \prod\{f(c) : X \wedge Y \leq_k c\} = \prod\{f(\{a \wedge b\}) : X \leq_k a, Y \leq_k b\} =$$
$$= \prod\{f(a) \wedge f(b) : X \leq_k a, Y \leq_k b\} = \prod\{f(a) : X \leq_k a\} \wedge \prod\{f(b) : Y \leq_k b\} =$$
$$= f(X) \wedge f(Y).$$

Similarly we can establish that $f(X \vee Y) = f(X) \vee f(Y)$. As for the operation $\otimes$, we proceed as follows:

$$f(X \otimes Y) = \prod\{f(c) : X \otimes Y \leq_k c\} = \prod\{f(a) : X \leq_k a\} \otimes \prod\{f(b) : Y \leq_k b\} =$$
$$= f(X) \otimes f(Y). ■$$

Now we can formulate an effect of the above construction important for the logical developments below:

**Lemma 3.17** *If $X \notin D_1$ (= {1,T}) in a minimal info-algebra A, then there is a homomorphism $f:A \rightarrow 2^{set}$ such that $f(X) \notin D_1$ in $2^{set}$.*

*Proof:* Since $X \notin D_1$, there is an a with $X \leq_k a$ and such that a ≠ 1, and we can find a filter F in $\mathbf{L_A}$ which omits a.The homomorphism $f:A \rightarrow \mathbf{2^{set}}$ associated with this filter maps X either on 0 (if $X \cap F = \varnothing$) or on ⊥ (when $X \cap F \neq \varnothing$). Anyway, $f(X) \notin D_1$. ■

It would be a natural next step to formulate a representation theorem for info–algebras as sub-algebras of the set expansions of frame lattices. Along the lines of the results in the preceding sections one needs to define first the notion of $\mathbf{2^{set}}$ model on a frame F = <W,...> as a mapping $\phi$: W×Fml→$\mathbf{2^{set}}$ where for each w∈W, $\phi$(w,A)∈ Hom($\mathcal{L}$,$\mathbf{2^{set}}$) for the minimal signature. $|A|_1$ denotes the set {w : $\phi$(w, A) = 1}, $|A|_0$, $|A|_\perp$, $|A|_\top$ have similar meanings. A *singleton a* in the frame F is a partition of W, i.e., $|a|_1 = W \setminus |a|_0$, in other words singletons are *exact* truth values.

We write $A \leq_\phi B$ if $\forall w(\phi(w,A) \leq_k \phi(w,B))$. Now let $\|A\|$ = {a: a is a singleton in F and $A \leq_\phi a$}. The idea is to establish that $\|.\|$ is from Hom($\mathcal{L}$, $(F^+)^{\mathbf{set}}$). Disappointingly enough this is true only for the *internal* part of the language.

**Lemma 3.18** *The mapping $\|.\|$ is from Hom($\mathcal{L}_0$, $(F^+)^{set}$), where $\mathcal{L}_0$ is the language without ⊗.*

*Proof:* To begin with, note that $|A|_\top \neq \varnothing$ iff $\|A\| = \varnothing$ – that takes care of the situation when there are occurrences of T in the valuations of A or B. Observe also that $|A \wedge B|_1 = |A|_1 \cap |B|_1$, $|A \wedge B|_0 = |A|_0 \cup |B|_0$, $|A \vee B|_1 = |A|_1 \cup |B|_1$ and $|A \vee B|_0 = |A|_0 \cap |B|_0$. Using these we can prove, e.g. $\|A \wedge B\| = \|A\| \wedge \|B\|$: $\|A \wedge B\|$ = {c: $A \wedge B \leq_\phi c$}, but $A \wedge B \leq_\phi c$ means that $|c|_1 \supseteq |A|_1 \cap |B|_1$ and $|c|_0 \supseteq |A|_0 \cup |B|_0$. Let now a, b be the singletons with $|a|_1 = c \cup |A|_1$ and $|b|_1 = c \cup |B|_1$. It is easy to see that $A \leq_\phi a$ and $B \leq_\phi b$, and that $a \cap b = c$. Thus $\|A\| \wedge \|B\| \supseteq \|A \wedge B\|$. To justify the opposite inclusion note that if $A \leq_\phi a$ and $B \leq_\phi b$, then $A \wedge B \leq_\phi a \wedge b$.

The case with ∨ is left to the reader. ■

**Remark** The above construction can be cast in a slightly different form in order to reveal its kinship to a very well-known idea in many-valued logic: the notion of *supervaluations.*

Call a mapping $\psi$: W×Fml→$2^{\mathbf{set}}$ a supervaluation for $\phi$ if:

1. $\phi \leq_k \psi$;
2. $\psi$ is *exact*, i.e. for variables p, $\psi(w,p) \in \{0,1\}$.

Let $\|A\|_s = \{|A|_1^{\psi}$: $\psi$ is a supervaluation for $\phi\}$. The claim that $\|A\|_s = \|A\|$ for $A \in Fml(\mathcal{L}_0)$ is easily justified by the fact that $\{|A|_1^{\psi}, |A|_0^{\psi}\}$ is a singleton.

The difference with the tradition lies in the way supervaluations are used: while customarily the family of supervaluations for a given $\phi$ is converted into a valuation $\bar{\phi}$, which in our case would look as follows:

$$\bar{\phi}(w,A) = \mathbf{T}, \text{ if there are no supervaluations;}$$

and in the presence of supervaluations,

$$\bar{\phi}(w,A) = \begin{cases} 1 & \text{if } \forall\psi(\psi(w,A) = 1) \\ 0 & \text{if } \forall\psi(\psi(w,A) = 0) \\ \bot & \text{otherwise,} \end{cases}$$

our usage avoids such a conversion and keeps the family of supervaluations as a new generalized truth value.

***External info-algebras*** An info-algebra **A** is called *external* , if two unary operations ■ and ♦ can be defined in **A** satisfying:

1. $\blacksquare\prod\{X_i : i \in I\} = \bigwedge\{\blacksquare X_i : i \in I\}$;
2. $\blacklozenge\prod\{X_i : i \in I\} = \bigvee\{\blacklozenge X_i : i \in I\}$;
3. $\blacksquare a = \blacklozenge a = a$, for singletons.

All relevant properties of the external modalities in set expansions (as documented in Lemma 3.9) can be derived from the above definition (which presupposes that $\mathbf{L_A}$ is a complete lattice).

**Remark** As a general remark to the idea of set expansion: it should have become clear by now that admitting arbitrary sets of lattice elements as generalized truth values, insisting at the same time that this move is caused by incompleteness of information, uncertainty of data, or vagueness of predicates, etc., is somewhat inconsistent: to define an *arbitrary* set X requires very detailed information about the individual members of X., which seems implausible in circumstances when one *lacks* the relevant knowledge. Thus *restricted* classes of such generalized truth values seem to be a more realistic way of modeling imperfect epistemic situations (see the Conclusion for a discussion).

As an example of restrictions that arise from specific imperfections of data let us consider a frame F = <W, ...> where the available knowledge permits us to discern different possible worlds only up to certain equivalence relation $\approx$, so the only subsets of W one "can be aware of" are unions of equivalence classes $[w] = \{w': w \approx w'\}$ with respect to the *indiscernibility* relation $\approx$ (called *rough* sets). For a set $V \subseteq W$ denote by $V_1$ the set $\{w: [w] \subseteq V\}$ and let $V_0 = \{w: [w] \cap V \neq \emptyset\}$ – these are respectively the biggest rough set inside V and the biggest rough set including V. An observer having the above limitations can know the "real" truth value $\|A\|$ only to contain $\|A\|_1$ and to be contained in $\|A\|_0$, so any set between these two bounds would be a possible "real" truth value for him, i.e., the generalized interpretation of A would be $\{U: \|A\|_1 \subseteq U \subseteq \|A\|_0\}$. Note that in such a setting the existence of the operations ■ and ♦ is a natural consequence: $■\{U: \|A\|_1 \subseteq U \subseteq \|A\|_0\} = \|A\|_1$ and $♦\{U: \|A\|_1 \subseteq U \subseteq \|A\|_0\} = \|A\|_0$.

The next section is devoted to the algebraic study of such special subfamilies of the set expansions which lead us back to the realm of bilattices.

## 4 Intervals

Let us first review briefly the notion of an interval in a lattice **L**. Recall that a subset X of L is called *connected* (or *convex*) if:

$$x,y \in X \text{ and } x \leq z \leq y \Rightarrow z \in X.$$

A subset X is a *closed interval* in the lattice, if $X = \{x: a \leq x \leq b\}$ for some $a,b \in L$, in this case X is denoted by [a,b]. Note that a closed interval is a bounded set, and furthermore it is a bounded convex set.

***Generalized intervals*** In a bounded lattice **L** we can consider *generalized (closed) intervals* as pairs [a,b] where the condition for a proper interval $a \leq b$ does not necessarily hold. On the set of all generalized intervals one can introduce the following operations:

$$[a,b] \wedge [c,d] = [a \wedge c, b \wedge d]$$
$$[a,b] \vee [c,d] = [a \vee c, b \vee d]$$
$$[a,b] \otimes [c,d] = [a \wedge c, b \vee d]$$
$$[a,b] \oplus [c,d] = [a \vee c, b \wedge d]$$
$$0 = [0,0] \; 1 = [1,1]$$
$$\bot = [0,1] \; \top = [1,0]$$

$[a,b] \leq_t [c,d]$ if $[a,b] \wedge [c,d] = [a,b]$

$[a,b] \leq_k [c,d]$ if $[a,b] \otimes [c,d] = [a,b]$.

**Lemma 4.1** *1. $[a,b] \otimes [b,d]$ is the smallest interval that contains $[a,b]$ and $[c,d]$ in case they are proper intervals, i.e. it is equal to $[[a,b] \cup [c,d]]$;*

*2. if $[a,b] \oplus [c,d]$ is proper, then it is the intersection of $[a,b]$ and $[c,d]$;*

*3. $[a,b] \leq_t [c,d]$ iff $a \leq c$ and $b \leq d$;*

*4. restricted to the proper intervals $\leq_k$ is the inverse inclusion $\supseteq$.*

**Proposition 4.2** *If **L** is a distributive bounded lattice, then the set of all generalized intervals in **L** equipped with the above operations (denoted from now on by $\mathbf{L}^2$) is a distributive bilattice.*

In $\mathbf{L}^2$ a conflation operation and the external modalities can be defined by:

$-[a,b] = [b,a]$

$\blacksquare[a,b] = [a,a]$

$\blacklozenge[a,b] = [b,b]$.

In this way the notions of consistent and exact truth values become available in $\mathbf{L}^2$. Let us observe that an element $[a,b]$ is a consistent value in $\mathbf{L}^2$ iff $[a,b]$ is a proper interval, and also an element of $\mathbf{L}^2$ is exact iff it is of the form $[a,a]$.

**Example 10** In the possible worlds example intervals can be simulated by two forcing relations $\models_1$ and $\models_0$ with the suitable truth conditions:

$w \models_1 A \wedge B$ iff $w \models_1 A$ and $w \models_1 B$;

$w \models_0 A \wedge B$ iff $w \models_0 A$ and $w \models_0 B$;

the same for the disjunction, etc. For the external modalities one has:

$w \models_1 \blacksquare A$ iff $w \models_1 A$ ;

$w \models_1 \blacklozenge A$ iff $w \models_0 A$;

$w \models_0 \blacksquare A$ iff $w \models_1 A$ ;

$w \models_0 \blacklozenge A$ iff $w \models_0 A$.

These conditions stem from the interpretation of mappings $\phi(w,A)$ into **4** (viewed as $\mathbf{2}^2$) as defining the forcing relations $\models_1$ and $\models_0$ by demanding that $w \models_1 A$ if $T \leq_t \phi(w,A)$, i.e. if $\phi(w,A) \in D_1$, and that $w \models_0 A$ if $\perp \leq_t \phi(w,A)$, i.e. $\phi(w,A) \in D_0$.

The relationship between $\mathbf{L}^2$ and $\mathbf{L}\times\mathbf{L}$ is clear in the presence of negation: $\mathbf{L}^2 \cong \mathbf{L}\times\mathbf{L}$ by the map $f(\langle a,b\rangle) = [a,\sim b]$, but in general they are not isomorphic – an interesting example of this being pseudo Boolean algebras.

**Remark** In a classical possible worlds model the above isomorphism determines the following equivalences:

$$w \models_1 A \text{ iff } w \models A \text{ and } w \models_0 A \text{ iff } w \not\models A.$$

Also $\mathbf{[0,1]}^2$ is just another representation of $\mathbf{[0,1]}\times\mathbf{[0,1]}$ (as was mentioned in [2]) since $\mathbf{[0,1]}$ is a de Morgan lattice.

***External bilattices*** A bilattice **B** is called *external*, if one can define two unary operations on **B** – ■ and ♦, having the properties listed in Lemmata 2.14 and 2.15 (items 2 and 3). A different representation theorem for external bilattices can be obtained from the $\mathbf{L}^2$ construction.

**Theorem 4.3** *If* ***B*** *is a distributive external billatice, then it can be represented as* $\boldsymbol{L}^2$ *for a distributive lattice* ***L***.

*Proof:* Take **L** to be {x: ■x = x} = {x: ♦x = x}, i.e. the set of exact values, with the restriction of $\leq_t$ on it. Clearly **L** is a distributive bounded lattice. Define now a map $f\colon \mathbf{B}\to\mathbf{L}^2$ by setting f(x) = [■x,♦x].

The claim is that f maps **B** isomorphically onto $\mathbf{L}^2$. This is (or should be already) a routine check using the identities for ■ and ♦. ■

Note that if x is a consistent element of **B** (with respect to the conflation definable in **B** by means of (4) of Lemma 2.15) then ■x $\leq_t$ ♦x, so the generalized interval corresponding to x is *proper*.

**Remark** This theorem is useful in those cases where there is no de Morgan negation available in **L**. In particular it is applicable to pseudo Boolean algebras.

Let us in conclusion describe a connection between certain elements of the set expansions of distributive lattices and the interval construction. To this purpose consider $\mathbf{L}^{\mathbf{set}}$ and the following map:

$$f(X) = [\inf X, \sup X].$$

**Proposition 4.4** *1. Restricted to elements of $L^{set}$ different from T, f is an info-algebra homomorphism into the set of consistent elements of $L^2$ when the underlying lattice is complete and completely distributive.*

*2. Restricted to finite non-empty subsets of L, f is an info-algebra homomorphism into the set of consistent elements of $L^2$ (here no conditions are imposed on L).*

*Proof.* Let us show for example that f(X∨Y)=f(X)∨f(Y).

It is easy to demonstrate (using either the complete distributivity or the finiteness of X and Y) that:

inf(X∨Y) = inf{x∨y: x∈X,y∈Y} = inf{x∨infY: x∈X} = infX∨infY;

sup(X∧Y) = sup{x∧y:x∈X,y∈Y} = sup{x∧supY: x∈X} = supX∧supY,

etc.

Thus f(X∨Y) = [inf(X∨Y), sup(X∨Y)] = [infX∨infY, supX∨supY] = [infX, supX] ∨ [infY, supY] = f(X)∨f(Y).

The other cases can be checked similarly. ■

So the map f is an *info-algebra* homomorphism of $L^{set}$ \ {T} into the set of consistent elements of the billatice $L^2$. In fact it is a map *onto* since every proper interval is a non–empty subset of L and f is the identity over such subsets of L.

*Part II*

## LOGICAL ASPECTS

It was pointed out in the Introduction that there are different ways to use bilattices as semantic tools. In this part we explore the possibility to treat them (as well as the info-algebras) in the traditional fashion of algebraic logic: as logical matrices semantically defining logical systems, i.e., as a generalization of the truth-table method used in the classical logic.

***Generalized matrices*** Let us recall some definitions and some basic facts from the theory of propositional logics (cf. [8,61]), restricted to our current needs, e.g., we presuppose only a finite number of finitary logical connectives.

A propositional language $\mathcal{L}$ over an infinite (countable in our case) set of variables Var($\mathcal{L}$) is an absolutely free algebra of some signature (with the above restrictions), freely generated by Var($\mathcal{L}$). We assume that the operations include conjunction and disjunction, the two constants 0,1 and eventually other constants and operations $o_i$. The elements of this algebra are called *formulae* and form a set Fml($\mathcal{L}$), so $\mathcal{L} = \langle \mathrm{Fml}(\mathcal{L}), \wedge, \vee, \ldots, 0, 1\rangle$. Reference to $\mathcal{L}$ will be dropped from now on, whenever possible.

A mapping C: $\wp(\mathrm{Fml}) \rightarrow \wp(\mathrm{Fml})$ is a *consequence operation* (note that this is the monotone case, as will be throughout this section), if the following conditions are satisfied for all subsets Γ, Δ of Fml:

(1) $\Gamma \subseteq C(\Gamma)$;
(2) $C(\Gamma) = C(C(\Gamma))$;
(3) $\Gamma \subseteq \Delta$ implies $C(\Gamma) \subseteq C(\Delta)$.

A consequence operation C is *compact* if for every Γ:

$$C(\Gamma) = \bigcup \{C(\Delta) : \Delta \subseteq \Gamma \text{ and } \Delta \text{ is finite}\}.$$

A *generalized matrix* $\mathcal{M} = \langle \mathbf{A}, D, H\rangle$ for $\mathcal{L}$ is a triple where:

1. **A** is an algebra similar to the language $\mathcal{L}$ – the truth–value space;
2. D is subset of the truth–value space – the elements of D are the *designated* truth values;
3. $H \subseteq \mathrm{Hom}(\mathcal{L}, \mathbf{A})$ – its elements are called *admissible valuations*.

A matrix M is called *standard* when H = Hom ($\mathcal{L}$,**A**). Every class of matrices ***K*** determines a consequence operation $C_{\boldsymbol{K}}$:

$$A \in C_{\boldsymbol{K}}(\Gamma) \text{ iff } \forall\, \mathcal{M} = \langle \mathbf{A}, D, H\rangle \in \boldsymbol{K}\ \forall h \in H \text{ (if } h[\Gamma] \subseteq D, \text{ then } h(A) \in D).$$

For singleton classes $\{\mathcal{M}\}$ we write simply $C_{\mathcal{M}}$. As a rule instead of $A \in C_{\boldsymbol{K}}(\Gamma)$ we write $\Gamma \models_{\boldsymbol{K}} A$, or just $\Gamma \models A$ when there is no danger of confusion.

A *propositional logic* **S** is a pair $<\mathcal{L},C>$ where C is a consequence operation in the language $\mathcal{L}$ of **S**. A class ***K*** is called a *semantics* for the logic $<\mathcal{L},C>$, if $C = C_{\boldsymbol{K}}$.

*Finitely–approximable logics* **S** are characterized by a class of finite matrices, i.e. there is a class ***K*** of finite matrices which is a semantics for **S**. *Finite* logics (or *finite-valued* logics) are determined by a finite class of finite matrices.

In this part we are planning to present a variety of logical systems arising from the truth spaces considered in the previous sections in a single unified framework by proving theorems of the following kind: *the system S is characterized by a class of algebras (bilattices, info–algebras, etc.), viewed as generalized logical matrices.*

Within this framework one is confronted with several *choices* which determine the logical system:

- the choice of language, i.e., what bilattice or info-algebra operations should be considered *logical*, as opposed to others that are *computational* in character;
- the choice of D – the distinguished truth values;
- the choice of the class H – the admissible valuations, i.e. what types of homomorphisms are considered relevant.

With respect to the first mentioned choice there are several approaches. The *liberal* approach is to consider *all* operations available in the investigated class as equally logical, so the propositional language is to have the same signature as the class itself, i.e. all operations have corresponding logical connectives. We are going to give several examples of this approach, e.g., the logic of all bilattices with all operations, the logic of info–algebras, etc.

A second, more *restrictive*, approach is to put down criteria by which an operation can be judged to be logical or not. Let us formulate a few criteria as an example:

1) A logical operation in a bilattice should preserve acquired information about truth values – assuming this, we arrive at the requirement of *k–monotonicity*. Such a criterion excludes for example the conflation and the external modalities as candidates for logical operations.

2) One may insist on *conservativity* of a logical operation in the sense that when applied to exact truth values it should yield exact values – this criterion excludes the k–operations $\oplus$ and $\otimes$, but admits conflation and the external modalities.

3) An even less restrictive requirement is to demand the operations to *preserve consistency*, i.e., when applied to consistent truth values the operation

should give consistent truth values – the operations ⊕ and conflation are excluded in this case.

The problem with the set of distinguished truth values is in fact a version of the more general question of what conditions should be met in order to recognize a statement as "true". A first thing that comes to mind is that the intended meaning of the elements of bilattices, set expansions or info–algebras implies that $1 \in D$, so the simplest decision is $D = \{1\}$, i.e., to recognize as valid only such inferences that preserve the property of "being only true". But, besides being a not very happy choice technically, the restriction of D to $\{1\}$ is not easily justifiable. In fact the inclusion of $\top$ in every respect is just as rational. Thus a more reasonable choice of D would have $1, \top \in D$. In a bilattice though D should include with every member also all elements that are "truer" – this leads to the choice

$$D_1 = \{x: \top \leq_t x\}.$$

Such a set of distinguished truth values emphasizes the positive foundations to accept something as true; if we have *all the reasons* to accept a statement as true we do accept it. Another possible choice would favour elements that are *not refutable*, so we would have

$$D_0 = \{x: \bot \leq_t x\}.$$

This time, if we have *no reasons to refute* a statement, we accept it.

In set expansions $D_1$ and $D_0$ can be viewed as the sets of elements X satisfying respectively $\forall x \in X(\ x=1)$ and $\exists x \in X(x=1)$. Unfortunately the former definition leads to some technical complications and destroys the duality between the two choices. A naturally better option is $D_0 = \{X:\sup X = 1\}$ – dual to $\{X:\inf X = 1\}$ (= $D_1$).

Restrictions concerning the set of admitted homomorphisms can include such requirements as e.g.:

- H is the set of all *consistent* valuations, i.e. functions whose range contains only consistent elements, or
- H is the set of valuations into info–algebras that contain only *finite* sets in their ranges, etc.

## 5 Logical systems related to bilattices

Let us fix some terminology: if a logic is defined by a class of generalized logical matrices with no restrictions on the admissible homomorphisms we speak of a *standard* system, if H is restricted to the homomorphisms with consistent values, then we use the term *consistent* logics; if it contains only finite sets as values (this in the case of set expansions), then the logic is *finitary*. If the set D of distinguished truth values is $D_1$, we speak of *positive* logics, if it is $D_0$ – we say that the logic is *negative.* If the signature of the language and the algebras of the class coincide, we speak of a *full* logic, otherwise we use different adjectives showing which operations in the algebras have language connectives as counterparts.

A general remark should be made at this point: formulae of the considered languages can be viewed as terms and equations A = B can happen to be identities in the classes of algebras determining the consequence operation. In such cases both A$\models$B and B$\models$A hold. The converse implication is not valid in general.

***The standard logics of bilattices*** We start with the logic determined by the class of all distributive bilattices. The propositional language $\mathcal{L}$ is assumed to contain four propositional constants 0,1, $\bot$,T and four binary connectives $\vee$, $\wedge$, $\oplus$ and $\otimes$. It is to be interpreted in distributive bilattices viewed as *standard* logical matrices with $D_1 = \{x: T \leq_t x\}$ as their set of distinguished truth values (thus we deal with the full positive standard logic).



**Lemma 5.1** *The consequence operation $\models$ determined by the above choices has the following properties:*

*0. If $0\in\Gamma$, $\bot\in\Gamma$ or $A\in\Gamma$, then $\Gamma\models A$*
*$\Gamma\models 1$; $\Gamma\models T$*
*1. $\Gamma\models A$ and $\Gamma\models B$ implies $\Gamma\models A\wedge B$*
*$\Gamma, A\models C$ or $\Gamma, B\models C$ implies $\Gamma, A\wedge B\models C$*
*2. $\Gamma\models A$ or $\Gamma\models B$ implies $\Gamma\models A\vee B$*
*$\Gamma, A\models C$ and $\Gamma, B\models C$ implies $\Gamma, A\vee B\models C$*
*3. $\Gamma\models A$ and $\Gamma\models B$ implies $\Gamma\models A\otimes B$*
*$\Gamma, A\models C$ or $\Gamma, B\models C$ implies $\Gamma, A\otimes B\models C$*
*4. $\Gamma\models A$ or $\Gamma\models B$ implies $\Gamma\models A\oplus B$*
*$\Gamma, A\models C$ and $\Gamma, B\models C$ implies $\Gamma, A\oplus B\models C$*

(Here $\Gamma$ is a set of formulae, A, B, C – formulae; we write $\Gamma$, A instead of $\Gamma\cup\{A\}$, etc.)

*Proof:* In order to check (0) – (4) assume that **B** is a distributive bilattice and $h\in Hom(\mathcal{L},\mathbf{B})$.

For (0) note that no h maps 0 or $\bot$ into $D_1$, that 1 and T are in $D_1$ and that if h maps $\Gamma$ into $D_1$, then A being a member of $\Gamma$ implies that $h(A)\in D_1$, too. Skipping (1) we consider now the more interesting second part of (2). Assume that $\Gamma$, A$\models$ C and $\Gamma$, B$\models$ C and reason from the contrary to establish the conclusion. Assume that **B** is the bilattice providing the counterexample h which maps $\Gamma$ and A$\vee$B into $D_1$ but $h(C)\notin D_1$. Since **B** may happen to be unsuitable for our purposes (having elements x and y such that $x\vee y\in D_1$ while neither of them is a designated truth value), we transfer this counterexample to **4** resorting to the fact that there is a homomorphism f:**B**→**4** which maps $D_1$ into {1,T} (= $D_1$ of **4**), but f(h(C))∉$D_1$. The composition $h_1$ = f;h is from Hom($\mathcal{L}$,**4**). Clearly $h_1$ maps Γ and A∨B into $D_1$, but $h_1$(C)∉$D_1$. Now by the assumption we have $h_1$(A) and $h_1$(B) outside {1,T}. On the other hand $h_1$(A∨B) = $h_1$(A)∨$h_1$(B), and we get a contradiction since in **4** the disjunction of elements outside {1,T} is outside {1,T}, too.

The establishing of (3) runs as follows: h maps Γ into $D_1$, in particular h(A) and h(B) belong to $D_1$, i.e., $T \leq_t h(A)$ and $T \leq_t h(B)$, but then by the monotonicity of ⊗ we have $T \leq_t h(A) \otimes h(B)$. For the second part note that from $T \leq_t h(A \otimes B)$ it follows by monotonicity of ⊕, that $T \oplus h(A) \leq_t (h(A) \otimes h(B)) \oplus h(A) = h(A)$, i.e. $T \leq_t h(A)$, and analogously $T \leq_t h(B)$.

Case (4) is similar to (2). ■

For the presentation of this and subsequent logical systems we choose *sequential style* calculi. For our purposes it is sufficient to adopt the view that sequents are of the form Γ |– A, where Γ is a finite set of formulae, A – a formula. Thus our systems are inherently intuitionistic (having the restriction of single formula in the right-hand side).

**Definition 5.2** A sequent Γ |– A is a *basic* sequent, or an *axiom*, if one of the three below hold:

1. A∈Γ,
2. 0∈Γ or ⊥∈Γ,
3. A = 1 or A = T.

**Rules** With each connective of $\mathcal{L}$ two types of rules are associated, governing association of formulae in the left–hand and the right hand side respectively:

$$(\wedge|\!-)\qquad \frac{\Gamma, A \mid\!- C}{\Gamma, A\wedge B \mid\!- C} \qquad \frac{\Gamma, B \mid\!- C}{\Gamma, A\wedge B \mid\!- C}$$

$$(|\!-\wedge)\qquad \frac{\Gamma \mid\!- A; \quad \Gamma \mid\!- B}{\Gamma \mid\!- A\wedge B}$$

(∨|–) $$\frac{\Gamma, A \vdash C; \quad \Gamma, B \vdash C}{\Gamma, A\vee B \vdash C}$$

(|–∨) $$\frac{\Gamma \vdash A}{\Gamma \vdash A\vee B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A\vee B}$$

(⊗|–) $$\frac{\Gamma, A \vdash C}{\Gamma, A\otimes B \vdash C} \qquad \frac{\Gamma, B \vdash C}{\Gamma, A\otimes B \vdash C}$$

(|–⊗) $$\frac{\Gamma \vdash A; \quad \Gamma \vdash B}{\Gamma \vdash A\otimes B}$$

(⊕|–) $$\frac{\Gamma, A \vdash C; \quad \Gamma, B \vdash C}{\Gamma, A\oplus B \vdash C}$$

(|–⊕) $$\frac{\Gamma \vdash A}{\Gamma \vdash A\oplus B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A\oplus B}$$

A sequent is provable if it can be derived from basic sequents by means of the rules.

**Lemma 5.3** *If a sequent Π–A is provable, then Π=A.*

*Proof:* Induction on the height of the derivation tree of the sequent. For basic sequents see (0) of the previous lemma, the rest of the items there take care of the induction step. ■.

**Remark** The present formulation makes familiar structural rules such as *weakening*, *contraction*, and *cut* redundant in the sense that they are admissible rules in the system. In later systems formulated with the cut rule it is not always clear whether it is eliminalbe.

Let Γ now be an arbitrary set of formulae. We define [Γ] as {B:for some finite subset $\Gamma_0$ of Γ the sequent $\Gamma_0 \vdash B$ is provable}. Call Δ a theory if [Δ] = Δ. The set of all formulae Fml is an example of a theory – the *trivial* theory. A non–trivial theory Δ is *prime* if $A\vee B\in\Delta \Rightarrow A\in\Delta$ or $B\in\Delta$ and $A\oplus B\in\Delta \Rightarrow A\in\Delta$ or $B\in\Delta$.

**Lemma 5.4** *Let Δ be a prime theory, then the following hold:*

*0. T, I ∈ Δ, 0,⊥∉ Δ;*
*1. A∧B∈Δ iff A∈Δ and B∈Δ;*
*2. A∨B∈Δ iff A∈Δ or B∈Δ;*
*3. A⊗B∈Δ iff A∈Δ and B∈Δ;*
*4. A⊕B∈Δ iff A∈Δ or B∈Δ.*

**Lemma 5.5** *If A∉[Γ], then there exists a prime theory Δ⊇Γ such that A∉Δ.*

*Proof:* The argument is standard: the family of all non–trivial theories containing Γ and omitting A is a Zorn family and therefore has maximal elements. It is easy to show for a maximal element Δ that Δ is a prime theory, for example B∨C∈Δ implies B∈Δ or C∈Δ since if the contrary is assumed, we would have both [Δ,B] and [Δ, C] not in the family and so A∈[Δ,B] and A∈[Δ,C], which easily implies that A∈[Δ], contrary to the assumption. ■

**Theorem 5.6** *If Γ⊨A, then A∈[Γ].*

*Proof:* This is the so–called *completeness* property. In particular it means that if Γ⊢A is not a provable sequent, then it is not semantically valid (i.e. we use contraposition). Let A∉[Γ], then by the lemma above there is a prime theory Δ containing Γ and such that A∉Δ. Having this Δ we can define a valuation h:$\mathcal{L}$→ **4** mapping Γ into {1,T} = $D_1$ and A outside $D_1$ thus showing that Γ⊭ A. For example, if we set for a propositional variable p:

$$h(p) = \begin{cases} 0 \text{ if } p\notin\Delta \\ 1 \text{ if } p\in\Delta \end{cases}$$

and extend this to a homomorphism into **4**, the by induction on the complexity of a formula B it can be shown that:

$$h(B)\in D_1 \text{ iff } B\in\Delta.$$

Indeed the induction steps are justified by the corresponding clauses of Lemma 5.4, e.g.: h(C∧B)∈$D_1$ iff h(C)∧h(B)∈$D_1$ iff (by the properties of **4**) h(C)∈$D_1$ and h(B)∈$D_1$ iff (by the induction hypothesis) C∈Δ and B∈Δ iff C∧B∈Δ; h(C⊕B)∈$D_1$ iff h(C)⊕h(B)∈$D_1$ iff (by the properties of **4**) h(C)∈$D_1$ or h(B)∈$D_1$ iff (by the induction hypothesis) C∈Δ or B∈Δ iff C⊕B∈Δ, etc. ■

**Remarks** 1. With the same success we might have used any other function h satisfying h(p)∈$D_1$ iff p∈Δ, e.g., h(p) = ⊥ if p∉Δ, h(p) = T if p∈Δ.

2. Note that **4** is adequate for the logic of all distributive bilattices, i.e. [Γ] = $C_{4^+}$(Γ), where $4^+$ = <**4**, $D_1$, Hom($\mathcal{L}$,**4**)>.

3. Thus in fact the logic of all distributive bilattices is a finite (four-valued) logic, it is compact, decidable and so on.

The full negative standard logic of all bilattices (defined by matrices of the kind $\mathcal{B}$= <**B**, $D_0$, Hom($\mathcal{L}$,**4**)> with $D_0 = \{x: \bot \leq_t x\} = \{x: x \leq_k 1\}$) is in its basic structure quite similar to the positive one. The new consequence operation has the properties from Lemma 5.1 with the following differences: in (0) T and ⊥ interchange places, and in (3) and (4) ⊕ and ⊗ exchange places (for example instead of (3) one has Γ⊨ A and Γ⊨ B implies Γ⊨ A⊕B and Γ, A⊨ C or Γ, B⊨ C implies Γ, A⊕B⊨ C). These properties are established now with a reference to the existence of homomorphisms into **4** mapping prescribed elements outside $D_0$ – a consequence of the theory of *dual* bi-filters.

Corresponding changes have to be made in the rules and in the notion of a basic sequent: Γ⊢A is an axiom in the new system iff (1) - (3) from Definition 5.2 hold with T and ⊥ interchanging places. The rules concerning ⊕ and ⊗ undergo similar interchanges. This leads to changes in the properties of prime theories (Lemma 5.4, where in item (0) we have now ⊥, 1∈Δ, 0,T∉Δ, and in items (3) and (4) ⊕ and ⊗ exchange roles) but the proof of the completeness theorem is just as simple as that of Theorem 5.6:

$$[\Gamma] = C_{\mathcal{4}}(\Gamma), \text{ where } \mathcal{4} = \langle \mathbf{4}, D_0, \mathrm{Hom}(\mathcal{L},\mathbf{4})\rangle.$$

Thus the negative and the positive full standard logics of all distributive bilattices are *k*-dual versions of each other.

***The logics of bilattices with negation*** Let us consider an extended language $\mathcal{L}$ which contains the unary connective ¬. Correspondingly the relevant class of bilattices consists now of all distributive bilattices with negation. The sequent system for the full standard positive logic is obtained by adding to the basic system the rules for sequents containing negation.

**Rules for ¬:**

$$(\neg\wedge\vdash) \qquad \frac{\Gamma,\neg A\vdash C; \quad \Gamma,\neg B\vdash C}{\Gamma, \neg(A\wedge B)\vdash C}$$

$$(\vdash\neg\wedge) \qquad \frac{\Gamma\vdash \neg A}{\Gamma\vdash \neg(A\wedge B)} \qquad \frac{\Gamma\vdash \neg B}{\Gamma\vdash \neg(A\wedge B)}$$

$$(\neg\vee\vdash)\quad \frac{\Gamma, \neg A \vdash C}{\Gamma, \neg(A\vee B) \vdash C} \qquad \frac{\Gamma, \neg B \vdash C}{\Gamma, \neg(A\vee B) \vdash C}$$

$$(\vdash\neg\vee)\quad \frac{\Gamma \vdash \neg A;\ \ \Gamma \vdash \neg B}{\Gamma \vdash \neg(A\vee B)}$$

$$(\neg\otimes\vdash)\quad \frac{\Gamma, \neg A \vdash C}{\Gamma, \neg(A\otimes B) \vdash C} \qquad \frac{\Gamma, \neg B \vdash C}{\Gamma, \neg(A\otimes B) \vdash C}$$

$$(\vdash\neg\otimes)\quad \frac{\Gamma \vdash \neg A;\ \ \Gamma \vdash \neg B}{\Gamma \vdash \neg(A\otimes B)}$$

$$(\neg\oplus\vdash)\quad \frac{\Gamma, \neg A \vdash C;\ \ \Gamma, \neg B \vdash C}{\Gamma, \neg(A\oplus B) \vdash C}$$

$$(\vdash\neg\oplus)\quad \frac{\Gamma \vdash \neg A}{\Gamma \vdash \neg(A\oplus B)} \qquad \frac{\Gamma \vdash \neg B}{\Gamma \vdash \neg(A\oplus B)}$$

$$(\neg\neg\vdash)\quad \frac{\Gamma, A \vdash C}{\Gamma, \neg\neg A \vdash C} \qquad (\vdash\neg\neg)\quad \frac{\Gamma \vdash A}{\Gamma \vdash \neg\neg A}$$



**Lemma 5.7** *If $\sqcap$–A is provable, then $\sqcap$=A.*

*Proof:* The proof is analogous to the proof of Lemma 5.1 and makes use of the identities from Fact 2.4 and the proposition about homomorphisms of negated bilattices: if F is a prime bi-filter in a bilattice with negation, then the pair (F,¬F) determines a homomorphism into **4** (recall that ¬F is a prime bi-ideal in this case).■

Again we can extend the left–hand side of any unprovable sequent $\Gamma \vdash A$ into a prime theory $\Delta$ with $A \notin \Delta$. To the properties of prime theories from Lemma 5.4 we must add the following:

5. $\neg(A\wedge B)\in\Delta$ iff $\neg A\in\Delta$ or $\neg B\in\Delta$;
6. $\neg(A\vee B)\in\Delta$ iff $\neg A\in\Delta$ and $\neg B\in\Delta$;

7. $\neg(A\otimes B)\in\Delta$ iff $\neg A\in\Delta$ and $\neg B\in\Delta$;
8. $\neg(A\oplus B)\in\Delta$ iff $\neg A\in\Delta$ or $\neg B\in\Delta$;
9. $\neg\neg A\in\Delta$ iff $A\in\Delta$.

**Theorem 5.8** *If $\Gamma\models A$, then $A\in[\Gamma]$.*

*Proof:* The proof is similar to the previous completeness proof. Given a prime theory $\Delta$ we define a function from Var into **4**:

$$h(p) = \begin{cases} \top, \text{ if } p\in\Delta, \neg p\in\Delta \\ 1, \text{ if } p\in\Delta, \neg p\notin\Delta \\ 0, \text{ if } p\notin\Delta, \neg p\in\Delta \\ \perp, \text{ if } p\notin\Delta, \neg p\notin\Delta \end{cases}$$

For the extension of h to a homomorphism from Hom($\mathcal{L}$,**4**) we can show (with the help of the properties of prime theories) that the above is preserved for an arbitrary formula B:

$$h(B) = \begin{cases} \top, \text{ if } B\in\Delta, \neg B\in\Delta \\ 1, \text{ if } B\in\Delta, \neg B\notin\Delta \\ 0, \text{ if } B\notin\Delta, \neg B\in\Delta \\ \perp, \text{ if } B\notin\Delta, \neg B\notin\Delta \end{cases}$$

The assumption that $\Delta$ does not contain A guarantees that $h(A)\notin D_1$, while h maps $\Gamma$ into $D_1$. Thus $\Gamma\models A$ is refuted (in the smallest possible bilattice with negation – **4**).
■

**Remarks** 1. In view of Theorem 2.5 the above system is the full positive logic of all squares **L**×**L**.

2. Again $[\Gamma] = C_{4^+}(\Gamma)$, where **4** is considered with its negation, so this logic is four-valued and hence all the familiar consequences: compactness, decidability, etc.

The negative counterpart of the logic with negation can be obtained by adding to the basic negative system a family of rules for the negation *k-dual* to the family above, i.e. with $\oplus$ and $\otimes$ exchanging places, for example ($\vdash\neg\otimes$) becomes now

$$\frac{\Gamma\vdash\neg A}{\Gamma\vdash\neg(A\otimes B)} \qquad \frac{\Gamma\vdash\neg B}{\Gamma\vdash\neg(A\otimes B)}$$

This system is complete with respect to $\mathit{4}^{\neg}$.

***The logics of external bilattices***Let us consider the class of all external bilattices as logical matrices for a language containing ■ and ♦ and study its full positive logic. Since the set of distinguished truth values is $D_1$ we have ■A|=A and A|=■A, so the corresponding rules should be:

(■|–) $\dfrac{\Gamma, A \vdash C}{\Gamma, \blacksquare A \vdash C}$ (|–■) $\dfrac{\Gamma \vdash A}{\Gamma \vdash \blacksquare A}$.

For ♦ we have a bunch of rules (parallel to the rules concerning negation):

(♦∧|–) $\dfrac{\Gamma, \blacklozenge A \vdash C}{\Gamma, \blacklozenge(A \wedge B) \vdash C}$ $\dfrac{\Gamma, \blacklozenge B \vdash C}{\Gamma, \blacklozenge(A \wedge B) \vdash C}$

(|–♦∧) $\dfrac{\Gamma \vdash \blacklozenge A; \quad \Gamma \vdash \blacklozenge B}{\Gamma \vdash \blacklozenge(A \wedge B)}$

(♦∨|–) $\dfrac{\Gamma, \blacklozenge A \vdash C; \quad \Gamma, \blacklozenge B \vdash C}{\Gamma, \blacklozenge(A \vee B) \vdash C}$

(|–♦∨) $\dfrac{\Gamma \vdash \blacklozenge A}{\Gamma \vdash \blacklozenge(A \vee B)}$ $\dfrac{\Gamma \vdash \blacklozenge B}{\Gamma \vdash \blacklozenge(A \vee B)}$

(♦⊕|–) $\dfrac{\Gamma, \blacklozenge A \vdash C}{\Gamma, \blacklozenge(A \oplus B) \vdash C}$ $\dfrac{\Gamma, \blacklozenge B \vdash C}{\Gamma, \blacklozenge(A \oplus B) \vdash C}$

(|–♦⊕) $\dfrac{\Gamma \vdash \blacklozenge A}{\Gamma \vdash \blacklozenge(A \oplus B)}$ $\dfrac{\Gamma \vdash \blacklozenge B}{\Gamma \vdash \blacklozenge(A \oplus B)}$

(♦⊗|–) $\dfrac{\Gamma, \blacklozenge A \vdash C; \quad \Gamma, \blacklozenge B \vdash C}{\Gamma, \blacklozenge(A \otimes B) \vdash C}$

$$(\vdash\blacklozenge\otimes)\quad \frac{\Gamma\vdash\blacklozenge A;\ \ \Gamma\vdash\blacklozenge B}{\Gamma\vdash\blacklozenge(A\otimes B)}$$

$$(\blacklozenge\blacklozenge\vdash)\quad \frac{\Gamma,\blacklozenge A\vdash C}{\Gamma,\blacklozenge\blacklozenge A\vdash C}\qquad (\blacklozenge\blacksquare\vdash)\quad \frac{\Gamma,\blacksquare A\vdash C}{\Gamma,\blacklozenge\blacksquare A\vdash C}$$

$$(\vdash\blacklozenge\blacklozenge)\quad \frac{\Gamma\vdash\blacklozenge A}{\Gamma\vdash\blacklozenge\blacklozenge A}\qquad (\vdash\blacklozenge\blacksquare)\quad \frac{\Gamma\vdash\blacksquare A}{\Gamma\vdash\blacklozenge\blacksquare A}$$

**Theorem 5.9** *$\Gamma\vdash A$ is provable iff $\Gamma\models A$.*

*Proof:* The correctness part is routine, for the completeness we note that prime theories $\Delta$ have here the following properties – additional to the ones from Lemma 5.4:

5'. $\blacksquare B\in\Delta$ iff $B\in\Delta$;
6'. $\blacklozenge(B\wedge C)\in\Delta$ iff $\blacklozenge B\in\Delta$ and $\blacklozenge C\in\Delta$;
7'. $\blacklozenge(B\vee C)\in\Delta$ iff $\blacklozenge B\in\Delta$ or $\blacklozenge C\in\Delta$;
8'. $\blacklozenge(B\otimes C)\in\Delta$ iff $\blacklozenge B\in\Delta$ or $\blacklozenge C\in\Delta$;
9'. $\blacklozenge(B\oplus C)\in\Delta$ iff $\blacklozenge B\in\Delta$ and $\blacklozenge C\in\Delta$;
10'. $\blacklozenge\blacksquare B\in\Delta$ iff $\blacksquare B\in\Delta$;
11'. $\blacklozenge\blacklozenge B\in\Delta$ iff $\blacklozenge B\in\Delta$.

Given a prime theory $\Delta$ we define a map h:

$$h(p)=\begin{cases}\top, & \text{if } \blacksquare p\in\Delta, \blacklozenge p\notin\Delta\\ 1, & \text{if } \blacksquare p\in\Delta, \blacklozenge p\in\Delta\\ 0, & \text{if } \blacksquare p\notin\Delta, \blacklozenge p\notin\Delta\\ \bot, & \text{if } \blacksquare p\notin\Delta, \blacklozenge p\in\Delta\end{cases}$$

The extension of h to all formulae is easily shown to satisfy the same conditions with an arbitrary formula B instead of the variable p, so if $A\notin\Delta$, then $h(A)\notin D_1$, while members of $\Delta$ (among them all $B\in\Gamma$) are mapped to $D_1$. ■

**Remark** External bilattices are in fact of the kind $\mathbf{L}^2$, so the positive logic of generalized intervals in distributive lattices coincides with $<\mathcal{L},C_{4^+}>$, where **4** is viewed as the bilattice $\mathbf{2}^2$.

For the negative logic ■ and ♦ interchange roles, since we have the following fact:

A has a value in $D_0$ iff ♦A is evaluated as 1.

Therefore the rules which have to be added to the basic negative standard logic of bilattices are:

• rules for ♦ to replace the rules for ■:

$$(\blacklozenge\vdash)\quad \frac{\Gamma, A\vdash C}{\Gamma, \blacklozenge A\vdash C} \qquad (\vdash\blacklozenge)\quad \frac{\Gamma\vdash A}{\Gamma\vdash \blacklozenge A}.$$

• dual versions of the remaining rules, i.e., versions in which everywhere ♦ and ■ change places, as well as ⊕ and ⊗. For example such a rule would be:

$$(\blacksquare\oplus\vdash)\quad \frac{\Gamma, \blacksquare A\vdash C}{\Gamma, \blacksquare(A\oplus B)\vdash C} \quad \frac{\Gamma, \blacksquare B\vdash C}{\Gamma, \blacksquare(A\oplus B)\vdash C}.$$

Adjusting the notion of a prime theory to the new inference system a proof of a completeness theorem is readily available along the lines of previous proofs: the full negative standard logic of external bilattices is $<\mathcal{L}, C_{\mathcal{F}}>$.

***The logics of de Morgan bilattices*** As pointed out in Section 4, if **L** is a de Morgan lattice, then $\mathbf{L}^2 \cong \mathbf{L}\times\mathbf{L}$. Thus the class of bilattices which are external and at the same time have a negation (we call them here *de Morgan bilattices*) defines logics that combine the features of the logics of both superclasses. In bilattices of the above kind, where negation and conflation commute, the two external modalities are interdefinable, so in order to axiomatize their full positive standard logic we choose a language with ¬ and ■ and treat the other connectives (like ♦ or conflation –) as abbreviations.

Here we encounter for the first time the effect of the fact that modalized formulae (i.e. formulae with one of the external modalities as the principle connective) assume only exact values, e.g., ¬■B and ■B cannot have simultaneously their truth values in $D_1$. Thus the positive system has an augmented notion of a basic sequent:

Γ⊢A is an axiom, if also ¬■B∈Γ and ■B∈Γ, for some formula B. The rules are the rules of the basic logic plus the rules for ¬ and the two rules concerning ■. The differences in the notion of axiom are reflected in an additional property of prime theories:

if ■B∈Δ, then ¬■B∉Δ.

From a prime theory Δ we can obtain a valuation by the same definition as in the case of pure logic with negation:

$$h(p) = \begin{cases} T, \text{ if } p\in\Delta, \neg p\in\Delta \\ 1, \text{ if } p\in\Delta, \neg p\notin\Delta \\ 0, \text{ if } p\notin\Delta, \neg p\in\Delta \\ \bot, \text{ if } p\notin\Delta, \neg p\notin\Delta \end{cases}$$

The proof that this extends to a full homomorphism has just one new moment - the case of ■B:

h(■B) = T iff ■B∈Δ and ¬■B∈Δ (both sides are impossible);

h(■B) = 1 iff ■B∈Δ and ¬■B∉Δ (note here that ■B∈Δ iff B∈Δ, and that B∈Δ implies ¬■B∉Δ; then use the induction hypothesis);

The case h(■B) = 0 iff ■B∉Δ and ¬■B∈Δ and h(■B) = ⊥ iff ■B∉Δ and ¬■B∉Δ are checked similarly. Otherwise the proof proceeds in the same way as above.

**Remark** The system we have just proven complete is also the positive logic of the probabilistic bilattice **[0,1]**×**[0,1]**.

For the negative logic (i.e. the logic defined by $D_0$) it is more convenient to consider ♦ as basic connective and ■ as an abbreviation. Consequently we make the following changes in the negative standard full logic of bilattices with negation:

- extend the language;
- extend the notion of axiom by adding the clause that Γ⊢A is a basic sequent if for some B, {♦B,¬♦B}⊆Γ;
- add the rules (♦⊢) and (⊢♦).

For the system just described we have a completeness proof similar to the one above and exploiting the relations of modalized formulae and prime theories.

***The logics of intuitionistic bilattices*** Our next example of a logic will be the system defined by all bilattices of the kind **L**×**L**, where **L** is a pseudo–Boolean algebra, considered as standard logical matrices with $D_1$ = {x: T$\leq_t$x}, i.e. the positive standard full logic. For the lack of a better name we call such bilattices *intuitionistic*. Let us recall that in pseudo–Boolean algebras there is a binary operation of *relative pseudo–complementation* a→b (which generates also a *pseudo–negation* ~a = a→0). So in **L**×**L** we can consider alongside the negation ¬ some additional operations:

<a,b>→<c,d> = <a→c,a∧d>;

~x = x→0 (so that ~<a,b> = <~a,a>);

x↔y = (x→y)∧(y→x) (so that <a,b>↔<c,d> = <a↔c,(a∧d)∨(c∧b)>).

Note also that <a,b>∈$D_1$ iff a = 1.

**Remark** We have chosen to lift the implication from the pseudo Boolean algebra to the corresponding intuitionistic bilattice by means of the above definition following one of the possible intuitions about forcing and rejecting an implication: it shoud be forced whenever forcing the antecedent guarantees that the consequent will be forced; it should be rejected in all cases when the antecedent is forced but the consequent is rejected. Accordingly we presuppose possible worlds models with truth-conditions as follows:

$w \models A \to B$ iff $\forall w'(w \leq w'$ and $w' \models A \Rightarrow w' \models B)$;

$w =\!\mid A \to B$ iff $w \models A$ and $w =\!\mid B$.

Let us warn the readers that the model here are intuitionistic, i.e., $\models$ and $=\!\mid$ are both monotone with respect to $\leq$:

$w \models A$ and $w \leq w' \Rightarrow w' \models A$;

$w =\!\mid A$ and $w \leq w' \Rightarrow w' =\!\mid A$.

The signature of the language will contain $\wedge, \vee, \otimes, \oplus$, and $\neg$, the constants (of which all except 0 to be viewed as abbreviations), and also $\to$, ~ and $\leftrightarrow$ (the latter two viewed as abbreviations). For such a choice of connectives we have a nice property of the consequence operation:

**Lemma 5.10** *$\to$ is a suitable implication, i.e. $\Gamma, A \models B$ iff $\Gamma \models A \to B$.*

*Proof:* Assume $\Gamma$, $A \models B$ and that h: $\mathcal{L} \to \mathbf{L} \times \mathbf{L}$ maps $\Gamma$ into D. Consider h(A→B) = h(A)→h(B) and let h(A) = <a,b>, h(B) = <c,d>. Note that if h(A→B) is not a member of $D_1$, then its first projection a→c ≠ 1, so not a≤c and therefore not <a,b> $\leq_t$ <c,d>. Using the existence of prime bi-filters containing <a,b> but omitting <c,d> and the corresponding homomorphisms into **4** = **2**×**2** we can find an $h_1$:$\mathcal{L} \to$**4** mapping $\Gamma$ and A into $D_1$, but keeping the value of B outside $D_1$. In the opposite direction we use the fact that in any pseudo Boolean algebra **L**, a→c = 1 and a = 1 implies c = 1. ■

Having a semantic consequence operation together with an implication suitable for it, one can use Hilbert style systems based on axioms and just one rule – *Modus Ponens* . In the present case we propose the following axiomatic system:

0. All intuitionistically valid schemata for → and 0;

1. $A \wedge B \to A$, $A \wedge B \to B$, $A \to (B \to A \wedge B)$
   $A \to A \vee B$, $A \to A \vee B$, $(A \to C) \to ((B \to C) \to (A \vee B \to C))$
2. $A \otimes B \to A$, $A \otimes B \to B$, $A \to (B \to A \otimes B)$
   $A \to A \oplus B$, $A \to A \oplus B$, $(A \to C) \to ((B \to C) \to (A \oplus B \to C))$

3. $\neg\neg A \leftrightarrow A$
   $\neg(A \to B) \leftrightarrow A \wedge \neg B$

$\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$
$\neg(A \vee B) \leftrightarrow \neg A \wedge \neg B$

4. $\neg(A \otimes B) \leftrightarrow \neg A \otimes \neg B$
$\neg(A \oplus B) \leftrightarrow \neg A \oplus \neg B$

**Remark** The first projection of any $h \in Hom(\mathcal{L}, L \times L)$ is a homomorphism into **L** with respect to the "intuitionistic" part of the language (containing 0,1 and the connectives $\wedge, \vee, \rightarrow$). Thus a restricted formula A is a bilattice tautology iff it is an intuitionisic theorem.

To check that all axioms are bilattice tautologies is a routine matter. Let us give only one example: $\neg(A \rightarrow B) \leftrightarrow A \wedge \neg B$. Let $h(A) = \langle a,b \rangle$, $h(B) = \langle c,d \rangle$, then $h(\neg(A \rightarrow B)) = \langle a \wedge d, a \rightarrow c \rangle$, so when $h(\neg(A \rightarrow B)) \in D_1$ one has $a \wedge d = 1$. Therefore $h(A) \in D_1$ and $h(\neg B) \in D_1$. Conversely, $h(A \wedge \neg B) \in D_1$ implies $a = 1$, $d = 1$, and consequently $h(\neg(A \rightarrow B)) \in D_1$.

Among the theorems of the system are all truth-table equivalences for the constants, e.g. $\neg T \leftrightarrow T$, etc. We mention also $\neg{\sim}A \leftrightarrow A$, which shows an interesting interaction of the two negations in the system

**Theorem 5.11** *The proposed system is complete, i.e., A is a theorem if A is a bilattice tautology.*

*Proof:* Let us consider the set W of all prime theories $\Delta$ ordered by set–theoretical inclusion as an intuitionistic frame F (the canonical frame). On F we define a model:

$$\Delta \models p \text{ iff } p \in \Delta$$
$$\Delta =\!\mid p \text{ iff } \neg p \in \Delta$$

By induction on the complexity of a formula B one can prove the important *truth lemma*:

$$\Delta \models B \text{ iff } B \in \Delta$$
$$\Delta =\!\mid B \text{ iff } \neg B \in \Delta$$

Take the implication as an example. Prime theories $\Delta$ have two features relating to $\rightarrow$:

$$B \in \Delta, B \rightarrow C \in \Delta \Rightarrow C \in \Delta,$$
$$\forall \Delta' \supseteq \Delta\ (B \in \Delta' \Rightarrow C \in \Delta') \Rightarrow B \rightarrow C \in \Delta.$$

The first is obvious, the second is established by the familiar trick of extending a theory which does not contain $B \rightarrow C$ to a prime theory containing B but not C. Assume now that $\Delta \models B \rightarrow C$. This implies $\forall \Delta' \supseteq \Delta (\Delta' \models B \Rightarrow \Delta' \models C)$, so by the induction hypothesis $\forall \Delta' \supseteq \Delta (B \in \Delta' \Rightarrow C \in \Delta')$ and $B \rightarrow C \in \Delta$. If $B \rightarrow C \in \Delta$, then $\forall \Delta' \supseteq \Delta (B \in \Delta' \Rightarrow C \in \Delta')$ and using the induction hypothesis $\forall \Delta' \supseteq \Delta (\Delta' \models B \Rightarrow \Delta' \models C)$, i.e. $\Delta \models B \rightarrow C$.

As for =|, we reason as follows: $\Delta$=|B→C implies $\Delta$|=B and $\Delta$ =|C which is equivalent modulo the induction hypothesis to B∧¬C∈$\Delta$. But by the axioms 4 (since if E↔$E_1$ is a theorem, then E and $E_1$ are simultaneously in or out of a theory) ¬(B→C)∈$\Delta$. In the opposite direction the fact that ¬(B→C)∈$\Delta$ implies that B∧¬C∈$\Delta$, so B∈$\Delta$ and ¬C∈$\Delta$ and so $\Delta$|=B and $\Delta$=|C, which gives us the desired $\Delta$=|B→C.

Having the truth lemma we can define easily a homomorphism which refutes A as a tautology. Since A is not a theorem there is a prime theory $\Delta$ such that A∉$\Delta$. Let h(B) be <$\|B\|_1$,$\|B\|_0$>, where $\|B\|_1$ = {$\Delta$: $\Delta$|=B} and $\|B\|_0$ = {$\Delta$: $\Delta$=|B}. Clearly $\|A\|_1 \neq W$, so h(A)∉$D_1$, thus A is not a tautology in the bilattice **B**($F^+$), where $F^+$ is the pseudo Boolean algebra of all *cones* in F.■

**Remark** 1. This is the first example of a logic which is not *finite*, but only finitely approximable (as can be established by means of a filtration method, but we leave this aside for lack of space).

2. The implication is also a first example of a connective which is not *extensional* in the sense that the truth conditions for it do not concern only the current possible world (classical modalities have similar behavior).

3. Considering formulae as terms we can claim now that the following are equivalent:

(a) A = B is an identity in the class of all intuitionistic bilattices;
(b) A↔B and ¬A↔¬B are both theorems of the system.

The negative version of the above logic can be obtained by *k*-dualizing the system (which is not quite a trivial enterprise in this case): first of all we need a operation to play the role of → in the changed circumstances.To this end we propose a kind of dual implication x ⇨ y with the following definition:

<a,b> ⇨ <c,d> = <b∨c,~b∧d>.

The intuition behind it: A ⇨ B is assertable in a situation if the available knowledge is either enough to assert the consequent or enough to reject the antecedent; A ⇨ B has to be rejected by the available information if it guarantees that the antecedent will never be rejected and rejects the consequent. More formally, in intuitionistic relational models:

w|= A ⇨ B iff w=|A or w|=B;
w=| A ⇨ B iff ∀w'(w≤w' ⇒ w'≠|A) and w'=|B.

**Lemma 5.12** *The operation ⇨ is a suitable implication.*

*Proof:* • A, A ⇨ B|= B: let h(A) = <a,b>, h(B) = <c,d>. h(A)∈ $D_0$ iff b = 0, h(A⇨ B)∈ $D_0$ iff ~b∧d = 0, but clearly this implies d = 0, so h(B)∈ $D_0$.

• A|=B implies |= A ⇨ B: if A ⇨ B is not a tautology, then for some bilattice and some valuation h, h(A ⇨ B)∉ $D_0$, i.e. ~b∧d ≠ 0; in pseudo Boolean algebras the latter is equivalent to not d≤b and therefore not <a,b>$\leq_t$ <c,d> – we can find a dual prime bi-filter defining an appropriate homomorphism into **4** which would contradict the fact A|=B. ■

After changes in the language where now ⇨ is a basic connective instead of →, we have to change the axiom schemata: groups (0), (1) and (3) retain the same form (⇨ replacing →) while the schemata in groups (2) and (4) are dualized, i.e. ⊕ and ⊗ exchange places. The resulting logical system is complete and finitely approximable.

***The logics of the external intuitionistic bilattices*** The class {$\mathbf{L}^2$: **L** is a pseudo Boolean algebra} defines logical systems different from the ones just discussed. The signature of the class contains, besides the external modalities ■ and♦, an implication ⊃ lifted from the underlying lattice to the bilattice by means of the following definition:

$$[a,b]\supset[c,d] = [b\rightarrow c, a\rightarrow d].$$

Such a choice can be explained by the intuition of necessary and possible truths (represented respectively by the left and right boundaries of the generalized intervals): b→c represents faithfully the cases when an implication is necessarily true, while a→d gives the cases of possible truth. In intuitionistic possible worlds models this intuition is rendered by the following truth conditions concerning the two forcing relations $|=_1$ and $|=_0$:

$$w \models_1 A\supset B \text{ iff } \forall w'\geq w(w'\models_0 A \Rightarrow w'\models_1 B);$$

$$w \models_0 A\supset B \text{ iff } \forall w'\geq w(w'\models_1 A \Rightarrow w'\models_0 B).$$

Let us start with the full standard positive logic. Abbreviating ■x⊃y as x→y (with the hope that this will not cause confusion with the pseudo Boolean algebra operation), we have

$$[a,b]\rightarrow[c,d] = [a\rightarrow c, a\rightarrow d],$$

and the following fact: *→ is a suitable implication.* In view of this fact we present a Hilbert style axiomatization. *Modus ponens* is the only inference rule and the axiom schemata are grouped as in the previous case:

0. all intuitionistically valid schemata in the language of → and 0.

1. A∧B→A, A∧B→B, A→(B→A∧B)

$A \to A \vee B,\ A \to A \vee B,\ (A \to C) \to ((B \to C) \to (A \vee B \to C))$

2. $A \otimes B \to A,\ A \otimes B \to B,\ A \to (B \to A \otimes B)$
$A \to A \oplus B,\ A \to A \oplus B,\ (A \to C) \to ((B \to C) \to (A \oplus B \to C))$

3. $\blacksquare A \leftrightarrow A$
$\blacksquare(A \supset B) \leftrightarrow (\blacklozenge A \to \blacksquare B)$
$\blacklozenge(A \supset B) \leftrightarrow (\blacksquare A \to \blacklozenge B)$
$\blacklozenge(A \wedge B) \leftrightarrow (\blacklozenge A \wedge \blacklozenge B)$
$\blacklozenge(A \vee B) \leftrightarrow (\blacklozenge A \vee \blacklozenge B)$
$\blacklozenge \blacksquare A \leftrightarrow \blacksquare A$
$\blacklozenge \blacklozenge A \leftrightarrow \blacklozenge A$

4. $\blacklozenge(A \otimes B) \leftrightarrow \blacklozenge A \vee \blacklozenge B$
$\blacklozenge(A \oplus B) \leftrightarrow \blacklozenge A \wedge \blacklozenge B$

Let us list some theorems of the system:

$$\blacksquare(A \wedge B) \leftrightarrow \blacksquare A \wedge \blacksquare B$$
$$\blacksquare(A \vee B) \leftrightarrow \blacksquare A \vee \blacksquare B$$
$$\blacksquare(A \otimes B) \leftrightarrow \blacksquare A \vee \blacksquare B$$
$$\blacksquare(A \oplus B) \leftrightarrow \blacksquare A \wedge \blacksquare B$$

**Theorem 5.13** *A formula A is a theorem iff it is a tautology in the class of all external intuitionistic bilattices.*

*Proof:* Consider A which is a theorem – it is easy to check that A is a tautology. If A is not a theorem, then one can find a prime theory not containing A. The prime theories of the present logic ordered by inclusion form the canonical frame F of the system. The canonical model on F is defined by the map $\phi(\Delta, p)$ that assigns to each prime theory and a variable an element of **4** in the following way:

$$\phi(\Delta, p) = \begin{cases} \top, \text{ if } \blacksquare p \in \Delta, \blacklozenge p \notin \Delta \\ 1, \text{ if } \blacksquare p \in \Delta, \blacklozenge p \in \Delta \\ 0, \text{ if } \blacksquare p \notin \Delta, \blacklozenge p \notin \Delta \\ \bot, \text{ if } \blacksquare p \notin \Delta, \blacklozenge p \in \Delta \end{cases}$$

It is already routine to check that $\phi(\Delta, p)$ extends to a homomorphism of the full language into **4** with the same properties as above – this is based on the features of prime theories which augment those from Theorem 5.4 and from Theorem 5.9 (concerning $\to$) with clauses for $\supset$:

$$\blacksquare(B \supset C) \in \Delta \text{ iff } \forall \Delta' \supseteq \Delta\ (\blacklozenge B \in \Delta' \Rightarrow \blacksquare C \in \Delta');$$
$$\blacklozenge(B \supset C) \in \Delta \text{ iff } \forall \Delta' \supseteq \Delta\ (\blacksquare B \in \Delta' \Rightarrow \blacklozenge C \in \Delta').$$

In the bilattice $(F^{+})^2$ the formula A cannot be valid since it is not forced everywhere.■

**Remark** The completeness theorem allows us to describe identities in the class of all external intuitionistic bilattices: A = B is such an identity iff ■A↔■B and ♦A↔♦B are theorems. Since the logic is decidable (this is a corollary of its axiomatizability and finite approximability, the latter can be established by a filtration method) we have a decision procedure for testing identities.

A suitable implication for the negative logic of external intuitionistic bilattices is ♦A⊃B:

$$\blacklozenge[a,b]\supset[c,d] = [b\to c, b\to d].$$

Since $x \in D_0$ iff its right boundary is 1, it is easy to see that the necessary changes to be made in the positive system in order to obtain the negative one are:

- replace everywhere → with the new implication;
- leave groups (0) and (1) otherwise unchanged;
- dualize the rest, i.e. interchange ♦ and ■, as well as ⊕ and ⊗.

***The logics of Boolean bilattices*** In this subsection we treat very briefly the class of bilattices of the kind **L**×**L** or $\mathbf{L}^2$ (where **L** is a Boolean algebra). Here the whole variety of different implications introduced up to now reduces to two specimen:

$$\neg\blacksquare x \vee y \ ( = x\to y = \blacksquare x \supset y) \text{ and}$$
$$\neg\blacklozenge x \vee y \ ( = x \Rightarrow y = \blacklozenge x \supset y).$$

For the positive logic we take a language with the basic bilattice connectives (including negation) and the former implication (which is obviously suitable). The axiom system is obtained from that for intuitionistic bilattices by the addition of a single new schema – *the Peirce formula*: ((A→B)→A)→A. Note that the external modalities are definable:

$$\blacksquare x = \neg{\sim}x \ ({\sim}\, x \text{ is } x\to 0)\ ;\ \blacklozenge x = \neg\blacksquare\neg x.$$

The negative logic is formulated with the latter implication – it is obtained in the same way from the negative logic of intuitionistic bilattices: by adding the Peirce formula.

**Remark** Almost any other way of extending the basic logic from intuitionistic to classical would do.

## *6 Logical systems related to info–algebras*

Info-algebras lack the symmetries of bilattices and consequently the logical systems determined by classes of info-algebras lack many of the nice properties of the bilattice logics. Nevertheless info–logics display some interesting features and we devote this section to the study of several examples.

***The standard positive logic of all info-algebras*** Our basic system will be based on a language which includes the operations $\wedge, \vee, \otimes$ and *has no propositional constants*. We restrict the language in this way with simplicity of presentation in mind (the addition of the constants does not change the results but complicates the system of rules and the proofs). We consider the info–algebras as standard matrices with the set $D_1 = \{\top, 1\}$ as the set of distinguished truth values. Let us point out that this semantics has the following property: there are no tautologies in the language, i.e. for no formula A, $\emptyset \models A$ (e.g., for any A one can always find a valuation h with $h(A) = \bot$).

Call a formula *internal*, if $\otimes$ does not occur in it. A set $\Gamma$ is internal, if all its members are internal formulae. The systems will have as basic sequents expressions of the form $\Gamma \vdash A$, where $A \in \Gamma$. The rules concerning $\vee$ and $\otimes$ are the same as in the bilattice case. Conjunction though poses a problem: while $(\vdash\wedge)$ is OK, the rules $(\wedge\vdash)$ are not correct when interpreted in info–algebras (because of the possibility to assign $A \wedge B$ a value in $D_1$ keeping the value of A outside $D_1$, as for example is the case with the valuation h defined for p and q as $h(p) = \top$ and $h(q) = 0$, so $h(p\wedge q) \in D_1$ but $h(q) \notin D_1$). Therefore we have to put up with weaker rules $(\wedge\vdash)^+$:

$$(\wedge\vdash)^+ \qquad \frac{\Gamma, A \vdash C}{\Gamma, A\wedge B \vdash C} \qquad \frac{\Gamma, B \vdash C}{\Gamma, A\wedge B \vdash C}$$

where the plus sign marks the restriction on the type and variables of the formulae which appear in the bottom sequents: A is internal and $Var(A) \supseteq Var(B)$, for the left rule and B is internal and $Var(B) \supseteq Var(A)$, for the right one. Such a weakening calls for additional compensatory rules. In the first place we need the *cut rule*:

$$(\text{Cut}) \qquad \frac{\Gamma \vdash A; \ \Gamma, A \vdash B}{\Gamma \vdash B}$$

but also the following *distributivity* rules:

$$(\wedge\vee\vdash) \qquad \frac{\Gamma, (A\wedge C)\vee(B\wedge C) \vdash D}{\Gamma, (A\vee B)\wedge C \vdash D}$$

$$(\vdash\wedge\vee)\quad \frac{\Gamma\vdash (A\wedge C)\vee(B\wedge C)}{\Gamma\vdash (A\vee B)\wedge C}$$

$$(\wedge\otimes\vdash)\quad \frac{\Gamma, (A\wedge C)\otimes(B\wedge C)\vdash D}{\Gamma, (A\otimes B)\wedge C\vdash D}$$

$$(\vdash\wedge\otimes)\quad \frac{\Gamma\vdash (A\wedge C)\otimes(B\wedge C)}{\Gamma\vdash (A\otimes B)\wedge C}$$

$$(\wedge\wedge\vdash)\quad \frac{\Gamma, (A\wedge B)\wedge C\vdash D}{\Gamma, (A\wedge C)\wedge B\vdash D}$$

$$(\vdash\wedge\wedge)\quad \frac{\Gamma\vdash (A\wedge C)\wedge(B\wedge C)}{\Gamma\vdash (A\wedge B)\wedge C}$$

**Lemma 6.1** *The resulting system is correct: if $\Gamma\vdash A$ is provable, then $\Gamma\models A$.*

*Proof:* The proof is by straightforward checking. The only more exciting rules to treat are the additional distributivity rules, since most of them do not correspond to identities in the info-algebras. Let us do as an example the $(\wedge\vee)$-rules. For $(\wedge\vee\vdash)$ it is sufficient to establish that $(A\vee B)\wedge C\models(A\wedge C)\vee(B\wedge C)$. Take a valuation h such that $h((A\vee B)\wedge C))\in D_1$. If the value is T, then at least one of h(A), h(B) or h(C) is T, but then $h((A\wedge C)\vee(B\wedge C))$ = T, too.For the other possibility, let the value be 1. This means that $(a\vee b)\wedge c = 1$ for any $a\in h(A)$, $b\in h(B)$, $c\in h(C)$, and implies $c = 1$ and $a\vee b = 1$. But then for a typical member of $h((A\wedge C)\vee(B\wedge C))$ – $(a\wedge c_1)\vee(b\wedge c_2)$ – we have $(a\wedge c_1)\vee(b\wedge c_2) = a\vee b = 1$. For $(\vdash\wedge\vee)$ we have to check whether $(A\wedge C)\vee(B\wedge C)\models (A\vee B)\wedge C$ – but this is easy: skipping the case of occurrence of T, we consider a typical member of the left-hand side $(a\wedge c_1)\vee(b\wedge c_2) = 1$ and moreover $(a\wedge c)\vee(b\wedge c) = 1$ for any a,b,c from the appropriate sets. Applying the distributive law in the underlying lattice we obtain $(a\vee b)\wedge c = 1$ for a typical member of the right-hand side. ■

**Remark** A slightly more sophisticated counter-example involving A = p⊗q and B= p for which A∧B|≠A (as witnessed by the assignment h(p) = T, h(q) = 0) shows that the restriction to internal formulae in $(\wedge|-)^{+}$ is indeed necessary.

The next technical lemma is left entirely to the diligent reader (its proof relies on such provable sequents as A⊗B|–A; A⊗B|–B; A,B|–A⊗B; A∧(B⊗C)|–(A∧B)⊗(A∧C); (A∧B)⊗(A∧C)|–A∧(B⊗C), etc.).

**Lemma 6.2** *For any Γ there exists an internal $\Gamma_0$ such that $[\Gamma_0] = [\Gamma]$.*

The lemma shows that questions of the type "Is Γ|– A provable?" can be reduced to the same questions about internal formulae and sets of internal formulae.

Here is another list of provable sequents – to be used in the lemma that follows:

A∧B|– A∨B, A∧B|– B∧A,
A∧(B∧C)|–(A∧B)∧C, (A∧B)∧C|–A∧(B∧C),
((A∧C)∧(B∧C))∧D|–((A∧B)∧C)∧D),
((A∧B)∧C)∧D|–((A∧C)∧B)∧D,
((A∧C)∨(B∧C))∧D|– ((A∨B)∧C)∧D,
((A∨B)∧C)∧D|–((A∧C)∨(B∧C))∧D.



The space permits one proof as an example:

$$\dfrac{\dfrac{\dfrac{A\wedge B\vdash A\wedge B}{A\wedge B\vdash (A\wedge B)\vee(B\wedge B)}\,(\vdash\vee)}{A\wedge B\vdash (A\vee B)\wedge B}\,(\vdash\wedge\vee) \qquad \dfrac{A\vee B\vdash A\vee B}{(A\vee B)\wedge B\vdash A\vee B}\,(\wedge\vdash)^{+}}{A\wedge B\vdash A\vee B}\,(\text{Cut})$$

Unfortunately the derivations known to the author depend crucially on applications of the cut rule, so the cut elimination property of the above system is an open problem.

**Lemma 6.3** *For internal formulae $B_1, \ldots, B_n$, A, if $B_1, \ldots, B_n$ |– A is provable, then $B_1 \wedge C, \ldots, B_n \wedge C$|– A∧C is also provable, for any C.*

*Proof:* By induction on the height of the derivation tree. The case of axioms is clear, so we need to check the induction step, proving that if the top sequent in an application of a rule satisfies the above property, then the bottom sequent also satisfies this property. The notorious fate of such proofs notwithstanding, we present just a sample of the simplest cases. In general the derivations use (Cut) and depend on the provable sequents shown above.

Let $B_1, \ldots, B_n \vdash A$ be obtained by an application of $(\wedge\vdash)^+$, i.e. $B_n = B\wedge D$, $Var(B) \supseteq Var(D)$ and

$$\frac{B_1, \ldots, B \vdash A}{B_1, \ldots, B_n \vdash A}\ (\wedge\vdash)^+$$

By the induction hypothesis: $B_1\wedge C, \ldots, B\wedge C \vdash A\wedge C$ is provable.

But then the following is a proof of what is needed:

$$\frac{\dfrac{B_1\wedge C, \ldots, B\wedge C \vdash A\wedge C}{B_1\wedge C, \ldots, (B\wedge C)\wedge D \vdash A\wedge C}\ (\wedge\vdash)^+}{B_1\wedge C, \ldots, (B\wedge D)\wedge C \vdash A\wedge C}\ (\wedge\wedge\vdash)$$

Let us also consider the case when the last applied rule is $(\vdash\wedge)$. Now $A = A_1\wedge A_2$ and the application is:

$$\frac{B_1, \ldots, B_n \vdash A_1; \quad B_1, \ldots, B_n \vdash A_2}{B_1, \ldots, B_n \vdash A_1\wedge A_2}$$

By the induction hypothesis: $B_1\wedge C, \ldots, B_n\wedge C \vdash A_1\wedge C$ and $B_1\wedge C, \ldots, B_n\wedge C \vdash A_2\wedge C$ are provable. But then

$$\frac{\dfrac{B_1\wedge C, \ldots, B_n\wedge C \vdash A_1\wedge C; \quad B_1\wedge C, \ldots, B_n\wedge C \vdash A_2\wedge C}{B_1\wedge C, \ldots, B_n\wedge C \vdash (A_1\wedge C)\wedge(A_2\wedge C)}\ (\vdash\wedge)}{B_1\wedge C, \ldots, B_n\wedge C \vdash (A_1\wedge A_2)\wedge C}\ (\vdash\wedge\wedge)$$

and we are done.

One more rule as the last applied one: (∧∧⊢). In this case $B_n = (B_1 \wedge B_2) \wedge B_3$ and by the induction hypothesis $B_1 \wedge C, \ldots, ((B_1 \wedge B_3) \wedge B_2) \wedge C \vdash A_1 \wedge C$ is provable. The following derivation gets the desired result:

$$\frac{B_1 \wedge C, \ldots, ((B_1 \wedge B_3) \wedge B_2) \wedge C \vdash A_1 \wedge C; \ (B_1 \wedge B_2) \wedge B_3) \wedge C \vdash (B_1 \wedge B_3) \wedge B_2) \wedge C \text{ (see above)}}{B_1 \wedge C, \ldots, ((B_1 \wedge B_2) \wedge B_3) \wedge C \vdash A_1 \wedge C} \text{(Cut)}$$

The rest of the cases are left to the reader. ■

**Theorem 6.4** *If $\Pi = A$, then $A \in [\Gamma]$.*

*Proof:* We follow an already familiar path with some minor deviations: assuming without loss of generality that A and Γ are *internal* and A∉Γ, we find a maximal theory Δ among the theories that extend Γ and omit A. Such a theory need not be prime, but it still does the job because it turns out to be *relatively* prime, namely with respect to the class of internal formulae built up from the propositional variables occurring in A.

Call the variables of A *significant.* A significant formula B is such that Var(B)⊆Var(A).

**Lemma 6.5** *For significant internal formulae B and C:*

*1. B∧C∈Δ iff B∈Δ and C∈Δ.*

*For any formulae B and C:*

*2. B∨C∈Δ iff B∈Δ or C∈Δ*

*3. B⊗C∈Δ iff B∈Δ and C∈Δ.*

*Proof:* The establishing of (2) and (3) is routine. Let us check (1) which differs from the standard case. If B∈Δ and C∈Δ, then by (⊢∧) B∧C∈Δ. In the opposite direction: if B∧C∈Δ, then by one of the listed provable sequents B∨C∈Δ and so either B or C belong to Δ. Let B∈Δ. Now, C∉Δ means that A∈[Δ,C], so for some internal formulae $D_1, \ldots, D_m \in \Delta$:

$D_1, \ldots, D_m$, C⊢ A is provable.

Applying Lemma 6.3 we get that $D_1 \wedge B, \ldots, D_m \wedge B$, C∧B⊢ A∧B is also provable. All formulae of the left-hand side are from Δ. Thus A∧B∈Δ. Since B is significant, A∧B⊢A is provable, so A∈Δ – a contradiction with the assumptions on Δ. Therefore C∈Δ, too. ■

Having a theory Δ with the above properties we can define a function h: Var → $2^{set}$ :

$$h(p) = \begin{cases} 1, & \text{if p is significant and } p \in \Delta \\ 0, & \text{if p is significant and } p \notin \Delta \\ \top, & \text{if p is not significant} \end{cases}$$

Note first that the extension of h to a homomorphism $\mathcal{L} \to \mathbf{2}^{\mathbf{set}}$ maps all internal insignificant formulae to $\top$. For significant ones it can be established by induction that

$$h(B) = 1 \text{ iff } B \in \Delta.$$

To conclude the proof of the completeness theorem we note that A is significant and so $h(A) \notin D_1$, while all members of $\Gamma$ are mapped onto an element of $D_1$. ■

**Remarks** 1. Again we have found a simple algebra adequate for the system – the logic of all info–algebras coincides with the logic of $\mathbf{2}^{\mathbf{set}}$.

2. The admission of the constants 0, 1, $\top$, $\bot$ to the language while not changing the rules will necessitate changes in the notion of a basic sequent. Axioms will have to include also $\Gamma \vdash A$ where one of the following holds:

1. A is an *internal* formula with an occurrence of $\top$;
2. $A = 1$;
3. $0, \bot \in \Gamma$;
4. $A = B \wedge C$ and $\{0 \wedge B,\ 0 \wedge C, \bot \wedge B, \bot \wedge C\} \cap \Gamma \neq \emptyset$.

3. The full standard *negative* logic of info-algebras, defined by choosing as distinguished truth values $D_0 = \{X\colon 1 \in X\}$ poses the first major setback to our program: this logic is *not* anything like a dual to the above system. Let us start with the observation that the info-algebra $\mathbf{2}^{\mathbf{set}}$ is not adequate for this particular consequence relation any more: for example $A \vee A \models A$ in $\mathbf{2}^{\mathbf{set}}$, but not so in the set expansion of the four element Boolean algebra (with elements $a, b \neq 0, 1$) as shown by the valuation h for which $h(A) = \{a,b\} \notin D_0$ while $h(A \vee A) = \{a,b,1\} \in D_0$. This and similar counterexamples demonstrate the incorrectness of several rules, e.g., $(\vdash \vee)$ or the distributivity rules. Although the logic can be axiomatized and shown to be finitely approximable (but not finite) we leave that matter to another paper. The problem lies in the origin of the "nice" properties of the elements of $D_1$ and $D_0$ – in the former case $\inf X = 1$ is equivalent to $X \in D_1$, while in the latter $\sup X = 1$ (the "real" dual) is weaker than $X \in D_0$. Below we consider several systems with the weaker condition on $D_0$.

***The positive logic of info–algebras with negation*** Consider now the class of info-algebras with underlying de Morgan lattices. Its full positive standard logic is an extension of the system in the previous sub-section:

- the language has an additional connective ~;
- the notion of basic sequent is augmented to incorporate a restricted version of the *Duns Scot law*: Γ|–A is an axiom, if A is internal and also B∧~B∈Γ, for some formula B with Var(B)⊆Var(A);
- new rules concerning ~, are added, taken from the bilattice case – some of them *verbatim* as (~∧|–), (|–~∧), (|–~∨), (~⊗|–), (|–~⊗), but the rule (~∨|–) has a new outlook:

$$(\sim\vee|\!-)\qquad \frac{\Gamma,\ \sim A\wedge\sim B\ |\!-\ C}{\Gamma,\ \sim(A\vee B)\ |\!-\ C}.$$

**Lemma 6.6** *The resulting system is correct.*

*Proof:* The correctness of the new rules is obvious. As for the new axiom: if h(B∧~B)∈$D_1$, then h(B) = T and therefore for some variable p, h(p) = T, because, if h(B) is non-empty, then it has members a∧~a which cannot be 1. Since Var(B)⊆Var(A) and A is internal, h(A) = T, also. ■

**Lemma 6.7** *For any set Γ there exists an internal* $\Gamma_0$ *such that* $[\Gamma_0] = [\Gamma]$.

**Lemma 6.8** *If* $B_1, \ldots, B_n$ *|– A is provable, then* $B_1\wedge C, \ldots, B_n\wedge C$*|– A∧C is also provable (under the same conditions as above).*

*Proof:* The induction step now requires checking of the added rules. Let us do an example in which the last applied rule is (~∨|–). In this case $B_n$ = ~(B∨D) and by the induction hypothesis $B_1$∧C,..., (~B∧~D)∧C|–A∧C. The following sequent is provable: ~(B∨D)∧C|–(~B∧~D)∧C. Applying (Cut) we get $B_1$∧C,..., ~(B∨D)∧C|–A∧C – the desired result.

All other details are left to the reader. ■

**Theorem 6.9** *If Γ|=A, then A∈[Γ].*

*Proof:* Once again we can assume without loss of generality that we deal exclusively with internal formulae. Let in particular A be an internal formula such that A∉[Γ]. Just as in the proof of Theorem 6.4 we can d Γ to a maximal theory Δ, for which Γ⊆Δ and A∉Δ. This theory turns out to be relatively prime with respect to internal significant formulae, i.e. to formulae B with Var(B)⊆Var(A).

Note now that in this case: p is a significant variable implies $p\wedge{\sim}p\notin\Delta$, because $p\wedge{\sim}p\in\Delta$ would mean that A is also from $\Delta$ (recall that $p\in Var(A)$ and A is internal). Otherwise we have to add to the properties of relatively prime theories (from Lemma 6.5) some clauses concerning the negation (B,C – significant):

4. ${\sim}(B\wedge C)\in\Delta$ iff ${\sim}B\in\Delta$ or ${\sim}C\in\Delta$ (B,C – internal);
5. ${\sim}(B\vee C)\in\Delta$ iff ${\sim}B\in\Delta$ and ${\sim}C\in\Delta$;
6. ${\sim}(B\otimes C)\in\Delta$ iff ${\sim}B\in\Delta$ and ${\sim}C\in\Delta$ (in fact not needed in the proof);
7. ${\sim}{\sim}B\in\Delta$ iff $B\in\Delta$.

From $\Delta$ we can define a mapping h by setting h(p) = T for insignificant variables and for significant ones:

$$h(p) = \begin{cases} 1, \text{ if } p\in\Delta, {\sim}p\notin\Delta \\ 0, \text{ if } p\notin\Delta, {\sim}p\in\Delta \\ \perp, \text{ if } p\notin\Delta, {\sim}p\notin\Delta \end{cases}$$

Clearly the extension of h to a homomorphism assigns T to insignificant internal formulae B, while for the significant ones by induction on their complexity one can prove:

$$h(B)\in D_1 \text{ iff } B\in\Delta. \blacksquare$$

***The positive logic of intuitionistic info-algebras*** Our final example in this section will be the logic determined by the class of all info-algebras $\mathbf{L}^{set}$ where **L** is a pseudo-Boolean algebra. In this case the language has an internal operation of implication $\supset$ and an internal pseudo-negation ~ (${\sim}A = A\supset 0$). The system extends the basic info-algebra logic with rules for the implication and an additional class of basic sequents similar to the case of info-algebras with negation: $\Gamma\vdash A$ is an axiom, if A is internal and also $B,{\sim}B\in\Gamma$, for some formula B with $Var(B)\subseteq Var(A)$.

The rules for $\supset$ include:

$$(\supset\vdash)^{+} \qquad \frac{\Gamma\vdash A;\ \ \Gamma, B\vdash C}{\Gamma, A\supset B\vdash C}$$

with the restrictions: C is internal, $Var(A)\subseteq Var(C)$ and a series of distributivity rules which compensate the absence of any suitable $(\vdash\supset)$-type rule :

$$(\supset\wedge\vdash) \qquad \frac{\Gamma, (C\supset A)\wedge(C\supset B)\vdash D}{\Gamma, C\supset(A\wedge B)\vdash D}$$

$(\vdash\supset\wedge)$ $$\frac{\Gamma\vdash (C\supset A)\wedge(C\supset B)}{\Gamma\vdash C\supset(A\wedge B)}$$

$(\supset\vee\vdash)$ $$\frac{\Gamma, (A\supset C)\wedge(B\supset C)\vdash D}{\Gamma, (A\vee B)\supset C\vdash D}$$

$(\vdash\supset\vee)$ $$\frac{\Gamma\vdash (A\supset C)\wedge(B\supset C)}{\Gamma\vdash (A\vee B)\supset C}$$

$(\supset\otimes\vdash)$ $$\frac{\Gamma, (C\supset A)\otimes(C\supset B)\vdash D}{\Gamma, C\supset(A\otimes B)\vdash D} \qquad \frac{\Gamma, (A\supset C)\otimes(B\supset C)\vdash D}{\Gamma, (A\otimes B)\supset C\vdash D}$$

$(\vdash\supset\otimes)$ $$\frac{\Gamma\vdash (C\supset A)\otimes(C\supset B)}{\Gamma\vdash C\supset(A\otimes B)} \qquad \frac{\Gamma\vdash (A\supset C)\otimes(B\supset C)}{\Gamma\vdash (A\otimes B)\supset C}$$

$(\supset\supset\vdash)$ $$\frac{\Gamma, (A\wedge B)\supset C\vdash D}{\Gamma, (A\supset B)\supset C\vdash D}$$

$(\vdash\supset\supset)$ $$\frac{\Gamma\vdash (A\wedge B)\supset C}{\Gamma\vdash (A\supset B)\supset C}$$

A list of useful provable sequents would contain, e.g., $B, B\supset C\vdash B\wedge C$, $\sim(B\supset C)\vdash\sim\sim B\wedge\sim C$, $\sim\sim B\wedge\sim C\vdash\sim(B\supset C)$, $\sim(B\vee C)\vdash\sim B\wedge\sim C$, $\sim B\wedge\sim C\vdash\sim(B\vee C)$, $\sim B\vee\sim C\vdash\sim(B\wedge C)$, etc., besides sequents like $C\supset(A\otimes B)\vdash(C\supset A)\otimes(C\supset B)$, $(C\supset A)\otimes(C\supset B)\vdash C\supset(A\otimes B)$, $(A\otimes B)\supset C\vdash(A\supset C)\otimes(B\supset C)$, $(A\supset C)\otimes(B\supset C)\vdash(A\otimes B)\supset C$, etc., needed to show that just as in the previous cases one can concentrate exclusively on internal formulae when dealing with problems of derivability and semantic consequence: an analog of lemmata 6.2 and 6.7 holds here, too. A counterpart of lemmata 6.3 and 6.8 also holds for the present case. Thus in the proof of the completeness theorem we tread a familiar path.

**Theorem 6.10** *The positive logic of intuitionistic info-algebras is complete with respect to the semantic consequence relation.*

*Proof:* We need the construction of relatively prime theories used in the previous two proofs. Starting from a unprovable *internal* sequent $\Gamma \vdash A$ one can find a theory $\Delta_0$ maximal among theories containing $\Gamma$ and omitting A. $\Delta_0$ has three nice properties with respect to significant formulae B, C:

1. $B \wedge C \in \Delta$ iff $B \in \Delta$ and $C \in \Delta$;
2. $B \vee C \in \Delta$ iff $B \in \Delta$ or $C \in \Delta$;
3. if $B \in \Delta$ and $B \supset C \in \Delta$, then $C \in \Delta$.

For (3) recall that $B, B \supset C \vdash B \wedge C$ is provable, so if $B \in \Delta$ and $B \supset C \in \Delta$, then $B \wedge C \in \Delta$. For significant B and C this implies $C \in \Delta$.

Now we define a frame (in fact a generated subframe of the canonical frame) $F = \langle W, \subseteq \rangle$, where $W = \{\Delta: \Delta_0 \subseteq \Delta$ and $\Delta$ is relatively prime w.r.t. the significant formulae$\}$. Thus the elements of W satisfy (1) – (3) above. The pseudo-Boolean algebra $F^+$ of all *cones* in W (with operations $a \cup b, a \cap b, a \rightarrow b$ and $\neg a = a \rightarrow \emptyset$) will be used in the spirit of Lemma 3.18: setting

$$\phi(\Delta,p) = \begin{cases} \top, \text{ if } p \in \Delta, \neg p \in \Delta \\ 1, \text{ if } p \in \Delta, \neg p \notin \Delta \\ 0, \text{ if } p \notin \Delta, \neg p \in \Delta \\ \bot, \text{ if } p \notin \Delta, \neg p \notin \Delta \end{cases}$$

we extend $\phi(\Delta,p)$ to a member of Hom($\mathcal{L}_0$, $\mathbf{2}^{\mathbf{set}}$); having this $\phi$ we are able to define a mapping $\|.\|$ by

$$\|B\| = \{a: |B|_1 \subseteq a, |B|_0 \subseteq \neg a\},$$

where $|B|_1 = \{\Delta \in W: \phi(\Delta,B) = 1\}$, $|B|_0 = \{\Delta \in W: \phi(\Delta,B) = 0\}$, and then to prove that when restricted to significant formulae $\|.\|$ is a homomorphism into $(F^+)^{set}$, establishing thereby the fact that $\Gamma \not\models A$ since $\Gamma \subseteq \Delta$ for all members of W (thus for $B \in \Gamma$ one has $\|B\| = \{W\} = 1$ in $(F^+)^{set}$ or $\|B\| = \top$) but obviously $\|A\| \neq \{W\}$, since $A \notin \Delta_0$. We need to check whether:

1. $\|B \wedge C\| = \|B\| \cap \|C\|$;
2. $\|B \vee C\| = \|B\| \cup \|C\|$;
3. $\|B \supset C\| = \|B\| \rightarrow \|C\|$.

Leaving (1) and (2) to the reader, we treat the third equality: let us for example prove that $\|B\| \rightarrow \|C\| \subseteq \|B \supset C\|$, i.e., that $b \in \|B\|$ and $c \in \|C\| \Rightarrow b \rightarrow c \in \|B \supset C\|$. To this end we first demonstrate that $|B \supset C|_1 \subseteq b \rightarrow c$, in other words that

$$\phi(\Delta, B \supset C) = 1 \Rightarrow \Delta \models b \rightarrow c.$$

We reason from the contrary: let $\phi(\Delta,B\supset C) = 1$ but $\Delta|\neq b\to c$, thus $\exists\Delta'\supseteq\Delta(\Delta'|=b$ and $\Delta'|\neq c)$. Now $\Delta'|=b$ implies $\sim B\notin\Delta'$, while $\Delta'|\neq c$ implies $C\notin\Delta'$. Therefore we can extend $[\Delta', B,\sim C]$ to an element of $W - \Delta''$. Since $\Delta\subseteq\Delta''$ we have $B\supset C\in\Delta''$ – together with $B\in\Delta''$ this yields $\{C, \sim C\}\subseteq\Delta''$. This is a contradiction since C is significant.

Our second problem is $|B\supset C|_0 \subseteq \neg(b\to c)$, i.e., whether

$$\phi(\Delta,B\supset C) = 0 \Rightarrow \Delta|= \neg(b\to c) .$$

Reason as follows: assuming the contrary, i.e., that $\phi(\Delta,B\supset C) = 1$ but $\Delta|\neq\neg(b\to c)$. Now we have a $\Delta'\supseteq\Delta$ such that $\Delta'|=b\to c$. Recalling that $\sim(B\supset C)|\!-\!-\!-\sim B\wedge\sim C$ is provable and that $\phi(\Delta,B\supset C) = 0$ forces $\sim(B\supset C)\in\Delta$, it is clear that one can produce a $\Delta''\supseteq\Delta'$ such that $B\in\Delta''$, $\sim C\in\Delta''$ which would obviously contradict the fact that $\Delta''|=b\to c$.

The opposite inclusion is established by similar reasoning. ■

## 7 *Finitary, consistent and other restricted systems*

We devote this section to the study of logics which arise when in the general algebraic scheme for the consequence operation the set of admissible valuations Hom($\mathcal{L}$,A) is replaced by smaller families H of homomorphisms.

***Finitary logics of info-algebras*** As a first example we treat classes of generalized logical matrices based on info-algebras with H = {h: h(A) is a finite set for all A}. Clearly any finitary map h:Var→**A** can be extended to a unique homomorphism h∈H.

The positive finitary logic of all info-algebras coincides with the logic of all info-algebras (presented above). The interesting news here is the possibility of treating without complications a negative version of the logic, which is defined by the set of distinguished truth values $D^- = \{X: \sup X = 1\}$, i.e., $D^-$ consists of the elements $\{x_1, ..., x_m\}$ of A for which $x_1\vee ...\vee x_m = 1$.

For the axiomatization of the logic we need the following:

- the notion of an axiom taken unchanged from the positive case;
- we keep the rule (Cut);
- the rules for conjunction are taken without any restriction;
- (∨|–) is the same, but examples like $p|\neq p\vee q$ (consider h(p) = 1, h(q) = T) show that (|–∨) is not correct and has to be altered to a weaker rule:

$$(|\!-\vee)^- \qquad \frac{\Gamma\,|\!-A}{\Gamma\,|\!-A\vee B} \qquad \frac{\Gamma\,|\!-B}{\Gamma\,|\!-A\vee B},$$

where we have familiar requirements – $Var(B) \subseteq Var(A)$ and A is internal, for the left rule, and $Var(A) \subseteq Var(B)$ and B is internal – for the right;

• as should be expected the new rules for ⊗ are dual to the previous ones:

$$(\otimes \vdash) \quad \frac{\Gamma, A \vdash C; \quad \Gamma, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

$$(\vdash \otimes) \quad \frac{\Gamma \vdash A}{\Gamma \vdash A \otimes B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \otimes B};$$

• the added distributivity rules concern ∨ as a main connective:

$$(\vee\wedge \vdash) \quad \frac{\Gamma, (A \vee C) \wedge (B \vee C) \vdash D}{\Gamma, (A \wedge B) \vee C \vdash D}$$

$$(\vdash \vee\wedge) \quad \frac{\Gamma \vdash (A \vee C) \wedge (B \vee C)}{\Gamma \vdash (A \wedge B) \vee C}$$

$$(\vee\otimes \vdash) \quad \frac{\Gamma, (A \vee C) \otimes (B \vee C) \vdash D}{\Gamma, (A \otimes B) \vee C \vdash D}$$

$$(\vdash \vee\otimes) \quad \frac{\Gamma \vdash (A \vee C) \otimes (B \vee C)}{\Gamma \vdash (A \otimes B) \vee C}$$

$$(\vee\vee \vdash) \quad \frac{\Gamma, (A \vee B) \vee C \vdash D}{\Gamma, (A \vee C) \vee B \vdash D}$$

$$(\vdash \vee\vee) \quad \frac{\Gamma \vdash (A \vee C) \vee (B \vee C)}{\Gamma \vdash (A \vee B) \vee C}$$

One can easily check now that for each formula A there is a finite set of internal formulae $\{D_1, ..., D_m\}$ for which $A \vdash D_1 \otimes ... \otimes D_m$ and $D_1 \otimes ... \otimes D_m \vdash A$. This fact together with the corresponding semantic one: $A \models D_1 \otimes ... \otimes D_m$ and $D_1 \otimes ... \otimes D_m \models A$ reduce problems of provability of sequents and consequence relations to such problems in the domain of internal formulae: Γ|–A iff for some i, $\Gamma|\!-D_i$, Γ|=A iff for some i, $\Gamma|\!=D_i$.

**Lemma 7.1** *The resulting system is correct: if Γ|–A is provable, then Γ|=A.*

*Proof:* Standard. When checking for example the correctness of (∨|–) we need the proposition about homomorphisms into $\mathbf{2}^{\mathbf{set}}$ defined by prime filters F in the lattice underlying a given info-algebra and the fact that such homomorphisms map any set X with supX∈F into $D^-$ (because of the finiteness of X). ■

The proof of the completeness theorem mimics the proofs offered above: if Γ|–A is not provable (assume without loss of generality that they are both internal) extend Γ to a maximal theory Δ omitting A. Call a variable p *significant* , if it occurs in an internal formula B∈Δ. For Δ we have:

1. B∧C∈Δ iff B∈Δ and C∈Δ (for any formulae B and C);
2. B∨C∈Δ iff B∈Δ or C∈Δ (for significant internal B and C);
3. B⊗C∈Δ iff B∈Δ or C∈Δ (for any formulae B and C).

While (1) and (3) are routine, (2) needs some attention – the implication from left to right depends on the unrestricted rule (∨|–) and is standard; the converse implication is checked as follows: assume B∈Δ, then if C∈Δ, we have B∧C∈Δ and in view of the provability of B∧C|–B∨C, so we are done. If C∉Δ, then since C is significant, there is an *internal* formula D with Var(C)⊆Var(D) such that D∈Δ and consequently B∧D∈Δ. On one hand:

$$\dfrac{\dfrac{B\wedge D|\!-B\wedge D}{B\wedge D|\!-(B\wedge D)\vee C}\,(|\!-\vee)^- \qquad (B\wedge D)\vee C|\!-(B\vee C)\wedge(D\vee C)\ \text{(provable sequent)}}{B\wedge D|\!-(B\vee C)\wedge(D\vee C)}\,(\text{Cut})$$

On the other hand

$$\dfrac{B\vee C|\!-B\vee C}{(B\vee C)\wedge(D\vee C)|\!-B\vee C}\,(\wedge|\!-)$$

and applying (Cut) again, we obtain B∧D|–B∨C which is what we need.

Defining for variables p:

$$h(p) = \begin{cases} 1, & \text{if p is significant and } p\in\Delta \\ 0, & \text{if p is significant and } p\notin\Delta \\ \top & \text{otherwise,} \end{cases}$$

and extending it to a homomorphism from Hom($\mathcal{L}$,$\mathbf{2}^{\mathbf{set}}$), we can see that for significant internal formulae B:

$B \in \Delta$ iff $h(B) \in D^-$.

To nonsignificant formulae h assigns T. All members of Γ get values which are "true". Consider our formula A – either it is significant and then its value is 0, or it is non-significant and then its value is T. Anyway A is not "true" according to h. Thus we have established:

**Theorem 7.2** *The logic is complete: if Γ|=A, then A∈[Γ].*

The finitary logics of the class of *external info-algebras* coincides in fact with the finitary logics of *all* info-algebras (in a language extended with ■ and ♦) since finite sets X have always supX and infX. Now we can identify $D_1$ with {X: ■X = 1} and $D_0$ with {X: ♦X = 1}. Thus the positive finitary logic extends the basic info-algebra systems with *all* the rules concerning ■ and ♦ from the positive bilattice case (the list before Theorem 5.9). The correctness of the additions follows easily from the fact that modalized formulae cannot have T as value. For the proof of completeness we need the machinery of relatively prime theories developed above, but fortunately there are no unexpected complications.

The negative logic extends the basic finitary negative system with the *k*-dualized versions of the just cited rules and can be proven complete with respect to the finitary info–algebra matrices with $D_0$.

*Consistent logics* Under the term *consistent* we understand here logical systems that are defined semantically by classes of matrices with the following requirement on H⊆Hom(£,A): *the values in the range of any h∈H are consistent.*

In the bilattice case this means that in general only classes of bilattices with conflation are considered (with the exception of the case of the class of intuitionistic bilattices where the notion of consistency makes sense even without the conflation), while in any info-algebra consistent are all elements different from T. The restriction causes changes in the language of the logics: T and ⊕ are dropped for obvious reasons, conflation is not considered for the same reasons.

In view of the above remark the *basic positive consistent logic* for bilattices is the logic of all external bilattices (in a language without T and ⊕, with $D_1$ shrunk to {1}). It is easily checked that all consistent assignments validate nA|=♦A, so an addition is needed to the rules:

$$(\mathrm{cons})^+ \qquad \frac{\Gamma \vdash \blacksquare A}{\Gamma \vdash \blacklozenge A} \qquad \frac{\Gamma, \blacklozenge A \vdash C}{\Gamma, \blacksquare A \vdash C}\,.$$

It turns out that this is sufficient to obtain a complete logical system. In the proof (as well as in the following proofs) a role similar to the role of **4** is played by **3** =

<{0,1,⊥},∧, ∨,⊗, 0,1>: if Γ⊢A is not provable, then it can be semantically refuted in **3**.

The negative consistent logic of all external bilattices is obtained by similar deletions and additions:

- the language is restricted, the unnecessary rules dropped;
- new rules are added:

$$(\mathrm{cons})^{-} \quad \frac{\Gamma\vdash \blacklozenge A}{\Gamma\vdash \blacksquare A} \qquad \frac{\Gamma,\blacksquare A\vdash C}{\Gamma,\blacklozenge A\vdash C}.$$

The next example are the logics of de Morgan bilattices. Here the situation is quite similar – we have to add the consistency rules to the suitably restricted versions of the original logical systems. Let us note that in the positive case the effect of the added rules can be alternatively achieved by extending the notion of an axiom to include also: Γ⊢A is a basic sequent if also {B,¬B}⊆Γ for some formula B.

As for the external intuitionistic bilattices: their consistent logics are obtained by adding the axiom schema ■A→♦A.

Let us consider now the class of all intuitionistic bilattices as logical matrices with H = *the consistent valuations* (here this means that if h(A) = <a,b>, then a∧b = 0). They define a positive logic which is axiomatized by adding the scheme ¬A→(A→B) to the original system (formulated in the restricted language).

It is interesting to observe how the change from positive to negative affects the formal outlook of the systems: the negative consistent logic of intuitionistic bilattices is axiomatized by adding the *law of excluded middle*: A∨¬A.

**Remark** Dropping ⊗ from the language, too (i.e. considering the usual set of logical connectives), we arrive at a famous logic – *the logic of strong, or constrictive, negation* – invented by Vorob'yev and Nelson [41, 62]. Thus we have another completeness proof for this logic (cf. also the Conclusion).

In the info-algebra situation the consistent valuations validate rules as (∧⊢), as well as (⊢∨) in the negative case, without any restrictions (so in the corresponding consistent logics the distributivity rules are redundant). Thus we have the following

**Proposition 7.3** *The posirtive consistent logic of all info-algebras coincide with the positive consistent logic of all bilattices; the negative finitary consistent logic of info-¬algebras coincides with the negative consistent logic of all bilattices. The same is true for the case of algebras with negation.*

The only difficulty we encounter here is when the consistent logic of external intuitionistic bilattices is compared with the consistent logic of set expansions of pseudo-Boolean algebras. It is not known whether these two coincide. The problem lies in the fact that the mapping f defined in Proposition 4.4 is not a homomorphism with respect to the intuitionistic implication and for example f(♦(A→B)) = f(■A)→f(♦B) does not hold.

***Majority logic*** We conclude our exposition with an example of a semantics brought about by considerations in the spirit of the probabilistic treatment of plausibility of propositions. Recall that in the probabilistic framework a statement A is considered *plausible*, if $p(A)\geq\alpha$, for some fixed number $\alpha$ (for example $\alpha = 0.5$ or $\alpha = 0.95$) and where p is a *probability distribution*, i.e. $p:\mathcal{L}\to[0,1]$.

Let **L** be a bounded linear order. Consider in the class of bilattices **L**×**L** the following set of designated truth values:

$$D^{maj} = \{\langle a,b\rangle : a\geq b\}.$$

The intuition behind $D^{maj}$ is related to the view that in circumstances when all data is comparable with respect to truth-content it is reasonable to assume something as true if the positive arguments outweigh the negative ones. It seems that this idea is in direct correlation with the so-called *majority principle*. Stated somewhat vaguely the majority principle insists that, if the majority of the outcomes of a process are favorable (on some *linear* scale!), then the process as a whole is to be assumed as favorable. Imagine for example a group of experts giving opinion on the truth of a statement. According to the majority principle if the experts who believe the statement true are more than those who reject it, then it is rational to assume that the proposition is true or at least plausible.

Almost immediately one can see that such a semantics is at conflict with logical inference as we know it. The well-known *lottery paradox* (cf. Kyburg [38]) demonstrates that by an example of a finite set of highly plausible statements whose conjunction is not only implausible, but simply false. Nevertheless, there are some intriguing moments in the majority semantics which we want to describe briefly (a restricted version of the semantics – a classical language interpreted in **[0,1]**×**[0,1]**, was studied in [3]).

Although there are difficulties with the axiomatization of the logic, the notion of *tautology* admits a nice syntactic characterization. Let us note first that $\mathit{4}^{maj} = \langle \mathbf{4}, Hom(\mathcal{L},\mathbf{4}), D^{maj} \rangle$ is a characteristic matrix for the majority logic, so the notion of a tautology is decidable.

Call a variable or a negated variable a *literal*, a disjunction of literals – a *clause*, a conjunction of clauses – a *proposition*, a sum of propositions – a *datum*. It is easily provable that any formula is equivalent to a product of data (using the facts about de

Morgan bilattices). Call two literals *opposite* if they have the form p and ¬p; two clauses $C_1$ and $C_2$ are *connected* if they have a common variable (say p) occurring in opposite literal in the two clauses (say ¬p in $C_1$ and p in $C_2$).

The proof of the next lemma is very similar to the one that can be found in [3] so we omit it.

**Lemma 7.4** *1. A clause is a majority tautology iff two opposite literals occur in it.*
*2. A proposition is a majority tautology iff all its clauses are majority tautologies and they are pairwise connected.*

## CONCLUSION

Let us first briefly recapitulate our findings, namely the logics we have axiomatized:

- the standard full (positive and negative) logics of all distributive bilattices, i.e. of all algebras of the form $\mathbf{L_1}\times\mathbf{L_2}$ when $\mathbf{L_1},\mathbf{L_2}$ are distributive lattices; of all distributive bilattices with negation, i.e. of all $\mathbf{L}\times\mathbf{L}$ where $\mathbf{L}$ is a distributive lattice; of external bilattices, i.e. of all $\mathbf{L}^2$, where $\mathbf{L}$ is a distributive lattice; of de Morgan bilattices, i.e. of all $\mathbf{L}\times\mathbf{L}$ (= $\mathbf{L}^2$) where $\mathbf{L}$ is a lattice with negation; of all intuitionistic bilattices, i.e. of all $\mathbf{L}\times\mathbf{L}$, where $\mathbf{L}$ is a pseudo-Boolean algebra; of external intuitionistic bilattices, i.e. of all $\mathbf{L}^2$, where $\mathbf{L}$ is a pseudo-Boolean algebra; of Boolean bilattices, i.e. of all $\mathbf{L}\times\mathbf{L}$ (= $\mathbf{L}^2$) where $\mathbf{L}$ is a Boolean algebra;
- the standard positive full logics of all info-algebras, all info-algebras with negation, all set expansions of pseudo-Boolean algebras;
- the finitary logics (positive and negative) of all info-algebras, all info-algebras with negation, etc.
- the consistent versions of all the systems mentioned above.

All these logics are new – a fact due mainly to the presence of new connectives: $\otimes$, $\oplus$, etc., but even in the case of a language containing only traditional operators some systems appear in print for the first time, in particular the systems related to intuitionistic semantics.

### 8 *Historical Survey*

Our aim here is to point out examples of the three main constructions discussed in the preceeding sections:

- the bilattice construction (which appears either as $\mathbf{L_1}\times\mathbf{L_2}$ and different subspaces or in the form of generalized relational models);
- the set expansions;
- the interval construction (which also has two forms: algebraic – subspaces of $\mathbf{L}^2$ and relational – models with two forcing relations).

We want to demonstrate that the ideas behind these constructions were manifest in a variety of fields and brought to life by different background intuitions. Another purpose of the survey is to test the classificational power of these constructions by trying to present known semantics and logical systems as particular cases of the families studied in the present paper.

***The fundamental bilattice construction*** It is difficult to date the first occurrence of this construction, but at least it should be placed no later than the important examples below.

*The logic of strong negation* This is a logic that has its roots in the intuition shared by constructivists like Markov, Nelson, etc. (cf. [41]) that there is an essential difference between refuting a sentence by *reductio as absurdum* on one hand, and by constructing a counterexample on the other. The latter method gives rise to a new notion of a negation - a *strong* one (since it implies the former ordinary negation). Vorob'ev [62] was the first to axiomatize the idea presenting a calculus which extended the intuitionistic propositional logic with axioms concerning the new unary connective. Rasiowa developed in [45] an adequate algebraic semantics for Vorob'ev system: special de Morgan lattices (the *quasi Boolean algebras* of the Polish tradition) called N-lattices. Later Monteiro continued these investigations in [40] giving a representation theorem for N-lattices. Nevertheless up to the middle of the seventies there was no satisfactory semantics from the view-point of the constructive intuitions. Then (simultaneously!) Vakarelov [57] and Gurevich [35] published papers which contained similar (but not identical) models for the Vorob'ev calculus: Vakarelov gave an algebraic pairs construction (actually the consistent part of an intuitionistic bilattice $L \times L$), while Gurevich worked with Kripke models with two independent forcing relations, one for asserting and one for strongly rejecting a statement – essentially again the consistent part of a frame intuitionistic bilattice. Vakarelov also considered a relational semantics (anticipated in [40]) in an attempt to capture the idea of a counterexample. Recently the pairs construction was again used in investigations of N-lattices (Sendlewski [52]). The classical logic of strong negation (i.e. the consistent logic of all Boolean bilattices) is an extemsion of Kleene's three-valued consequnece relation. Moreover it coincides with Lukasiewicz' theree-valued logic, since Lukasiewicz' implication is definable as $(A \to B) \wedge (\neg B \to \neg A)$. Actually, when $\otimes$ is present the system of connectives is functionally complete in **3** (by a result of [54] Lukasiewicz' implication and the constant $\perp$ are already complete, but $\perp = x \otimes \neg x$ in **3**) – this was noted in [58].

*Generalized Kripke models*Kamp's paper [] was one of the first to discuss in details the logic of partial Kripke frames. There one can find a completeness result concerning the basic logic of such frames. This logic is in fact the consistent logic of all frame bilattices in a language without the knowledge connectives and coincides (depending on the connectives under consideration) with ....

An important later development – Veltman's *Data Semantics* [59], incorporated general models based on frames $F = \langle W, \leq \rangle$ with $\leq$ a partial order where *all chains have top elements.* Veltman's language included $\wedge, \vee, \neg, \to$ and two unary connectives called **may** and **must** respectively. Truth conditions for $\wedge, \vee, \neg$ were the usual ones (cf. Example 3) while the truth conditions for $\to$ coincided with those for the intuitionistic implication of Theorem 5.11. For **may** and **must** he set:

$w \models$ **may**$A$ iff $\exists w'(w \leq w'$ and $w' \models A)$;
$w =\!| $ **may**$A$ iff $\forall w'(w \leq w' \Rightarrow w' \not\models A)$;
$w \models$ **must**$A$ iff $\forall w'(w \leq w' \Rightarrow w' \not=\!| A)$;
$w =\!|$ **must**$A$ iff $\exists w'(w \leq w'$ and $w' =\!| A)$.

Introducing $\Box$ as the ordinary modality related to $\leq$ and considering the frame bilattice **B**(F) Veltman's operations have the following algebraic definitions:

$$x \rightarrow y = \Box(\neg \blacksquare x \vee y);$$

$$\mathbf{may}x = \blacksquare\Diamond x;$$

$$\mathbf{must}x = \blacklozenge\Box x .$$

Veltman admitted only valuations that assigned *stable* sets, i.e. *cones,* to variables which was rendered by $\Box h(p) = h(p)$. Thus Veltman's logic is in fact a bilattice modal logic determined by a particular class of frames (with the relevant restrictions on the admissible valuations).

***The set expansions*** In an early attempt Vakarelov [56] explored certain schema for obtaining relative semantics, later developed and applied to various nonclassical systems in [26,27,30,31,58] . Put very briefly, the schema consisted in the following: take a propositional language $\mathcal{L}$ and let $\mathcal{L}_1$ be any other language with counterparts to all the connectives of $\mathcal{L}$ (and possibly some additional ones). Assume that *Sem* is a semantics for $\mathcal{L}_1$, i.e. that for $s \in$ *Sem* we may in principle decide whether a formula $\alpha$ is *true at s* (denoted by $s \models \alpha$) or not, transfer *Sem* to formulas of $\mathcal{L}$ by means of (finite) sets of $\mathcal{L}_1$ formulae using *interpretation functions* i which assign to each $\mathcal{L}$ formula a set of $\mathcal{L}_1$ formulas, the following condition being satisfied: *if* $o(A_1,...,A_n)$ *is a connective of* $\mathcal{L}$, *then* $i(o(A_1,...,A_n) = \{o(\alpha_1,...,\alpha_n) : \alpha_k \in i(A_k), k=1,...,n\}$. Let *Int* denote the set of all interpretation functions and $Int_0$ – the set of those interpretation functions which do not contain the empty set in their ranges. Call a pair (s,i) *interpretation index.* Formulae in $\mathcal{L}$ can be evaluated at an index according to one of the following rules (but there certainly are other possibilities for evaluation, some of which were considered in [30, 31], among them the majority strategy according to which A is accepted, if the majority of members of i(A) are true):

$$A \text{ is true}_1 \text{ at } (s,i) \text{ iff } \forall \alpha \in i(A)\ s \models \alpha ;$$

$$A \text{ is true}_0 \text{ at } (s,i) \text{ iff } \exists \alpha \in i(A)\ s \models \alpha ;$$

For $\Gamma$ – a set of $\mathcal{L}$-formulae and A – an $\mathcal{L}$-formula say that A is an $(Sem, Int)_n$ consequence of $\Gamma$ (n= 0, 1), if for all indices (s,i), if all $B \in \Gamma$ are $\text{true}_n$ at (s,i), then A is also $\text{true}_n$ at (s,i). There are several intuitions behind the schema. For example, $\text{truth}_1$ can be associated with the notion of *disambiguation* (treated by Lewis in [39]): *a proposition is assumed true, if all its possible disambiguations are true.* In general disambiguations are formulated in a language different from the original one, but on the other hand they follow closely the structure of the proposition disambiguated. For $\text{truth}_0$ one has the notion of *justification* : a statement may be considered true iff there is at least one true justification of this statement (cf. [30,31]) . The justifications of a statement can be formulated in a completely different language, but the conditions upon the interpretation functions presuppose a very strict correspondence between the propositional structure of a statement and the structure of its justifications.

When applied to the classical propositional language (equipped with the ordinary **2**–semantics) the schema gives consequence relations related to some three– and four–valued logics (cf. [28]). For instance $(Sem_0, Int_0)_1$ is Kleene's original three-valued consequence relation (coinciding with Lukasiewicz' for the basic language, cf. [56]), in [26] it was proved that $(Sem_0, Int_0)_0$ is the consequence operation of the three–valued logic of Priest [48]. $(Sem_0, Int)_1$ and $(Sem_0, Int)_0$ were studied in [28] where the corresponding logics were formalized in a natural deduction style.

*Discussion of the intuitions behind sets of truth values*The notion of *sets of truth values as generalized truth values* has same origin as the bilattice construction: "X is the generalized truth value of A" means nothing more than "all we know at present is $\|A\| \in X$". Maximal possible knowledge corresponds to singletons, defective (contradictory, nonsensical) knowledge leads to an empty set of possible truth values. On this path we are immediately confronted with the problem: how are the sets of possible values X, Y, Z *given*? For example they can be thought as given *directly* (enumerated, etc.) or they can be "represented" by certain conditions defining the sets (these conditions are usually restrictions on the possible truth values). Now the question arises as to the language in which such conditions are formulated, how are they verified, etc. Although quite important, especially in applications, we leave their detailed analysis aside due to lack of space.

We can think of the information concerning the X's as of a family of restrictions (primitive restrictions) on the elements of the sets. The *consensus* approach would combine two families of restrictions in such a way that all restrictions that do not appear in both families would be dropped, so we would be left with only the restrictions common to X and Y. Now this guarantees that we get $X \cup Y$ as a result. A similar argumentation for the intersection though is not so conclusive – perhaps this is the cause of the troubles with $\oplus$ in the algebraic treatment of set expansions. Such difficulties show that the unrestricted notion of a set of truth values is not the suitable generalization in treating defective information.

***Intervals*** The idea of an interval in a truth space as a representation of the current knowledge about the truth value can be viewed as a specialization of the above arbitrary set expansions. It appears that quite similar justifications for the introduction of intervals can be traced in a variety of approaches to reasoning with imperfect information. Below we list some recent examples:

*Algebraic approaches*Garcia, Moussavi and Font developed in [21, 23] a logic based on an algebraic semantic described in Fitting [19] as reflecting intuitions about two kinds of unknowns: one representing "temporary lack of knowledge which is expected to be resolved within the system's time-space bounds" and the other– a permanently unknown truth valued (for some reasons). In [19] the resulting algebra is given as the consistent part of $3^2$, but actually in [21] the authors introduce their values as elements of the set expansion $3^{set}$. It is an interesting exercise to compare their reasons for dropping the set {0,1} out of the truth space with the arguments for considering set expansions in general.

*Probability approaches*Under this title we classify attempts to represent the uncertainty/plausibility of knowledge and inference by assigning a probabilistic measure to statements, the so-called *probability distributions*, with the idea that the greater the probability p(A) of a proposition, the more plausible it is, etc. (for a systematic modern exposition cf. Chapter 5 of [25]). Many thought that a unique value is not realistic and turned to probability *intervals* discussed by Dempster [10] and Shafer [53] among others. In Gardenfors [24] we find the note that the two limiting probabilities $p_*(A)$ and $p^*(A)$ must be interconnected with the following relation:

$$p_*(A)=1-p^*(\neg A).$$

thus a version of the rule $\neg[a,b] = [1-b,1-a]$.

Earlier, some people working in fuzzy set theory felt uneasy with the possibility to know the *exact* numerical value which a fuzzy predicate assigns to a particular object, so among the proposals for a more quantitatively realistic picture was the idea of *interval valued fuzzy sets* – functions assigning to elements of a domain E not numbers but open intervals $(a,b)\subseteq[0,1]$ (cf. e.g., [2] where the isomorphism between the consistent parts of $\mathbf{[0,1]\times[0,1]}$ and $\mathbf{[0,1]^2}$ was mentioned).

In the same vein, but in another field – *AI*:, Sandewall [49] proposed to consider intervals of real numbers [a,b] as representatives of *what we know* about the truth value of a proposition evaluated by "fuzzy" methods. He also explicitly defined the k-order as inverse set inclusion.

On a more qualitative level intervals appear also as the so-called *conditional objects* (cf. e.g. Dubois&Prade [11]) as in the tradition of treating uncertainty via conditional statements, conditional probabilities, etc., cf. [9,34]. A conditional object in a Booleam algebra of statements is a syntactic construction **b|a** with an incomplete semantics: **b|a** is meant to convey that, if a is true, then b is also true, but is undefined, if a is false. In the cited works one can find a partial ordering of conditional objects: **b|a** ≤ **c|d** iff $a\wedge b\leq c\wedge d$ and $\neg a\vee b\leq\neg c\vee d$ (and other properties as well) pointing to the interpretation:

$$\mathbf{b|a} = [a\wedge b, \neg a\vee b],$$

under which the order of conditional objects is just the interval t-order.

*Intervals and possible worlds models* In [28] a family of intuitionistic three–valued logics was studied arising from Kripke semantics based on models with two forcing relations – one included in the other. These were in fact the logics of the consistent parts of the external intuitionistic bilattices (in a restricted language).

In another instance of Kripke models with two forcing relations, Pequeno and Buchsbaum considered in [45] the *logic of epistemic inconsistency* with a semantics which turns out to be a version of the bilattices of the kind $\mathbf{L}^2$ where $\mathbf{L}$ is a Boolean algebra. Their possible worlds models have the forcing relations $\models_{min}$ and $\models_{max}$ and their language has the connectives $\wedge, \vee, \neg, \Rightarrow$, and ? - a unary connective. While $\wedge, \vee$, and $\neg$ have a familiar interpretation, $\Rightarrow$ and ? satisfy the following truth conditions:

$$w \models_{min} A \Rightarrow B \text{ iff } w \not\models_{max} A \text{ or } w \models_{min} B;$$

$$w \models_{max} A \Rightarrow B \text{ iff } w \not\models_{max} A \text{ or } w \models_{max} B;$$

$$w \models_{max} A? \text{ iff } \exists w'(w' \models_{max} A);$$

$$w \models_{min} A? \text{ iff } \forall w'(w' \models_{min} A).$$

In algebraic terms the semantics of Pequeno and Buchsbaum boils down to interpretation in the frame bilattices (or in general bilattices of the kind $\mathbf{L}^2$ where $\mathbf{L}$ is a Boolean algebra) with the usual understanding of $\wedge, \vee, \neg$ and $[a,b] \Rightarrow [c,d] = [b \rightarrow c, b \rightarrow d]$, thus $x \Rightarrow y = \neg \blacklozenge x \vee y$ (a familiar operation); $[a,b]? = [\Box a, \Diamond b]$, where $\Box$ and its dual $\Diamond$ corresponds to the universal binary relation $W \times W$. Pequeno and Buchsbaum admit only valuations that assign to variable *exact* truth values, i.e. values of the form $[a,a]$. Their definition of validity of a formula in a model: $M \models A$ iff $\forall w \models_{max} A$, implies that the logic is an extension of the negative *exact* logic of all frame bilattices $\mathbf{B}(F)$ with the universal modality – because $x?$ is definable in the bilattice language, e.g. as $(\Box x \wedge \top) \vee (\Diamond x \wedge \bot)$ or as $(\Box x)_1 \oplus (\Diamond x)_0$.

***Final remarks*** In the previous sections we described the nearest to the classical propositional logic possibilities for logical systems where the information manipulated is not complete and consistent. These simple many-valued logics are all based in fact on the same picture on which the classical logic rests: statements have their truth values determined according to their structure.

Of course among the most important problems of this domain in logic are the problems of how to "use" the lack of knowledge (partiality, contradictions, etc.) once it has appeared in an epistemic endeavor, and how to justify the rationality of the inference rules employed, assuming that they are satisfactory in their effectiveness aspect. They are still very far from a satisfactory solution.

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