WILEY SERIES IN PROBABILITY
AND MATHEMATICAL STATISTICS

WILEY SERIES IN PROBABILITY
AND MATHEMATICAL STATISTICS

Probability and Mathematical Statistics

ANDERSON · The Statistical Analysis of Times Series
ANDERSON · An Introduction to Multivariate Statistical Analysis
BARLOW, BARTHOLOMEW, BREMNER, and BRUNK · Statistical Inference Under Orler Restrictions
BARNETT · Comparative Statistical Inference
BHATTACHARYYA and JOHNSON · Statistical Concepts and Methods
BILLINGSLEY · Probability and Measure
CASSEL, SÄRNDAL, and WRETMAN · Foundations of Inference in Survey Sampling
DE FINETTI · Probability, Induction and Statistics
DE FINETTI · Theory of Proability, Volume I
DE FINETTI · Theory of Probability, Volume II
DOOB · Stochastic Processes
FELLER · An Introduction to Probability Theory and Its Applications, Volume I, *Third Edition,* Revised
FELLER · An Introduction to Probability Theory and Its Applications, Volume II, *Second Edition*
FISZ · Probability Theory and Mathematical Statistics, *Third Edition*
FRASER · Statistics—An Introduction
FULLER · Introduction to Statistical Time Series
HANNAN · Multiple Time Series
HANSEN, HURWITZ, and MADOW · Sample Survey Methods and Theory, Volumes I and II
HARDING and KENDALL · Stochastic Geometry
HOEL · Introduction to Mathematical Statistics, *Fourth Edition*
ISAACSON and MADSEN · Markov Chains
KAGAN, LINNIK, and RAO · Characterization Problems in Mathematical Statistics
KENDALL and HARDING · Stochastic Analysis
LAHA and ROHATGI · Probability Theory
LARSON · Introduction to Probability Theory and Statistical Inference, *Second Edition*
LARSON · Introduction to the Theory of Statistics
LEHMANN · Testing Statistical Hypotheses
MATHERON · Random Sets & Integral Geometry
MATTHES, KERSTAN, and MECKE · Infinitely Divisible Point Processes
PARZEN · Modern Probability Theory and Its Applications
PURI and SEN · Nonparametric Methods in Multivariate Analysis
RAGHAVARAO · Constructions and Combinatorial Problems in Design of Experiments
RAO · Linear Statistical Inference and Its Applications, *Second Edition*
ROHATGI · An Introduction to Probability Theory and Mathematical Statistics
SCHEFFE · The Analysis of Variance
SEBER · Linear Regression Analysis

Probability and Mathematical Statistics (Continued)

WILKS · Mathematical Statistics
WILLIAMS · Diffusions, Markov Processes, and Martingales, Volume I: Foundations
ZACKS · The Theory of Statistical Inference

Applied Probability and Statistics

BAILEY · The Elements of Stochastic Processes with Applications to the Natural Sciences
BAILEY · Mathematics, Statistics and Systems for Health
BARNETT and LEWIS · Outliers in Statistical Data
BARTHOLOMEW · Stochastic Models for Social Processes, *Second Edition*
BARTHOLOMEW and FORBES · Statistical Techniques for Manpower Planning
BECK and ARNOLD · Parameter Estimation in Engineering and Science
BENNETT and FRANKLIN · Statistical Analysis in Chemistry and the Chemical Industry
BHAT · Elements of Applied Stochastic Processes
BLOOMFIELD · Fourier Analysis of Time Series: An Introduction
BOX · R. A. Fisher, The Life of a Scientist
BOX and DRAPER · Evolutionary Operation: A Statistical Method for Process Improvement
BOX, HUNTER, and HUNTER · Statistics for Experimenters: An Introduction to Design, Data Analysis, and Model Building
BROWN and HOLLANDER · Statistics: A Biomedical Introduction
BROWNLEE · Statistical Theory and Methodology in Science and Engineering, *Second Edition*
BURY · Statistical Models in Applied Science
CHAMBERS · Computational Methods for Data Analysis
CHATTERJEE and PRICE · Regression Analysis by Example
CHERNOFF and MOSES · Elementary Decision Theory
CHOW · Analysis and Control of Dynamic Economic Systems
CLELLAND, deCANI, BROWN, BURSK, and MURRAY · Basic Statistics with Business Applications, *Second Edition*
COCHRAN · Sampling Techniques, *Third Edition*
COCHRAN and COX · Experimental Designs, *Second Edition*
COX · Planning of Experiments
COX and MILLER · The Theory of Stochastic Processes, *Second Edition*
DANIEL · Biostatistics: A Foundation for Analysis in the Health Sciences, *Second Edition*
DANIEL · Application of Statistics to Industrial Experimentation
DANIEL and WOOD · Fitting Equations to Data
DAVID · Order Statistics
DEMING · Sample Design in Business Research
DODGE and ROMIG · Sampling Inspection Tables, *Second Edition*
DRAPER and SMITH · Applied Regression Analysis
DUNN · Basic Statistics: A Primer for the Biomedical Sciences, *Second Edition*
DUNN and CLARK · Applied Statistics: Analysis of Variance and Regression
ELANDT-JOHNSON · Probability Models and Statistical Methods in Genetics

continued on back

The Algebra of Random Variables

The Algebra of Random Variables

M. D. SPRINGER

University of Arkansas

John Wiley & Sons

New York • Chichester • Brisbane • Toronto

Library of Congress Cataloging in Publication Data

Springer, Melvin Dale, 1918-
The algebra of random variables.

(Wiley series in probability and mathematical statistics)
Bibliography: p.
Includes index.
1. Random variables. 2. Integral transforms.
I. Title.

QA273.S698 519.2 78-9315
ISBN 0-471-01406-0

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

To

Wilma and Beth

Preface

The development of modern statistics and probability theory is focused largely on random variables, that is, variables whose probability distributions exist. That random variables play a dominant role in statistics and probability theory is evident when one considers the origin of various theorems and basic distributions in these fields. The central limit theorem, for example, deals entirely with the addition and subtraction of random variables. Again, the derivations of the Student-Fisher t and Snedecor F distributions are basically exercises in obtaining the distribution of quotients of certain independent random variables, and the solution to a standard problem in multivariate statistical analysis [14, p. 237] depends on a knowledge of the distribution of products of independent beta random variables. In the field of sampling techniques, Cochran [57] states that "the distribution of the ratio estimate y/x has proved annoyingly intractable because both y and x vary from sample to sample. The known theoretical results fall short of what we would like to know for practical applications." The desired theoretical results are, however, within the grasp of integral transform methods.

Although the integral transform method (i.e., the characteristic function) has long been used in studying sums of random variables—for example, in the case of the central limit theorem—it was not until 1948 that Epstein, in his paper "Some Applications of the Mellin Transform in Statistics" [92], pointed out the advantages of using the Mellin transform in deriving the standard statistical distributions involving products or quotients of independent random variables. And though integral transform methods are very useful in the derivation of statistical and probability theory requiring the application of algebraic operations to random variables, the application of such methods extends far beyond the theoretical domain. This is particularly true since the electronic computer has made possible the solution of many problems that heretofore defied solution simply because of prohibitive time requirements. In fact, it is in the applied field that the need for, and power of, an algebra of random variables in reaching exact solutions efficiently becomes most evident. Moreover, there are many problems whose exact solution cannot otherwise be achieved. It is both interesting and significant that such problems occur in numerous and diverse fields such as engineering, mathematics, economics, operations research, and psychology, to mention just a few representative areas. A number of examples given in Chapter 1 bear this out.

Many engineers and scientists are confronted with problems whose solution requires application of algebraic operations to random variables, yet there is presently no source supplying a unified and self-contained presentation of the methodology. It seems timely, therefore, that such a book be provided, and that it be designed to meet the needs of individuals in the following categories:

1. Advanced undergraduate and graduate students who are not majors in mathematical statistics and wish to take a one-semester course to learn how to apply algebraic operations to random variables. The material covered would probably consist of the unstarred chapters, sections, and problems, and possibly Appendices A, B, D, and E.
2. Advanced undergraduate and graduate students in statistics who wish to take a one-semester course dealing with the algebra of random variables, including the more theoretical aspects (starred chapters and sections) and the more difficult (starred) problems.
3. Graduate students who wish to take a reading course dealing with the algebra of random variables. Such a course could be tailored to the material cited in either of the categories above.
4. Individuals (graduate students, researchers, faculty members, etc.) who desire to participate in a seminar dealing with the algebra of random variables.
5. Individuals who wish to acquire a capability in the use of the algebra of random variables on their own, without recourse to a specific course or seminar on the subject.
6. Participants in seminars in industry.

The minimal required background for an individual in any of these categories consists of familiarity with the elementary aspects of differential and integral calculus, and with the basic methods of statistical inference and of distribution theory. Essentially, this is equivalent to two semesters of calculus and a two-semester course in basic statistics. However for individuals in any of the categories named who wish to include Chapters 6, 7, 9, Appendices C, F, and particularly the more difficult (starred) exercises, somewhat more maturity in statistics is desirable and probably necessary.

To avoid limiting the use of the book to those who have had a formal course in complex variables, I have included a relatively concise chapter presenting the basic elements of complex variables—definitions, concepts, and theorems—that form the basis for the transform methods and are necessary for an understanding and appreciation of the methodology. (For example, to understand why the condition that the Mellin transform be analytic in a particular strip is sufficient to ensure the uniqueness of a derived probability distribution, one must understand the definition of an integral transform and of Laurent's expansion.) The chapter is self-contained, and, in my judgment, is entirely comprehensible to an interested reader who understands the elementary aspects of differentiation and integration of real variables but has had no previous knowledge of complex variables. To require the interested reader to enroll in a course in introductory complex variables would be an inefficient, unnecessary, and burdensome method for acquiring the relevant information, since much of the material in such a course is not required by the reader and would probably discourage many potential readers from learning how to understand and apply transform methods to real world problems. By including such a chapter, I am confident that the size of the reading audience and the total benefit imparted thereto will be increased. At the same time, material that is primarily of theoretical interest* and is not required for application of the methods to practical problems, is included in starred sections, chapters, and appendices; thus it may be omitted without disrupting the continuity of the basic material necessary for the reader's ability to understand and apply the methods.

In short, I have endeavored to produce a book that will be self-contained, useful, interesting, and challenging, both to the student or analyst with the specified moderate mathematical background and to the mathematical statistician with theoretical training and interest. More specifically, my object has been to present, in a self-contained book, methods for utilizing integral transforms in the addition, subtraction, multiplication, and division of random variables, and in the analysis of algebraic functions of random variables. I believe that if these methods are clearly expounded and made available in a self-contained text, they will be utilized not only by mathematical statisticians but by engineers, scientists, researchers, and analysts alike, to solve many hitherto insoluble problems, both theoretical and applied, of the types cited in Chapter 1.

Many people have contributed to this book, both directly and indirectly. Contribution of subject matter is acknowledged throughout the text and in the references. I am particularly indebted to Dr. John L. Imhoff, head of

*For example, algebraic functions of independent H-function variables, the proof of Jordan's lemma, distribution problems in statistics, and various more difficult problems.

the Department of Industrial Engineering, University of Arkansas at Fayetteville, for his assistance and encouragement throughout the writing of this book, and to Dr. Aubrey E. Harvey, Coordinator of Research, for a small grant provided by the Graduate School during the summer of 1971.

I also express my gratitude to Mrs. Vicki Asfahl, Mrs. Donna Jones, Mrs. Marilu S. Zillman, and Mrs. Carol Lancaster for their expertise, patience, and efficiency in typing, and to my wife and daughter for bearing with me during the many long hours spent at home working on the manuscript. Finally, it is my pleasant duty to express my appreciation to Dr. Samuel Kotz for his thorough reading of the entire manuscript and for the corrections and improvements that resulted from this review.

MELVIN D. SPRINGER

Fayetteville, Arkansas
August 1978

Contents

The Algebra of Random Variables

CHAPTER 1

Introduction

1.1 HISTORICAL DEVELOPMENT

Since the early 1900s, the problem of deriving the distribution of sums of random variables (r.v.'s) has received a great deal of attention, and systematic procedures for determining such distributions have been well developed. Detailed discussions concerning sums of independent random variables (i.r.v.'s) have been given by many authors, such as Aroian [16], Wintner [411], Levy [205], Lukacs [222,225], and Cramer [73,74], to mention just a few.

The problem of determining the distribution of products and quotients of r.v.'s, however, has not been given the same extensive treatment. During the period 1929–1942, Craig [66,69,71] made some of the early investigations into the distribution of the product and quotient of two random variables, concentrating on normal variables. At about the same time Geary [117] derived his widely used approximation for the quotient of two normal variables, and in 1939 Huntington presented proofs for four theorems [155] resulting in a mathematical formulation for finding the distribution of the sum, difference, product, and quotient of two r.v.'s. Other early contributions to the theory of products and quotients of r.v.'s were made by Rietz [308], Haldane [140], Curtiss [75], and Aroian [17]. Most of the work, however, concentrated on products and quotients of two r.v.'s with specific probability density functions (p.d.f.'s).

Epstein [92] was the first to point out, in 1948, that the Mellin integral transform was a natural and powerful tool for analyzing products and quotients of i.r.v.'s. He derived directly and with ease the p.d.f.'s of the Student-Fisher t and F statistics, as well as the p.d.f. of two standardized normal i.r.v.'s. His work, however, was limited to quotients and products of 2. In 1954 Jambunathan [160] derived the distribution of the products of beta and of gamma i.r.v.'s for certain special cases. By means of a logarithmic transformation, Schulz-Arenstorf and Morelock determined the p.d.f. of the product of n uniform i.r.v.'s [323]. Also in 1959 Levy

[206, 207], posed the question of the construction of a general theory of multiplication of i.r.v.'s and derived some results for products of 2. In 1962 Zolotarev [420] began the construction of a general theory of multiplication of i.r.v.'s, analogous to the theory of addition of i.r.v.'s. His program focused on a sequence of theorems, stated without proof, that show both the similarity to and difference from the results for addition of i.r.v.'s. Other established results relating to product distributions concern isolated phases of the subject. They are cited at the appropriate points in the development of the algebra of r.v.'s.

Notwithstanding the scope of these various papers, the status of the theory of product distributions had not progressed to the point of deriving a general method for developing the p.d.f. of the distribution of products of more than two i.r.v.'s. The gap was largely closed in 1966 by Springer and Thompson [353, 354], who presented a general method for deriving the p.d.f. of the product of n i.r.v.'s that are not necessarily nonnegative nor identically distributed. They derived and identified the p.d.f.'s of the products of certain r.v.'s (e.g., beta, gamma, and Gaussian) as Meijer G-functions [358]. Later Carter [47, 48], showed that the p.d.f. of the product, power, or quotient of H-function variables is an H-function variable, and he developed an expression for the p.d.f. of a polynomial in H-function variables. Carter also identified many of the basic distributions in statistics and probability as H-functions, such as the gamma, beta, Weibull, half-normal, chi-square, and Rayleigh distributions. Prasad [294] established two theorems for deriving the Mellin transform of a function from its Laplace transform, and vice versa, which are often useful in determining the p.d.f.'s of algebraic functions of i.r.v.'s. Among the most recent contributions in the applied area are the books by Giffin [121] and Muth [267], both directed toward problems in operations research and certain areas of engineering. Giffin discusses the use of transforms in connection with some of the probabilistic models used in systems analysis (particularly linear systems) and operations research (particularly applications in queueing theory). However he treats only very briefly the important problem of the evaluation of inversion integrals by residues, relying mostly on the use of partial fraction expansions (when applicable) and the use of tables of integral transforms. Muth's book deals only with the Laplace and z transforms.

1.2 THE NATURE AND SCOPE OF THE ALGEBRA OF RANDOM VARIABLES

Since almost all problems in the real world are probabilistic, rather than deterministic, the solutions to many important problems in diverse fields require the ability to analyze r.v.'s. One may, for example, build and test a system in order to analyze the reliability of the system, but the reliability as calculated from test data is at best only an estimate of the true reliability. Such estimates are subject to random variations that cannot be controlled. However one can often determine the p.d.f. of these reliability estimates, and from these conclusions may be reached concerning the true but unknown system reliability. If the variable is not a composite one (such as a sum, difference, product, or quotient), its analysis is relatively straightforward. Yet on numerous occasions the r.v. is composite. Thus in the reliability analysis of a system consisting of n independent subsystems arranged in series, one usually has test data for each available subsystem before the system is assembled. The analysis of the reliability of such a system on the basis of subsystem test data involves the determination of the distribution of the product of n i.r.v.'s (see, e.g., ref. 356). In other situations, the r.v. whose distribution is required may be a sum, difference, quotient, or algebraic function of component r.v.'s whose p.d.f.'s are known. Such p.d.f.'s are obtainable through the use of integral transforms, but the only case that has received extensive systematic treatment (via characteristic functions) is that involving sums and differences of r.v.'s. In this book, the scope of integral transform methods is extended to cover the whole spectrum of the algebra of r.v.'s, thereby permitting the determination of the distribution not only of sums and differences, but also of products, quotients, and general algebraic functions of r.v.'s.

In the case of addition of r.v.'s, the problem is this: given n r.v.'s X_i with p.d.f.'s $f_i(x_i)$, $-\infty < x_i < \infty$ find the p.d.f. $g(w)$ of the r.v.

$$W = \sum_{i=1}^{n} X_i.$$

Likewise, it may be necessary to find the p.d.f. $h(y)$ of the r.v.

$$Y = \prod_{i=1}^{n} X_i,$$

which is a problem in the multiplication of either independent or dependent r.v.'s. Often both these problems can be directly solved through the use of integral transforms. The operations of addition and subtraction of r.v.'s are analytically equivalent, being achieved through the use of either the Laplace or the Fourier integral transform, the latter being equivalent to the characteristic function. Similarly, multiplication and division of r.v.'s are analytically equivalent operations, inasmuch as the quotient Y/X of two r.v.'s X and Y is equivalent to the multiplication of the two r.v.'s Y and $1/X$. Operations involving powers and roots of r.v.'s can also be directly analyzed by application of the basic laws of operation for Mellin transforms.

In the application of algebraic operations to r.v.'s, there are two broad classes: the class for which all the r.v.'s are independent, and that for which they are not. The importance of the independence condition stems from the equivalence, in the case of i.r.v.'s, of repeated convolution of the p.d.f.'s in the transform to successive multiplication of the integral transforms. In the event of dependence among the r.v.'s, this property no longer holds, and the evaluation of the inversion integral now entails evaluating a multiple, rather than a single, integral. Although the material in this book involves both independent and dependent r.v.'s, it is concerned primarily with the treatment of i.r.v.'s.

Unlike the Mellin transform, the Fourier transform (characteristic function) is readily adaptable to change of unit and origin of the component r.v.'s. For this reason, noncentrality poses no problem in the addition or subtraction of r.v.'s, but it considerably complicates the procedure of deriving the distribution of products and quotients. The Fourier and Mellin transforms also differ insofar as range of variable is concerned, the former being defined over the doubly infinite range $(-\infty,\infty)$ and the latter over the singly infinite range $(0,\infty)$. Therefore when multiplication or division of i.r.v.'s having doubly infinite range is to be carried out, it is necessary to partition each i.r.v. into two components, one for which the p.d.f. is nonnegative on the interval $(0,\infty)$ and zero elsewhere, and another for which the p.d.f. is nonnegative on the interval $(-\infty,0)$ and zero elsewhere. If such i.r.v.'s are symmetric about the origin, the derivation of the distribution of the product or quotient becomes equivalent to that for i.r.v.'s having a singly infinite range.

For many practical problems, the r.v.'s involved are restricted to nonnegative values. For such cases the Laplace transform and inversion integral may be used to determine the p.d.f. of the sum of these r.v.'s. The procedure for deriving the p.d.f. $g(w)$ of the sum

$$W=\sum_{i=1}^{n} X_i$$

of n i.r.v.'s X_i with p.d.f.'s $f_i(x_i)$ is completely analogous to that for deriving the p.d.f. of the product of these r.v.'s. Specifically, the Laplace transform of $g(w)$ is the product of the Laplace transforms of $f_i(x_i)$, knowledge of which permits the determination of $g(w)$ by evaluating the inversion integral along the Bromwich path.

The more general problem of deriving the p.d.f. of algebraic functions of i.r.v.'s may now also be analyzed. For example, one may wish to analyze the r.v.

$$Y = X_1 + \frac{X_2 + X_3}{X_4},$$

where the i.r.v.'s X_i have specified p.d.f.'s $f_i(x_i)$, $i=1,2,3,4$. This analysis is considerably simplified if one can convert the Laplace (or Fourier) transform of $g_1(u)$, where $U = X_2 + X_3$, into a Mellin transform, and then convert the Mellin transform of $g_2(v)$, $V = U/X_4$, into a Laplace (or Fourier) transform. For then the Laplace (or Fourier) transform of the p.d.f. $h(y)$ corresponding to Y can be directly determined without first having determined the p.d.f.'s $g_1(u)$ and $g_2(v)$. Because of the utility of such transform conversion capabilities, methods for the conversion of Laplace (or Fourier) transforms to Mellin transforms, and vice versa, as developed by Prasad [294], are presented.

It is interesting to note the manner in which the Mellin integral transform enters into the characteristic function of the p.d.f. of sums (or differences) of products (or quotients) of i.r.v.'s [364; 366, pp. 486–487]. Specifically, as shown in Chapter 5, the p.d.f. $h(w)$ of the sum

$$W = X_1 Y_1 + X_2 Y_2 + \cdots + X_n Y_n$$

of products $X_j Y_j$, where X_j and Y_j are mutually i.r.v.'s for all $j = 1, 2, \ldots, n$, has the characteristic function

$$F_t(h(w)) = \sum_{s=1}^{\infty} \frac{(it)^{s-1}}{(s-1)!} \prod_{j=1}^{n} M_s(f_j(x_j)) M_s(g_j(y_j)).$$

Such operational techniques play an important role in deriving and analyzing the p.d.f. of sums of products and quotients of i.r.v.'s.

When transforms and/or inversion integrals cannot be evaluated either in closed form or in exact series form, numerical procedures such as the fast Fourier transform (FFT) and numerical inversion of the Laplace transform may be useful. Frequently other p.d.f.'s (e.g., the beta and Laguerre p.d.f.'s) yield satisfactory approximations.

Since the concept of infinite divisibility is only marginally (and mainly formally) connected with the algebra of random variables, it is not discussed here. The theoretical developments in this area are well covered in the monographs by Linnik [210], Lukacs [225], and Ramachandran [301], and in a number of papers by various authors (see, e.g., Chapters 5, 6, 8, and 9 of Lukacs's monograph for specific authors and their contributions).

1.3 AREAS OF APPLICATION

Although *The Algebra of Random Variables* is established primarily on a mathematical basis and its applications are mainly statistical, these applications nevertheless occur in a wide variety of areas. When considered with applications to theoretical aspects of statistics discussed in Chapter 9, the examples of applications that follow indicate the present and potential usefulness of *The Algebra of Random Variables* in both theoretical and applied areas.

A simple but practical problem requiring the analysis of products of i.r.v.'s concerns signal amplification. If n amplifiers are connected in series and if X_i denotes the amplification of the ith amplifier, the analysis of the total amplification $Y = X_1 \cdot X_2 \cdots X_n$ is basically a problem in the analysis of products of i.r.v.'s [214]. Again, certain problems in the physical sciences connected with the theory of spin-stabilized rockets require the use of the distribution of the product of a central chi-square and a noncentral chi-square r.v. [13]. In medicine, a hospital administrator may wish for planning purposes to analyze the distribution of the hospital costs for accident victims. Since the number of accidents in a period, the number of days spent in the hospital, and the total cost per patient-day are all r.v.'s (i.e., nondeterministic), the distribution of the total hospital cost is the distribution of a product of three r.v.'s [214]. Also, in the treatment of various military operations research (MOR) problems related to radar discrimination [255], it is necessary to analyze the product and quotient of Bessel function variables. Again, in a recent study [38] on the detection of radar targets of unknown Doppler frequency, the likelihood ratio test led to a consideration of an r.v.

$$W = \sum_{i=1}^{n} X_i$$

consisting of a sum of i.r.v.'s, each having a Pareto distribution. Further applications involving sums of Pareto variables occur in economic studies, where the Pareto distribution is used to describe the distribution of incomes [237]. Hill and Buck [149] have shown how the zeta and Laplace transforms (defined in Section 2.8) may be employed to considerable advantage in the modeling and analysis of economic situations involving, respectively, discrete and continuous time series of cash flows. Their statement that "the application of this methodology in general practice awaits the development of techniques which are specifically geared to engineering economics and a comprehensive demonstration of the methodology" indicates the potential power of integral transform methods in economic analyses.

Applications of *The Algebra of Random Variables* abound in the field of engineering, and only a few are mentioned here. One such application was made by Webb [394] in 1965 in connection with attempts to improve the detection of weak signals masked by noise. Mathematically, the problem reduces to the derivation and analysis of the distribution of the product of diode detector wave forms. Again, in the area of system reliability, the evaluation of confidence limits for the reliability of systems composed of independent subsystems, using only subsystem test data, is of particular importance for complex systems. The solution to this problem requires the analysis of products of i.r.v.'s [356, 357]. Closely related to the reliability problem is that of system availability, which involves the combined analysis of "up time" and "down time" of a system. As has recently been shown [38, 376], the analysis of system availability reduces to the basic problem of the analysis of products of beta and Euler i.r.v.'s. [38, 344, p. 3].

Encompassing both system reliability and availability is the concept of system effectiveness. In 1963, the Air Force Systems Command (AFSC) deemed the problem of evaluating weapons systems effectiveness sufficiently important to warrant the formation of the Weapons System Effectiveness Industry Advisory Committee (WSEIAC) for the express purpose of providing "technical guidance and assistance to the AFSC in the development of a technique to apprise management of current and predicted weapon system effectiveness at all phases of weapon system life." The findings of the committee, published by the Air Force in five parts [393] in 1965, presented an effectiveness evaluation technique, together with methods and procedures for predicting and measuring system effectiveness. It is of particular significance here that the technique is focused on a definition of system effectiveness that expresses the effectiveness (E) as a product of three random variables [393, (2)a, b]: system availability (A), system dependability (D), and system capability (C). That is, $E = ADC$; thus in the final analysis the ability to analyze the effectiveness of an Air Force weapons system by this technique is contingent on one's ability to carry out a mathematical analysis of a random variable E, which is a product of three random variables A, D, and C. Applications of the algebra of r.v.'s to aerospace problems of other types have been discussed by Donahue [83, 84].

Learning machines, adaptive machines, and automatic pattern recognition are general descriptions for a wide variety of estimation, detection, and classification tasks that are encountered in modern engineering, and to which *The Algebra of Random Variables* has some existing, and considerable potential, application. Particular undertakings that are representative of these tasks are the reproduction of signals at the terminal of a radio communication link; the detection of radar targets; the reading by machine of typed or handwritten language; the recognition of spoken language by automatic machines; the measurement of variables in a production control problem; and the automatic identification of features in two-dimensional optical displays such as photographs and television pictures.

A simple example [1] of an r.v. that is an algebraic function of r.v.'s arises in connection with the analysis of the manufacturing cost of a product, where n units of the product are to be produced on a machine tool in a mass production plant. The set-up cost (C_s), the machine time (t_i) for the ith unit, and the cost per unit machine time (C_m) are r.v.'s governed by specified probability laws. The manufacturing cost per unit

$$C = \frac{C_s + C_m \Sigma_i^n t_i}{n}$$

is a rational function of the r.v.'s C_s, C_m, and t_i, and its analysis requires the knowledge of the distribution of sums of products of i.r.v.'s.

In the field of mechanical engineering, there has recently been some attempt to account explicitly for randomness in internal structural qualities, which has led to the application of probabilistic models in analyzing the statics of random beams. Stark and Shukla [365] have shown that a variety of well-known expressions for concentrated loads and moments for different support conditions have an underlying polynomial-type random model, and these authors have utilized Fourier and Mellin transforms in the resultant analysis.

Perhaps the newest application of these methods is in the area of nonlinear filtering. As one author [274] has stated:

> This application [the multiplicative processing of images] is motivated very directly, because image formation is predominantly a multiplicative process.... Applications of multiplicative filtering which may have potential are compensators for channel fading, systems for simultaneous amplitude and phase modulation and detection, automatic gain controls for other than audio application, a.c. and d.c. power regulators, and radar signal processing.

In the area of stochastic geometric programming, Stark [367] has recently utilized the Mellin transform to encode randomness in the constraint and objective function coefficients using the substituted dual function, which enabled him to obtain statistical moments and the probability distribution of the optimal objective value. Although integral transforms appear to have considerable potential along these lines, to date very little work has been done involving the application of integral transforms to problems in the field of stochastic geometric programming.

These few present and potential applications of *The Algebra of Random Variables* are sufficient to indicate the wide applicability of such an algebra to real world problems. Equally basic, of course, is the role of integral transforms in both the theoretical and applied areas of statistics, as Chapter 9 bears out.

CHAPTER 2

Differentiation and Integration in the Complex Plane

2.1 INTRODUCTION

The use of integral transforms to derive probability density and distribution functions of sums, products, quotients, and—more generally—algebraic functions of i.r.v.'s, involves the evaluation of contour integrals in the complex plane. Usually the evaluation of such integrals is accomplished by application of the residue theorem, which entails the evaluation of residues. The procedure for evaluating residues is in itself very simple, involving nothing more difficult than the evaluation of an nth order derivative of a function $f(z)$ of a complex variable. To understand the theory and application of *The Algebra of Random Variables*, one need only understand some basic rudiments of complex variables, particularly those relating to differentiation and integration in the complex plane. Included in this category are the concepts of a complex number, a complex variable, a complex function, a pole, a residue, an analytic function, a complex integral, a contour integral, an integral transform, an inversion integral, and the statements of a few basic theorems relevant to the evaluation of contour integrals. This chapter, designed to provide these rudiments, requires on the part of the reader only a knowledge of the elementary elements of differential and integral calculus.

2.2 COMPLEX NUMBERS AND VARIABLES

A complex number z is a number of the form $z = x + iy$, where x and y are real numbers and i is the imaginary unit defined by $i = \sqrt{-1}$. The real numbers x and y are called, respectively, the real and imaginary components of z, and are often denoted by the symbols $\mathrm{R}(z)$ and $\mathrm{I}(z)$. Any two complex numbers that differ only in the sign of their imaginary parts constitute a conjugate pair, and either is said to be the conjugate of the other. Thus if $z=x+iy$, then its conjugate is $\bar{z}=x-iy$. Note that the product $z\bar{z}=x^2+y^2$ is a purely real number.

A complex number z can be represented geometrically either by the point P (Fig. 2.2.1), whose abscissa and ordinate are, respectively, the real and imaginary components of the given number, or by the vector $\mathbf{r}=OP$, which joins the origin to this point. This xy plane, in which complex numbers are represented geometrically, is variously referred to as the complex plane or the Argand diagram. This vector has two important attributes in addition to its components x and y, namely, its length $r=|z|=\sqrt{x^2+y^2}$ and its direction angle $\theta=\tan^{-1}(y/x)$. Clearly, since $x=r\cos\theta$, $y=r\sin\theta$, and $e^{i\theta}=\cos\theta+i\sin\theta$, the complex number $z=x\pm iy$ can be written in the polar form

$$z=r(\cos\theta\pm i\sin\theta), \tag{2.2.1}$$

as well as in the exponential form

$$z=re^{\pm i\theta}. \tag{2.2.2}$$

The length r, as indicated previously, is the absolute value or modulus of z, and the angle θ is the amplitude or argument of z. The product of a complex number and its conjugate is the square of the length of the vector;

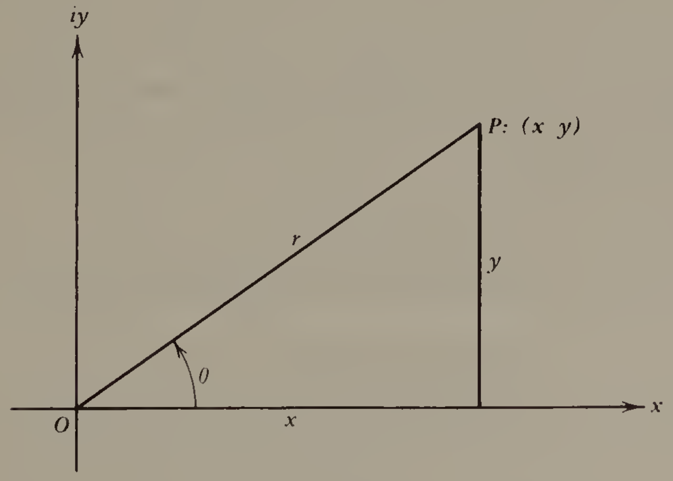


Fig. 2.2.1 The complex plane.

that is,

$$z\bar{z}=|z|^2$$
$$=r^2.$$

Finally, if x and y are real variables, $z=x+iy$ is defined to be a complex variable.

2.3 FUNCTIONS OF A COMPLEX VARIABLE

As is well known, y is defined to be a function of the real variable x if for each value of x there corresponds one or more values of y, and this is expressed by writing $y=f(x)$. Similarly, if $z=x+iy$ and $w=u+iv$ are two complex variables, and if for each value of z in some portion of the complex plane one or more values of w are defined, then w is said to be a function of z, which is expressed symbolically in the form $w=f(z)$. Also, w is a single-valued or multiple-valued function of z, respectively, according as only one value or more than one value of w corresponds to a given value of z.

The regions which are involved in contour integration are usually defined by one or more simple closed curves, the latter being defined as a curve which completely bounds a finite section of a plane or surface. These regions may be closed, open, simply connected or multiply connected. If the region includes all the points of its boundary curve or curves, it is said to be *closed*. If it contains none of its boundary points, it is called an open region. Also, if any simple closed curve which can be drawn in a region R can enclose only points in R, then R is said to be *simply connected*. Alternatively, if a simple closed curve can be drawn in R which can enclose points which are not in R, then R is said to be *multiply connected*.

Since the concepts of limit and continuity are frequently utilized in connection with *The Algebra of Random Variables*, the reader will now be reminded of their definitions.

Definition[1] If $f(z)$ is a single-valued function of z, and w_0 is a complex constant, and if for every $\varepsilon>0$ there exists a positive number $\delta(\varepsilon)$ such that

$$|f(z)-w_0|<\varepsilon$$

for all z such that $0<|z-z_0|<\delta(\varepsilon)$, then w_0 is said to be the limit of $f(z)$

[1]From *Advanced Engineering Mathematics*, by C. R. Wylie. Copyright 1951 by McGraw-Hill. Used with permission of McGraw-Hill Book Company.

as $z \to z_0$. Symbolically,

$$\lim_{z \to z_0} f(z) = w_0.$$

Closely associated with the concept of a limit is that of continuity. A single-valued function of z is defined to be *continuous at a point*,[2] z_0, if each of the following conditions is met:

1. $f(z_0)$ exists.
2. $\lim_{z \to z_0} f(z)$ exists.
3. $\lim_{z \to z_0} f(z) = f(z_0)$.

If a function $f(z)$ is continuous at every point of a region R, it is said to be continuous throughout R.

An important theorem on continuous functions is stated below without proof. The proof is found in any standard text on functions of complex variables.

Theorem 2.3.1 A necessary and sufficient condition that

$$f(z) = u(x,y) + iv(x,y)$$

be continuous is that the real functions $u(x,y)$ and $v(x,y)$ be continuous.

One type of function of a complex variable of particular importance in *The Algebra of Random Variables* is that of exponential type or order. A function $f(x)$ is said to be of exponential type or order as x tends to infinity, provided some constant "a" exists such that the product $e^{-ax}|f(x)|$ is bounded for all x greater than some finite number X. This means that $f(x)$ cannot grow more rapidly than Me^{-ax} as $x \to \infty$, where M is some constant [350, p. 2]. In terms of order notation, $f(x) = O(e^{ax})$ (see, e.g., ref, 226, p. 1).

Various statements and theorems concerning the limiting values of complex functions can be simplified through the use of the following order symbols.

[2] From *Advanced Engineering Mathematics*, by C. R. Wylie. Copyright 1951 by McGraw-Hill. Used with permission of McGraw-Hill Book Company.

Order symbols O and o

(a) The symbol O. Consider two functions $f(z)$ and $g(z)$ of a complex variable z, where $g(z) \neq 0$, for all z in a region R of the complex plane, and let z and z_0 be points in R. If there exists a number A independent of z such that $|f(z)/g(z)| \leqslant A$ for all z in R, one says that [226, p. 1]

$$f(z) = O(g(z)) \qquad \text{as} \quad z \to z_0 \text{ in } R.$$

In the special case dealing with functions of a real variable, the region R consists of the real line.

Example 2.3.1 Let

$$f(z) = \frac{1 - \cos z}{z}$$

$$= \frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - + \cdots$$

(Note that $\cos z \equiv 1 - z^2/z! + z^4/4! - z^6/6! + - \cdots$ for $\leqslant z \leqslant \frac{\pi}{2}$.)

Then $f(z) = O(z)$ as $z \to 0$

Example 2.3.2 Let $f(z) = e^{-z}$. Then $f(z) = O(z)$ as $z \to \infty$ for $0 < |z| < \infty$.

(b) The symbol o: If $\lim f(z)/g(z) \to 0$ as $z \to z_0$ in R, then one says [226, p. 1] $f(z) = o(g(z))$ as $z \to z_0$ in R.

Example 2.3.3 Let $f(z) = 1 - \cos z = z^2/2! - z^4/4! + z^6/6! - + \cdots$. Then $f(z) = o(z)$ as $z \to 0$.

Example 2.3.4 Let $f(z) = z$. Then $f(z) = o(z^2)$ as $z \to \infty$.

2.4 ANALYTIC FUNCTIONS

Since the concept of an analytic function is basic to the methods for the evaluation and application of contour integrals, it is important for the reader to bear in mind the simple definition of an analytic function. Specifically, if $w = f(z)$ possesses a derivative at $z = z_0$ and at every point in some neighborhood[3] of z_0, then $f(z)$ is said to be *analytic at* $z=z_0$, and z_0 is called a *regular* point of the function. If a function $f(z)$ is not analytic at $z=z_0$, but if every neighborhood of z_0 contains points at which $f(z)$ is analytic, then z_0 is called a *singular* point of $f(z)$. A function that is analytic at all points of a region R is said to be *analytic in* R and is variously referred to as an analytic function, a regular function, or a holomorphic function [161, p. 11]. If a complex function is analytic for all finite values of the variable (i.e., in the complex plane), it is called an entire function [161]. It can be shown that an analytic function $f(z)$ of a complex variable has continuous derivatives of all orders.

Since the derivative of a function of a complex variable forms the basis for the definition of an analytic function, it is important to understand the definition and a few of the more important properties of such a derivative. The derivative of a function of a complex variable $w=f(z)$ is defined as

$$\frac{dw}{dz}\equiv w'\equiv f'(z)\equiv \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}, \tag{2.4.1}$$

which is conceptually and symbolically identical to the definition of the derivative of a function of a real variable. Furthermore, the general theory of limits is valid for complex variables as well as for real variables, so that formulas for the differentiation of functions of a real variable will have identical counterparts in the domain of complex numbers. It should be pointed out, however, that the existence of the derivative $f(z)$ requires that the limit of the difference quotient (2.4.1) be the same regardless of how Δz approaches zero. Fortunately there are some simple necessary and sufficient conditions for the existence of the derivative of a function of a complex variable, such as the Cauchy-Riemann equations in Theorem 2.4.1.

Theorem 2.4.1[4] If u and v are real single-valued functions of x and y that with the four partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ are continuous throughout a region R, then the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\qquad\text{and}\qquad\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \tag{2.4.2}$$

are both necessary and sufficient conditions that $f(z)=u(x,y)+v(x,y)$ be

[3]The neighborhood of a point is the interior of some geographic figure (usually a square or circle) that contains the point.
[4]From *Advanced Engineering Mathematics*, by C. R. Wylie. Copyright 1951 by McGraw-Hill. Used with permission of McGraw-Hill Book Company.

analytic in R. In this case, the derivative of $f(z)$ is given by either

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \text{or} \qquad f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \tag{2.4.3}$$

2.5 INTEGRALS: REAL AND COMPLEX

The definitions of real and complex integrals are conceptually analogous in that both are based on the limit of a sum. The main difference between the conventional (Riemann) integral and a complex integral is that the former involves integrating a real function $f(x)$ over the x-axis and the latter involves integrating a complex function $f(z)$ over a curve in the complex plane. This curve could, of course, be the x-axis, but frequently it is a closed contour such as a semicircle. The analogy between the two definitions is now shown.

One first of all recalls that the Riemann integral introduced in basic calculus courses is defined as the limit of a sum. Specifically, let $f(x)$ be a function of the real variable x, continuous over the interval (a,b). Divide this interval into subintervals whose lengths are $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, and choose points, one in each subinterval, whose abscissas are $x_1, x_2, \ldots, x_n$, respectively. Then the limit of the sum

$$\sum_{i=1}^{n} f(x_i)\Delta x_i$$

as n increases without limit in such a way that each subinterval approaches zero as a limit, is called the definite (Riemann) integral of $f(x)$. Symbolically,

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)\,dx. \tag{2.5.1}$$

One usually thinks of this real (Riemann) integral as the area bounded above by the curve $y=f(x)$, below by the x-axis, and on the two sides by the vertical lines $x=a$ and $x=b$. In the case of a complex integral, no such interpretation necessarily exists. Throughout this book, the integral (2.5.1) is regarded strictly as the limit of a sum. When viewed in this light, the definition of the Riemann integral in the real plane is very similar to that of the integral of a complex function evaluated over a contour in the complex plane, as we now show.

Actually, it is a simple matter to extend formally the definition of an integral from the real domain to the complex plane. Specifically, let $f(z)$ be any function of z, not necessarily analytic, and let C be a curve of finite length connecting the points A and B (Fig. 2.5.1). Let the points z_i, $i=1,2,\ldots,n$ divide the curve C into $n-1$ intervals, and set

$$\Delta z_i = z_i - z_{i-1}.$$

Also, let z_i' be any point of the arc $z_{i-1}z_i$. Then the limit of the sum

$$\sum_{i=1}^{n} f(z_i')\Delta z_i \tag{2.5.2}$$

as n goes to infinity in such a way that the length of every chord Δz_i approaches zero, is defined to be the *integral of the complex function* $f(z)$ *along* C,

$$\int_C f(z)\,dz,$$

where C may be either an open or a closed curve. If it is a closed curve (i.e., completely encloses a given region) the integral is called a (complex) contour integral; otherwise, it is usually referred to as a (complex) line integral.

Although $f(z)$ was not required to be an analytic function in the foregoing definition of a complex integral, it is important to realize that if $f(z)$ is analytic within a simply connected region R, then the line integral

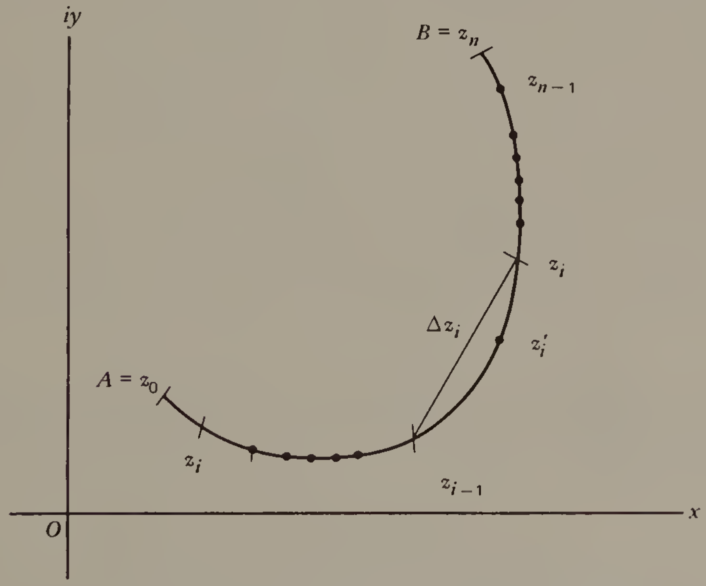


Fig. 2.5.1 Integration in the complex plane.

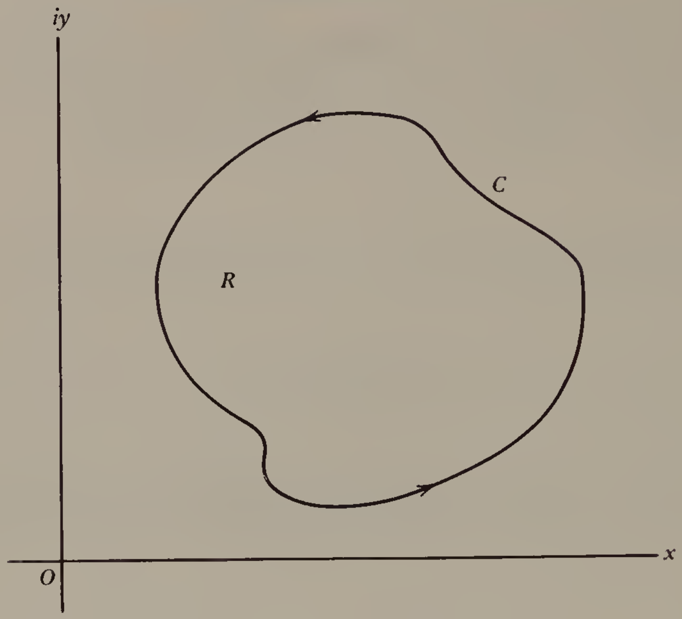


Fig. 2.5.2 Positive direction of integration over closed contour C.

between any two points A and B (Fig. 2.5.1) of R is independent of the path. It should also be remembered that when a function $f(z)$ is integrated over a closed contour, the sign of the integral will be positive or negative, depending on the direction of integration over the contour. The integral is defined to be positive when the contour is traversed in such a way that the region enclosed by the contour is always to one's left. (Fig. 2.5.2). This is sometimes referred to as the left-hand rule [349, p. 94].

Perhaps the most fundamental and far-reaching result in the theory of analytic functions is the famous theorem of Cauchy, which is now stated without proof, since the proof is readily available in any textbook on functions of complex variables. This theorem is important insofar as *The Algebra of Random Variables* is concerned primarily because it is needed to prove the residue theorem, which is fundamental in the evaluation of inversion integrals. The evaluation of inversion integrals is the key to the determination of the distribution of sums, differences, products, quotients, and more generally, algebraic functions of i.r.v.'s, as we see later.

Theorem 2.5.1 (Cauchy's theorem)[5]. If $f(z)$ is analytic at all points within and on a closed curve C, and if $f'(z)$ is continuous throughout this closed region R, then

$$\int_C f(z)\,dz = 0.$$

[5]From *Advanced Engineering Mathematics*, by C. R. Wylie. Copyright 1951 by McGraw-Hill. Used with permission of McGraw-Hill Book Company.

2.6 THE LAURENT EXPANSION OF A FUNCTION OF A COMPLEX VARIABLE

The most important theorem associated with *The Algebra of Random Variables* is the residue theorem, which is almost always used to evaluate the inversion integral yielding a desired p.d.f. A fundamental element in the derivation of the residue theorem is the Laurent expansion which, together with the residue theorem, we now briefly discuss.

It is well known [113] that a complex function $f(z)$ may be expanded in a Taylor series about any point inside a region within which the function is analytic. In many situations it is most helpful to be able to expand a complex function $f(z)$ about a point where, or in the neighborhood of which, the function is not analytic. This necessitates a new type of expansion known as Laurent's series, since the method of Taylor series is clearly inapplicable in such cases. Laurent's expansion furnishes a representation that is valid in the annular ring (Fig. 2.6.1) bounded by two concentric circles, provided the function that is being expanded is analytic everywhere between the two circles. As with a Taylor series, the function may have singular points outside the larger circle, but unlike a Taylor series, it may also have singular points *within* the inner circle. The result is that negative as well as positive powers of $(z-a)$ now appear in the Laurent expansion. The result is stated more specifically in the following theorem.

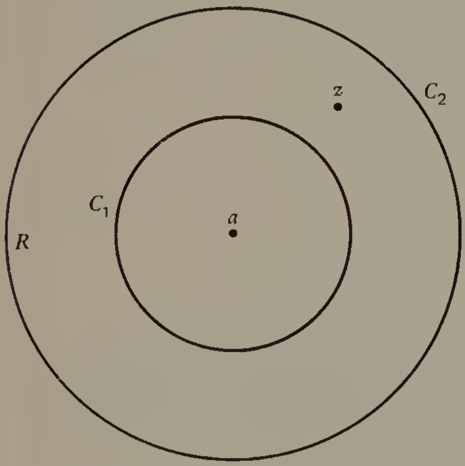


Fig. 2.6.1 Annular ring for Laurent's expansion.

Theorem 2.6.1 (Laurent's series).[6] If $f(z)$ is analytic throughout the closed region R, bounded by two concentric circles C_1 and C_2, then at any point in the annular ring bounded by the circles, $f(z)$ is expressible in

[6]From *Advanced Engineering Mathematics*, by C. R. Wylie. Copyright 1951 by McGraw-Hill. Used with permission of McGraw-Hill Book Company.

series form as

$$f(z)=\sum_{j=-\infty}^{\infty} a_j(z-a)^j, \tag{2.6.1}$$

where a is the common center of the circles, and

$$a_j=\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{j+1}}, \tag{2.6.2}$$

each integral being taken in the counterclockwise sense around any curve C, lying within the annulus and encircling the inner boundary. Furthermore, this series is unique (see, e.g., refs. 415, 150, pp. 209–211).

Note particularly that when $j=-1$ in (2.6.2) one obtains

$$a_{-1}=\frac{1}{2\pi i}\int_C f(z)\,dz. \tag{2.6.3}$$

That is, the integral of a complex function $f(z)$ evaluated over any curve C lying inside the annulus and encircling the inner boundary can be obtained by observing or otherwise determining the coefficient a_{-1} in the term $a_{-1}(z-a)^{-1}$ of the Laurent expansion of $f(z)$. This coefficient a_{-1} is called the residue of $f(z)$ at that singular point or pole. The following example is illustrative.

Example 2.6.1 Find the Laurent expansion of

$$f(z)=\frac{1}{z(1-z)^3}$$

about any point inside the annulus shown in Fig. 2.6.2.

In deriving the Laurent expansion of a function $f(z)$ about a given point z in the region R, one may use any convenient algebraic manipulation suggested by the function. For since the Laurent expansion of a function over a given annulus is *unique* [191, p. 689, Exercise 10], if a Laurent expansion is found by any process, it is necessarily a valid Laurent series. Thus in this example it is advantageous to write $f(z)$ in the form

$$f(z)=\frac{1}{(z-1)^3}\left[\frac{1}{1+(z-1)}\right]=\frac{1}{(z-1)^3}[1+(z-1)]^{-1}$$

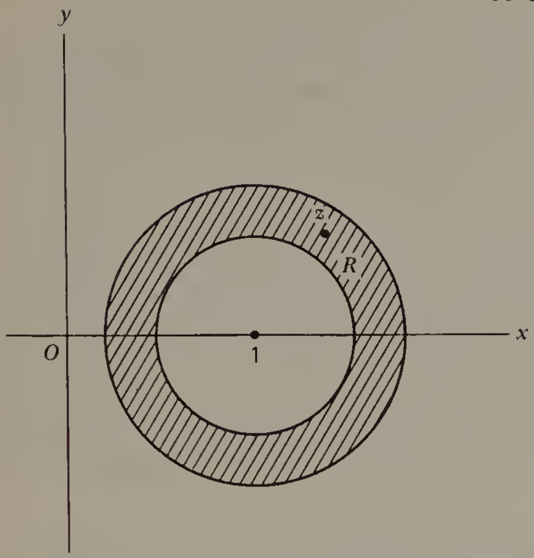


Fig. 2.6.2 Annulus for the Laurent expansion of $f(z)=z^{-1}(1-z)^{-3}$.

and then to apply the binomial expansion to the second factor. The result is

$$f(z)=\frac{1}{(z-1)^3}\left[1-(z-1)+(z-1)^2-(z-1)^3+\cdots\right]$$

$$=\frac{1}{(z-1)^3}-\frac{1}{(z-1)^2}+\frac{1}{z-1}-1+(z+1)-\cdots, \qquad (2.6.4)$$

which is the desired Laurent expansion, in which it is observed that $a_{-1}=1$. It is apparent from (2.6.4) that the numerical value of the coefficient a_{-1} may also be obtained by evaluating $(d^2/dz^2)[(z-1)^3 f(z)]$ at $z=1$. That is,

$$a_{-1}=\frac{1}{2}\left(\frac{d^2}{dz^2}\left[(z-1)^3\left(\frac{1}{z}\right)\frac{1}{(1-z)^3}\right]\right)\Bigg|_{z=1}$$

$$=\frac{1}{z^3}\Bigg|_{z=1}$$

$$=1.$$

Either way, it follows from Theorem 2.6.1 that

$$a_{-1}=\frac{1}{2\pi i}\int_C\frac{1}{z(1-z)^3}dz=1,$$

or equivalently,

$$\int_C f(z)dz=2\pi i, \qquad (2.6.5)$$

where $f(z)=1/[z(1-z)^3]$. Thus by means of differentiation—rather than integration—it has been established that the integral in (2.6.5) has the value $2\pi i$. This is a most important result, which is exploited shortly in establishing the residue theorem.

In summary, the Laurent expansion of a function $f(z)$ is intimately connected with singular points (i.e., points at which the function fails to be analytic). If $z=a$ is a singular point of the function $f(z)$, and if there exists a circle with center at a in which there are no other singular points of $f(z)$, then $z=a$ is called an *isolated singular point*. Now if $z=a$ is an isolated singularity of $f(z)$, then $f(z)$ can be expanded in a Laurent series around $z=a$ and inside an annulus whose inner radius can be made to approach zero.

If when $f(z)$ is expanded about an isolated singular point, all negative powers of $(z-a)$ after the mth are missing, $f(z)$ is said to have *a pole of order m at $z=a$*, and the sum of the terms with negative powers

$$\frac{a_{-m}}{(z-a)^m}+\cdots+\frac{a_{-2}}{(z-a)^2}+\frac{a_{-1}}{(z-a)} \tag{2.6.6}$$

is called the *principal part of $f(z)$ at $z=a$*.[7] It may be that the Laurent expansion of $f(z)$ contains an infinite number of negative powers of $(z-a)$, in which case the point $z=a$ is called an *essential singularity* of the function. For example, the function

$$\begin{aligned} f(z)&=e^{1/z}\\ &=1+\frac{1}{z}+\cdots+\frac{1}{3!z^3}+\cdots+\frac{1}{n!z^n}+\cdots \end{aligned}$$

has an essential singularity at $z=0$. However *most functions encountered in practical applications* do not have Laurent expansions of this type (i.e., expanded about points that are essential singularities). For the most part, they are functions for which the highest order pole in their Laurent expansion is of finite order. An example of such a function is the one given in Example 2.6.1, which has a pole of order 3 at $z=1$, the principal part of $f(z)$ at $z=1$ being

$$\frac{1}{(z-1)^3}-\frac{1}{(z-1)^2}+\frac{1}{z-1}.$$

In this case the residue at the pole $z=1$ is $a_{-1}=1$.

[7]From *Advanced Engineering Mathematics*, by C. R. Wylie. Copyright 1951 by McGraw-Hill. Used with permission of McGraw-Hill Book Company.

As has already been pointed out, the residue of a function $f(z)$ at a pole can be obtained by deriving the Laurent expansion of $f(z)$ and observing the coefficient a_{-1}. However, there is an easier way. For if $f(z)$ has a pole of order m at an isolated singular point $z=a$, then for any z inside an annular ring centered on the point a (Fig. 2.6.1), one can write

$$(z-a)^m f(z) = a_{-m} + a_{-m+1}(z-a) + a_{-m+2}(z-a)^2 + \cdots$$
$$+ a_{-2}(z-a)^{m-2} + a_{-1}(z-a)^{m-1} + a_0(z-a)^m + \cdots,$$

from which it is clear (since $f(z)$ is assumed to be analytic inside the annular ring) that

$$a_{-1} = \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z)\right]\Bigg|_{z=a}. \tag{2.6.7}$$

That is, instead of actually obtaining the Laurent expansion of $f(z)$ to determine the residue, one need only observe the order m of the highest order pole (singularity) of $f(z)$, multiply $f(z)$ by $(z-a)^m$, and evaluate the $(m-1)$st derivative of this product at $z=a$. This accounts for the usefulness of the residue theorem presented in Section 2.7.

2.7 THE RESIDUE THEOREM

Because of the importance of the residue theorem in expediting the evaluation of inversion integrals, the theorem now proved. The proof is based on both Cauchy's theorem and the Laurent expansion of a complex function $f(z)$.

Theorem 2.7.1 (the residue theorem).[8] If C is a closed curve and $f(z)$ is analytic within and on C except at a countable number of singular points in the interior of C, then

$$\int_C f(z)\,dz = 2\pi i \sum_j^n R_j, \tag{2.7.1}$$

where $R_1, R_2, \ldots, R_n$ are the residues of $f(z)$ at the n poles within C.

PROOF. Consider a simple closed curve C in whose interior the function $f(z)$ has isolated singularities (Fig. 2.7.1). If around each singular point one

[8]From *Advanced Engineering Mathematics*, by C. R. Wylie. Copyright 1951 by McGraw-Hill. Used with permission of McGraw-Hill Book Company.

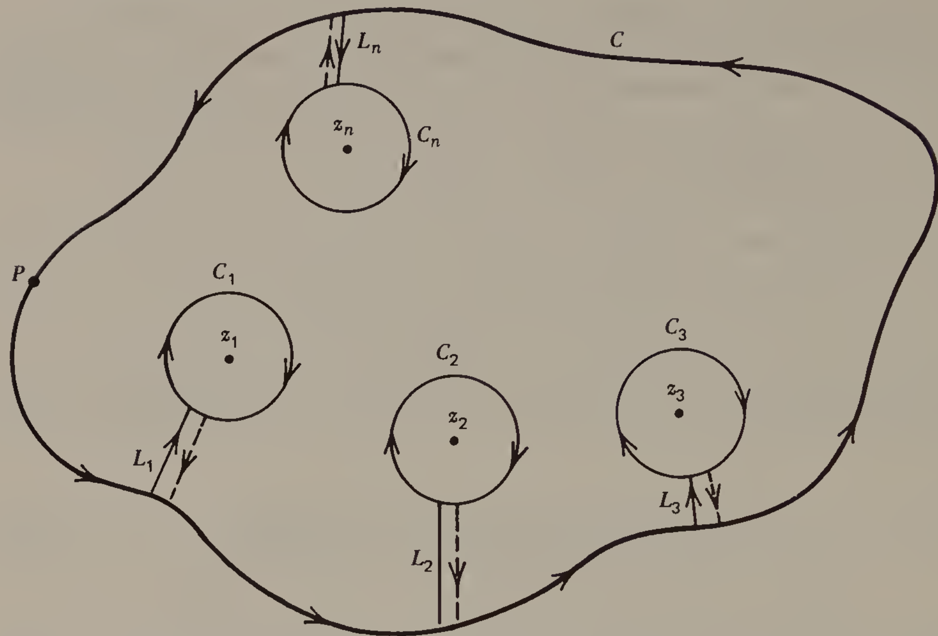


Fig. 2.7.1 Closed contour over which $f(z)$ is integrated to prove the residue theorem.

draws a circle so small that it encloses no other singular points, these circles, together with the curve C, constitute the boundary of a multiply connected region in which $f(z)$ is everywhere analytic and to which Cauchy's theorem is applicable. That is, the curves C, C_j, L_j, $j=1,2,\ldots,n$ constitute a closed curve on and inside which $f(z)$ is everywhere analytic, so that Cauchy's theorem may be applied. Furthermore, if one traverses C, C_j, and L_j, $j=1,2,\ldots,n$ in the direction indicated (i.e., utilizing the left-hand rule) necessarily returning from C_j to C over the same lines L_j in the opposite direction (as indicated by the dashed lines, which actually coincide with the solid lines), the contributions to the integral over the lines L_j cancel. Therefore, as a consequence of Cauchy's theorem,

$$\frac{1}{2\pi i}\int_C f(z)\,dz+\frac{1}{2\pi i}\int_{C_1} f(z)\,dz+\cdots+\frac{1}{2\pi i}\int_{C_n} f(z)\,dz=0. \quad (2.7.2)$$

If one now reverses the direction of integration around each of the circles and changes the sign of each of the integrals to compensate, (2.7.2) can be written in the form

$$\frac{1}{2\pi i}\int_C f(z)\,dz=\frac{1}{2\pi i}\int_{C_1} f(z)\,dz+\cdots+\frac{1}{2\pi i}\int_{C_n} f(z)\,dz, \quad (2.7.3)$$

where all integrals are now taken in the counterclockwise sense. Since the integrals on the right-hand side of (2.7.3) are, by definition, just the residues of $f(z)$ at the various isolated singularities (poles) within C, the residue theorem as stated earlier follows.

The residue theorem is important because if $f(z)$ is analytic at all points on and inside a closed contour C, except at a finite or countable number of isolated singular points inside C, then its Laurent expansion in the neighborhood of each isolated singular point exists. Furthermore, since a function is uniquely determined [150, pp. 209–211] by its Laurent expansion, each integral

$$\int_{C_j} f(z)\,dz, \qquad j=1,2,\ldots,n$$

on the right-hand side of (2.7.3) is *uniquely* determined by its residue at the relevant singular point or pole. Consequently, the integral

$$\int_C f(z)\,dz$$

on the left-hand side of (2.7.3) is *uniquely* determined by the sum of the residues at the poles.

The requirement that the singular points be isolated is crucial, for if a singular point is not isolated but is essential, the residue a_{-1} in the Laurent expansion of $f(z)$ cannot be obtained by differentiation.

The following example illustrates the power and utility of the residue theorem.

Example 2.7.2 Use the residue theorem to evaluate the integral of the function in Example 2.6.1, namely,

$$f(z)=\frac{1}{z(1-z)^3} \tag{2.7.4}$$

around the circle $|z|=\frac{3}{2}$.

There are two poles of the function inside the contour of integration. Specifically, there is a pole of order 1 at $z=0$ and a pole of order 3 at $z=1$. The residues at $z=0$ and $z=1$ are, respectively,

$$R_1=a_{-1}=\left.\left[z\left(\frac{1}{z(1-z)^3}\right)\right]\right|_{z=0}=1$$

and

$$R_2 = a_{-1} = \frac{1}{2!}\left[\frac{d^2}{dz^2}\left((1-z)^3 \frac{1}{z(1-z)^3}\right)\right]\Bigg|_{z=1}$$

$$= \frac{1}{2}\left(\frac{2}{z^3}\right)\Bigg|_{z=1}$$

$$= 1.$$

Hence

$$\int_C \frac{dz}{z(1-z)^3} = 2\pi i(R_1 + R_2)$$

$$= 4\pi i,$$

where C is the circle $|z| = \frac{3}{2}$.

2.8 INTEGRAL TRANSFORMS

The basis for analyzing distributions of sums, differences, products, quotients, powers—and more generally, algebraic functions—of continuous r.v.'s is the integral transform. Thus the Laplace transform provides the means for deriving and analyzing the distribution of sums of nonnegative r.v.'s. On the other hand, if the r.v.'s may take on both positive and negative values, the Fourier and bilateral Laplace transforms are the appropriate tools for deriving and analyzing the p.d.f. of their sums and differences. Similarly, the Mellin integral transform constitutes the counterpart of the Laplace integral transform in deriving and analyzing the distribution of products and quotients of nonnegative r.v.'s. To derive the distribution of products and quotients of n r.v.'s that may take on both positive and negative values, one must utilize a modified Mellin integral transform. This section defines these transforms and transform pairs. Chapters 3 to 6 cover the methods for their utilization in deriving sums, products, quotients, and algebraic functions of r.v.'s. Also, Appendix D.2 gives a table of integral transforms of basic p.d.f.'s. The following section discusses inverse transforms (i.e., inversion integrals).

The aforementioned integral transforms, each corresponding to a function $f(x)$, are now defined, together with transform pairs. A different notation (r,s,it) is used to identify the complex variable involved in the various transforms, since in the later discussion involving Prasad's theorem (Chapter 5), it is necessary for the three complex variables r, s, and it to maintain their separate identities.

2.8.1 Integral Transforms[9] and Transform Pairs

Laplace Transform of $f(x)$

If $f(x)$ is a real function and is defined and single valued almost everywhere for $x \geqslant 0$, with x a real variable, and is such that the integral

$$\int_0^\infty |f(x)| e^{-kx}\, dx \tag{2.8.1}$$

converges for some real value k, then $f(x)$ is said [382, p. 1] to be Laplace transformable, and

$$L_r(f(x)) = \int_0^\infty e^{-rx} f(x)\, dx \tag{2.8.2a}$$

is the Laplace transform of $f(x)$, where r is a complex variable. The inverse Laplace transform or inversion integral (discussed in the following section) is

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{rx} L_r(f(x))\, dx, \tag{2.8.2b}$$

which, together with (2.8.2a), constitutes a transform pair. Equation 2.8.2b determines $f(x)$ uniquely, if $L_r(f(x))$ is analytic in a strip consisting of the portion of the plane to the right of (and including) the Bromwich path $(c - i\infty, c + i\infty)$, the latter denoting the straight line

$$\lim_{a\to\infty} \overline{PQ} = \lim_{a\to\infty} (c - ia, c + ia)$$

(See Fig. 2.9.1*a*.)

Fourier Transform of $f(x)$

If $f(x)$ is a real function that is defined and single valued almost everywhere for $-\infty < x < \infty$, with x a real variable and is such that the integral

$$\int_{-\infty}^{\infty} |f(x)| e^{ikx}\, dx \tag{2.8.3}$$

[9]For convenience and brevity, the term "transform" is used hereafter instead of "integral transform."

converges for some real value of k, then $f(x)$ is said to be Fourier transformable [52, p. 312], and

$$F_t(f(x)) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx \tag{2.8.4}$$

is the Fourier transform of $f(x)$. The expression $F_t(f(x))$ is also called the characteristic function of $f(x)$, and e^{itx} is called the kernel [225, pp. 10–11].

The definition (2.8.4) of the Fourier transform is not always adopted. For example, Churchill [52], Ditkin and Purdnikov [79], and Trantner [382] use (2.8.4), and Titchmarsh [381], Whittaker and Watson [403], Erdelyi [95], and others use the definition

$$F_t(f(x)) = \int_{-\infty}^{\infty} e^{-itx} f(x)\,dx. \tag{2.8.5}$$

It is really immaterial which of the two definitions is used. For corresponding to any transform, there is an inverse transform (the inversion integral discussed in the following section); the two constitute what is called a transform pair. Thus

$$F_t(f(x)) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx \tag{2.8.5a}$$

and

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} F_t(f(x))\,dt \tag{2.8.5b}$$

constitute a transform pair, as do also

$$F_t(f(x)) = \int_{-\infty}^{\infty} e^{-itx} f(x)\,dx \tag{2.8.6a}$$

and

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} f(x)\,dx. \tag{2.8.6b}$$

Changing from one transform pair to the other merely changes the location of the poles in the corresponding complex Fourier transform (Appendix E) from a strip in the right half-plane (RHP) to one in the left half-plane (LHP), or vice versa. It also changes the Bromwich contour from a left-hand contour C_L to a right-hand contour C_R, or vice versa (Fig.

2.9.1*b*). Since evaluation of the inversion integral leads to the same result in each case, either transform pair may be used. For Fourier transforms, the definition (2.8.4) is used throughout this book, since then the Fourier transform is identical with the characteristic function, which is so well known among statisticians and probabilists. The transform pair then is as defined in (2.8.5a, b). In either case, (2.8.5b) and (2.8.6b) will determine $f(x)$ uniquely if the Fourier transform is analytic in a relevant strip containing the Bromwich path.

It should be pointed out that when Laplace transforms are used in this book, the transform pair is used in which the kernel of the transform is e^{-rx} and that of the inverse transform or inversion integral is e^{rx}, since this is standard procedure.

The Complex Fourier (Bilateral Laplace) Transform of f(x)

As Lathi has pointed out [200], the Fourier transform, as defined by (2.8.5a), may be considered as a tool for representing an arbitrary function $f(x)$ as a continuous sum of exponential functions of the form e^{-itx}, as given by (2.8.5b). The complex part of the exponent of the kernel—namely, it—is a purely imaginary number, that is, a complex number that is limited to the imaginary axis. In general, however, it is possible (and sometimes desirable) to represent a function $f(x)$ by a continuous sum of exponentials of the form e^{-rx} that leads to the complex Fourier transform pair:

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-rx}\mathcal{F}_r(f(x))\,dr \tag{2.8.7a}$$

$$\mathcal{F}_r(f(x))=\int_{-\infty}^{\infty} e^{rx}f(x)\,dx, \tag{2.8.7b}$$

where $r=x+iy$. The *complex Fourier transform* (2.8.7b) is also known as the *bilateral* (or two-sided) *Laplace transform.*

An important property of the bilateral Laplace transform is that it can be expressed in terms of two unilateral Laplace transforms. To show that this property holds, consider any function $f(x)$, $-\infty<x<\infty$, whose bilateral transform exists (i.e., for which the integral (2.8.7b) is absolutely convergent). The p.d.f. $f(x)$ can be decomposed into two components corresponding to negative and positive values of x:

$$\begin{aligned} f(x)&=f^-(x), \qquad -\infty<x\leqslant 0\\ &=f^+(x), \qquad 0\leqslant x<\infty. \end{aligned}$$

Then

$$\mathfrak{F}_r(f^-(x)) = \int_{-\infty}^{0} e^{rx} f^-(x)\,dx$$

$$= \int_0^{\infty} e^{-rx} f^-(-x)\,dx$$

$$= L_r(f^-(-x)), \qquad -\infty < x \leqslant 0$$

$$= L_{-r}(f^-(x)), \qquad 0 \leqslant x < \infty \tag{2.8.8a}$$

$$\mathfrak{F}_r(f^+(x)) = \int_0^{\infty} e^{rx} f^+(x)\,dx$$

$$= L_r(f^+(x)), \qquad 0 \leqslant x < \infty, \tag{2.8.8b}$$

where $L_{-r}(f^-(x))$, $L_r(f^+(x))$ are unilateral Laplace transforms. That is, $\mathfrak{F}_r(f^-(x))$ is obtained by finding the unilateral Laplace transform $L_r(f^-(-x))$ and replacing r by $-r$, whereas $\mathfrak{F}_r(f^+(x))$ is identical with $L_r(f^+(x))$, since $f^+(x)$ involves only nonnegative values of x. (See Examples E.1 and E.2 of Appendix E.)

Finally, as shown by Lathi and others,[10] if (and only if) the region in which both $\mathfrak{F}_r(f^-(x))$ and $\mathfrak{F}_r(f^+(x))$ converge includes the imaginary axis, the ordinary Fourier transform exists and is equivalent to the complex Fourier transform obtained by replacing the purely imaginary number it by a general complex number r. Furthermore, all the terms of the transform (2.8.8a) represented by LHP poles correspond to the component $f^-(x)$, and all the terms of the transform (2.8.8b) represented by RHP poles correspond to $f^+(x)$.

Mellin Transform of $f(x)$

If $f(x)$ is a real function that is defined and single valued almost everywhere for $x \geqslant 0$, with x a real variable, and is such that

$$\int_0^{\infty} x^{k-1} |f(x)|\,dx$$

converges for some real value k, then $f(x)$ is said to be *Mellin transformable*

[10]See Appendix E.

and

$$M_s(f(x)) = \int_0^\infty x^{s-1} f(x)\,dx, \tag{2.8.9}$$

where s is a complex number, is the Mellin transform [380, p. 2; 347, p. 41] of $f(x)$. The inverse Mellin transform or inversion integral (discussed in the following section) is

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} M_s(f(x))\,dx, \tag{2.8.10}$$

which, together with (2.8.9), constitutes a transform pair. Equation 2.8.10 determines $f(x)$ uniquely, if the Mellin transform is analytic in the relevant strip.

It is clear that the Fourier and Laplace transforms are of the exponential type. That the Mellin integral transform is also of the exponential type becomes evident if one notes that any nonnegative variable x is expressible in the form $x = e^{\ln x}$. Thus

$$\begin{aligned} M_s(f(x)) &= \int_0^\infty x^{s-1} f(x)\,dx \\ &= \int_0^\infty e^{(s-1)\ln x} f(x)\,dx, \end{aligned} \tag{2.8.11}$$

which is clearly a function of the exponential type. This is of considerable importance when Jordan's lemma is invoked in the process of inverting a Mellin transform to obtain the p.d.f. $f(x)$.

The Z and Zeta Transforms

The generating function (discussed in Chapter 3) is a well-known tool for deriving the probability distribution of sums of discrete i.r.v.'s, and it is closely related to the Z-transform. Specifically, the Z-transform and generating function are power series transformations of the probability mass function $p(x_k)$ of an integral-valued (discrete) r.v. x_k, $k = 0, 1, 2, \ldots, n$, into a function of the complex variable Z. The power series transformation with positive exponents, namely,

$$F(Z) = \sum_{k=0}^{\infty} p(x_k) Z^k \tag{2.8.12}$$

is called the *generating function*, and the power series transformation with negative exponents,

$$F(Z) = \sum_{k=0}^{\infty} p(x_k) Z^{-k}, \tag{2.8.13}$$

is known as the Z-transform [26].

As was stated above, the generating function is used to derive the probability distribution of sums of discrete i.r.v.'s. In such derivations, the notation customarily used by probabilists and statisticians is here employed to denote the generating function, namely,

$$A(s) = \sum_{k=0}^{\infty} a_k s^k. \tag{2.8.14}$$

It should be pointed out that s is not an r.v. but is rather a tool for extracting the probabilities a_k, which are counterparts of $p(x_k)$, $k = 0, 1, 2, \ldots$, in (2.8.12) and (2.8.13). More is said about the generating function (2.8.14) in Chapter 3 in connection with sums of discrete i.r.v.'s.

The Z-transform frequently has applications in connection with discrete time series. Generally speaking, discrete time series may be described [170] by a function $f(nT)$, where T is a constant-length time interval and n is an integer. The usual form of the Z-transform in such cases is [149]

$$Z\{f(nT)\} = \sum_{n=0}^{\infty} f(nT) Z^{-n}, \tag{2.8.15}$$

where Z is a complex variable. As is well known, the Laplace and Z-transform variables are related by the equation

$$Z = e^{jT}, \tag{2.8.16}$$

so that the Z-transform is expressible in the form

$$Z\{f(nT)\} = \sum_{n=0}^{\infty} f(nT)(e^{jT})^{-n}. \tag{2.8.17}$$

A lesser-known transform, the zeta transform, also has considerable application relative to discrete variable time series as utilized in economic models. It is defined [149] as

$$Z\{f(nT)\} = \sum_{n=0}^{\infty} f(nT)(1 + zT)^{-n}, \tag{2.8.18}$$

where z is again a complex variable. It is also well known in the literature of transform theory that the relationship between the Z and zeta transform variables is

$$Z = 1 + zT. \tag{2.8.19}$$

Accordingly, the substitution of Z, as given by (2.8.19), into (2.8.15) generates (2.8.18), showing the interrelation between these transforms and furnishing a direct means of finding the zeta transform from a Z-transform table.

Zeta transforms have application in economic modeling. In particular, zeta transforms serve in the economic analysis of discrete time series in a manner similar to the Laplace transform methodology for continuous cash flow functions as described by Buck and Hill [149].

The Hankel Transform

Of less importance than the aforementioned transforms in theoretical and applied statistics is the Hankel transform [276]. Lord [215–217] has pointed out that the characteristic function of the sum of n independent random vectors when the vectors have spherical distributions in s dimensions, is a Hankel transform. Specifically, when an s-dimensional random vector X has a spherical distribution with $p(r)\,dr$ for the probability of $r<|X|<r+dr$, one can define a characteristic function (Hankel transform)

$$\Phi(\rho) = (2\pi)^{\frac{-s}{2}} \rho^{\frac{-s}{2}+1} \int_0^\infty r^{\frac{s}{2}} J_{(s/2)-1}(r\rho) p(r)\,dr \tag{2.8.20}$$

whose inversion yields

$$p(r) = 2^{(-s/2)+1} \left[\Gamma\left(\frac{s}{2}\right)\right]^{-1} \int_0^\infty (r\rho)^{s/2} J_{(s/2)-1}(r\rho)\Phi(\rho)\,d\rho, \tag{2.8.21}$$

where $J_{(s/2)-1}(\cdot)$ is the Bessel function of the first kind of order $s/2-1$ (see Appendix D.1). The Hankel transform has considerable application in certain problems of mathematical physics [382].

The Walsh-Hadamard Transform

The Walsh-Hadamard transform of probability mass functions of discrete i.r.v.'s has recently received some attention [296] in a variety of engineering applications (image coding, communication theory, etc.). Pearl [280] points out that whereas the Fourier basis (exponential functions) constitutes the natural representation for systems with translational symmetry, the Walsh transform is the natural representation of systems with dyadic symmetry, so that stochastic systems with dyadic symmetry benefit most from the properties of Walsh transform analysis and the computational advantages it offers. Pearl cites some applications in the areas of information theory and pattern recognition. The reader is referred to Pearl's paper [279] for a precise definition of the Walsh transform.

Logarithmic Transformation

It is instructive to observe the relationship between the Mellin transform of the p.d.f. $f(x)$, $x>0$, and the Fourier transform of the p.d.f. $g(y)$ of the transformed variable $y=\ln X$, $-\infty<y<\infty$. Since

$$g(y)=e^{y}f(e^{y}),$$

it follows from the definition of the Mellin transform that

$$\begin{aligned} M_s(f(x)) &= \int_0^{\infty} x^{s-1}f(x)\,dx \\ &= \int_{-\infty}^{\infty} e^{y(s-1)}g(y)\,dy, \end{aligned}$$

where s is a complex number. Setting $s-1=-it$, t real, one obtains the Fourier transform of $g(y)$. Thus the Fourier transform of $g(y)$ can be obtained from the Mellin transform of $f(x)$ by replacing the argument s in the Mellin transform by $1-it$. Conversely, one can obtain the Mellin transform of $f(x)$ from the Fourier transform of $g(y)$ by replacing it in the Fourier transform of $g(y)$ by $1-s$.

2.8.2 Some Important Properties of Integral Transforms

In each of the following cases, it is assumed that each transform pair exists within the region of convergence, and $F_t(f(x))$ is defined by (2.8.5).

1. Linearity property.

Laplace: $L_r(c_1 f_1(x)+c_2 f_2(x))=c_1 L_r(f_1(x))+c_2 L_r(f_2(x))$
Fourier: $F_t(c_1 f_1(x)+c_2 f_2(x))=c_1 F_t(f_1(x))+c_2 F_t(f_2(x))$
Mellin: $M_s(c_1 f_1(x)+c_2 f_2(x))=c_1 M_s(f_1(x))+c_2 M_s(f_2(x))$

2. First translation or shifting property.

Laplace: $L_r(e^{ax}f(x)) = L_{r-a}(f(x))$
Fourier: $F_t(e^{ax}f(x)) = F_{t+ia}(f(x))$
Mellin: $M_s(x^{-a}f(x)) = M_{s-a}(f(x))$

3. Second translation or shifting property.

Laplace: $L_r(f(x-a)) = e^{-ar}L_r(f(x)), \qquad x > a$
Fourier: $F_t(f(x-a)) = e^{-iat}F_t(f(x))$

4. Scaling.

Laplace: $L_r(f(ax)) = \frac{1}{a}L_{r/a}(f(x))$
Fourier: $F_t(f(ax)) = \frac{1}{a}F_{t/a}(f(x))$
Mellin: $M_s(f(ax)) = a^{-s}M_s(f(x))$

5. Transforms of derivatives.

Laplace: $L_r(f^{(n)}(x)) = r^nL_r(f(x)) - r^{n-1}f(0) - r^{n-2}f'(0) \cdots - rf^{(n-2)}(0) - f^{(n-1)}(0)$
if $f(x)$ and the derivatives $f'(x), \ldots, f^{(n-1)}(x)$ are continuous for $0 \leqslant x \leqslant X$ and of exponential order for $x > X$, and $f^{(n)}(x)$ is piecewise continuous [350, p. 2] for $0 \leqslant x \leqslant X$, where X is some finite value of x.
Fourier: $F_t(f^{(n)}(x)) = i^nt^nF_t(f(x))$
Mellin: $M_s(f^{(n)}(x)) = (-1)^n(s-1)\cdots(s-(n-1))(s-n)M_{s-n}(f(x))$

6. Transforms of integrals.

Laplace: $$L_r\left[\int_0^x f(u)\,du\right] = \frac{L_r(f(x))}{r}$$
Fourier: $$F_t\left[\int_0^x f(u)\,du\right] = \frac{F_t(f(x))}{it}$$

7. Multiplication by x^n.

Laplace: $L_r(x^nf(x)) = (-1)^nL_r^{(n)}(f(x))$
Fourier: $F_t(x^nf(x)) = (i)^nF_t^{(n)}(f(x))$
Mellin: $M_s(x^nf(x)) = M_{s+n}(f(x))$

8. Division by x.

Laplace: $$L_r\left(\frac{f(x)}{x}\right) = \int_r^\infty L_r(f(u))\,du \qquad \left(\text{provided } \lim_{x\to 0}\frac{f(x)}{x} \text{ exists}\right)$$

Fourier: $F_t\left(\dfrac{f(x)}{ix}\right)=\int_t^\infty F_t(f(u))\,dt$ (provided $\lim\limits_{x\to 0}\dfrac{f(x)}{x}$ exists)

Mellin: $M_s\left(\dfrac{f(x)}{x}\right)=M_{s-1}(f(x))$ (provided $\lim\limits_{x\to 0}\dfrac{f(x)}{x}$ exists)

9. Periodic functions. Let $f(x)$ have period $T>0$ so that $f(x+T)=f(x)$. Then we can write

Laplace: $L_r(f(x))=\dfrac{\int_0^T e^{-rx}f(x)\,dx}{1-e^{-rT}}$

Fourier: $F_t(f(x))=\dfrac{\int_0^T e^{-itx}f(x)\,dx}{1-e^{-itT}}$

10. Exponentiation.

$M_s(f(x^a))=a^{-1}M_{s/a}(f(x))$

2.8.3 The Mellin Integral Transform of a Function of a Complex Random Variable[11]

The great majority of cases—both theoretical and applied—involving the use of the Mellin transform in connection with products, quotients, and powers of i.r.v.'s concern Mellin transforms of real variables. However the Mellin integral transform may be equally well defined for functions $f(z)$ of complex r.v.'s z, as Kotlarski has shown [186]. He considered a bivariate r.v. (R,Φ) taking values (r,ϕ) on the half-plane

$$0<r<\infty, \qquad -\infty<\phi<\infty$$

and associated with the complex number

$$z=Re^{i\Phi}.$$

For the function of the variable z to have a one-to-one correspondence with the angle Φ, only principal values of Φ, denoted by Φ^*, are used; that is,

$$\Phi^*\equiv\Phi \bmod 2\pi.$$

The bivariate r.v. (R,Φ^*) then takes its values (r,ϕ^*) on the half-strip

$$0<r<\infty, \qquad -\pi<\phi^*\leqslant\pi.$$

[11]The results in this section are based on the papers by Kotlarski [186] and Brock and Krutchkoff [40] and are reprinted here with permission of the Executive Editor of *The Annals of Statistics*.

Kotlarski defined the Mellin transform of $f(z)$ in terms of R and ϕ, as

$$M_{u,v}(h(r,\phi)) = E[R^u e^{iv\Phi}],$$

where u and v are complex numbers. He also pointed out that

$$M_{it,v}(h(r,\phi)) = \psi_{(\log R,\Phi)}(t,v),$$

where $\psi_{(\log R,\Phi)}(t,v)$ is the characteristic function of the joint distribution of the bivariate r.v. $(\log R, \Phi)$:

$$\psi_{(\log R,\Phi)}(it) = E\left[\exp(i(t\log R + v\Phi))\right].$$

Brock and Krutchkoff [40] consider a complex r.v. $Z = Re^{i\Phi}$ such that the density of (R,Φ) is given by

$$f(r,\phi) = \frac{|bc_4| c_2^{c_1}(1-a^2)^{1/2}}{2\pi m\Gamma(c_1)} r^{c_1c_4-1}\lambda^{c_1c_3-1}\exp(-c_2\lambda^{c_3}r^{c_4}),$$

where all parameters are real, $r>0$, $m\pi/|b|<\phi<m\pi|b|$, $\lambda = 1 - a\sin(b\theta + \alpha)$, $|a|<1$, $b\neq 0$, $c_1>0$, $c_2>0$, $c_4\neq 0$, m is a natural number, and $0\leqslant \alpha < 2\pi$, and $f(r,\phi)$ is zero otherwise. They then point out that the generalized Mellin transform is

$$h(s,t) = -(1-a^2)^{1/2}\Gamma\left(\frac{s}{c_4}+c_1\right)\sin\left(\frac{mt\pi}{b}\right)\left[4\pi m c_2^{s/c_4}\Gamma(c_1)\Gamma\left(c_3\frac{s}{c_4}+1\right)\right]^{-1}$$
$$\times \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} H_{jl}\left[\left(\frac{t}{b}\right)\sin^j\alpha + ij\sin^{j-1}\alpha\cos\alpha\right],$$

where

$$H_{jl} = \frac{(-1)^{mj}\Gamma(2l+j+c_3 s/c_4+1)\Gamma(j/2+t/2b)\Gamma(j/2-t/2b)a^{2l+j}}{2^{2l}\Gamma(j+1)\Gamma(j/2+t/2+l+1)\Gamma(j/2-t/2+l+1)}$$

and $t\neq 0, \pm b, \pm 2b, \ldots,$ and $s/c_4 + c > 0$. (For any integer k, $h(s,kb)$ is evaluated by taking the limit of $h(s,t)$ as $t\to kb$.)

Brock and Krutchkoff show that when one sets $a=\alpha=0$, the density function $f(r,\phi)$ becomes

$$f(r,\phi) = \frac{|bc_4| c_2^{c_1}}{2\pi m\Gamma(c_1)} r^{c_1c_4-1}\exp(-c_2 r^{c_4}),$$

where $r>0$, $-m\pi/|b|<\phi\leqslant m\pi/|b|$, $c_1>0$, $c_2>0$, $c_4\neq 0$, $b\neq 0$, and m is a natural number. Then, for selected values of c_1, c_2, and c_4, $f(r,\phi)$ becomes the p.d.f. of the product of Weibull-uniform, chi-uniform, and gamma-uniform r.v.'s.

Levy [207] has also given a tool for treating products of complex r.v.'s. However the literature dealing with Mellin transforms of functions of complex variables appears to be quite sparse, probably because their use in theory and application is relatively infrequent, as compared with the extensive use of Mellin transforms of real variables. This being the case, the subject is not further discussed; it is mentioned here merely for the sake of completeness.

2.9 COMPLEX INVERSION INTEGRALS

If the integral transform of a function $f(x)$ is known, the function $f(x)$ can be obtained by evaluating the corresponding inversion integral. The reader may wonder whether the inversion integral of a function is ever known when the function itself is not known. Actually, there are frequent instances when such is the case: that is, when one knows the integral transform of $f(x)$ before he knows what $f(x)$ is. This situation exists, for example, when one wishes to derive the p.d.f. of a sum, mean, difference, product, quotient—or more generally, an algebraic function—of specific i.r.v.'s, in which case the integral transform of the desired p.d.f. is expressible in terms of the integral transforms of the p.d.f.'s of the specific component r.v.'s, which integral transforms in many cases are known.

Since later chapters contain frequent applications of these types of transform and their associated inversion integrals, they are defined here and the nature of their application relative to *The Algebra of Random Variables* is briefly discussed, to permit the reader to see them in their proper perspective before considering the details of their application.

If the transforms defined by (2.8.2a), (2.8.8a, b) and (2.8.9) are analytic in a relevant strip containing the Bromwich path, there are reciprocal formulas (inversion integrals) that determine the p.d.f. $f(x)$ uniquely. Specifically, the corresponding inversion integrals are, respectively,

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{rx}L_r(f(x))\,dr \tag{2.9.1}$$

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-rx}\mathfrak{F}_r(f^-(x))\,dr \qquad -\infty<x<0, \tag{2.9.2}$$

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-rx}\mathfrak{F}_r(f^+(x))\,dr \qquad 0<x<\infty, \tag{2.9.3}$$

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}M_s(f(x))\,ds \qquad 0<x<\infty. \tag{2.9.4}$$

The reciprocal formulas are valid for all x where $f(x)$ is continuous and the path of integration is any line parallel to the imaginary axis and lying within the strip of analyticity of the relevant integral transform. Although it is extremely important that the inversion integral determines the p.d.f. uniquely, if the associated integral transform is analytic in the relevant strip or half-plane, the reader is reminded that this condition is sufficient and is not to be construed as necessary.

It should perhaps be pointed out why this condition of analyticity is sufficient for the uniqueness of $f(x)$ as obtained from the inversion integral. This sufficiency exists because the analyticity of the transform (in the relevant strip) ensures that the integrand of the inversion integral is expressible as a Laurent expansion, which expansion is always unique. The inversion integral may then be evaluated by the method of residues, as we show later. However, unless at least one pole exists, the integrand of the inversion integral is not expressible as a Laurent series, and the simple method of residures can no longer be used.

In the event that the characteristic function contains poles, the complex Fourier (or bilateral Laplace) transform involving the general complex variable r, can be used advantageously in connection with the residue theorem to obtain $f(x)$ from the inversion integral (2.8.5b). If the characteristic function contains no poles, but is absolutely integrable, the ordinary Fourier inversion integral (2.8.5b) can be evaluated directly as a function of the real variable t, usually in a straightforward manner, to obtain $f(x)$ uniquely. The derivation of the distribution of the mean of n normal i.r.v.'s (Section 9.1.1) is a case in point.

The question now arises as how best to evaluate the integrals (2.9.1) through (2.9.4) over a Bromwich path $(c-i\infty, c+i\infty)$, assuming that the relevant transforms contain poles (which is usually the case). It would be particularly convenient if one could find a closed contour enclosing the poles of the integrand and over which the integral would be equal to that over the Bromwich path, since then the residue theorem could conveniently be applied. Jordan's lemma, proved in Appendix A, states conditions under which the integrals (2.9.1) through (2.9.4), when evaluated over the closed contour $C_L = QKLMPQ$ or $C_R = QPTQ$ (Fig. 2.9.1), depending on the location of the poles, yield results identical with those obtained by integrating over the Bromwich path $(c-i\infty, c+i\infty)$. This will occur, of course, when the contribution to the contour integral from integrating over the arcs $QKLMP$ and PTQ is zero. The importance of Jordan's lemma stems from the fact that in the great majority of situations requiring the evaluation of inversion integrals, the requisite conditions of the lemma are satisfied, thereby simplifying the evaluation of these integrals. Throughout this book, the closed contour consisting of the circular arc and chord, as the latter approaches infinity, is referred to as the *Bromwich contour* and should not be confused with the *Bromwich path*,

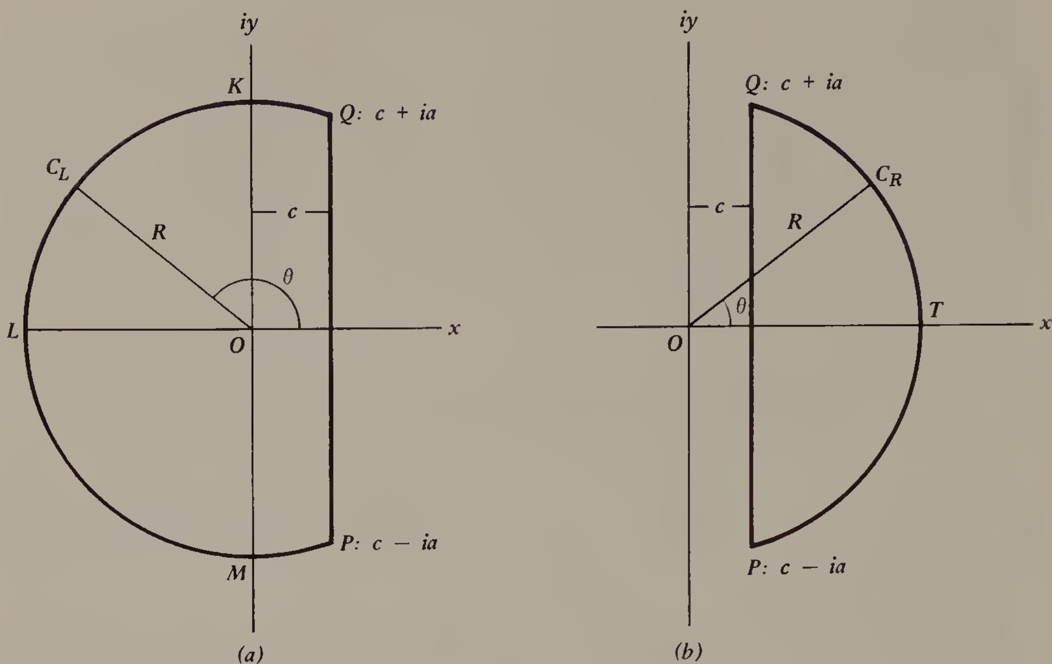


Fig. 2.9.1 Bromwich contours used in evaluating integrals over the Bromwich path $(c-i\infty, c+i\infty)$.

which is the *infinite extension of chord PQ*, resulting when the radius of the circle approaches infinity (Fig. 2.9.1a,b).

It should be pointed out that in the statement and proof of Jordan's lemma, the letter s denotes a complex variable and, to be consistent with the previous notation, the s *should* be replaced by r when applied to Laplace and complex Fourier transforms and inversion integrals.

Jordan's Lemma

(a) (As applied to C_R, Fig. 2.9.1b). If $f(s)\to 0$ uniformly with regard to $\arg s$ as $|s|\to\infty$ when $-\pi/2 \leqslant \arg s \leqslant \pi/2$ and, if $f(s)$ is analytic when $|s|\to k$ and $-\pi/2 < \arg s < \pi/2$, then

$$\lim_{a\to\infty}\int_{PTQ} e^{-ms} f(s)\,ds = 0, \tag{2.9.5}$$

where k and m are positive real constants.

(b) (As applied to C_L, Fig. 2.9.1a). If $f(x)\to 0$ uniformly with regard to $\arg s$ as $|s|\to\infty$ when $\pi/2 \leqslant \arg s \leqslant \frac{3}{2}\pi$, and if $f(s)$ is analytic when

$|s| \to k$ and $\pi/2 \leqslant \arg s \leqslant \frac{3}{2}\pi$, then

$$\lim_{a\to\infty} \int_{QKLMP} e^{ms} f(x)\,dx = 0, \tag{2.9.6}$$

where k and m are real positive constants.

As already mentioned, the proof of this lemma is given in Appendix A. The crucial part of the lemma is the condition that $f(s)$ approach zero uniformly with respect to $\arg s$ as $|s| = R \to \infty$. Actually, it is easily shown that this condition will be satisfied if one can find constants $M > 0$, $k > 0$ such that on the relevant circular arc ($QKLMP$ or PTQ),

$$|f(s)| < \frac{M}{R^k}, \tag{2.9.7}$$

where $s = Re^{i\theta}$. For if two constants $M > 0$, $k > 0$ can be found such that the inequality (2.9.7) is satisfied, then for any $\varepsilon > 0$ one can find a value R_0 depending on ε but independent of $\arg s$ such that $M/R^k < \varepsilon$ whenever $R > R_0$. Hence as $R \to \infty$, $|f(s)|$ approaches zero *independently* of $\arg s$, from which it follows that $f(s)$ is uniformly convergent with respect to $\arg s$. For future reference, this fact is stated in the form of a theorem.

Theorem 2.9.1. If one can find constants $M > 0$, $k > 0$ such that on the circumference of a circle (where $s = Re^{i\theta}$) with center at the origin and radius R (Fig. 2.9.1)

$$|f(s)| < \frac{M}{R^k},$$

then $f(s)$ is uniformly convergent with respect to $\arg s$.

Appendix A also shows that the value of the integral over each of the arcs QK and MP approaches zero as R tends to infinity.

Furthermore, it can be shown (but is not proved here) that the condition (2.9.7) always holds if

$$f(s) = \frac{P(s)}{Q(s)}, \tag{2.9.8}$$

where $P(s)$ and $Q(s)$ are polynomials and the degree of $P(s)$ is less than the degree of $Q(s)$. Recall that the Bromwich path is chosen such that all the poles of $f(s)$ (the counterpart of $f(s)$ in transform and inversion problems being the corresponding transform) lie on the same side of the path: to the left when the Bromwich contour is C_L, and to the right when the Bromwich contour is C_R. For then the conditions of Jordan's lemma relative to the kernel $e^{-(\ln x)s}$ will be satisfied for the contour C_L when $0<x<1$, and for the contour C_R when $1\leqslant x<\infty$.

It should also be pointed out that the condition of Jordan's lemma, relative to C_L, that $f(s)$ be analytic when both $|s|\to k$ and $\pi/2\leqslant \arg s\leqslant \frac{3}{2}\pi$ requires that all poles of $f(s)$ be at a finite distance from the origin. Similarly, when the circular arc is C_R, the condition that $f(s)$ must be analytic when both $|s|\to k$ and $-\pi/2\leqslant \arg s\leqslant \pi/2$ again requires that all poles of $f(s)$ be at a finite distance from the origin. In either case, this condition may be removed if, for example, the poles of $f(s)$ are countable and are spaced at intervals along the real axis, as Appendix F demonstrates.

In establishing Jordan's lemma in Appendix A, one finds it helpful to utilize the two theorems stated and proved below.

Theorem 2.9.2a. The inequality $\sin\theta \geqslant 2\theta/\pi$ holds for $0\leqslant\theta\leqslant\pi/2$.

PROOF. This theorem is readily proved by utilizing the fact that for $0\leqslant\theta\leqslant\pi/2$, the function of $y=\sin\theta$ is concave downward and is always above the straight line $y=2\theta/\pi$, except at the end points $\theta=0$ and $\theta=\pi/2$, where the two functions have identical values. That is,

$$\sin\theta > \frac{2\theta}{\pi}, \quad 0<\theta<\frac{\pi}{2}$$

$$\sin\theta = \frac{2\theta}{\pi}, \quad \theta=0, \theta=\frac{\pi}{2}.$$

Thus

$$\sin\theta \geqslant \frac{2\theta}{\pi}, \quad 0\leqslant\theta\leqslant\frac{\pi}{2}.$$

Theorem 2.9.2b. The inequality $\cos\theta \geqslant 1-2\theta/\pi$ holds for $0\leqslant\theta\leqslant\pi/2$.

PROOF. The proof of this theorem parallels that of Theorem 2.9.2a. In particular, note that for $0\leqslant\theta\leqslant\pi/2$, the function $y=\cos\theta$ is concave downward and is always above the straight line $y=1-2\theta/\pi$, except at the end points $\theta=0$ and $\theta=\pi/2$, where the two functions have identical values. Specifically,

$$\cos\theta > 1 - \frac{2\theta}{\pi}, \quad 0 < \theta < \frac{\pi}{2}$$

$$\cos\theta = 1 - \frac{2\theta}{\pi}, \quad \theta = 0, \theta = \frac{\pi}{2}$$

Hence

$$\cos\theta \geqslant 1 - \frac{2\theta}{\pi}$$

EXERCISES

2.1 Perform the indicated operations

(a) $4-3i+(-6+7i)$. *Ans.* $-2+4i$.
(b) $(3-i)(1+3i)$. *Ans.* $6+8i$.
(c) $\dfrac{-3+3i}{4-2i}$. *Ans.* $\dfrac{3i-9}{10}$.
(d) $\dfrac{5-i}{4+5i}$. *Ans.* $\dfrac{15-29i}{41}$.
(e) $|5-12i||5+12i|$. *Ans.* 169.
(f) $\left|\dfrac{1}{1+3i} - \dfrac{1}{1-3i}\right|$. *Ans.* $\dfrac{3}{5}$.

2.2 Express the following complex numbers in polar form.

(a) $3+3i$. *Ans.* $z = 3\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 3\sqrt{2}\, e^{i\pi/4}$.

(b) $1-\sqrt{3}\,i$. *Ans.* $z = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = 2e^{\pi i/3}$.

(c) $-3-3\sqrt{3}\,i$. *Ans.* $z = 6\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 6e^{(4/3)(\pi i)}$.

2.3 Express each of the following functions of the complex variable z in the form $u(x,y)+iv(x,y)$.

(a) $f(z)=z^2$. *Ans.* $u(x,y)=x^2+y^2$; $v(x,y)=2xy$.

(*b*) $f(z)=\dfrac{1}{1-z}$. *Ans.* $u(x,y)=\dfrac{1-x}{(1-x)^2+y^2}$.

$v(x,y)=\dfrac{y}{(1-x)^2+y^2}$.

(*c*) $f(z)=e^{4z}$. *Ans.* $u(x,y)=e^{4x}\cos 4y$

$v(x,y)=e^{4x}\sin 4y$

(*d*) $f(z)=\ln z^2$. *Ans.* $u(x,y)=2\ln\sqrt{x^2+y^2}$.

$v(x,y)=2\arctan\dfrac{y}{x}$.

2.4 Give the Taylor series expansion for the function $f(x)=1/(1-x)$ of the real variable x, and state the interval of convergence.

Ans. $f(x)=\sum_{j=0}^{\infty} x^j$; converges for $|x|<1$.

In Exercises 2.5, 2.6, and 2.7, obtain the Laurent series in z about the relevant poles. State the location and order of each pole.

2.5 $f(z)=\dfrac{e^z}{(z-1)^2}$.

Hint. Let $z-1=u$. Then

$$\frac{e^z}{(z-1)^2}=e\left(\frac{e^u}{u^2}\right)=\frac{e}{u^2}\left(1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\cdots\right).$$

Now make the substitution $u=z-1$ to obtain the Laurent expansion in z.

2.6 $f(z)=\dfrac{z}{(z+1)(z+3)}$.

Hint. To find the Laurent expansion about the pole $z=-1$, let $z+1=u$ and express $f(z)$ as a function of u, recalling that

$$\frac{1}{1+u/2}=1-\frac{u}{2}+\left(\frac{u}{2}\right)^2-\left(\frac{u}{2}\right)^3+\cdots+(-1)^n\left(\frac{u}{2}\right)^n+\cdots$$

2.7 $f(z)=\dfrac{1}{z(z+2)^3}$.

Hints. (*a*) For the expansion about $z=0$, write $1/[z(z+2)^3]=1/[8z(1+z/2)^3]$ and expand $(1+z/2)^{-3}$ using the binomial theorem.
(*b*) For the expansion about $z=-2$, use the substitution $u=z+2$ and expand $1/(u-2)$ in series form.

2.8 Find the residues at the pole $z=1$ in Exercise 2.5 by differentiation.

Ans. e.

2.9 Find the residues at the relevant poles in Exercise 2.6 by differentiation.

Ans. At $z=-1$ residue $=-\frac{1}{2}$.
At $z=-3$ residue $=\frac{3}{2}$.

2.10 Find the residues at the relevant poles in Exercise 2.7 by differentiation.

Ans. At $z=0$ residue $=\frac{1}{8}$.
At $z=-2$ residue $=-\frac{1}{4}$.

2.11 Verify the Laplace transform given in Table D.2.1 for each of the following p.d.f.'s. (Note that in item c the bilateral Laplace transform is appropriate.)

(*a*) $f(x)=1, \quad 0\leqslant x\leqslant 1$ (uniform)
(*b*) $f(x)=\lambda e^{-\lambda x}, \quad 0\leqslant x<\infty$ (exponential)
(*c*) $f(x)=\dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}, \quad -\infty<x<\infty$ (normal)

2.12 Verify the characteristic function given in Table D.2.1 for each of the p.d.f.'s in Exercise 2.11.

2.13 Verify the Mellin transform given in Table D.2.1 for each of the p.d.f.'s in Exercise 2.11.

2.14 Invert the following Laplace transforms:

(*a*) $L_r(f(x))=\dfrac{1}{r-a}$. *Ans.* $f(x)=e^{ax}$.
(*b*) $L_r(f(x))=\dfrac{1}{r^{n+1}}$. *Ans.* $f(x)=\dfrac{x^n}{\Gamma(n+1)}$.

(*c*) $L_r(f(x))=\dfrac{1}{r^2+a^2}$. *Ans.* $f(x)=\dfrac{\sin ax}{a}$.

(*d*) $L_r(f(x))=\dfrac{5r+4}{r^3}$. *Ans.* $f(x)=5x+2x^2$.

2.15 Invert the following Fourier transforms or characteristic functions (use (2.8.5a, b)).

(*a*) $F_t(f(x))=\dfrac{a}{a-it}$. *Ans.* $f(x)=ae^{-ax}$, $0\leqslant x<\infty$.

(*b*) $F_t(f(x))=\exp\left(itu-\dfrac{t^2\sigma^2}{2}\right)$. *Ans.* $f(x)=\dfrac{1}{\sigma\sqrt{2\pi}}\times\exp\left\{\dfrac{-[(x-u)/\sigma]^2}{2}\right\}$, $-\infty<x<\infty$.

(*c*) $F_t(f(x))=\dfrac{1}{(1-2it)^{m/2}}$, m even. *Ans.* $f(x)=\dfrac{x^{(m/2)-1}}{2^{m/2}\Gamma(m/2)}e^{-x/2}$, $x\geqslant 0$.

2.16 Invert the following Mellin transforms.

(*a*) $M_s(f(x))=\lambda^{-s+1}\Gamma(s)$. *Ans.* $f(x)=\lambda e^{-\lambda x}$, $x\geqslant 0$.

(*b*) $M_s(f(x))=\dfrac{\lambda^{-s+1}\Gamma(s+a-1)}{\Gamma(a)}$. *Ans.* $f(x)=\dfrac{\lambda}{\Gamma(a)}(\lambda x)^{a-1}e^{-\lambda x}$, $0\leqslant x<\infty$.

(*c*) $M_s(f(x))=\dfrac{\alpha+1}{s+\alpha}$. *Ans.* $f(x)=(\alpha+1)x^{\alpha}$, $0\leqslant x\leqslant 1$, α real.

CHAPTER 3

The Distribution of Sums and Differences of Random Variables

3.1 THE FOURIER CONVOLUTION AS THE DISTRIBUTION OF A SUM

Though it may not often be regarded as such the Fourier convolution

$$g(w)=\int_{-\infty}^{\infty} f_1(w-x_2)f_2(x_2)dx_2 \tag{3.1.1a}$$

$$=\int_{-\infty}^{\infty} f_1(x_1)f_2(w-x_1)dx \tag{3.1.1b}$$

of two functions $f_1(x_1)$ and $f_2(x_2)$ is actually, except for possibly a constant factor, the p.d.f. $g(w)$ of the sum $W=X_1+X_2$ of the i.r.v.'s X_1 and X_2. To see that this is so, consider the transformation

$$\begin{aligned} w&=x_1+x_2,\\ x_2&=x_2 \end{aligned} \tag{3.1.2}$$

which, when solved inversely for x_1 and x_2, gives

$$\begin{aligned} x_1&=w-x_2,\\ x_2&=x_2. \end{aligned} \tag{3.1.3}$$

By means of the inverse transformation (3.1.3), the joint probability element

$$f(x_1,x_2)dx_1\,dx_2=f(x_1)f(x_2)dx_1\,dx_2 \tag{3.1.4}$$

is transformed into the joint probability element $f(w,x_2)dx_2\,dw$:

$$g(w,x_2)dx_2\,dw = f_1(w-x_2)f_2(x_2)|J|\,dx_2\,dw, \tag{3.1.5}$$

J being the Jacobian of the inverse transformation (3.1.3):

$$\begin{aligned} J &= \begin{vmatrix} \dfrac{\partial x_1}{\partial w} & \dfrac{\partial x_1}{\partial x_2} \\ \dfrac{\partial x_2}{\partial w} & \dfrac{\partial x_2}{\partial x_2} \end{vmatrix} \\ &= \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \\ &= 1. \end{aligned} \tag{3.1.6}$$

Integrating out the variable x_2 in (3.1.5) yields the p.d.f. of the sum $w = x_1 + x_2$, namely,

$$f(w) = \int_{-\infty}^{\infty} f_1(w-x_2)f_2(x_2)dx_2 \tag{3.1.7a}$$

$$= \int_{-\infty}^{\infty} f_1(x_1)f_2(w-x_1)dx_1, \tag{3.1.7b}$$

which is also recognized as the Fourier convolution of $f_1(x_1)$ and $f_2(x_2)$.

In the same way, one can readily show that the p.d.f. $g(w)$ of the difference $W = X_1 - X_2$ of two i.r.v.'s X_1 and X_2 with p.d.f.'s $f_1(x_1)$ and $f_2(x_2)$ is the Fourier convolution

$$g(w) = \int_{-\infty}^{\infty} f_1(w+x_2)f_2(x_2)dx_2 \tag{3.1.8a}$$

$$= \int_{-\infty}^{\infty} f_1(x_1)f_2(w+x_1)dx_1. \tag{3.1.8b}$$

To establish this result, one utilizes the transformation

$$\begin{aligned} w &= x_1 - x_2 \\ x_2 &= x_2, \end{aligned} \tag{3.1.9}$$

whose inverse is expressible as

$$x_1 = x_2 + w,$$
$$x_2 = x_2, \tag{3.1.10}$$

having the Jacobian

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial w} & \dfrac{\partial x_1}{\partial x_2} \\ \dfrac{\partial x_2}{\partial w} & \dfrac{\partial x_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} +1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= +1. \tag{3.1.11}$$

Again, the joint probability element

$$f(x_1, x_2) dx_1 dx_2 = f_1(x_1) f_2(x_2) dx_1 dx_2 \tag{3.1.12}$$

is transformed into the joint probability element

$$g(w, x_2) dx_2 dw = f_1(w + x_2) f(x_2) |J| dx_2 dw. \tag{3.1.13}$$

On integrating out x_2, one obtains the p.d.f. of the difference $w = x_1 - x_2$, namely,

$$g(w) = \int_{-\infty}^{\infty} f_1(w + x_2) f_2(x_2) dx_2 \tag{3.1.14a}$$

$$= \int_{-\infty}^{\infty} f_1(x_1) f_2(w + x_1) dx_1, \tag{3.1.14b}$$

which again is seen to be a Fourier convolution of $f_1(x_1)$ and $f_2(x_2)$.

As specific examples of Fourier convolutions, consider the p.d.f.'s of the sum and the difference of two independent and identically distributed uniform r.v.'s X_1 and X_2 with p.d.f.

$$f_i(x_i) = 1, \qquad 0 \leqslant x_i \leqslant 1, \qquad i = 1, 2. \tag{3.1.15}$$

From (3.1.1a) it follows that the sum convolution is

$$g(w)=\int_{\substack{\text{Range}\\ \text{of } x_2}} f_1(w-x_2)f(x_2)dx_2, \tag{3.1.16}$$

where $w=x_1+x_2$. Given (3.1.15), we see that (3.1.16) becomes

$$g(w)=\int_0^w dx_2=w, \qquad 0\leqslant w\leqslant 1 \tag{3.1.16a}$$

$$=\int_{w-1}^{1} dx_2=2-w, \qquad 1\leqslant w\leqslant 2, \tag{3.1.16b}$$

which is represented graphically in Fig. 3.1.1. Note that the limits of integration in (3.1.16a) are governed by two constraints: (1) $x_1=w-x_2$ is necessarily nonnegative, since X_1 is a nonnegative r.v., which restricts the maximum value of x_2 to w; (2) $w=x_1+x_2$ takes on its smallest possible value (zero) when both x_1 and x_2 take on their smallest values, namely, zero. But since $w=x_1+x_2$ can also assume values as large as 2, another integration in addition to (3.1.16a) must be considered. Since $x_1=w-x_2\leqslant 1$, when $1\leqslant w\leqslant 2$ the variable x_2 may range from $w-1$ to 1 and still not violate the constraint $0\leqslant x_2\leqslant 1$. This accounts for the limits of integration in (3.1.16b).

In the same way the difference convolution $g(w)$, where $W=X_2-X_1$ is the difference of two identically distributed uniform i.r.v.'s each with p.d.f.

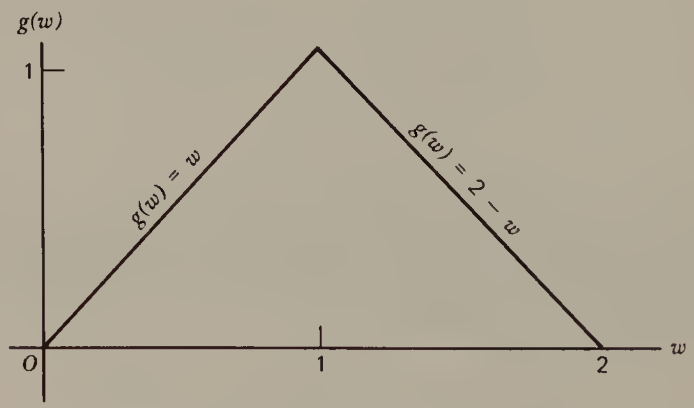


Fig. 3.1.1 P.d.f. of the sum of two uniform i.r.v.'s.

(3.1.15) is, from (3.1.14a),

$$g(w) = \int_{\substack{\text{range} \\ \text{of } x_2}} f_1(w+x_2) f_2(x_2) dx_2$$

$$= \int_{\substack{\text{Range} \\ \text{of } x_2}} dx_2.$$

As before, the question at this point is the range of integration of x_2. Since $w = x_2 - x_1$ and $0 \leqslant x_1 \leqslant 1$, it is clear that w is negative if and only if $x_2 < x_1$, and that w ranges from -1 (when $x_2 = 0$, $x_1 = 1$) to 0 (when $x_2 = x_1$). Moreover, since $x_2 = w + x_1$, x_2 has a minimum value of zero (when $x_1 = x_2 = 0$) and a maximum value of $w+1$ (which necessarily occurs when x_1 assumes its maximum value of one). Thus

$$g(w) = \int_0^{w+1} dx_2 = w+1, \qquad -1 \leqslant w \leqslant 0. \tag{3.1.17a}$$

On the other hand, w will be nonnegative if and only if $x_2 \geqslant x_1$, having its smallest nonnegative value (zero) when $x_2 = x_1$ and its largest value (one) when $x_2 = 1$, $x_1 = 0$. Again, since $x_2 = w + x_1$ and $x_1 \geqslant 0$, x_2 ranges from w (when $x_1 = 0$) to a maximum value of one, governed by the original constraint $0 \leqslant x_2 \leqslant 1$. (Note that it is possible for $x_2 = w + x_1$ to achieve the

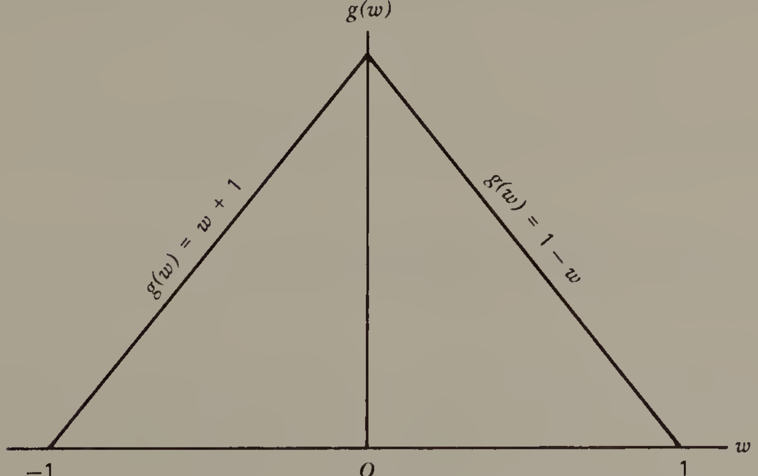


Fig. 3.1.2 P.d.f. of the difference of two uniform i.r.v.'s.

value 1, since when x_1 is 0, w will be 1 when x_2 is 1.) Thus

$$g(w)=\int_w^1 dx_2=1-w, \qquad 0\leqslant w\leqslant 1. \tag{3.1.17b}$$

Figure 3.1.2 plots these p.d.f.'s.

By repeated application of the Fourier convolution, one can, of course, derive the p.d.f. of the sum $W=\Sigma_{i=1}^n X_i$ of n i.r.v.'s, as a later section shows. There is, however, a simpler method that enables one to obtain the p.d.f. of the sum of n i.r.v.'s in one step. Not surprisingly, this method is based on the Fourier integral transform, which is the subject of the section that follows.

3.2 THE DISTRIBUTION OF SUMS AND DIFFERENCES OF CONTINUOUS INDEPENDENT RANDOM VARIABLES

The relationship between the Fourier transform and the Fourier inversion integral provides a powerful method for deriving the distribution of sums and differences of i.r.v.'s. Although the dual relationship between an integral transform and the corresponding inversion integral for a Fourier transform pair has been stated, the result is now restated as a theorem pertaining to cases in which $f(x)$ is a p.d.f. (and hence a function that is nonnegative, real, single valued—or single valued almost everywhere[12] —and absolutely integrable[13]).

Theorem 3.2.1 (the Fourier inversion theorem). If $f(x)$ is a p.d.f. defined over the range $(-\infty,\infty)$, the Fourier transform (characteristic function)

$$\begin{aligned} F_t(f(x)) &= E[e^{itx}] \\ &= \int_{-\infty}^{\infty} e^{itx}f(x)\,dx, \end{aligned} \tag{3.2.1}$$

where t is a real number, always exists. If the Fourier transform (characteristic function) is absolutely integrable over $(-\infty,\infty)$, or is analytic in some horizontal strip $-\alpha<it<\beta$, then the p.d.f. is uniquely determined by the

[12]A function is single valued almost everywhere (a.e.) if it is single valued at all points except those of a set of measure zero. A point set is said to be of measure zero if for any positive number ε there exists a finite or countably infinite set of intervals (open or closed), such that each point of the set is contained in at least one of the intervals and the sum of the measures of the intervals is less than ε.

[13]The function $f(x)$ is absolutely integrable if $\int_{-\infty}^{\infty}|f(x)|dx$ converges.

inversion integral

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}F_t(f(x))dt. \tag{3.2.1a}$$

This theorem has been proved by Lukacs [225] and others and is not proved here, except to show that the characteristic function always exists. This is readily apparent, since

$$\begin{aligned}\left|\int_{-\infty}^{\infty}e^{itx}f(x)\,dx\right| &\leq \int_{-\infty}^{\infty}|e^{itx}|f(x)\,dx\\ &= \int_{-\infty}^{\infty}f(x)dx\\ &=1,\end{aligned}$$

since $f(x)$ is a p.d.f. and

$$\begin{aligned}|e^{itx}| &= |\cos tx + i\sin tx|\\ &=\cos^2 tx+\sin^2 tx\\ &=1.\end{aligned}$$

The Fourier transform (3.2.1) may be considered to be a function of the real variable t or of the special (purely imaginary) complex variable $z=it$. If the Fourier transform contains no poles, the horizontal strip consists of the entire plane, in which case the inversion integral (3.2.1a) may be readily evaluated by integrating with respect to the real variable t over the real axis, or any other line parallel to the real axis (see (3.4.17) and (3.4.18) and Example 9.1.1).

On the other hand, if the Fourier transform (3.2.1) contains poles, it is usually more convenient to replace this transform by an equivalent complex Fourier (or bilateral Laplace) transform

$$\int_{-\infty}^{\infty}e^{rx}f(x)dx$$

whenever one such exists, where r is a complex number not restricted to purely imaginary values it. Whenever such an equivalent complex Fourier (bilateral Laplace) transform exists, one may replace the purely imaginary variable it in (3.2.1a) by the general complex number r and evaluate the resultant inversion integral over the Bromwich path $(-i\infty,i\infty)$. The question of when such an equivalent complex Fourier transform exists is therefore of considerable importance, and is stated in the theorem below. (The proof of this theorem is given in Appendix E.)

Theorem 3.2.2 (the complex Fourier or bilateral Laplace inversion theorem). If $f(x)$ is a real function that is defined and single valued almost everywhere for $-\infty<x<\infty$, and is absolutely integrable over the range $(-\infty,\infty)$, where

$$\begin{aligned} f^-(x)&=f(x), \qquad -\infty<x\leqslant 0\\ &=0, \qquad 0\leqslant x<\infty \end{aligned} \tag{3.2.2a}$$

$$\begin{aligned} f^+(x)&=0, \qquad -\infty<x\leqslant 0\\ &=f(x), \qquad 0\leqslant x<\infty, \end{aligned} \tag{3.2.2b}$$

then the complex Fourier transform $\mathfrak{F}_r(f(x))$ exists, where

$$\begin{aligned} \mathfrak{F}_r(f(x))&=\mathfrak{F}_r^-(f(x)), \qquad -\infty<x<0\\ \mathfrak{F}_r^-(f(x))&=\int_{-\infty}^{0} e^{rx}f^-(x)\,dx \end{aligned} \tag{3.2.3a}$$

$$\begin{aligned} \mathfrak{F}_r(f(x))&=\mathfrak{F}_r^+(f(x)), \qquad 0<x<\infty\\ \mathfrak{F}_r^+(f(x))&=\int_{0}^{\infty} e^{rx}f^+(x)\,dx. \end{aligned} \tag{3.2.3b}$$

Conversely, if $\mathfrak{F}_r^-(f(x))$ and $\mathfrak{F}_r^+(f(x))$ exist and are analytic, then the p.d.f.

$$\begin{aligned} f(x)&=f^-(x), \qquad -\infty<x<0\\ &=f^+(x), \qquad 0\leqslant x<\infty \end{aligned}$$

is uniquely determined by the inversion integrals

$$\begin{aligned} f^-(x)&=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-rx}\mathfrak{F}_r^-(f(x))\,dr, \qquad -\infty<x\leqslant 0\\ &=0, \qquad \text{elsewhere} \end{aligned} \tag{3.2.4a}$$

$$\begin{aligned} f^+(x)&=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-rx}\mathfrak{F}_r^+(f(x))\,dr, \qquad 0\leqslant x<\infty\\ &=0, \qquad \text{elsewhere.} \end{aligned} \tag{3.2.4b}$$

Theorems 3.2.1 and 3.2.2 are important because each ensures that a p.d.f. is uniquely determined by its characteristic function (Fourier transform), provided this characteristic function is analytic in a relevant strip. It should also be noted that when x is restricted to nonnegative values (i.e., when $f^{-}(x)=0,\ -\infty<x\leqslant 0$), the complex Fourier transform becomes identical with the ordinary Laplace transform. In such a case

$$\begin{aligned} L_r(f(x)) &= \mathfrak{F}_r^+(f(x)) \\ &= \mathfrak{F}_r(f(x)) \end{aligned} \tag{3.2.4c}$$

and

$$f(x)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-rx}L_r(f(x))\,dr. \tag{3.2.4d}$$

The problem of evaluating the inversion integrals (3.2.4a, b) still remains. In this connection, the reader is reminded that the poles of the complex Fourier transform in the inversion integral (3.2.4a) lie in the LHP. Hence when the conditions of Jordan's lemma are satisfied, the inversion integral (3.2.4a) taken over the Bromwich path $(-i\infty,i\infty)$, is equivalent to that taken over the left-hand (closed) Bromwich contour C_L (Fig. 2.9.1*a*) as $R\to\infty$, which is readily evaluated by application of the residue theorem. Similarly, when the conditions of Jordan's lemma are satisfied, the inversion integral (3.2.4b), taken over the Bromwich path $(-i\infty,i\infty)$, is equivalent to that taken over the right-hand (closed) Bromwich contour C_R (Fig. 2.9.1*b*) as $R\to\infty$. Thus the value of the inversion integral is again obtained by applying the residue theorem, since the relevant poles now lie in the RHP.

Actually, the Bromwich path in the inversion integrals (3.2.4a, b) is not necessarily restricted to the imaginary axis. If there is a region containing the imaginary axis and in which neither $\mathfrak{F}_r^-(f(x))$ nor $\mathfrak{F}_r^+(f(x))$ has any pole, the inversion integral (3.2.4a) may be evaluated over any Bromwich path $(c-i\infty,c+i\infty)$ contained in the above-named region, so long as the poles of $\mathfrak{F}_r^-(f(x))$ lie to the left of this Bromwich path. Similarly, the inversion integral (3.2.4b) may be evaluated over any Bromwich path $(c'-i\infty,c'+i\infty)$ contained in the above-named region, so long as the poles of $\mathfrak{F}_r^+(f(x))$ lie to the right of this Bromwich path. One such Bromwich path is, of course, the imaginary axis $(-i\infty,i\infty)$, which is satisfactory for both the inversion integrals (3.2.4a, b).

The preceding theorems indicate how to obtain the p.d.f. $(f(x))$ from its Fourier or bilateral Laplace transform. Once $f(x)$ is obtained, the distribution function $F(x)$ can, of course, always be determined as $\int_{-\infty}^{x} f(u)\,du$.

However $F(x)$ can also be directly derived from the Fourier transform of $f(x)$, as the following theorem shows. The proof can be found in a number of readily accessible texts (e.g., those by Lukacs [225] and Kendall and Stuart [178]) and is not given here.

Theorem 3.2.3 The characteristic function $F_t(f(x))$ of $f(x)$ uniquely determines the distribution function $F(x)$; more specifically [178, pp. 94–95],

$$F(x)=F(0)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1-e^{-ixt}}{it}\right)F_t(f(x))\,dt \tag{3.2.5}$$

$$=\frac{1}{2}+\frac{1}{2\pi}\int_{0}^{\infty}\frac{\left[e^{itx}F_{-t}(f(x))-e^{-itx}F_t(f(x))\right]}{it}\,dt, \tag{3.2.5a}$$

the form (3.2.5) being particularly convenient for use with nonnegative r.v.'s.

The reader is reminded that although the characteristic function $F_t(f(x))$ always exists and uniquely determines the *distribution* function $F(x)$ [178, p. 94], it does not necessarily determine the p.d.f. $f(x)$ uniquely. A sufficient condition that the p.d.f. $f(x)$ be uniquely determined by the inversion integral (3.2.1a) is that the characteristic function $F_t(f(x))$ be analytic in a strip containing the real axis. Or equivalently, the inversion integrals (3.2.4a, b) uniquely determine $f(x)$ if $\mathfrak{F}_r^-(f(x))$, $\mathfrak{F}_r^+(f(x))$ are analytic in a strip containing the imaginary axis. It should be remembered that this condition of analyticity is sufficient.

As has already been stated, the definition of the Fourier integral transform imposes no restrictions on the range of the variable, hence is directly applicable to the problem of deriving the p.d.f. of sums or differences that may assume either positive or negative values. The counterpart of the Fourier transform, defined only for p.d.f.'s of r.v.'s restricted to nonnegative values, is the ordinary Laplace transform. Although either the Fourier transform or the complex Fourier (bilateral Laplace) transform is the logical choice for analyzing sums and differences of i.r.v.'s that are not limited to nonnegative values, the ordinary Laplace transform can be used equally well to derive p.d.f.'s of sums or differences of nonnegative i.r.v.'s. This is clear from the dual relationship cited in Chapter 2 between the Laplace transform and the Laplace inversion integral. For the benefit of the reader and for convenience in future reference, this relationship is now stated in theorem form. Since the proof can be found in various texts on complex variables (e.g., Trantner [382, p. 1]), it is not given here.

Theorem 3.2.4 If $f(x)$ is a function of a real variable and is defined and single valued almost everywhere for $x \geqslant 0$, and is such that the integral

$$\int_0^\infty |f(x)| e^{-kx} dx$$

converges for some real value k, then $f(x)$ is said to be Laplace transformable, and

$$L_r(f(x)) = \int_0^\infty e^{-rx} f(x) dx \tag{3.2.6}$$

is the Laplace transform of $f(x)$, where r is a complex variable. Conversely, if the Laplace transform $L_r(f(x))$ is analytic and of order $O(r^{-k})$ in some half-plane $\mathrm{Re}(r) \geqslant c$, c, k real, $k > 1$, then $f(x)$ is uniquely determined by the inversion integral

$$f(x) = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{w - i\beta}^{w + i\beta} e^{rx} L_r(f(x)) dr \tag{3.2.7}$$

evaluated over any line $\mathrm{Re}(r) = w \geqslant c$ [52, p. 178].

Hereafter, for convenience, the inversion integral (3.2.7) is written in the form

$$f(x) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} e^{rx} L_r(f(x)) dr.$$

The importance of the foregoing theorems and corollary in deriving the p.d.f. of sums or differences of i.r.v.'s stems from the following fundamental theorem.

Theorem 3.2.5 If $X_1, X_2, \ldots, X_n$ are continuous i.r.v.'s with p.d.f.'s $f_j(x_j)$, $j = 1, 2, \ldots, n$, then the p.d.f. $g(w)$ of the sum

$$W = \sum_{j=1}^{n} X_j \tag{3.2.8}$$

is given by

$$g(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itw} \prod_{j=1}^{n} F_t(f_j(x_j)) dt. \tag{3.2.9}$$

PROOF. The Fourier transform of the p.d.f. $g(w)$ is, by definition,

$$F_t(g(w)) = E[e^{itw}]. \tag{3.2.10}$$

From the definition of w, (3.2.10) becomes

$$\begin{aligned} F_t(g(w)) &= E\left[e^{it\sum_{j=1}^{n} x_j}\right] \\ &= E\left[\prod_{j=1}^{n} e^{itx_j}\right], \end{aligned} \tag{3.2.11}$$

which, in view of the independence of the X_j's, is expressible as

$$\begin{aligned} F_t(g(w)) &= \prod_{j=1}^{n} E[e^{itx_j}] \\ &= \prod_{j=1}^{n} \int_{-\infty}^{\infty} f_j(x_j) e^{itx_j} dx_j \\ &= \prod_{j=1}^{n} F_t(f_j(x_j)). \end{aligned} \tag{3.2.12}$$

Then, if each of the Fourier transforms $F_t(f_j(x_j))$, $j = 1, 2, \ldots, n$ is analytic, it follows from Theorem 3.2.1 that

$$g(w) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-itw} \prod_{j=1}^{n} F_t(f_j(x_j)) dt, \tag{3.2.13}$$

which proves the theorem.

Similarly, if the i.r.v.'s X_j, $j = 1, 2, \ldots, n$, are nonnegative, the inversion integral (3.2.13) is expressible in the form

$$g(w) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{rw} \prod_{j=1}^{n} L_r(f_j(x_j)) dr. \tag{3.2.14}$$

In other words, $g(w)$ can be determined from a knowledge of the Fourier or bilateral Laplace transforms of the p.d.f.'s of the component variables in the sum (3.2.8). Therein lies the power of the Fourier and bilateral Laplace transforms in deriving the p.d.f. of sums of i.r.v.'s.

Consider now the important special case of a difference

$$W = X_1 - X_2 \tag{3.2.15}$$

of two i.r.v.'s with p.d.f.'s $f_1(x_1), f_2(x_2)$, respectively. The p.d.f. of W can be obtained from a knowledge of the Fourier transforms of $f_1(x_1)$ and $f_2(x_2)$, as Theorem 3.2.6 shows.

Theorem 3.2.6 If X_1 and X_2 are continuous i.r.v.'s with p.d.f.'s $f_j(x_j)$, $j=1,2$, then the p.d.f. $g(w)$ of the r.v.

$$W=X_1-X_2$$

is given by

$$g(w)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itw}F_t(f_1(x_1))F_t(f_2(-x_2))dt, \tag{3.2.16a}$$

or equivalently, by

$$g(w)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{-rw}\mathfrak{F}_r^+(f_1(x_1))\mathfrak{F}_r^-(f_2(-x_2))dr. \tag{3.2.16b}$$

PROOF. Denote the p.d.f. of the r.v. $X_2'=-X_2$ by $f_2'(x_2')$. Then

$$\begin{aligned} W&=X_1-X_2\\ &=X_1+X_2', \end{aligned} \tag{3.2.17}$$

and from Theorem 3.2.5

$$g(w)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itw}F_t(f_1(x_1))F_t(f_2'(x_2'))dt. \tag{3.2.18}$$

But it is clear that $f_2'(x_2')$ is simply a reflection of $f_2(x_2)$, that is,

$$f_2'(x_2')=f_2(-x_2)$$

and therefore

$$g(w)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itw}F_t(f_1(x_1))F_t(f_2(-x_2))dt, \tag{3.2.19}$$

which, since $f_1(x_1)$ and $f_2(x_2)$ are absolutely integrable, is also equivalent to (3.2.16b). This establishes the theorem.

The distribution function $G(w)$ of an r.v. W, defined as

$$G(w)=\int_{-\infty}^{w}g(w)dw \tag{3.2.20}$$

can be obtained by direct integration in (3.2.20) or through the use of Fourier transform of $g(w)$ by way of the relationship (3.2.5a). Thus the counterparts of (3.2.5) and (3.2.5a) are, respectively,

$$G(w)=G(0)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1-e^{-iwt}}{it}\right)F_t(g(w))\,dt \qquad (3.2.21a)$$

$$=\frac{1}{2}+\frac{1}{2\pi}\int_0^{\infty}\frac{e^{itw}F_{-t}(g(w))-e^{-itw}F_t(g(w))}{it}\,dt \qquad (3.2.21b)$$

The power of (3.2.21) lies in the fact that it enables one to determine the distribution function $G(w)$ without first obtaining the p.d.f. $g(w)$. This is the case, for example, when

$$W=\sum_{j=1}^{n}X_j$$

is the sum of n specified random variables X_j whose p.d.f.'s $f_j(x_j)$ are known.

Although it is not usually so done, one can, of course, obtain the p.d.f. $g(w)$ of the sum

$$W=\sum_{j=1}^{n}X_j$$

as an $(n-1)$ step Fourier convolution. For by (3.1.7a), the p.d.f. $g_N(w_N)$ of the sum $W_N=W_{N-1}+X_N$, $N\geqslant 2$, for two i.r.v.'s W_{N-1} and X_N with p.d.f.'s $g_{N-1}(W_{N-1})$ and $f_N(x_N)$ is the Fourier convolution

$$g_N(w_N)=\int_{-\infty}^{\infty}g_{N-1}(w_{N-1}+x_N)f_N(x_N)\,dx_N. \qquad (3.2.22)$$

Thus by evaluating (3.2.22) successively for $N=2,3,\ldots,n$ one obtains the p.d.f. of the sum of n i.r.v.'s by an $(n-1)$-step procedure. However one can achieve the same result—usually more conveniently—by evaluating the inversion integral (3.2.13).

3.2.1 Distribution of the Sum of Two Identically Distributed Uniform Independent Random Variables

In deriving the p.d.f. of sums of i.r.v.'s of either singly or doubly infinite range, the resultant p.d.f. consists of a single component that obtains for the entire range of the sum. As has already been indicated, however, when the i.r.v.'s are defined over a finite range, the resultant p.d.f. of the sum consists of different components, each of which is valid over a subinterval of the complete range. The present example illustrates this fact, as noted when the solution was obtained as a convolution (Section 3.1). The solution is now obtained by way of the Laplace transform, primarily to point out the manner in which the component p.d.f.'s are automatically determined by Jordan's lemma, in contrast to their determination by way of limits of integration in the convolution method.

Consider then, the derivation of the p.d.f. $g(w)$ of the sum $W = X_1 + X_2$ of two identically distributed uniform i.r.v.'s having the p.d.f.

$$f(x_j) = 1, \qquad 0 \leq x_j \leq 1, \qquad j = 1,2. \tag{3.2.23}$$

Note first that

$$\begin{aligned} L_r f(x_1) &= L_r(f(x_2)) \\ &= \int_0^1 e^{-rx} dx \\ &= \frac{1 - e^{-r}}{r}. \end{aligned} \tag{3.2.24}$$

Then

$$\begin{aligned} L(g(w)) &= \frac{(1 - e^{-r})^2}{r^2} \\ &= \frac{1 - 2e^{-r} + e^{-2r}}{r^2} \end{aligned} \tag{3.3.25}$$

and

$$g(w) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \frac{e^{rw}}{r^2} (1 - 2e^{-r} + e^{-2r}) dr, \tag{3.2.26}$$

where there is a single pole at $r = 0$ and the Bromwich contour is $C_L = QKLMPQ$ (Fig. 2.9.1a) corresponding to $c > 0$ (since the pole is at the origin and w is nonnegative). Write $g(w)$ in the form

$$g(w) = g_1'(w) + g_2'(w) + g_3'(w), \tag{3.2.27}$$

where

$$g_1'(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{rw}}{r^2}\,dr, \tag{3.2.28}$$

$$g_2'(w) = \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{rw}}{r^2}(2e^{-r})\,dr$$

$$= \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{2e^{(w-1)r}}{r^2}\,dr, \tag{3.2.29}$$

$$g_3'(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{rw}}{r^2}(e^{-2r})\,dr$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{(w-2)r}}{r^2}\,dr. \tag{3.2.30}$$

It is easily verified that the conditions of Jordan's lemma are satisfied for $g_1'(w)$ when $w>0$; for $g_2'(w)$ when $1 \leqslant w \leqslant 2$; and for $g_3'(w)$ when $w>2$. Hence

$$g_1'(w) = R_1, \qquad w>0,$$

while

$$g_2'(w) = R_2, \qquad 1 \leqslant w \leqslant 2$$

and

$$g_3'(w) = R_3, \qquad w>2$$

$$=0, \qquad \text{since} \quad 0 \leqslant w \leqslant 2,$$

where R_1, R_2, and R_3, respectively, are, the residues of the second order poles in the integrands of the integrals (3.2.28), (3.2.29), and (3.2.30). Then

$$R_1 = \frac{d}{dr}(e^{wr})\Big|_{r=0}$$

$$= w, \qquad 0 \leqslant w \leqslant 2 \tag{3.2.31}$$

$$R_2 = -2\frac{d.}{dr}(e^{(w-1)r})\Big|_{r=0}$$

$$= -2(w-1), \qquad 1 \leqslant w \leqslant 2. \tag{3.2.32}$$

Consequently,

$$g_1(w) = g_1'(w)$$

$$= w, \qquad 0 \leqslant w \leqslant 1 \tag{3.2.33}$$

$$g_2(w) = g_1'(w) + g_2'(w)$$

$$= w - 2w + 2$$

$$= 2 - w, \qquad 1 \leqslant w \leqslant 2 \tag{3.2.34}$$

which are precisely the results obtained from the evaluation of the corresponding convolution in Section 3.1.

The reader may surmise at this point that the p.d.f. of the sum

$$W = \sum_1^n X_j$$

of n uniform i.r.v.'s has n components, which is actually the case (see Exercise 3.12). In fact, the p.d.f. of

$$W = \sum_{j=1}^n X_j, \qquad n > 1,$$

and its derivatives of order k, $k = 1, 2, \ldots, n-2$, are continuous at $X = j$, $j = 1, 2, \ldots, n$, whereas its $(n-1)$st order derivative is discontinuous at these values of X [73, p. 245].

3.2.2 Distribution of the Sum of Independent Chi-Square Random Variables

One of the important distributions in both theoretical and applied statistics is the chi-square distribution with p.d.f.

$$f(x) = \frac{1}{2^{m/2}\Gamma(m/2)} x^{m/2-1} e^{-x/2}, \qquad 0 < x < \infty, \tag{3.2.35}$$

where the parameter m is called the number of degrees of freedom. It is well known that a chi-square variable possesses the reproductive property with respect to addition.[14] That is, the sum

$$W = \sum_{j=1}^n X_j \tag{3.2.36}$$

[14]An r.v. x is said to possess the reproductive property with respect to addition when the sum of n random values of x has the same type of distribution as does x.

of n independent random chi-square variables X_j having n_j degrees of freedom, has a chi-square distribution with

$$N_m = \sum_{j=1}^{n} n_j$$

degrees of freedom, as we now show.

Note that $X_j, j=1,2,\ldots,n$ is nonnegative, so that it is natural to use the Laplace transform of the p.d.f. (3.2.35), namely,

$$L_r(fx_j) = \int_0^\infty e^{-rx_j} x_j^{(n_j/2)-1} e^{-x_j/2} dx_j$$

$$= \frac{1}{(1+2r)^{n_j/2}}. \tag{3.2.37}$$

Then

$$L_r(g(w)) = \prod_{j=1}^{n} L_r(f(x_j))$$

$$= (1+2r)^{-\frac{1}{2}\sum_1^n n_j} \tag{3.2.38}$$

Since $L_r(g(w))$ is analytic in the strip corresponding to $\mathrm{Re}(r) > -\frac{1}{2}$, it determines $g(w)$ uniquely. Hence $g(w)$ has a chi-square distribution with $\Sigma_1^n n_j$ degrees of freedom, since $L_r(g(w))$ as given by (3.2.38) is identical with the Laplace transform of a chi-square distribution with $\Sigma_1^n n_j$ degrees of freedom (Table D.2.1, Appendix D.2, see also Exercise 9.29).

3.3 THE DISTRIBUTION OF THE SUM OF MIXTURES OF INDEPENDENT RANDOM VARIABLES

It bears stating at this point that the distribution of a sun of n identically and independently distributed r.v.'s is, for the most part, relatively easy to obtain by means of characteristic functions when each r.v. is of singly infinite range, or when each r.v. is of doubly infinite range. Moreover, if the r.v. possesses the reproductive property, the distribution may perhaps be more readily obtained by the moment-generating function, as Section 3.6 demonstrates.

Such is not the case, however, for the distribution of the sum of identically and independently distributed r.v.'s of finite range; nor is it true for the distribution of the sum of a mixture of n i.r.v.'s, some of finite range and some of infinite range, nor for the sum of a mixture of n i.r.v.'s, some of singly infinite range and some of doubly infinite range. The resultant p.d.f. (and d.f.) of the sum is then partitioned into components, each of which is valid for a specific range of values of the sum.

In such a case (except when the number of variables is large and the distribution of the sum is asymptotically normal), the distribution usually cannot be determined by other methods (such as the geometric and moment-generating function methods). It is in such an instance that the power of the integral transform method comes sharply into focus.

It is, of course, difficult to express in general form the component p.d.f.'s which correspond to the various subranges for the sum of mixtures of i.r.v.'s in general, since the notation for expressing the result becomes increasingly cumbersome as the number of i.r.v.'s of finite range increases, or when the ranges of such variables overlap. However the manner in which the integral transform automatically partitions the p.d.f. of the sum into component p.d.f.'s that are valid over specific subranges can be grasped from a consideration of the example in Section 3.3.1. (Exercises 3.15 and 3.16 are also illuminating.)

3.3.1 The Distribution of the Sum of a Uniform and an Exponential Random Variable

Let X_1 and X_2 be, respectively, uniform and exponential i.r.v.'s with p.d.f.'s

$$\begin{aligned} f_1(x_1) &= 1, & 0 \leqslant x_1 \leqslant 1 \\ f_2(x_2) &= e^{-x_2}, & 0 \leqslant x_2 < \infty \end{aligned}$$

and consider the p.d.f. $g(w)$ of the sum $W = X_1 + X_2$. This density function is obtained by inverting the Laplace transform $L_r(g(w))$, where

$$L_r(g(w)) = L_r(f_1(x_1))L_r(f_2(x_2)),$$

$$\begin{aligned} L_r(f_1(x_1)) &= \int_0^1 e^{-rx_1}dx_1 \\ &= \frac{1-e^{-r}}{r} \end{aligned}$$

$$\begin{aligned} L_r(f_2(x_2)) &= \int_0^\infty e^{-rx_2}e^{-x_2}dx_2 \\ &= \frac{1}{r+1} \end{aligned}$$

Then

$$g(w)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{rw}\frac{(1-e^{-r})}{r(r+1)}dr$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{rw}}{r(r+1)}dr-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{(w-1)r}}{r(r+1)}dr.$$

For each of these two integrals, the poles are all in the LHP (i.e., $\operatorname{Re}(r)\leqslant 0$), so that the conditions of Jordan's lemma are satisfied if, respectively, $w>0$ for the first integral and $w>1$ for the second integral.[15] That is,

$$\begin{aligned} g(w)&=g_1(w), && w\geqslant 0\\ &=g_2(w), && w\geqslant 1\\ &=0 && \text{elsewhere.}\end{aligned}$$

Specifically,

$$g_1(w)=\frac{1}{2\pi i}(2\pi i)\left[\left.\frac{e^{rw}}{r+1}\right|_{r=0}+\left.\frac{e^{rw}}{r}\right|_{r=-1}\right]$$

$$=1-e^{-w},\qquad w\geqslant 0$$

$$g_2(w)=\frac{1}{2\pi i}(2\pi i)\left[\left.\frac{e^{(w-1)r}}{r+1}\right|_{r=0}-\left.\frac{e^{(w-1)r}}{r}\right|_{r=-1}\right]$$

$$=-1+e^{-(w-1)},\qquad w\geqslant 1.$$

Or equivalently,

$$\begin{aligned} g(w)&=1-e^{-w}, && 0\leqslant w\leqslant 1\\ &=e^{-(w-1)}-e^{-w}, && 1\leqslant w<\infty\end{aligned}$$

[15]Although the conditions of Jordan's lemma are not satisfied when $w=0$, $w=1$, it can be readily shown that the values of the relevant contour integrals over the circular arc $C_L=QKLMPQ$ (Fig. 2.9.1*a*) approach zero as $R\to\infty$ for these two values of w, so that the residue theorem applies for all values $0\leqslant w<2$. Specifically, when $w=0$,

$$\lim_{r\to\infty}\frac{e^{wr}}{r(r+1)}<\lim_{r\to\infty}\frac{1}{|r||r+1|}=\lim_{R\to\infty}\frac{1}{R(R+1)}=0$$

and when $w=1$,

$$\lim_{r\to\infty}\frac{(1-e^{-1})e^{w-1}}{r(r+1)}<\lim_{r\to\infty}\frac{1-e^{-1}}{|r||r+1|}=\lim_{R\to\infty}\frac{1-e^{-1}}{R(R+1)}=0.$$

It is easily verified that

$$\int_0^\infty g(w)dw=\int_0^1 g_1(w)dw+\int_1^\infty g_2(w)\,dw$$
$$=1$$

and that $E[w]=E[x_1]+E[x_2]$.

The method can be extended to the derivation of the p.d.f. of a sum of more than two r.v.'s in a direct and straightforward manner, as the reader can verify by solving Exercises 3.16 and 3.19.

3.4* THE DISTRIBUTION OF SUMS AND DIFFERENCES OF DEPENDENT RANDOM VARIABLES

Up to this point, the discussion has been limited to the derivation of p.d.f.'s of sums and differences of *independent random variables*. When the independence condition is removed, the analysis becomes—in general—considerably more complicated, largely because the transform of the p.d.f.'s of sums of dependent r.v.'s is no longer expressible as the product of the transforms of the component r.v.'s. The impact of this condition can be noted in the following analysis of sums and differences of dependent r.v.'s.

3.4.1 Multivariate Fourier Transforms

Section 3.2 defined the Fourier transform or characteristic function of the univariate p.d.f. $f(x)$ as

$$F_t(f(x))=E[e^{itx}]=\int_{-\infty}^\infty e^{itx}f(x)\,dx. \tag{3.4.1}$$

It was also pointed out that if the Fourier transform $F_t(f(x))$ was analytic, then $f(x)$ could be obtained by evaluating the Fourier inversion integral

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^\infty e^{-itx}f(x)\,dx. \tag{3.4.2}$$

More generally, the Fourier transform (or *multivariate characteristic function*, hereafter abbreviated as m.c.f.) of the multivariate p.d.f. $f(x_1,x_2,\ldots,x_n)$ is defined [178, p. 107] as

$$F_{t_1,t_2,\ldots,t_n}(f(x_1,x_2,\ldots,x_n))=\int_{-\infty}^\infty\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty e^{it_1x_1+it_2x_2\cdots+it_nx_n}$$
$$\times f(x_1,x_2,\ldots,x_n)\,dx_1\,dx_2\cdots dx_n, \tag{3.4.3}$$

which is hereafter denoted by the simpler form $F(t_1,t_2,\ldots,t_n)$. If $F(t_1,t_2,\ldots,t_n)$ is an analytic function of $t_i, i=1,2,\ldots,n$ then $f(x_1,x_2,\ldots,x_n)$ can be obtained from $F(t_1,t_2,\ldots,t_n)$ by evaluating the multiple inversion integral

$$f(x_1,x_2,\ldots,x_n)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{1}^{n}e^{-itx_j}$$

$$\times F(t_1,t_2,\ldots,t_n)\cdot\prod_{1}^{n}dx_j. \tag{3.4.4}$$

As in the univariate case, an analytic m.c.f. defines the corresponding distribution uniquely. If the x_i, $i=1,2,\ldots,n$ are independent, the m.c.f. (3.4.3) is expressible as the product of n univariate characteristic functions (u.c.f.'s), namely,

$$F(t_1,t_2,\ldots,t_n)=\prod_{i=1}^{n}F_{t_i}(f_i(x_i)), \tag{3.4.5}$$

where $F_{t_i}(f(x_i))$ is given by (3.4.1) with $t=t_i$ and $x=x_i$. Similarly, the multivariate p.d.f. (3.4.4) is expressible as the product of n univariate inversion integrals; that is,

$$f(x_1,x_2,\ldots,x_n)=\prod_{j=1}^{n}\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}e^{-itx_j}F_{t_j}(f(x_j))\,dx_j\right]$$

$$=f_1(x_1)f_2(x_2)\cdots f_n(x_n). \tag{3.4.6}$$

However if the $x_j, j=1,2,\ldots,n$ are not independent, the m.c.f. is not expressible in the form (3.4.5), and neither is the multivariate p.d.f. $f(x_1,x_2,\ldots,x_n)$ expressible in the factored form (3.4.6).[16] In particular, the determination of the distribution of a sum of dependent r.v.'s requires first that one obtain $F(t_1,t_2,\ldots,t_n)$ from (3.4.3). Then, letting $t_i=t, i=1,2,\ldots,n$ in the resultant m.c.f. $F(t_1,t_2,\ldots,t_n)$ and evaluating the associated inversion integral, one obtains the p.d.f. $g(w)$ of the sum $W=\sum_{j=1}^{n}X_j$ of n dependent r.v.'s, namely,

$$g(w)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itw}F(t,t,\ldots,t)\,dt. \tag{3.4.7}$$

[16]That is, a necessary and sufficient condition for the independence of n variables is that their m.c.f. factorize into their individual characteristic functions [178, p. 357].

The p.d.f. $g(w)$ may also be obtained by evaluating the inversion integral

$$g(w)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{-rw}\mathfrak{F}(r,r,\ldots,r)\,dr \tag{3.4.7a}$$

corresponding to the complex Fourier transform $\mathfrak{F}_r(t,t,\ldots,t)$, where r is a complex number (see Section 2.8.1).

It is sometimes possible to transform dependent r.v.'s into independent variables, as when the joint p.d.f. is of the form $f(x_1,\ldots,x_n)=\text{const}\,e^{-Q}$, where Q is a quadratic form in $(x_1,\ldots,x_n)$ [32, pp. 127–143]. For example, the r.v.'s X_1 and X_2 with joint p.d.f.

$$f(x_1,x_2)=\frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x_1^2}{\sigma_1^2}-\frac{2\rho x_1x_2}{\sigma_1\sigma_2}+\frac{x_2^2}{\sigma_2^2}\right)\right],$$

$$|\rho|<1,\qquad -\infty<x_i<\infty,\qquad i=1,2 \tag{3.4.8}$$

are dependent (correlated) normal r.v.'s. Consider now the transformation

$$y_1=(1-\rho^2)^{-\frac{1}{2}}\left(\frac{x_1}{\sigma_1}-\frac{\rho x_2}{\sigma_2}\right)$$

$$y_2=\frac{x_2}{\sigma_2} \tag{3.4.9}$$

with Jacobian J, where

$$J^{-1}=\begin{vmatrix}\dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2}\\[2ex] \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2}\end{vmatrix}$$

$$=\begin{vmatrix}\dfrac{1}{\sigma_1(1-\rho^2)^{1/2}} & \dfrac{-\rho}{\sigma_2(1-\rho^2)^{1/2}}\\[2ex] 0 & \dfrac{1}{\sigma_2}\end{vmatrix}$$

$$=\frac{1}{\sigma_1\sigma_2(1-\rho^2)^{1/2}} \tag{3.4.10}$$

and

$$y_1^2+y_2^2=(1-\rho^2)^{-1}\left[\frac{x_1^2}{\sigma_1^2}-\frac{2\rho x_1x_2}{\sigma_1\sigma_2}+\frac{x_2^2}{\sigma_2^2}\right].$$

Application of the transformation to the joint p.d.f. (3.4.8) gives

$$\begin{aligned}g(y_1,y_2)&=f(x_1,x_2)|J|\\&=\frac{1}{2\pi}\exp\left[-\frac{1}{2}(y_1^2+y_2^2)\right]\\&=\left(\frac{1}{\sqrt{2\pi}}e^{-y_1^2/2}\right)\left(\frac{1}{\sqrt{2\pi}}e^{y_2^2/2}\right).\end{aligned}\tag{3.4.11}$$

Clearly the transformed variables are independent, even though the original variables were dependent.

3.4.2 The Distribution of the Sum of Two Dependent, Normal Random Variables

Since the reader may find it helpful to follow through the derivation of the distribution of the sum of two dependent, normal r.v.'s before proceeding to the general case for a sum of N, this section considers the sum of only two such r.v.'s. The general case is treated in the next section.

The first step is to derive the bivariate characteristic function (bivariate Fourier transform) of the joint (bivariate normal) p.d.f.

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x_1^2}{\sigma_1^2}-\frac{2\rho x_1x_2}{\sigma_1\sigma_2}+\frac{x_2^2}{\sigma_2^2}\right)\right],$$

$$|\rho|<1,\qquad -\infty<x_i<\infty,\qquad \sigma_i>0,\qquad i=1,2.\tag{3.4.12}$$

The bivariate Fourier transform or characteristic function is then obtained from (3.4.3) with $n=2$; that is,

$$F(t_1,t_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp\{i(x_1t_1+x_2t_2)\}}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}}$$

$$\times\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x_1^2}{\sigma_1^2}-2\rho\frac{x_1x_2}{\sigma_1\sigma_2}+\frac{x_2^2}{\sigma_2^2}\right)\right]dx_1\,dx_2.\tag{3.4.13}$$

Making the substitution

$$y_1 = x_1 - \sigma_1^2 t_1 - \rho\sigma_1\sigma_2 t_2,$$

$$y_2 = x_2 - \rho\sigma_1\sigma_2 t_1 - \sigma_2^2 t_2 \qquad (3.4.14)$$

and noting that the Jacobian J is 1, where

$$J = \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{vmatrix}^{-1} = 1$$

one finds

$$F(t_1,t_2) = \exp\left[-\frac{1}{2}\left(t_1^2\sigma_1^2 + 2t_1t_2\sigma_1\sigma_2\rho + t_2^2\sigma_2^2\right)\right]\frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}}$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{y_1^2}{\sigma_1^2} - \frac{2\rho y_1 y_2}{\sigma_1\sigma_2} + \frac{y_2^2}{\sigma_2^2}\right)\right]dy_1\,dy_2$$

$$= \exp\left[-\frac{1}{2}\left(t_1^2\sigma_1^2 + 2\rho_1\sigma_1\sigma_2 t_1 t_2 + t_2^2\sigma_2^2\right)\right]. \qquad (3.4.15)$$

If one now denotes the p.d.f. of the sum $w = x_1 + x_2$ by $g(w)$, the bivariate characteristic function of $g(w)$ is

$$F(t,t) = \exp\left[-\frac{1}{2}\left(\sigma_1^2 + 2\rho_1\sigma_1\sigma_2 + \sigma_2^2\right)t^2\right]. \qquad (3.4.16)$$

Then $g(w)$ is given by the inversion integral

$$g(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-itw)\exp\left[-\frac{1}{2}\left(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2\right)t^2\right]dt \qquad (3.4.17)$$

$$= \frac{\exp[-w^2/(2a^2)]}{2\pi a}\int_{-\infty}^{\infty} e^{-z^2/2}\,dz, \qquad (3.4.18)$$

where

$$z = at - \frac{iw}{a},$$

$$a^2 = \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2.$$

Clearly, the inversion integral (3.4.18) reduces to

$$g(w) = \frac{\exp\left[-w^2/2(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)\right]}{\sqrt{2\pi}\sqrt{\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2}} \qquad -\infty < w < \infty, \quad (3.6.19)$$

which is the well-known p.d.f. of the sum of two correlated normal r.v.'s with correlation coefficient ρ and variances σ_1^2 and σ_2^2. The p.d.f. of the sum of n correlated normal r.v.'s is obtained in the same manner. (See Exercise 3.17.)

3.4.3 The Distribution of the Sum of n Dependent, Normal Random Variables

The distribution of the sum of n dependent normal r.v.'s $X_1, X_2, \ldots, X_n$ is focused on the multivariate normal p.d.f.

$$f(x_1, x_2, \ldots, x_n) = \frac{|V^{-1}|^{1/2}}{(2\pi)^{n/2}} \exp\left\{-\frac{1}{2}(x-\mu)'\mathbf{V}^{-1}(x-\mu)\right\}, \quad (3.4.20)$$

where $\mathbf{V}$ is a symmetrical $(n \times n)$ matrix, namely, the variance-covariance matrix

$$\mathbf{V} = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 & \cdots & \rho_{2n}\sigma_2\sigma_n \\ - & - & - & - & - \\ - & - & - & - & - \\ \rho_{n1}\sigma_n\sigma_1 & \rho_{n2}\sigma_n\sigma_2 & \rho_{n3}\sigma_n\sigma_3 & \cdots & \sigma_n^2 \end{bmatrix}. \quad (3.4.21)$$

The usual symbols are employed. Specifically, a boldface capital letter denotes a matrix, a boldface lowercase letter a column vector, and a prime indicates transposition; thus a row vector is represented by a boldface, primed lowercase letter. Furthermore, $\mathbf{V}^{-1}$ is the inverse of the matrix $\mathbf{V}$, and ρ_{jk} is the correlation coefficient

$$\rho_{jk}=\frac{\mu_{jk}}{\sigma_j\sigma_k} \tag{3.4.22}$$

where

$$\begin{aligned}\mu_{jk}&=E[(x_j-\mu_j)(x_k-\mu_k)], \qquad (3.4.23)\\ \mu_j&=E[x_j],\\ \mu_k&=E[x_k]\\ \sigma_j^2&=E[(x_j-\mu_j)^2],\\ \sigma_k^2&=E[(x_k-\mu_k)^2],\end{aligned}$$

$$\mu=\begin{bmatrix}\mu_1\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \mu_n\end{bmatrix}.$$

The inverse $\mathbf{V}^{-1}$ is obtained as

$$\mathbf{V}^{-1}=\begin{bmatrix}\frac{V_{11}}{|V|} & \frac{V_{21}}{|V|} & \cdots & \frac{V_{n1}}{|V|}\\ \frac{V_{12}}{|V|} & \frac{V_{22}}{|V|} & & \frac{V_{n2}}{|V|}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{V_{1n}}{|V|} & \frac{V_{2n}}{|V|} & & \frac{V_{nn}}{|V|}\end{bmatrix}=\frac{\mathbf{V}'}{|V|}, \tag{3.4.24}$$

where $|V|$ is the determinant of $\mathbf{V}$ and

$$\mathbf{V}'=(V_{ji}) \tag{3.4.25}$$

is the adjoint of $\mathbf{V}$ in which V_{ji} denotes the cofactor of v_{ji}, the element in the jth row and ith column of the matrix $\mathbf{V}$. The matrix $\mathbf{V}$ is usually denoted by

$$\mathbf{V}=(V_{ij}), \tag{3.4.26}$$

where v_{ij} represents the element in the ith row and jth column. For the matrix (3.4.21), $v_{ij}=\rho_{ij}\sigma_i\sigma_j$.

Thus for the bivariate normal distribution utilized to determine the distribution of the sum of two dependent r.v.'s in Section 3.4.2,

$$\mathbf{V}=\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \tag{3.4.27}$$

$$|\mathbf{V}|=\sigma_1^2\sigma_2^2(1-\rho^2) \tag{3.4.28}$$

$$V_{11}=\sigma_2^2, \qquad V_{12}=-\rho\sigma_1\sigma_2, \qquad V_{21}=-\rho\sigma_1\sigma_2, \qquad V_{22}=\sigma_1^2. \tag{3.4.29}$$

Matrix 3.4.24 then becomes

$$\mathbf{V}^{-1}=\begin{bmatrix} \dfrac{1}{\sigma_1^2(1-\rho^2)} & \dfrac{-\rho}{\sigma_1\sigma_2(1-\rho^2)} \\ \dfrac{-\rho}{\sigma_1\sigma_2(1-\rho^2)} & \dfrac{1}{\sigma_2^2(1-\rho^2)} \end{bmatrix}. \tag{3.4.30}$$

It should perhaps be pointed out that the matrix (3.4.21) may be singular, in which case $\mathbf{V}$ has rank $r<n$. In this case, the p.d.f. (3.4.20) is reduced to r dimensions, which means that one or more variables are redundant.

By utilizing the fact that there exists an orthogonal[17] transformation [178, p. 347]; [32, p. 154, Exercise 1]

$$\mathbf{x}=\mathbf{By}, \tag{3.4.31}$$

which transforms the argument of the exponential function in (3.4.20) into a sum of squares $\sum_i a_i y_i^2$, where the coefficients a_i^2 are the latent roots or eigenvalues [161, p. 122] of V^{-1}, one can establish [178] that the m.c.f. of

[17]An orthogonal transformation is a linear transformation of the variables $x_1, x_2, \ldots, x_n$ of the form

$$y_i=\sum_{j=1}^{n} a_{ij}x_j, \qquad i=1,2,\ldots,n,$$

which leaves the quadratic form $x_1^2+x_2^2+\cdots+x_n^2$ invariant.

the multivariate density function (3.4.20) is

$$F(t_1,t_2,\ldots,t_n)=\exp\left(-\tfrac{1}{2}\mathbf{t}'\mathbf{V}\mathbf{t}\right)\exp(i\mathbf{t}'\boldsymbol{\mu}). \tag{3.4.32}$$

The p.d.f. $g(w)$ of the sum $W=\sum_{i}^{n} X_i$ is then given by the inversion integral

$$g(w)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itw}F(t,t,\ldots,t)\,dt \tag{3.4.33a}$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-rw}F(r,r,\ldots,r)\,dr. \tag{3.4.33b}$$

3.5 THE DISTRIBUTION OF THE SUM OF DISCRETE INDEPENDENT RANDOM VARIABLES

Among discrete i.r.v.'s, those assuming only integral values $k=0,1,2,\ldots,$ are of special importance, and here we consider the distribution of the sum of such r.v.'s. The derivation of the probability distribution of such sums is accomplished by the well-known method of generating functions, the latter being defined below. (The material in this section, with the exception of the example, is taken from Feller [99, pp. 248–252], and is reprinted with the permission of the publisher.)

Definition 3.5.1. Let $a_0,a_1,a_2,\ldots,$ be a sequence of real numbers. If

$$A(s)=a_0+a_1s+a_2s^2+\cdots \tag{3.5.1}$$

converges in some interval $-s_0<s<s_0$, then the function $A(s)$ is called the generating function of the sequence $\{a_j\}$.

It should be pointed out that s is not an r.v. but is merely a tool for generating the sequence $\{a_j\}$. If the sequence $\{a_j\}$ is bounded, a comparison with the geometric series shows that the series (3.5.1) converges at least for $|s|<1$.

Consider now the problem of determining the probability distribution of a sum of nonnegative, independent integral-valued r.v.'s with the probability distributions

$$\Pr[X=j]=a_j \tag{3.5.2a}$$

and

$$\Pr[Y=j]=b_j. \tag{3.5.2b}$$

The event $(X=j, Y=k)$ has probability $a_j b_k$. The sum

$$W = X + Y \tag{3.5.3}$$

is a new r.v., and the event $W=r$ is the union of the mutually exclusive events

$$(X=0, Y=r), \qquad (X=1, Y=r-1), \ldots, (X=r, Y=0). \tag{3.5.4}$$

Consequently, the distribution $c_r = P[W=r]$ is given by

$$c_r = a_0 b_r + a_1 b_{r-1} + a_2 b_{r-2} + \cdots + a_{r-1} b_1 + a_r b_0. \tag{3.5.5}$$

The operation (3.5.5), leading from the two sequences $\{a_k\}$ and $\{b_k\}$ to a new sequence $\{c_k\}$, is called a *convolution* according to the following definition [99, p. 250].

Definition 3.5.2. Let $\{a_k\}$ and $\{b_k\}$ be any two number sequences.[18] The new sequence $\{c_r\}$ defined by (3.5.5) is called the convolution of $\{a_k\}$ and $\{b_k\}$ and is denoted by

$$\{c_k\} = \{a_k\} * \{b_k\}. \tag{3.5.6}$$

The sequences $\{a_k\}$ and $\{b_k\}$ have generating functions

$$A(s) = \sum_k a_k s^k \tag{3.5.7a}$$

and

$$B(s) = \sum_k b_k s^k. \tag{3.5.7b}$$

The product $A(s) \cdot B(s)$ can be obtained by termwise multiplication of the power series for $A(s)$ and $B(s)$. Collecting terms with equal powers of s, one finds that the coefficient c_r of s^r in the expansion of $A(s) \cdot B(s)$ is given by (3.5.5). This result is expressed in Theorem 3.5.1.

Theorem 3.5.1 If $\{a_k\}$ and $\{b_k\}$ are sequences with generating functions $A(s)$ and $B(s)$, and c_k is their convolution, the generating function $C(s) = \sum_k c_k s^k$ is the product

$$C(s) = A(s) B(s). \tag{3.5.8}$$

If X and Y are nonnegative, integral-valued, mutually independent r.v.'s

[18]The sequences $\{a_k\}$ and $\{b_k\}$ are probability distributions usually in our applications, but they need not be.

with generating functions $A(s)$ and $B(s)$, their sum $W=X+Y$ has the generating function $A(s)\,B(s)$.

More generally, if $\{a_k\},\{b_k\},\{c_k\},\{d_k\},\ldots$, are any sequences, one can form the convolution $a_k * b_k$ and then the convolution of this new sequence with c_k, and so on. The generating function of $\{a_k\}*\{b_k\}*\{c_k\}*\{d_k\}$ is $A(s)\,B(s)\,C(s)\,D(s)$, which shows that the order in which the convolutions are performed is immaterial. For example, $\{a_k\}*\{b_k\}*\{c_k\}=\{c_k\}*\{b_k\}*\{a_k\}$, and so on. That is, the convolution is an associative and commutative operation, just as is the summation of r.v.'s.

In the study of the sum

$$W=X_1+X_2+\cdots+X_n \tag{3.5.9}$$

of n i.r.v.'s, the special case in which the X_j have a common distribution is of particular interest and importance. If $\{a_j\}$ is the common probability distribution of the X_j, the probability distribution of the sum (3.5.9) will be denoted by the symbol $\{a_j\}^{n*}$. Then

$$\{a_j\}^{2*}=\{a_j\}*\{a_j\}, \tag{3.5.10a}$$

$$\{a_j\}^{3*}=\{a_j\}^{2*}*\{a_j\}, \tag{3.5.10b}$$

and in general

$$\{a_j\}^{n*}=\{a_j\}^{(n-1)*}*\{a_j\}. \tag{3.5.10c}$$

That is, $\{a_j\}^{n*}$ is the sequence of probabilities whose generating function is $A^n(s)$. Specifically, $\{a_j\}^{1*}$ is the same as $\{a_j\}$, and $\{a_j\}^{0*}$ is defined as the sequence whose generating function is $A^0(s)=1$, that is, the sequence $(1,0,0,0,\ldots)$.

For example, the generating function for the binomial distribution $b(k;n,p)=\binom{n}{k}p^kq^{n-k}$ is

$$\sum_{k=0}^{n}\binom{n}{k}(ps)^kq^{n-k}=(q+ps)^n. \tag{3.5.11}$$

Since this generating function is the nth power of $(q+ps)$, we see that $b(k;np)$ is the distribution of the sum $Z=\sum_{j=1}^{n}X_j$ of n i.r.v.'s with the common generating function $(q+ps)$; each variable X_j assumes the value 0 with probability q and the value 1 with probability p. That is,

$$b(k;n,p)=\{b(k;1,p)\}^{n*} \tag{3.5.12}$$

is the sequence of probabilities that the sum of the number of successes in n Bernoulli trials is $k, k=0,1,2,\ldots,n$.

Example 3.5.1 illustrates the use of convolutions in determining the probability distribution of the sum of i.r.v.'s that do not have a common distribution.

Example 3.5.1. Two types of Bernoulli trial are carried out, in which the probabilities of success in a given trial are, respectively, p_1 and p_2. If $m=2$ Bernoulli trials of type 1 and $n=3$ Bernoulli trials of type 2 are carried out, derive the probability distribution of the sum $W=X+Y$ of the number of successes in $N=5$ independent trials.

From (3.5.5), the desired probability distribution is

$$P[W=r]=c_r$$

$$=\sum_{i=0}^{r} a_i b_{r-i}, \qquad r=0,1,2,\ldots,5, \tag{3.5.13}$$

where

$$a_i=\binom{2}{i} p_1^i q_1^{2-i}, \qquad i=0,1,2, \tag{3.5.14a}$$

$$b_j=\binom{3}{j} p_2^j q_2^{3-j}, \qquad j=0,1,2,3. \tag{3.5.14b}$$

Specifically,

$$c_0=a_0 b_0=q_1^2 q_2^3$$

$$c_1=a_0 b_1+a_1 b_0$$

$$=3p_2 q_1^2 q_2^2+2p_1 q_1 q_2^3$$

$$c_2=a_0 b_2+a_1 b_1+a_2 b_0$$

$$=3p_2^2 q_1^2 q_2+6p_1 p_2 q_1 q_2^2+p_1^2 q_2^3$$

$$c_3=a_0 b_3+a_1 b_2+a_2 b_1$$

$$=q_1^2 p_2^3+6p_1 p_2^2 q_1 q_2+3p_1^2 p_2 q_2^2$$

$$c_4=a_1 b_3+a_2 b_2$$

$$=2p_1 p_2^3 q_1+3p_1^2 p_2^2 q_2$$

$$c_5=a_2 b_3$$

$$=p_1^2 p_2^3.$$

3.6 MOMENT-GENERATING FUNCTIONS

It is clear that Fourier, Laplace, and Mellin transforms all have the properties of a moment-generating function (m.g.f.) of the p.d.f. $f(x)$ of an r.v. X, since

$$E[X^k]=\left[\frac{1}{i^k}\frac{d^k}{dt^k}F_t(f(x))\right]\Bigg|_{t=0}$$

$$E[X^k]=\left[(-1)^k\frac{d^k}{dr^k}L_r(f(x))\right]\Bigg|_{r=0}$$

[where $L_r(f(x))$ is defined by (2.8.2a) and $F_t(f(x))$ by (2.8.5).]

$$E[X^k]=[M_s(f(x))]\big|_{s=k+1}.$$

Furthermore, if these transforms are analytic in the relevant strip (as they usually are), they determine $f(x)$, hence its moments, uniquely.

However the familiar m.g.f. (here denoted by $M_x(t)$) for a p.d.f.[19] $f(x)$ is not an integral transform but is defined to be

$$M_x(t)=\int_{\substack{\text{range}\\ \text{of } X}} e^{tx}f(x)\,dx,$$

where t is a real (continuous) variable. (If t is regarded as a complex variable, then $M_x(t)$ is equivalent to the bilateral Laplace transform (2.8.7b).) The kth moment is then

$$\begin{aligned}
M_x(t)&=E[e^{tx}]\\
&=\int_{-\infty}^{\infty}e^{tx}f(x)\,dx\\
&=\int_{-\infty}^{\infty}\left[1+tx+\frac{(tx)^2}{2!}+\cdots+\frac{(tx)^n}{n!}+\cdots\right]f(x)\,dx\\
&=\int_{-\infty}^{\infty}f(x)\,dx+t\int_{-\infty}^{\infty}xf(x)\,dx+\frac{t^2}{2!}\int_{-\infty}^{\infty}x^2f(x)\,dx\\
&\quad+\cdots+\frac{t^n}{n!}\int_{-\infty}^{\infty}x^nf(x)\,dx+\cdots\\
&=\sum_{j=0}^{\infty}\frac{t^j}{j!}\mu_j'.
\end{aligned}$$

[19]The m.g.f. of a probability mass function $p(x)$ is

$$M_x(t)=\sum_{\text{All } x}e^{tx}p(x).$$

Thus

$$\mu_j' = \left[\frac{\partial^j}{\partial t^j} M_x(t) \right] \Bigg|_{t=0.}$$

Moment-generating functions are sometimes used to show that a statistic or r.v. has a particular p.d.f. For instance, there are some situations in which one can derive rather easily the m.f.g. of a p.d.f. without knowing the p.d.f. itself. If such an m.g.f. is identical with that of a known distribution—say, the chi-square distribution—then one concludes that the heretofore unknown p.d.f. is chi-square. A case in point is the p.d.f. of the sum of the squares of n i.r.v.'s, each having the standardized normal distribution. It is easy to derive the m.g.f. of this p.d.f. and to show that it is identical to that of a chi-square distribution with n degrees of freedom. One concludes, therefore, that the sum of the squares of n such i.r.v.'s has a chi-square distribution with n degrees of freedom. However in utilizing the m.g.f., one must be certain that the conditions specified in its definition are met. In particular, the m.g.f. must exist in an interval $|t| < h$, $h > 0$ [153, 2nd ed, p. 40]. Some examples given by Kotlarski [187] demonstrate some interesting deficiencies—not in the m.g.f. techniques, as he suggested—but in what happens when the conditions of a theorem or technique are not met [272]. The deficiency of the m.g.f. is that it does not always exist for a distribution. Also, as Kotlarski [187] points out, the domain of an m.g.f. depends on the distribution, whereas the domain of all characteristic functions is the same, namely, the real line.

In view of these limitations of the m.g.f., and since m.g.f.'s are commonly used, the following question arises: given that a set of constants are, in fact, the moments of a distribution, can any other distribution have the same set? In other words, does the given set of moments determine the distribution uniquely? This question is not relevant when the moments are determined from an analytic transform, in which case the p.d.f. and its moments are uniquely determined. The question is relevant, however, when one seeks to determine a p.d.f. $f(x)$ from a knowledge of its moments or cumulants as obtained from the moment-generating or cumulant-generating functions. The cumulant κ_k is the coefficient of $(it)^k/k!$ in the power series expansion of log $F_t(f(x))$, if an expression exists [178]. The cumulants bear a definite relation to the moments of a p.d.f. and in general, have properties that are more useful from the theoretical standpoint, as Section 8.10 reveals. At any rate, the answer to the aforementioned question of moments can be found in various theorems, three of which are given below without proof.[20] The proofs may be found in Stuart and Kendall [178, pp. 109–110].

[20]Reproduced by permission of the publishers, Charles Griffin & Company Ltd., of London and High Wycombe, from Kendall and Stuart, *Advanced Theory of Statistics*, Vol. I, 1st ed. 1958; 4th ed. 1977.

Theorem 3.6.1 A set of moments determines a distribution uniquely if the series $\Sigma_{j=0}^{\infty}\nu_j t^j/j!$ converges for some real nonzero t, where ν_j is the absolute jth moment about any origin.

A few simple but effective consequences of this theorem are stated in the following corollaries [178, p. 110].[21]

Corollary 1. The moments uniquely determine the distribution, if the upper limit of $\lim_{n\to\infty}\sqrt[n]{\nu}\ /n$ is finite.

Corollary 2. A sufficient condition that the moments determine a distribution uniquely is that $\lim_{n\to\infty}\sqrt[2n]{\mu'_{2n}}\ /2n$ be finite (a condition that enables one to disregard the absolute moments).

Corollary 3. The moments uniquely determine the distribution, if the range of the distribution is finite.

Two other criteria for answering the moment question, the first due to Carleman [46] and the second to Stieltjes [178, p. 111], are given in the following two theorems, stated here without proof.

Theorem 3.6.2 A set of moments determines a distribution uniquely if (when the range of the distribution is $(-\infty,\infty)$)

$$\sum_{j=0}^{\infty}\frac{1}{(\mu_{2j})^{1/(2j)}}$$

diverges. For the limits 0 to ∞, the corresponding series is

$$\sum_{j=0}^{\infty}\frac{1}{(\mu_{2j})^{1/(2j)}}$$

Theorem 3.6.3 If there exists a p.d.f. $f(x)$, the moments determine it uniquely if, for limits $-\infty$ to $+\infty$ and some fixed x_0,

$$f(x)<M|x|^{\beta-1}\exp(-\alpha|x|^{\lambda})\qquad\text{for}\quad |x|>x_0,\qquad M,\beta,\alpha,>0,\qquad \lambda\geqslant 1$$

and for limits 0 to ∞,

$$f(x)<M|x|^{\beta-1}\exp(-\alpha|x|^{\lambda})\qquad\text{for}\quad |x|>x_0,\qquad M,\beta,\alpha,>0,\qquad \lambda\geqslant \tfrac{1}{2}.$$

[21]Reproduced by permission of the publishers, Charles Griffin & Company Ltd., of London and High Wycombe, from Kendall and Stuart, *Advanced Theory of Statistics*, Vol. I, 1st ed., 1958; 4th ed., 1977.

Example 3.6.1 Show that the normal distribution

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{-x^2}{2\sigma^2}\right), \qquad -\infty<x<\infty$$

is uniquely determined by its moments.

SOLUTION[22] Since the distribution is symmetrical about the origin, all odd order moments are zero (i.e., $\mu'_{2r+1}=0$). It can be shown (e.g., by partial integration) that all even order moments exist; specifically,

$$\begin{aligned}\mu'_{2r}&=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}x^{2r}\exp\left(\frac{-x^2}{2\sigma^2}\right)dx\\&=\frac{\sigma^{2r}}{2^r}\frac{(2r)!}{r!}, \qquad r=0,1,2,\ldots.\end{aligned}$$

Consider, then, the evaluation of

$$\lim_{n\to\infty}\frac{\sqrt[2n]{\mu'_{2n}}}{2n}=\lim_{n\to\infty}\left(\frac{1}{2n}\right)\left(\frac{\sigma^{2n}(2n)!}{2^n n!}\right)^{\frac{1}{2n}} \tag{3.6.1}$$

Using Stirling's formula for evaluating large factorials, one has

$$\lim_{n\to\infty} n!=\sqrt{2\pi n}\, n^n e^{-n},$$

which when used in (3.6.1) yields

$$\begin{aligned}\lim_{n\to\infty}\frac{\sqrt[2n]{\mu'_{2n}}}{2n}&=\lim_{n\to\infty}\frac{\sigma}{n\sqrt{2}}\left[\frac{e^{-2n}(2n)^{2n}\sqrt{4\pi n}}{e^{-n}n^n\sqrt{2\pi n}}\right]^{1/2n}\\&=\lim_{n\to\infty}\frac{\sigma}{\sqrt{2e}\,n^{1/2}}\\&=\lim_{n\to\infty}\frac{\sigma}{\sqrt{2en}}\\&=0.\end{aligned}$$

[22]Reproduced by permission of the publishers, Charles Griffin & Company Ltd., of London and High Wycombe, from Kendall & Stuart, *Advanced Theory of Statistics*, Vol. 1, 1st ed., 1958; 4th ed., 1977.

Hence it follows from Corollary 2 that the normal distribution is uniquely determined by its moments.

It can also be shown that the moments of many of the basic distributions in statistics (e.g., the chi-square, beta, uniform, and Rayleigh distributions) determine the distribution uniquely (see Exercise 3.6). Because of this, the m.g.f. often can be used to advantage, as we now show. First, however, it is helpful to state some important operational rules relevant to the m.g.f. $M_x(t)$ of the p.d.f. $f(x)$:

$$M_x(t) = E[e^{tx}] \qquad \text{(definition of m.g.f.)}$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \qquad \text{(provided this integral exists).}$$

$$M_{cx}(t) = E[e^{tcx}]$$

$$= M_x(ct) \qquad (c = \text{a constant}), \tag{3.6.2}$$

$$M_{x+c}(t) = E[e^{(x+c)t}]$$

$$= e^{ct} M_x(t) \qquad (c = \text{a constant}), \tag{3.6.3}$$

$$M_{\sum_j^n x_j}(t) = E[e^{(x_1 + x_2 + \cdots + x_n)t}]$$

$$= \prod_{j=1}^{n} E[e^{tx_j}] \qquad \text{(assuming the } x_j \text{ are independent),}$$

$$M_{\bar{x}}(t) = M_{(1/n)\Sigma_1^n x_j}(t)$$

$$= \prod_{j=1}^{n} M_{x_j}\left(\frac{t}{n}\right). \tag{3.6.4}$$

Example 3.6.2 Use the m.g.f. to derive the p.d.f. $g(w)$ of the mean $w = \frac{1}{n}\Sigma_{j=1}^n x_j$ of n normal i.r.v.'s with p.d.f.'s

$$f_j(x_j) = \frac{1}{\sigma_j \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x_j - \mu}{\sigma}\right)^2 \right], \qquad -\infty < x_j < \infty.$$

One begins by showing that the normal p.d.f. with mean μ and variance σ^2 has the m.g.f.

$$M_x(t) = e^{\mu t} + \frac{\sigma^2 t^2}{2}.$$

To establish this result, note that, by definition,

$$\begin{aligned} M_x(t) &= E\left[e^{tx}\right] \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx, \end{aligned}$$

which reduces to

$$\begin{aligned} M_x(t) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2 - 2\mu\sigma^2 t - \sigma^4 t^2\right] dx \\ &= \exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right] \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma^2}(x-(\mu+\sigma t))^2\right] dx \\ &= \exp\left[\mu t + \sigma^2 \frac{t^2}{2}\right]. \end{aligned}$$

Since it has previously been proved that a normal distribution is uniquely determined by its moments, it follows that a necessary and sufficient condition that a distribution be normal with mean m and variance v is that its m.g.f. be of the form

$$M_x(t) = \exp\left(mt + v\frac{t^2}{2}\right). \tag{3.6.5}$$

Thus w will be normally distributed, if and only if $g(w)$ has an m.g.f. of the form of (3.6.5). Now, from (3.6.4), $g(w)$ has the m.g.f.

$$\begin{aligned} M_w(t) &= \prod_{j=1}^{n} M_{x_j}(t) \\ &= \prod_{i=1}^{n} \exp\left[\mu_j t + \sigma_j^2 \frac{t^2}{2}\right] \\ &= \exp\left[\left(\sum_{j=1}^{n} \mu_j\right) t + \left(\sum_{1}^{n} \sigma_j^2\right) \frac{t^2}{2}\right]. \end{aligned}$$

which is of the form (3.6.5). Hence $g(w)$ is normally distributed with mean $\Sigma_1^n \mu_j = n\mu$ and variance $\Sigma_1^n \sigma_j^2 = n\sigma^2$.

In concluding this discussion of m.g.f.'s, one should point out that since the Mellin transform generates the moments of a function per se, it is sometimes more convenient to use than the traditional m.g.f. or the Fourier or Laplace transforms, all of which require the use of derivatives. The Mellin transform is also a natural m.g.f. for determining the moments of the distribution of products and quotients of i.r.v.'s, whereas the characteristic function, the Laplace transform, or the "traditional" m.g.f. are the appropriate methods for obtaining the moments of the distribution of sums and differences of i.r.v.'s.

EXERCISES

3.1 State the generating function that generates the probabilities of the number of spots on the upper face of a die when it is tossed once.

3.2 Use the generating function to determine the probabilities of the possible sums of the spots on the upper faces of two fair dice.

3.3 Let X be the number of nickles drawn from a box containing two nickles and two dimes in a draw of two coins. State the generating function for the probabilities $P_r[X=j], j=0,1,2$.

3.4 One of the numbers 2 and 3 is chosen at random with the probability of $\frac{1}{2}$ for each, and one of the numbers 2, 3, 4 is chosen at random with probabilities $\frac{1}{3}$ each. Determine the generating function that generates the probabilities $P_r[S=j], j=4,5,6,7$, where S denotes the sum of the two numbers chosen.

3.5 Let X be the number of trials required to obtain the first success in a sequence of independent Bernoulli trials in which the probability of success at each trial is p (geometric distribution).

(*a*) Find the generating function for the r.v. X.
(*b*) Find $E[X]$.
(*c*) Find the generating function for the r.v. $X+2$.

3.6 Show that the moments of each of the following distributions determine the distribution uniquely.

(*a*) The uniform distribution

$$f(x)=1, \qquad 0 \leqslant x \leqslant 1$$
$$=0, \qquad \text{otherwise.}$$

(*b*) The chi-square distribution with m degrees of freedom:

$$f(x)=\frac{1}{2^{m/2}\Gamma(m/2)}x^{m/2-1}e^{-x/2}, \qquad 0 \leqslant x < \infty$$
$$=0, \qquad \text{otherwise.}$$

(*c*) The beta distribution

$$f(x)=\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}, \qquad 0 \leqslant x \leqslant 1$$
$$=0, \qquad \text{otherwise.}$$

(*d*) The Rayleigh distribution

$$f(x)=xe^{-x^2/2}, \qquad 0 \leqslant x < \infty$$
$$=0, \qquad \text{otherwise.}$$

Hint. (*a*), (*c*) Use Corollary 3 of Section 3.6. (*b*), (*d*) Use the Mellin transform to obtain the moments of the distribution, and then apply any of the theorems or corollaries stated in Section 3.6 that are appropriate.

3.7 Find the moments of the standardized normal distribution

$$f(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \qquad -\infty < x < \infty$$

by means of (*a*) the m.g.f. and (*b*) the Mellin transform.

3.8 Find the moments of the Cauchy distribution

$$f(x)=\frac{1}{\pi(1+x^2)}, \qquad -\infty < x < \infty$$

by using (*a*) the Mellin transform and (*b*) the m.g.f.

3.9 Let x_1, x_2, and x_3 be three r.v.'s with joint p.d.f.

$$f(x_1,x_2,x_3)=\frac{6}{(1+x_1+x_2+x_3)^4}, \qquad x_1>0, \qquad x_2>0, \qquad x_3>0$$
$$=0, \qquad \text{otherwise.}$$

Find the p.d.f. $g(w)$ of the sum $w=x_1+x_2+x_3$.

$$\textit{Ans.} \quad g(w)=\frac{3w^2}{(1+w)^4}, \qquad w>0$$
$$=0 \qquad \text{otherwise.}$$

3.10 Derive the p.d.f. $g(w)$ of the difference $w=x_1-x_2$ of two identically distributed exponential i.r.v.'s each having p.d.f. $f(x_1)=e^{-x_i}$, $x_i \geqslant 0$, $i=1,2$.

$$\textit{Ans.} \quad g(w)=\frac{e^{-w}}{2}, \qquad 0\leqslant w<\infty$$
$$=\frac{e^{w}}{2}, \qquad -\infty<w\leqslant 0.$$

3.11 Show that if for $k=1,2,\ldots,n$, X_k is a Cauchy r.v. with parameter b_k, having the p.d.f.

$$f(x_k)=\frac{1}{\pi b_k\left[1+(x_k/b_k)^2\right]}, \qquad -\infty<x_k<\infty,$$

then

$$W=\sum_{k=1}^{n} X_k$$

is a Cauchy r.v. with parameter $\Sigma_{k=1}^{n} b_k$, where the X_k are assumed to be independent.

3.12 Prove that the p.d.f. $g(w)$ of the sum $w=\Sigma_{j=1}^{n} x_j$ of n identically distributed uniform i.r.v.'s each having p.d.f.

$$f(x_j)=1, \qquad 0\leqslant x_j\leqslant 1$$
$$=0, \qquad \text{elsewhere}$$

is

$$g(w)=\frac{1}{(r-1)!}\left\{w^{r-1}-\binom{r}{1}(w-1)^{r-1}+\binom{r}{2}(w-2)^{r-1}\right.$$

$$\left.-\binom{r}{3}(w-3)^{r-1}+\cdots+\cdots+(-1)^{r-1}\binom{r}{r-1}(w-(r-1))^{r-1}\right\},$$

$$r-1<w<r, \qquad r=1,2,\ldots,n.$$ (Cramer, [73, p. 245])

3.13* Let $x_1, x_2, \ldots, x_n$ be n i.r.v.'s, each of which is distributed uniformly (rectangularly). Without loss of generality, one may assume that the variable x_j, $j=1,2,\ldots,n$ is distributed uniformly between $\pm a_j$ with mean 0, with p.d.f.

$$f_j(x_j)=\frac{1}{2a_j}, \qquad -a_j\leqslant x_j\leqslant a_j$$

$$=0, \qquad |x_j|\geqslant a_j.$$

Prove that the distribution function $F(w)$ of the sum $W=\sum_{j=1}^n X_j$ is

$$F(w)=\sum\frac{(-1)^{s_1+s_2+\cdots+s_n}\left[(w+a_1+a_2+\cdots+a_n)/2-\Omega\right]^n}{n!a_1a_2\ldots a_n},$$

where $\Omega\equiv s_1a_1+s_2a_2+\cdots+s_na_n\leqslant\dfrac{w+a_1+a_2+\cdots+a_n}{2}$

and the summation is to be extended over all $s_1, s_2, \ldots, s_n$ that can only have two values, 0 and 1, till $(w+a_1+a_2+\ldots a_n)/2-\Omega$ is not negative.

(Mitra, 1970)

(The probability distribution of the sum of i.r.v.'s, each of which is distributed uniformly between different ranges, is of both theoretical and practical importance in many branches of science, particularly in numerical analysis.)

3.14* Show that when the number n of uniform i.r.v.'s with p.d.f.'s given in Exercise 3.13 is large, the asymptotic density function $g(w)$ of the sum $W=\sum_{j=1}^n X_j$ is given by

$$g(w)=\left[\frac{3}{2\pi(a_1^2+a_2^2+\cdots+a_n^2)}\right]^{1/2}\exp\left(-\frac{3}{2}\frac{w^2}{a_1^2+\cdots+a_n^2}\right).$$

(Mitra, 1970)

3.15 Find the p.d.f. $g(w)$ of the sum

$$W = X_1 + X_2$$

of an exponential and a beta r.v. having, respectively, the p.d.f.'s

$$\begin{aligned} f_1(x_1) &= e^{-x_1}, \qquad 0 \leqslant x_1 < \infty \\ &= 0, \qquad \text{elsewhere} \\ f_2(x_2) &= 6x_2(1-x_2), \qquad 0 \leqslant x_2 \leqslant 1 \\ &= 0, \quad \text{elsewhere,} \end{aligned}$$

assuming that X_1 and X_2 are independent. Check your result by showing that

$$\int_0^\infty g(w)\,dw = 1$$

and

$$E[W] = E[X_1] + E[X_2].$$

3.16* Determine the p.d.f. $g(w)$ of the sum

$$W_x = X_1 + X_2 + X_3,$$

where X_1, X_2, X_3 are, respectively, uniform, exponential, and chi-square (with 4 degrees of freedom) i.r.v.'s with p.d.f.'s

$$\begin{aligned} f_1(x_1) &= 1, \qquad 0 \leqslant x_1 \leqslant 1 \\ &= 0, \qquad \text{elsewhere} \\ f_2(x_2) &= e^{-x_2}, \qquad 0 \leqslant x_2 < \infty \\ &= 0, \qquad \text{elsewhere} \end{aligned}$$

$$\begin{aligned} f_3(x_3) &= \frac{1}{4} x_3 e^{(-x_3)/2}, \quad 0 \leqslant x_3 < \infty \text{(chi-square distribution} \\ &\qquad\qquad\qquad \text{with 4 degrees of freedom)} \\ &= 0, \qquad\qquad \text{elsewhere.} \end{aligned}$$

Ans. $$\begin{aligned} g(w) &= 1 - e^{-w} - we^{-w/2}, \qquad 0 \leqslant w \leqslant 1 \\ &= e^{-(w-1)} + (w-1)e^{-(w-1)/2} - e^{-w} - we^{-w/2}, \qquad w \geqslant 1 \\ &= 0, \quad \text{elsewhere} \end{aligned}$$

Verify that

$$\int_0^{\infty} g(w)\,dw = 1$$

and

$$E[W] = E[X_1] + E[X_2] + E[X_3].$$

3.17 Find the p.d.f. of the sum of n correlated normal r.v.'s with joint p.d.f. (3.4.20).

3.18 Find the p.d.f. $g(w)$ of the sum $W = X_1 + X_2$ of two i.r.v.'s X_1 and X_2 having p.d.f.'s

$$\begin{aligned} f_1(x_1) &= 1, && -1 \leqslant x_1 \leqslant 0 \\ &= 0 && \text{elsewhere} \\ f_2(x_2) &= e^{-x_2} && 0 \leqslant x_2 < \infty \\ &= 0 && \text{elsewhere.} \end{aligned}$$

$$\textit{Ans.}\quad \begin{aligned} g(w) &= 1 - e^{-(w+1)}, && -1 \leqslant w \leqslant 0 \\ &= e^{-w} - e^{-(w+1)}, && 0 \leqslant w < \infty \end{aligned}$$

3.19 Find the p.d.f. $g(w)$ of the sum $W = X_1 + X_2 + X_3$ of three i.r.v.'s having p.d.f.'s

$$\begin{aligned} f_1(x_1) &= 1, && -1 \leqslant x_1 \leqslant 0 \\ &= 0, && \text{elsewhere} \\ f_2(x_2) &= e^{-x_2}, && 0 \leqslant x_2 < \infty \\ &= 0, && \text{elsewhere} \\ f_3(x_3) &= \frac{1}{\sqrt{2\pi}} e^{-x_3^2}, && -\infty < x_3 < \infty \end{aligned}$$

Verify that $\int_{-\infty}^{\infty} g(w)\,dw = 1$ and that

$$E[W] = E[X_1] + E[X_2] + E[X_3].$$

3.20 Show that (3.2.5) and (3.2.5a) are equivalent. (See [178, pp. 94–97].)

CHAPTER 4

The Distribution of Products and Quotients of Random Variables

4.1 THE MELLIN CONVOLUTION AND ITS RELATION TO PRODUCT DISTRIBUTIONS

The importance of the Mellin integral transform in studying the p.d.f. of products of i.r.v.'s is not surprising when one notes that the p.d.f. of the product $Y=X_1X_2$ of two nonnegative i.r.v.'s with p.d.f.'s $f_1(x_1)$ and $f_2(x_2)$ is expressible as a Mellin convolution, namely,

$$h_2(y)=\int_0^\infty \frac{1}{x_2} f_1\left(\frac{y}{x_2}\right) f_2(x_2)\,dx_2 \tag{4.1.1a}$$

$$=\int_0^\infty \frac{1}{x_1} f_1(x_1) f_2\left(\frac{y}{x_1}\right) dx_1. \tag{4.1.1b}$$

To establish this fact, consider the transformation of variables

$$Y=X_1X_2, \qquad X_2=X_2, \tag{4.1.2}$$

which, on solving inversely for X_1 and X_2, yield

$$X_1=\frac{Y}{X_2}, \qquad X_2=X_2. \tag{4.1.3}$$

The transformation (4.1.2) transforms the joint p.d.f. $f(x_1,x_2)=f_1(x_1)f_2(x_2)$ into $g(y,x_2)$, where

$$g(y,x_2)=f_1\left(\frac{y}{x_2}\right) f_2(x_2)|J|, \tag{4.1.4}$$

J being the Jacobian of the inverse transformation (4.1.3):

$$J=\begin{vmatrix} \dfrac{\partial x_1}{\partial y} & \dfrac{\partial x_1}{\partial x_2} \\ \dfrac{\partial x_2}{\partial y} & \dfrac{\partial x_2}{\partial x_2} \end{vmatrix}$$

$$=\begin{vmatrix} \dfrac{1}{x_2} & -\dfrac{y}{x_2^2} \\ 0 & 1 \end{vmatrix}$$

$$=\frac{1}{x_2}. \tag{4.1.5}$$

Integrating out the variable x_2 in (4.1.4), one obtains the p.d.f. of y, namely,

$$h_2(y)=\int_0^\infty \frac{1}{x_2} f_1\left(\frac{y}{x_2}\right) f_2(x_2)\,dx_2, \tag{4.1.6}$$

which is precisely the Mellin convolution of $f_1(x_1)$ and $f_2(x_2)$. That is, the Mellin convolution of two p.d.f.'s $f_1(x_1)$ and $f_2(x_2)$ is precisely the p.d.f. $h(y)$ of the product r.v. $Y=X_1X_2$.

Similarly, the p.d.f. of the quotient $Y=X_1/X_2=(X_1)(1/X_2)$ of two nonnegative i.r.v.'s with p.d.f.'s $f_1(x_1)$ and $f_2(x_2)$ is expressible as the Mellin convolution

$$h_2(y)=\int_0^\infty x_2 f_1(yx_2) f_2(x_2)\,dx_2 \tag{4.1.7}$$

of $f_1(x_1)$ and $g_2(1/x_2)$. As in the case of products, this can be established by utilizing a transformation, specifically,

$$Y=\frac{X_1}{X_2}, \qquad X_2=X_2, \tag{4.1.8}$$

the inverse of which is

$$X_1=YX_2, \qquad X_2=X_2. \tag{4.1.9}$$

Since the Jacobian of the transformation (4.1.9) is

$$J=\begin{vmatrix} \dfrac{\partial x_1}{\partial y} & \dfrac{\partial x_1}{\partial x_2} \\ \dfrac{\partial x_2}{\partial y} & \dfrac{\partial x_2}{\partial x_2} \end{vmatrix}$$

$$=\begin{vmatrix} x_2 & y \\ 0 & 1 \end{vmatrix}$$

$$=x_2, \tag{4.1.10}$$

the joint p.d.f. $f(x_1,x_2)=f_1(x_1)f_2(x_2)$ is transformed into $g(y,x_2)$, where

$$\begin{aligned} g(y,x_2)&=f_1(yx_2)f_2(x_2)|J| \\ &=x_2 f_1(y,x_2)f_2(x_2). \end{aligned} \tag{4.1.11}$$

On integrating (4.1.11) with respect to x_2, one obtains the Mellin convolution

$$\begin{aligned} h_2(y)&=\int_0^\infty g(y,x_2)\,dx_2 \\ &=\int_0^\infty x_2 f_1(yx_2)f_2(x_2)\,dx_2, \end{aligned} \tag{4.1.12}$$

which is the p.d.f. of the quotient r.v. $Y=X_1/X_2$. That the p.d.f. $h_2(y)$ in (4.1.12) is in fact the Mellin convolution of $f_1(x_1)$ and $g_2(1/x_2)$ perhaps becomes more apparent if one expresses (4.1.12) in the equivalent form

$$h_2(y)=\int_0^\infty \frac{1}{(1/x_2)} f_1\left(\frac{y}{1/x_2}\right) f_2(x_2)\,dx_2. \tag{4.1.13}$$

The variable x_2 in both (4.1.12) and (4.1.13) may, of course, be interchanged with the variable x_1 without affecting the convolution of the pair of functions involved.

As specific examples of Mellin convolutions, consider the p.d.f.'s of the quotient and product of two independent and identically distributed uniform r.v.'s X_1 and X_2, where

$$f_i(x_i)=1, \qquad 0<x_i<1, \qquad i=1,2. \tag{4.1.14}$$

For the product convolution, one has, from (4.1.1a)

$$h_2(y)=\int_0^\infty \frac{1}{x_2} f_1\left(\frac{y}{x_2}\right) f_2(x_2)\,dx_2. \tag{4.1.15}$$

Note that the lower limit of integration of x_2 is determined from the relation

$$x_1=\frac{y}{x_2}, \qquad 0 \leqslant x_1 \leqslant 1, \qquad 0 \leqslant x_2 \leqslant 1.$$

That is, since $x_1 \leqslant 1$, the lower limit on x_2 is y. Also, since $x_1 \leqslant 1$ and $x_2 \leqslant 1$, the maximum value of $y=x_1x_2$ is 1. Hence the value of the integrand in (4.1.15) is zero for $x_2<y$ and for $x_2>1$, so that (4.1.15) becomes

$$\begin{aligned} h_2(y) &= \int_y^1 \frac{dx_2}{x_2} \\ &= (\ln x_2)\big|_y^1 \\ &= \ln 1 - \ln y \\ &= -\ln y, \qquad 0 \leqslant y \leqslant 1. \end{aligned} \tag{4.1.16}$$

Similarly, for the quotient convolution, one has, from (4.1.12),

$$h_2(y)=\int_0^\infty x_2 f_1(yx_2) f_2(x_2)\,dx_2. \tag{4.1.17}$$

Since $y=x_1/x_2$, it follows that $x_2 \leqslant x_1/y = x_1(1/y)$. That is,

$$x_2 \leqslant \frac{1}{y}, \tag{4.1.18}$$

since $x_1 \leqslant 1$. However (4.1.18) is valid only if $y \geqslant 1$, since when $y<1$, (4.1.18) violates the necessary constraint $x_2 \leqslant 1$. If $y<1$, (4.1.18) must be replaced by the inequality

$$x_2 \leqslant 1. \tag{4.1.19}$$

Furthermore, since x_1 and x_2 are nonnegative, the integrand in (4.1.17) is zero when $x_2 < 0$. Consequently, the p.d.f. (4.1.17) has two components:

$$h_2(y) = \int_0^1 x_2\,dx_2$$

$$= \frac{1}{2}, \qquad 0 \leqslant y \leqslant 1 \tag{4.1.20a}$$

and

$$h_2(y) = \int_0^{1/y} x_2\,dx_2$$

$$= \frac{1}{2y^2}, \qquad y \geqslant 1. \tag{4.1.20b}$$

Thus the convolution of the quotient of two identically distributed uniform r.v.'s with p.d.f. (4.1.14) has the two components (4.1.20a, b) that hold when $y \geqslant 1$ and $0 \leqslant y \leqslant 1$, respectively. Figure 4.1.1 shows the convolution graphically.

It has been shown that the Mellin convolution of $f_1(x_1)$ and $f_2(x_2)$ is precisely the p.d.f. of the product r.v. $Y = X_1 X_2$. By $n-1$ repeated convolutions of the p.d.f.'s, one obtains the p.d.f. of the product $Y = \prod_{i=1}^{n} X_i$ of n i.r.v.'s X_i, as a subsequent section of this chapter proves. It is also shown that repeated convolution of the p.d.f.'s is equivalent in the transform to successive multiplication of the Mellin integral transforms—which considerably simplifies the derivation of the distribution of products of i.r.v.'s.

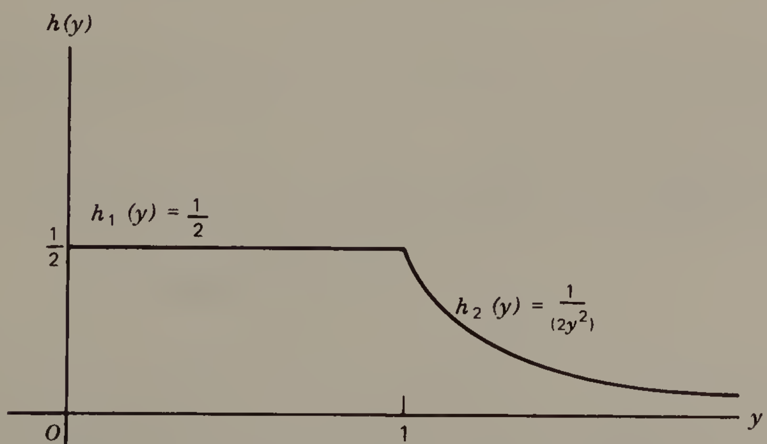


Fig. 4.1.1 Graphical representation of the p.d.f. of the quotient of two identically distributed uniform r.v.'s.

4.2 THE MELLIN INTEGRAL TRANSFORM AND INVERSION INTEGRAL

As we see shortly, the Mellin integral transform and its associated inversion integral play a fundamental role in the derivation of p.d.f.'s of products, quotients, and—more generally—algebraic functions of i.r.v.'s. Although the dual relationship between an integral transform and the corresponding inversion integral for a Mellin transform pair has been previously stated, for the benefit of the reader and for convenience in future reference, the result is now stated as a theorem.

Theorem 4.2.1. (The Mellin Transform and Inversion Theorem). If $f(x)$ is a real function that is defined and single valued almost everywhere for $x \geqslant 0$ and is absolutely integrable over the range $(0, \infty)$, then the Mellin transform

$$M_s(f(x)) = \int_0^\infty x^{s-1} f(x)\, dx \tag{4.2.1}$$

exists [347, p. 41]. Conversely, if the Mellin transform exists and is an analytic function of the complex variable s for $c_1 \leqslant \mathrm{Re}(s) \leqslant c_2$, where c_1 and c_2 are real, then the inversion integral

$$\frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{w-i\beta}^{w+i\beta} x^{-s} M_s(f(x))\, ds \tag{4.2.2}$$

evaluated along any line $c_1 \leqslant \mathrm{Re}(s) = w \leqslant c_2$ converges to the function $f(x)$ independently of w [347, p. 42].

Hereafter, the Mellin inversion integral (4.2.2) is written in the more convenient form:

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} M_s(f(x))\, ds. \tag{4.2.3}$$

As in the case of Fourier and Laplace inversion integrals, the line $(c - i\infty, c + i\infty)$ is referred to as the Bromwich path. The integrals (4.2.2) and (4.2.3) are important because under the stated conditions, a function $f(x)$ is uniquely determined by its Mellin transform $M_s(f(x))$, so that $f(x)$ can be obtained from a knowledge of its Mellin transform by means of (4.2.3).

As has already been shown, the Mellin convolution of two functions $f_1(x_1)$, $f_2(x_2)$, $0 \leqslant x_1 < \infty$, $0 \leqslant x_2 \leqslant \infty$, namely,

$$h_2(y) = \int_0^\infty \frac{1}{x_2} f_1\left(\frac{y}{x_2}\right) f_2(x_2)\,dx_2, \tag{4.2.4}$$

is the p.d.f. of the product $Y = X_1 X_2$ of the i.r.v.'s X_1 and X_2 with p.d.f.'s $f_1(x_1)$ and $f_2(x_2)$, respectively. Note also that by the definition (4.2.1) the Mellin transform of the product density function $h_2(y)$ is

$$\begin{aligned} M_s(h_2(y)) &= E[y^{s-1}] \\ &= E[(x_1 x_2)]^{s-1} \\ &= E[x_1^{s-1}]E[x_2^{s-1}] \\ &= M_s(f(x_1)) M_s(f(x_2)). \end{aligned} \tag{4.2.5}$$

Hence the p.d.f. $h_2(y)$ of the product $Y = X_1 X_2$ of two i.r.v.'s with p.d.f.'s $f_1(x_1)$ and $f_2(x_2)$ is the Mellin convolution whose Mellin transform is the product of the Mellin transforms of $f_1(x_1)$ and $f_2(x_2)$. Therefore, by Theorem 4.2.1, the Mellin convolution $h_2(y)$ may also be obtained by evaluating the inversion integral

$$h_2(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^{-s} \prod_{j=1}^{2} M_s(f_j(x_j))\,ds, \tag{4.2.6}$$

a fact which is exploited in deriving the p.d.f. of products of i.r.v.'s in the section that follows.

4.3 THE DISTRIBUTION OF PRODUCTS AND QUOTIENTS OF INDEPENDENT NONNEGATIVE RANDOM VARIABLES

The Mellin convolution (4.2.6) may be extended to include the p.d.f. $h_n(y)$ of the product $Y = X_1 X_2 \cdots X_n$ of n nonnegative i.r.v.'s. Thus by (4.2.4) the p.d.f. $h_3(y)$ of the product $Y = X_1 X_2 X_3$ of three nonnegative i.r.v.'s is the Mellin convolution

$$h_3(y) = \int_0^\infty \frac{1}{x_3} h_2\left(\frac{y}{x_3}\right) f_3(x_3)\,dx_3, \tag{4.3.1}$$

whose Mellin transform, from (4.2.5), is

$$\begin{aligned} M_s(h_3(y)) &= M_s(h_2(y))M_s(f_3(x_3)) \\ &= M_s(f_1(x_1))M_s(f_2(x_2))M_s(f_3(x_3)). \end{aligned} \tag{4.3.2}$$

Clearly $(n-1)$ successive applications of (4.2.4) and (4.2.5) for nonnegative i.r.v.'s leads to the general results

$$h_n(y)=\int_0^\infty \frac{1}{x_n} h_{n-1}\left(\frac{y}{x_n}\right) f_n(x_n)\,dx_n \tag{4.3.3}$$

and

$$M_s(h_n(y))=\prod_{i=1}^{n} M_s(f_i(x_i)). \tag{4.3.4}$$

Thus the p.d.f. $h_n(y)$ of the product of n i.r.v.'s can be obtained by evaluating (4.3.3) as an $(n-1)$-step Mellin convolution. Since the Mellin transform of $h_n(y)$ is given by (4.3.4), however, it follows from the inversion Theorem 4.2.1 that (4.3.3) is also equivalent to

$$h_n(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}\prod_{i=1}^{n} M_s(f_i(x_i))\,ds, \qquad c_1<c<c_2, \tag{4.3.5}$$

provided the $M_s(f_i(x_i))$, $i=1,2,\ldots,n$ are analytic in a strip $c_1<\mathrm{Re}(s)<c_2$, where c, c_1, and c_2 are real. In other words, $h_n(y)$ can be obtained directly from a knowledge of the Mellin transforms of the component r.v.'s by way of (4.3.5). In fact, since the evaluation of (4.3.5) is almost always easier, simpler, and quicker than the evaluation of the $(n-1)$-fold convolution (4.3.3), the p.d.f. $h_n(y)$ is usually derived from (4.3.5). Therein lies the utility of the Mellin integral transform in determining the p.d.f.'s of products of i.r.v.'s.

It should be pointed out that (4.3.4) is an immediate consequence of the definition (4.2.1) of the Mellin transform. For if $h(y)$ is the p.d.f. of $Y=\prod_{i=1}^{n}X_i$, where the X_i are i.r.v.'s with p.d.f.'s $f_i(x_i)$, then by the definition (4.2.1),

$$\begin{aligned} M_s(h(y)) &= E[y^{s-1}] = E[(x_1x_2\cdots x_n)^{s-1}] \\ &= E[(x_1^{s-1}x_2^{s-1}\cdots x_n^{s-1})] \\ &= \prod_{i=1}^{n} E[x_i^{s-1}] = \prod_{i=1}^{n} M_s(f(x_i)). \end{aligned}$$

The distribution function $H(y)$, defined as

$$H(y)=\int_0^y h(x)\,dx, \qquad 0\leqslant y<\infty, \tag{4.3.6}$$

can be obtained by direct integration in (4.3.6) or through use of the Mellin transform of $h(y)$. More specifically,

$$M_s(1-H(y))=s^{-1}M_{s+1}(h(y)), \tag{4.3.7}$$

so that

$$1-H(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}s^{-1}M_{s+1}(h(y))\,ds. \tag{4.3.8}$$

To establish (4.3.8), assume that the derivative $H'(y)=(d/dy)\mathrm{H}(y)=h(y)$ exists for $0<y<\infty$, and let

$$G(y)=1-H(y). \tag{4.3.9}$$

Then from the well-known relationship [95 (9), p. 307],

$$M_s(G'(y))=-(s-1)M_{s-1}(G(y)), \tag{4.3.10}$$

it follows that

$$-M_s(H'(y))=-(s-1)M_{s-1}(G(y)),$$

hence that

$$M_{s+1}(H'(y))=sM_s(G(y)). \tag{4.3.10a}$$

Or equivalently,

$$M_s(1-H(y))=\frac{M_{s+1}(h(y))}{s}. \tag{4.3.10b}$$

Inversion of the Mellin transform (4.3.10b) yields

$$H(y)=1-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}\frac{1}{s}M_{s+1}(h(y))\,ds, \tag{4.3.11}$$

which is identical with (4.3.8).

The Mellin integral transform may also be used to obtain the p.d.f. $h(y)$ of the quotient $Y=X_1/X_2$ of two nonnegative i.r.v.'s X_1 and X_2 with p.d.f.'s $f_1(x_1)$ and $f_2(x_2)$, respectively. To derive this quotient density function, write Y as the product of X_1 and $1/X_2$ and note that if W has the p.d.f. $f(w)$, and $U=W^a$, where a is a real number, has p.d.f. $g(u)$, then by definition the Mellin transform of the r.v. U is

$$M_s(g(u))=E[u^{s-1}]$$

$$=E[(w^a)^{s-1}]$$

$$=\int_0^\infty w^{a(s-1)} f(w)\,dw,$$

or equivalently,

$$M_s(g(u))=M_{as-a+1}(f(w)). \tag{4.3.12}$$

In particular, if $a=-1$, (4.3.12) states that the Mellin transform of the p.d.f. $g(1/w)$ of the reciprocal r.v. $1/W$ is the Mellin transform of $f(w)$ with s replaced by $-s+2$. Hence

$$M_s(h(y))=M_s(f_1(x_1))M_s(g(1/x_2)) \tag{4.3.13a}$$

$$=M_s(f_1(x_1))M_{-s+2}(f(x_2)), \tag{4.3.13b}$$

so that

$$h(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}M_s(f_1(x_1))M_{-s+2}(f_2(x_2))\,ds. \tag{4.3.14}$$

Finally,

$$H(y)=1-\int_{c-i\infty}^{c+i\infty}\left(\frac{y^{-s}}{s}\right)M_{s+1}(f_1(x_1))M_{-s+1}(f_2(x_2))\,ds. \tag{4.3.15}$$

4.4 THE DISTRIBUTION OF PRODUCTS AND QUOTIENTS OF CONTINUOUS STANDARDIZED NONNEGATIVE RANDOM VARIABLES: SPECIFIC CASES

As Chapter 3 pointed out, a change of origin and unit poses no problem in the analysis of sums of i.r.v.'s inasmuch as such changes can be absorbed in the Fourier or Laplace transforms. This is not the case, however, in the analysis of products of i.r.v.'s. A change in the unit or scale factor is reflected in the Mellin transform as indicated by the scaling property of Section 2.8.2, but there is no similar relationship enabling one to absorb in the Mellin transform the effect of a change or shift in the origin. For this reason, r.v.'s that are not everywhere positive—for example, identically distributed nonstandardized normal i.r.v.'s $N(\mu,\sigma)$[23]—and nonstandardized i.r.v.'s (regardless of whether they are nonnegative) must be treated separately. For example, the class of nonstandardized r.v.'s would include products of uniform r.v.'s with different ranges (a_i,b_i), $a_i>0$, $b_i>0$, as well as noncentral r.v.'s that are necessarily of doubly infinite range $(-\infty,\infty)$. This section discusses only the distribution of products and quotients of some important standardized continuous nonnegative r.v.'s.

4.4.1 Products of Independent Uniform Random Variables

Consider the product $Y=\prod_{i=1}^{n}X_i$ of n identically distributed uniform i.r.v.'s, each having p.d.f.

$$\begin{aligned} f(x_i)&=1, \qquad 0\leqslant x_i\leqslant 1, \qquad i=1,2,\ldots,n\\ &=0, \qquad \text{otherwise.} \end{aligned} \tag{4.4.1}$$

Let $h(y)$ denote the p.d.f. of the r.v. Y. Then

$$\begin{aligned} M_s(f(x_i))&=\int_0^1 x_i^{s-1}\,dx_i\\ &=\frac{1}{s} \end{aligned} \tag{4.4.2}$$

and

$$M_s(h(y))=\left(\frac{1}{s}\right)^n, \tag{4.4.3}$$

so that

$$\begin{aligned} h(y)&=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}(M_s(f(y)))^n\,ds\\ &=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^{-s}}{s^n}\,ds, \qquad c>0. \end{aligned} \tag{4.4.4}$$

[23]The symbol $N(\mu,\sigma)$ denotes a normal distribution with mean μ and standard deviation σ.

Appendix B establishes that the integral (4.4.4) satisfies the conditions of Jordan's lemma, hence may be evaluated by the residue theorem. That is,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^{-s}}{s^n}\,ds = \text{sum of residues at the poles}$$
$$= R_0,$$

where R_0 denotes the residue at the only pole, namely, the pole at $s=0$. Since it is a pole of order n,

$$R_0 = \frac{1}{(n-1)!}\,\frac{d^{n-1}}{ds^{n-1}}\left(s^n\frac{y^{-s}}{s^n}\right)\bigg|_{s=0}$$
$$= \frac{1}{(n-1)!}\left(y^{-s}\left(\ln\frac{1}{y}\right)^{n-1}\right)\bigg|_{s=0}$$
$$= \frac{(\ln 1/y)^{n-1}}{(n-1)!}.$$

Thus

$$h(y) = \frac{(\ln 1/y)^{n-1}}{(n-1)!}, \qquad 0<y\leqslant 1.$$

The p.d.f. $h(y)$ of the quotient $Y=X_1/X_2$ of two uniform i.r.v.'s each having the p.d.f. (4.4.1) has previously been derived as a Mellin convolution. It is now obtained by evaluating the relevant Mellin inversion integral (4.3.14). From (4.3.13b), we have

$$M_s(h(y)) = M_s(f(x_1))M_{-s+2}(f(x_2))$$
$$= \left(\frac{1}{s}\right)\left(\frac{1}{-s+2}\right), \tag{4.4.5}$$

so that $h(y)$ may be obtained by evaluating the inversion integral

$$h(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}\left(\frac{1}{s}\right)\left(\frac{1}{-s+2}\right)ds. \tag{4.4.6}$$

Appendix B shows that as a consequence of Jordan's lemma

$$\begin{aligned} h(y) &= h_1(y), \qquad 0 \leqslant y \leqslant 1 \\ &= h_2(y), \qquad 1 \leqslant y < \infty \end{aligned}$$

where

$$\begin{aligned} h_1(y) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^{-s}}{s(-s+2)}\, ds, \qquad 0 \leqslant y \leqslant 1 \\ &= \frac{1}{2\pi i} \int_{QKLMPQ} \frac{y^{-s}}{s(-s+2)}\, ds, \qquad 0 \leqslant y \leqslant 1 \end{aligned} \tag{4.4.7}$$

and

$$\begin{aligned} h_2(y) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^{-s}}{s(-s+2)}\, ds, \qquad 1 \leqslant y < \infty \\ &= \frac{1}{2\pi i} \int_{QPTQ} \frac{y^{-s}}{s(-s+2)}\, ds. \end{aligned} \tag{4.4.8}$$

In (4.4.7) the contour $C_L = QKLMPQ$ is a circular arc (Fig. 2.9.1a) with infinite chord having end points $c-ia$, $c+ia$, $a \to \infty$, and enclosing the origin $s=0$, where c is any value between 0 and 2 and the integration is carried out in the positive direction. In (4.4.8), the contour $C_R = QPTQ$ is again a circular arc with infinite chord having end points $c-ia$, $c+ia$, $a \to \infty$, this time enclosing the point $s=2$ (Fig. 2.9.1b). Jordan's lemma dictates (as shown in Appendix B) the contours in Fig. 2.9.1 and furthermore ensures that the contribution to the intervals (4.4.7) and (4.4.8) from the arcs $QKLMP$ and QPT, respectively, is zero, so that the residue theorem may be applied to obtain the values of the two integrals over the Bromwich path $(c-i\infty, c+i\infty)$. Specifically, (4.4.7) becomes

$$\begin{aligned} h_1(y) &= \frac{1}{2\pi i}\left[\int_{QKLMP} \frac{y^{-s}}{s(-s+2)}\, ds + \int_{PQ} \frac{y^{-s}}{s(-s+2)}\, ds\right] \\ &= 0 + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^{-s}}{s(-s+2)}\, ds \\ &= \frac{1}{2\pi i}(2\pi i) \left.\frac{y^{-s}}{-s+2}\right|_{s=0} \end{aligned}$$

or

$$h_1(y)=\tfrac{1}{2}, \qquad 0\leqslant y\leqslant 1. \tag{4.4.9}$$

Similarly, (4.4.8) becomes (Fig. 2.9.1b)

$$h_2(y)=\frac{1}{2\pi i}\left[\int_{PTQ}\frac{y^{-s}}{s(-s+2)}\,ds+\int_{QP}\frac{y^{-s}}{s(-s+2)}\,ds\right].$$

$$=\frac{1}{2\pi i}\left[\lim_{R\to\infty}\int_{QTP}\frac{y^{-s}}{s(s-2)}\,ds+\lim_{R\to\infty}\int_{PQ}\frac{y^{-s}}{s(s-2)}\right]ds. \tag{4.4.10}$$

From Jordan's lemma and the left-hand rule, it follows that

$$h_2(y)=0+\frac{1}{2\pi i}(2\pi i)\frac{y^{-s}}{s}\bigg|_{s=2}$$

$$=\frac{y^{-2}}{2}, \qquad 1\leqslant y<\infty.$$

These results necessarily agree with those obtained when the p.d.f. of the quotient $Y=X_1/X_2$ was derived as a Mellin convolution.

4.4.2 Products of Independent Beta Random Variables

Equations (6.3.7) and (6.4.7) show that the p.d.f. of the product of n beta i.r.v.'s having parameters a_i and b_i and p.d.f. (4.4.11) is an H-function with p.d.f.

$$h(y)=\prod_{j=1}^{n}\frac{\Gamma(a_j+b_j)}{\Gamma(a_j)}\mathbf{H}_{n,n}^{n,0}\left[y\left|\begin{matrix}(a_1+b_1-1,1),\ldots,(a_n+b_n-1,1)\\(a_1-1,1),\ldots,(a_n-1,1)\end{matrix}\right.\right], \qquad y>0$$

$$=0, \qquad \text{elsewhere.}$$

Or equivalently, since the Meijer G-function is a special case of the H-function,

$$h(y)=\prod_{j=1}^{n}\frac{\Gamma(a_j+b_j)}{\Gamma(a_j)}\mathbf{G}_{n,n}^{n,0}\left[y\left|\begin{matrix}a_1+b_1-1,\ldots,a_n+b_n-1\\a_1-1,\ldots,a_n-1\end{matrix}\right.\right].$$

And since $h(y)$ is an H-function r.v., it can be obtained in the exact series form given by (7.1.2), which is valid regardless of whether the parameters a_j,b_j are integers.

The derivation of the distribution of beta i.r.v.'s with p.d.f. (4.4.11) and integer parameters is considerably simpler than that for beta variables with noninteger parameters. Nevertheless, the derivation of this distribution illustrates the analytical nature and structure of product distributions and is carried out directly, rather than extracted from the general p.d.f. (7.1.1), of which it is a special case. The result is stated in the form of a theorem.

Theorem 4.4.1. The p.d.f. $h(y)$ of the product $Y=\prod_{y=1}^{n} X_i$ of n independent beta variables of the first kind

$$f_i(x_i)=\frac{1}{B(a_i,b_i)}x_i^{a_i-1}(1-x_i)^{b_i-1} \tag{4.4.11}$$

with integral parameters a_i, b_i is expressible in closed form as

$$h(y)=\sum_{k=1}^{m}\sum_{j=0}^{e_k-1}\frac{K_{kj}y^{d_k-1}(-\ln y)^{e_k-j-1}}{(e_k-1-j)!j!},$$

where

$$K_{k0}=\sum_{\substack{q=1\\ q\neq k}}^{m}(d_q-d_k)^{-e_q},$$

$$K_{kj}=\sum_{r=0}^{j-1}\sum_{\substack{q=1\\ q\neq k}}^{m}(-1)^{r+1}\frac{\binom{j-1}{r}r!e_q}{(d_q-d_k)^{r+1}}K_{k,j-1-r}, \qquad j>0,$$

and d_k denotes the m different integers that occur with multiplicity e_k among the $a_i-1, a_i, a_i+1, \ldots, a_i+b_i-2$ for $i=1,2,\ldots,n$.

PROOF. Consider n beta variables of the first kind with integral parameters a_i, b_i with p.d.f. (4.4.11). Since the Mellin integral transform of the p.d.f. (4.4.11) is

$$\begin{aligned} M(f_i(x)|s) &= \frac{B(s+a_1-1,b_i)}{B(a_i,b_i)} \\ &= \frac{\Gamma(a_i+b_i)\Gamma(s+a_i-1)}{\Gamma(a_i)\Gamma(s+a_i-1+b_i)}, \end{aligned} \tag{4.4.12}$$

it follows that

$$M_s(h(y))=\prod_{i=1}^{n}\frac{\Gamma(a_i+b_i)}{\Gamma(a_i)(s+a_i-1)(s+a_i)\cdots(s+a_i-2+b_i)} \tag{4.4.13}$$

and

$$h(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}M_s(h(y))\,ds. \tag{4.4.14}$$

The problem lies in evaluating the integral (4.4.14) in closed form. To simplify the evaluation, one can, without loss of generality, rewrite (4.4.13) in a form that shows the location and order of the distinct poles, namely,

$$M_s(h(y))=\prod_{i=1}^{n}\frac{(a_i+b_i-1)!}{(a_i-1)!(s+d_1-1)^{e_1}(s+d_2-1)^{e_2}\cdots(s+d_m-1)^{e_m}}, \tag{4.4.15}$$

where $(a_i-1)!=\Gamma(a_i)$, $(a_i+b_i-1)!=\Gamma(a_i+b_i)$, $d_k\neq d_j$, $k\neq j$, and there occur m distinct integers $d_1<d_2<\cdots<d_m$ of multiplicity $e_1,e_2,\ldots,e_m$, respectively, among the $a_i-1,a_i,\ldots,a_i+b_i-2$, for $i=1,2,\ldots,n$. If $M_s(h(y))$ as given by (4.4.15) is now substituted into the integral (4.4.14), the resultant inversion integral may be evaluated by the method of residues, since the conditions of Jordan's lemma are satisfied.[24] Specifically,

$$\begin{aligned} h(y)&=\left(\prod_{i=1}^{n}\frac{(a_i+b_i-1)!}{(a_i-1)!}\right)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^{-s}\,ds}{(s+d_1-1)^{e_1}\cdots(s+d_m-1)^{e_m}}\\ &=\left(\prod_{i=1}^{n}\frac{(a_i+b_i-1)!}{(a_i-1)!}\right)\sum_{k=1}^{m}R_k, \end{aligned}$$

where R_k is the residue at the kth pole of order e_k, $k=1,2,\ldots,m$, namely,

$$R_k=\frac{1}{(e_k-1)!}\left(\frac{d^{e_k-1}}{ds^{e_k-1}}(s+d_k-1)^{e_k}y^{-s}\prod_{q=1}^{m}(s+d_q-1)^{-e_q}\right)\Bigg|_{s=-(d_k-1)}. \tag{4.4.16}$$

Using Leibniz's rule for differentiation of products, one can write (4.4.16) in the form

$$R_k=y^{-s}\sum_{j=0}^{e_k-1}\frac{(-\ln y)^{e_k-1-j}}{(e_k-1-j)!j!}W^{(j)}(s)\Bigg|_{s=-(d_k-1)}, \tag{4.4.17}$$

[24]Chapter 6 shows that this is an H-function inversion integral, and Appendix F proves that the conditions of Jordan's lemma always hold for any H-function inversion integral.

where

$$W(s) = \prod_{\substack{q=1\\ q\neq k}}^{m} (s+d_q-1)^{-e_q}.$$

Differentiation of $\ln W(s)$ yields

$$W^{(1)}(s) = -W(s) \sum_{\substack{q=1\\ q\neq k}}^{m} \frac{e_q}{s+d_q-1},$$

to which Leibniz's rule for differentiation of products may be applied to obtain

$$W^{(j)}(s)\big|_{s=-(d_k-1)} = K_{kj}, \tag{4.4.18}$$

$$K_{k0} = \prod_{\substack{q=1\\ q\neq k}}^{m} (d_q-d_k)^{-e_q}, \tag{4.4.19}$$

$$K_{kj} = \sum_{r=0}^{j-1} \sum_{\substack{q=1\\ q\neq k}}^{m} (-1)^{r+1}\binom{j-1}{r}\frac{r!\,e_q K_{k,j-1-r}}{(d_q-d_k)^{r+1}}, \qquad j>0, \tag{4.4.20}$$

and $W^{(j)}(s)$ denotes the jth derivative of $W(s)$. Hence from (4.4.17) and (4.4.18) it follows that

$$h(y) = \left(\prod_{i=1}^{n} \frac{(a_i+b_i-1)!}{(a_i-1)!}\right) \sum_{k=1}^{m} \times \sum_{j=0}^{e_k-1} \frac{K_{kj}\, y^{d_k-1}(-\ln y)^{e_k-1-j}}{(e_k-1-j)!\,j!}. \tag{4.4.21}$$

The same result could have been obtained by expanding the Mellin transform (4.4.15) in partial fractions [355] and utilizing the fact [95, p. 343] that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^{-s}}{(s+p)^b}\,ds = \frac{y^p}{\Gamma(b)}\left(\ln\frac{1}{y}\right)^{b-1}, \qquad 0 \leq y \leq 1$$

$$= 0, \qquad \text{otherwise},$$

although the previous approach using recursive relationships is more suitable for computer utilization.

Theorem 4.4.2. The p.d.f. of the product $Y=\prod_{i=1}^{n}X_i$ of n identically distributed independent beta variables with p.d.f. (4.4.11) and integer parameters a,b has the closed form representation

$$h(y)=\left(\frac{(a+b-1)!}{(a-1)!}\right)^{n}\sum_{k=1}^{b}\sum_{j=0}^{n-1}K_{kj}y^{a+k-2}(-\ln y)^{n-1-j}, \tag{4.4.22}$$

where

$$K_{k0}=\prod_{\substack{q\neq k\\ q=1}}^{b}(d_q-d_k)^{-n} \tag{4.4.23}$$

and

$$K_{kj}=\sum_{\substack{q=1\\ q\neq k}}^{b}\sum_{r=0}^{j-1}(-1)^{r+1}\binom{j-1}{r}\frac{nr!}{(d_q-d_k)^{r+1}}K_{k,j-1-r},\qquad j>0. \tag{4.4.24}$$

PROOF. Since all the poles are of order n and $a_i=a$, $b_i=b$, $i=1,2,\ldots,n$, the result follows directly from Theorem 4.4.1 by putting $m=b+1$, $e_k=n$, $k=1,2,\ldots,m$; $d_q=a+q-2$, $q=1,2,\ldots,m$.

To illustrate the application of the procedure for determining the p.d.f. of a product of independent beta variables, Example 4.4.1 is presented.

Example 4.4.1. Determine the p.d.f. $h(y)$ of the product $Y=X_1X_2X_3$, of three i.r.v.'s x_i, $i=1,2,3$, whose p.d.f.'s are

$$f(x_i)=\frac{1}{B(a_i,b_i)}x_i^{a_i-1}(1-x_i)^{b_i-1},\qquad 0\leqslant x_i\leqslant 1,\qquad i=1,2,3$$

$$a_1=9,b_1=3;$$

$$a_2=8,b_2=3;$$

$$a_3=4,b_3=2.$$

On evaluating the constants K_{kj} in (4.4.24), one finds the density and distribution functions to be, respectively,

$$h(y)=\frac{3960}{7}y^3-1980y^4+99{,}000y^7+(374{,}220+356{,}400\ln y)y^8$$
$$-(443{,}520-237{,}600\ln y)y^9-\frac{198{,}000}{7}y^{10}$$

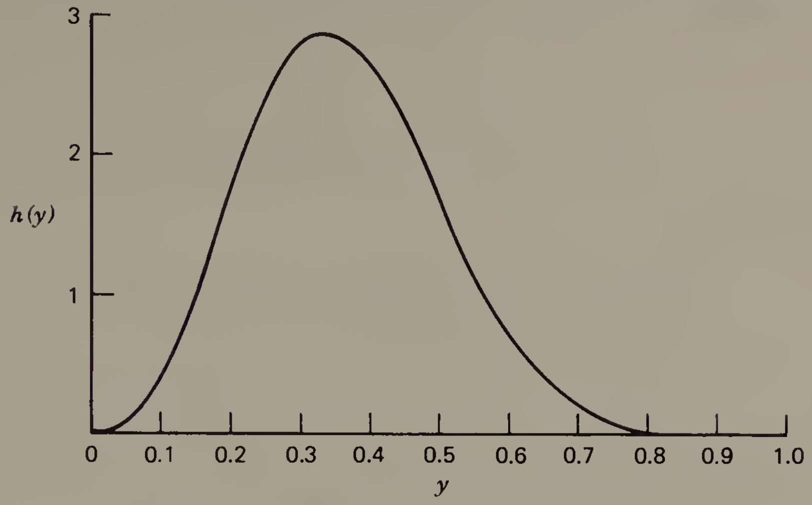


Fig. 4.4.1 P.d.f. of product of three beta r.v.'s.

and

$$H(y)=\frac{990}{7}y^4-396y^5+12{,}375y^8$$
$$+(37{,}180+39{,}600\ln y)y^9$$
$$-(46{,}728-23{,}760\ln y)y^{10}-\frac{18{,}000}{7}y^{11}.$$

Figure 4.4.1 plots $h(y)$.

4.4.3 The Product of Independent Gamma Random Variables

As Chapter 6 demonstrates, the gamma r.v. X_j with p.d.f.

$$f(x_j)=\frac{1}{\Gamma(b_j)}x_j^{b_j-1}e^{-x_j}, \qquad b_j>0, \qquad 0\leqslant x<\infty \tag{4.4.25}$$

is an H-function r.v. Then, from (6.4.9), the product $Y=\prod_{j=1}^n X_j$ of n gamma i.r.v.'s with p.d.f. (4.4.25) is an H-function r.v. with p.d.f.

$$h(y)=\prod_{j=1}^{n}\frac{1}{\Gamma(b_j)}\mathbf{H}_{0,n}^{n,0}\left[y|(b_1-1,1),\ldots,(b_n-1,1)\right], \qquad y>0,$$

which is expressible in the exact series form (7.1.2) given in Chapter 7. However, since in this case,

$$M_s(h(y))=\prod_{j=1}^{n}\left[\Gamma(b_j)\right]^{-1}\Gamma(s+b_j-1),$$

(7.1.1) simplifies considerably, reducing to the form

$$h(y)=\sum_{j=0}^{\infty}\sum_{k=1}^{n}R_{jk}(y,s)\bigg|_{s=-d_k-j}, \tag{4.4.26}$$

where [358, p. 724]

$$R_{jk}(y,s)=\frac{1}{(m_k-1)!}\frac{d^{m_k-1}}{ds^{m_k-1}}y^{-s}U^{(0)}(s,k)$$

and

$$U^{(0)}(s,k)=(s+d_k+j)^{m_k}\prod_{a=1}^{n}\left\{\Gamma^{m_a}(s+d_a)\prod_{t=0}^{p_a}(s+d_a+t)^{r_{ta}}\right\}.$$

Equation 4.4.26 expresses $h(y)$ as a sum of residues

$$R_{jk}(y,s)=\frac{y^{-s}}{(m_k-1)!}\sum_{i=0}^{m_k-1}\binom{m_k-1}{i}\left(\ln\frac{1}{y}\right)^{i}U^{(m_k-1-i)}(s,k)$$

at the poles $s=-d_k-j$, $k=1,2,\ldots,n$, $j=0,1,2,\ldots$. The computation of these residues is considerably expedited by using the algorithm [358, p. 725]

$$U^{(q+1)}(s,k)=\sum_{m=0}^{q}\binom{q}{w}U^{(q-m)}(s,k)V^{(m)}(s,k),\qquad q=0,1,2,\ldots, \tag{4.4.27}$$

where

$$\begin{aligned}V^{(m)}(s,k)&=m_k\psi^{(m)}(s+d_k+j+1)+(-1)^{m+1}(m+1)!(s+d_k+t)^{-m-1}\\&\quad+\sum_{t=0}^{p_k}(-1)^m m!(s+d_k+t)^{-(m+1)}+\sum_{a=1}^{n}m_a\psi^{(m)}(s+d_a)\\&\quad+\sum_{a=1}^{n}\sum_{t=0}^{p_a}(-1)^m m!(s+d_a+t)^{-(m+1)},\end{aligned}$$

and

$$\begin{aligned}\psi(s+a)&=\frac{d}{ds}\ln\Gamma(s+a)\\&=\frac{(d/ds)\Gamma(s+a)}{\Gamma(s+a)}\end{aligned} \tag{4.4.28}$$

is Euler's psi function, frequently referred to as the digamma function. This algorithm is derived by writing $U^{(0)}(s,k)$ in the form

$$U^{(0)}(s,k)=\left\{\frac{\Gamma(s+d_k+j+1)}{(s+d_k)\cdots(s+d_k+j-1)}\right\}^{m_k}\prod_{t=0}^{p_k}(s+d_k+t)^{r_{tk}}$$

$$\times\prod_{\substack{a=1\\a\neq k}}^{n}\Gamma^{m_a}(s+d_a)\left\{\prod_{t=0}^{p_a}(s+d_a+t)^{r_{ta}}\right\}$$

and differentiating $\ln U^{(0)}(s,k)$ with respect to s, giving

$$U^{(1)}(s,k)=U^{(0)}(s,k)V^{(0)}(s,k), \tag{4.4.29}$$

where

$$V^{(0)}(s,k)=m_k\psi(s+d_k+j+1)-m_k\sum_{t=0}^{j-1}\frac{1}{s+d_k+t}$$

$$+\sum_{t=0}^{p_k}r_{tk}(s+d_k+t)^{-1}+\sum_{\substack{a=1\\a\neq k}}^{n}m_a\psi(s+d_a)$$

$$+\sum_{a=1}^{n}\sum_{t=0}^{p_a}r_{ta}(s+d_a+t)^{-1}.$$

Application of Leibniz's rule for the differentiation of a product to (4.4.29) yields the algorithm (4.4.27).

Theorem 4.4.3. The p.d.f. of the product $Y=X_1X_2\cdots X_n$ of n *identically distributed* independent gamma variables each having p.d.f.

$$f(x)=\frac{1}{\Gamma(b)}x^{b-1}e^{-x},\qquad 0\leqslant x<\infty \tag{4.4.30}$$

is

$$h(y)=\sum_{j=0}^{\infty}\sum_{k=0}^{n-1}\sum_{m=0}^{k-1}(-1)^{k-m}(-\ln y)^{n-1-k}ny^{j}$$

$$\cdot\left\{\zeta(k-m,1)-\sum_{i=0}^{j-1}\frac{1}{(j-i)^{k-m}}\right\}U^{(m)}(s)\Bigg|_{s=-j}, \tag{4.4.31}$$

where

$$U^{(0)}(s)\Big|_{s=-j}=\frac{1}{(-1)^j j!},$$

and

$$U^{(k)}(s)\Bigg|_{s=-j}=\left[\frac{d^k}{ds^k}\left(\frac{\Gamma(s+b+j)}{(s+b-1)(s+b)\cdots(s+b-2+j)}\right)^n\right]\Bigg|_{s=-j}$$

can be expressed recursively as

$$U^{(k)}(s)\Bigg|_{s=-j}=\frac{n\sum_{m=0}^{k-1}(-1)^{k-1}(k-1)!}{m!(k-1-m)!}$$

$$\times\left\{\zeta(k-m,1)-\sum_{i=0}^{j-1}\frac{1}{(j-i)^{k-m}}\right\}U^{(m)}(s)\Bigg|_{s=-j}, \qquad j=1,2,\ldots$$

where $\zeta(\beta,a)$ is the Riemann zeta function defined by

$$\zeta(\beta,a)=\sum_{\alpha=0}^{\infty}(a+\alpha)^{-\beta}, \tag{4.4.32}$$

PROOF. From Table D.2, formula 7, the Mellin transform of the gamma p.d.f (4.4.30) is seen to be

$$M_s(f(x))=\frac{\Gamma(s+b-1)}{\Gamma(b)}, \qquad \operatorname{Re}(s)>-(b-1),$$

so that

$$M_s(h(y))=\frac{\Gamma^n(s+b-1)}{\Gamma^n(b)}$$

and

$$h(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^{-s}}{\Gamma^n(b)}\Gamma^n(s+b-1)\,ds.$$

Since the conditions of Jordan's lemma are satisfied,

$$h(y)=\sum_{j=0}^{\infty} R(y,n,j)$$

$$=\sum_{j=0}^{\infty}\left[\frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}\{(s+b-1+j)^n\Gamma^n(s+b-1)\}\right]\Bigg|_{s=-j}.$$

As before, these residues, when evaluated recursively through the use of Leibniz's rule for the differentiation of products, lead to (4.4.31). The details are given by Springer and Thompson [358].

The representation of $h(y)$ as given by Theorem 4.4.3 is particularly convenient for adaptation to electronic computers. Lomnicki [214] has expressed the representation in a different form but reducible to that of (4.4.31). Specifically, Lomnicki's counterpart of the p.d.f. (4.4.31) is

$$h(y)=\frac{y^{b-1}\phi_n(y)}{\Gamma^n(b)},$$

where

$$\phi_n(y)=\frac{1}{(n-1)!}\sum_{j=0}^{\infty}\frac{(-1)^{jn}y^j}{(j!)^n}$$
$$\times Z_{n-1}(A(s),A'(s),\ldots,A^{(n-2)}(s))|, \qquad s=-j \tag{4.4.33}$$

and

$$Z_{n-1}(A(s),A'(s),\ldots,A^{N-2}(s))$$
$$=\sum\frac{n![A^{(n_1-1)}(s)]^{k_1}[A^{(n_2-1)}(s)]^{k_2}\cdots[A^{(n_r-1)}(s)]^{k_r}}{(n_1!)^{k_1}(n_2!)^{k_2}\cdots(n_r!)^{k_r}(k_1!k_2!\cdots k_r!)} \tag{4.4.34}$$

and the sum above is extended to all the partitions of the number n such that $\sum_{j=1}^{r}m_jk_j=n$. The notation $A^0(s)$ denotes $A(s)$, and $A^{(k)}(s)$ signifies the kth derivative of $A(s)$, where by definition

$$A(s)=-\ell ny+n\psi(s+1)$$

and $\psi(s+1)$ is the Euler psi function previously defined in (4.4.28). Also, since $A(s)$ and its derivatives must be evaluated at $s=-j$ when used in the expression (4.4.34), one should note that

$$A^{(k)}(s)\big|_{s=-j}=A^{(k)}(-j)=n\left\{\psi^{(k)}(1)+r!\sum_{k=1}^{j}k^{-(k+1)}\right\}, \tag{4.4.35}$$

where

$$\psi^{(k)}(s+j+1)=(-1)^{k+1}k!\zeta(k+1,s+j+1) \tag{4.4.36}$$

is the kth derivative of the Euler psi function (or digamma function), usually referred to as the polygamma function, expressed in terms of the Riemann zeta function defined by (4.4.32). The evaluation of $\phi_n(y)$ is considerably simplified by the recursive relationships (4.4.35) and

$$Z_{m+1}(\cdot)=A(s)Z_m(\cdot)+Z'_m(\cdot). \tag{4.4.37}$$

Unfortunately, the procedure and recursive relationships apply only to the derivation of the p.d.f. $h(y)$ of products of n *identically distributed* gamma i.r.v.'s.

Using the same notation, Lomnicki also showed that the p.d.f. $h(y)$ of the product $Y=\prod_{i=1}^{n}X_i$ of n identically distributed Weibull i.r.v.'s each with p.d.f.

$$f(x)=\beta x^{\beta-1}\exp(-x^{\beta}), \qquad 0\leqslant x<\infty$$

is

$$h(y)=\beta y^{\beta-1}\phi_n(y^{\beta})$$

and that the p.d.f. of the product of n standardized normal i.r.v.'s $N(0,1)$ is

$$h(y)=(2\pi)^{-n/2}\phi_n\left(\frac{y^2}{2^n}\right).$$

4.4.4* The Distribution of the Product of Two Independent, Noncentral Beta Random Variables

Let X_1 and X_2 be two noncentral beta i.r.v.'s with parameters p_j, q_j, and λ_j; or equivalently, with p.d.f.'s

$$f(x_j)=\sum_{k=0}^{\infty}\frac{\Gamma(\{2k+p_j+q_j\}/2)\lambda_j^k}{\Gamma(\{q_j/2\})\Gamma(\{2k+p_j\}/2)k!}$$

$$\times e^{-\lambda_j}x_j^{(2k+p_j-2)/2}(1-x_j)^{(q_j-2)/2}, \qquad j=1,2, \qquad 0\leqslant x_j\leqslant 1.$$

What is the p.d.f. $g(y)$ of the product $Y=X_1X_2$?

Malik [234] derived this p.d.f. by the use of Mellin transforms. He noted that

$$M_s(g(y))=e^{-(\lambda_1+\lambda_2)}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{\lambda_1^m\Gamma\left(\frac{2m+p_1+q_1}{2}\right)\Gamma\left(\frac{2m+p_1+2s-2}{2}\right)}{\left(\frac{2m+p_1}{2}\right)\Gamma\left(\frac{2m+p_1+q_1+2s-2}{2}\right)m!}$$

$$\times\frac{\lambda_2^{k-m}\Gamma\left(\frac{2k-2m+p_2+q_2}{2}\right)\Gamma\left(\frac{2k-2m+p_2+2s-2}{2}\right)}{\Gamma\left(\frac{2k-2m+p_2}{2}\right)\Gamma\left(\frac{2k-2m+p_2+q_2+2s-2}{2}\right)(k-m)!}. \tag{4.4.38}$$

To find the p.d.f. $g(y)$, one must find the inverse Mellin transform of each term in the series (4.4.38), namely,

$$M^{-1}(M_m)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}$$

$$\times\frac{y^{-s}\Gamma\left(s+m+\frac{p_1}{2}-1\right)\Gamma\left(s+k-m+\frac{p_2}{2}+1\right)}{\Gamma\left(s+m+\frac{p_1}{2}+\frac{q_1}{2}-1\right)\Gamma\left(s+k-m+\frac{p_2}{2}+\frac{q_2}{2}-1\right)}\,ds.$$

Consul [59] has shown that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{-s}\Gamma(s+a)\Gamma(s+b)}{\Gamma(s+a+r)\Gamma(s+b+n)}\,ds=\frac{x^a(1-x)^{r+n-1}}{\Gamma(r+n)}$$

$$\times F(n,a-b+r;r+n;1-x),$$

where $F(\alpha,\beta;\gamma,x)$ is the hypergeometric function (see Appendix D.1).

Consequently,

$$M^{-1}(M_k)=\frac{y^{k+(p_1/2)-1}(1-y)^{(q_1/2)+(q_2/2)-1}}{\Gamma(q_1/2+q_2/2)}$$

$$\times\mathbf{F}\left(\frac{q_2}{2},2m-k+\frac{p_1}{2}-\frac{p_2}{2}+\frac{q_1}{2};\quad\frac{q_1}{2}+\frac{q_2}{2};\quad 1-y\right), \tag{4.4.39}$$

from which it follows that

$$g(y)=e^{-(\lambda_1+\lambda_2)}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{\lambda_1^m\lambda_2^{k-m}\Gamma\left(\frac{2m+p_1+q_1}{2}\right)\Gamma\left(\frac{2k-2m+p_2+q_2}{2}\right)}{\Gamma\left(\frac{2m+p_1}{2}\right)\Gamma\left(\frac{2k-2m+p_2}{2}\right)m!(k-m)!}$$

$$\times\frac{y^{m+(p_1/2)-1}(1-y)^{(q_1/2)+(q_2/2)-1}}{\Gamma\left(\frac{q_1}{2}+\frac{q_2}{2}\right)}$$

$$\times\mathbf{F}\left(\frac{q_2}{2},2m-k+\frac{p_1}{2}-\frac{p_2}{2}+\frac{q_1}{2};\quad\frac{q_1}{2}+\frac{q_2}{2};\quad 1-y\right). \tag{4.4.40}$$

As Malik has pointed out, this density function is also expressible in terms of a mixture of beta distributions, namely,

$$g(y)=e^{-(\lambda_1+\lambda_2)}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\sum_{r=0}^{\infty}\lambda_1^m\lambda_2^{k-m}$$

$$\times\frac{\Gamma\left(\frac{p_1}{2}+\frac{q_1}{2}+m\right)\Gamma\left(\frac{q_2}{2}+r\right)\Gamma\left(2m-k+\frac{p_1}{2}-\frac{p_2}{2}+\frac{q_1}{2}+r\right)}{\Gamma\left(2m-k+\frac{p_1}{2}-\frac{p_2}{2}+\frac{q_1}{2}\right)\Gamma\left(\frac{p_1}{2}+\frac{q_1}{2}+\frac{q_2}{2}+m+r\right)B\left(\frac{p_2}{2}+k-m,\frac{q_2}{2}\right)m!(k-m)!r!}$$

$$\times\frac{y^{m+(p_1/2)-1}(1-y)^{(q_1/2)+(q_2/2)+r-1}}{B\left(\frac{p_1}{2}+m,\frac{q_1}{2}+\frac{q_2}{2}+r\right)}, \tag{4.4.41}$$

where the sum over r comes from the hypergeometric function $F(.,.;.,x)$.

The distribution function of y is

$$F(y)=\int_0^y f(y)\,dy$$

$$=e^{-(\lambda_1+\lambda_2)}$$

$$\cdot\sum_{k=0}^{\infty}\sum_{m=0}^{k}\sum_{r=0}^{\infty}\frac{\lambda_1^m\lambda_2^{k-m}\Gamma\left(\frac{p_1}{2}+\frac{q_1}{2}+m\right)\Gamma\left(\frac{q_2}{2}+r\right)\Gamma\left(2m-k+\frac{p_1}{2}-\frac{p_2}{2}+\frac{q_1}{2}+r\right)}{\Gamma\left(2m-k+\frac{p_1}{2}-\frac{p_2}{2}+\frac{q_1}{2}\right)\Gamma\left(\frac{p_1}{2}+\frac{q_1}{2}+\frac{q_2}{2}+m+r\right)B\left(\frac{p_2}{2}+k-m;\frac{q_2}{2}\right)}$$

$$\cdot\frac{1}{m!(k-m)!r!}\times I_y\left(\frac{p_1}{2}+m,\frac{q_1}{2}+\frac{q_2}{2}+r\right).$$

where

$$I_y(a,b)=\frac{1}{B(a,b)}\int_0^y t^{a-1}(1-t)^{b-1}\,dt \tag{4.4.42}$$

is the incomplete beta function, which has been tabulated by Pearson [282]. Parenthetically, the incomplete beta function should not be confused with the well-known beta function

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}\,dx, \tag{4.4.42a}$$

which is expressible in terms of gamma functions as

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{4.4.42b}$$

where the gamma function is defined as

$$\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}\,dx. \tag{4.4.43}$$

Finally, by setting $\lambda_1=\lambda_2=0$ in (4.4.40), one obtains the p.d.f. of the product of two central beta variates:

$$g(y)=\frac{\Gamma\left(\frac{p_1}{2}+\frac{q_1}{2}\right)\Gamma\left(\frac{p_2}{2}+\frac{q_2}{2}\right)y^{(p_1/2)-1}(1-y)^{(p_1/2)+(q_2/2)-1}}{\Gamma\left(\frac{p_1}{2}\right)\Gamma\left(\frac{p_2}{2}\right)\Gamma\left(\frac{q_1}{2}+\frac{q_2}{2}\right)}$$

$$\times\mathbf{F}\left(\frac{q_2}{2};\frac{p_1}{2}-\frac{p_2}{2}+\frac{q_1}{2};\frac{q_1}{2}+\frac{q_2}{2};1-y\right),$$

where $F(a,b,c;z)$ is Gauss's hypergeometric function (Appendix D.1).

4.5 THE DISTRIBUTION OF PRODUCTS AND QUOTIENTS OF CONTINUOUS RANDOM VARIABLES THAT ARE NOT EVERYWHERE POSITIVE

4.5.1 Derivation of the Distribution of Products

To treat the more general problem of deriving the distribution of products of i.r.v.'s that may assume both positive and negative values, a procedure developed by Epstein [92] for the case of two variables and extended by Springer and Thompson [354] to n variables is presented. This extension is accomplished by partitioning a function $f_i(x_i)$, $-\infty<x_i<\infty, i=1,2,\dots,n$, into two components, namely,

$$f_i(x_i)=f_i^-(x_i)+f_i^+(x_i), \tag{4.5.1}$$

in which $f_i^-(x_i)$ vanishes identically except on the interval $-\infty<x_i\leqslant 0$, where $f_i^-(x_i)=f_i(x_i)$. Similarly, $f_i^+(x_i)$ is defined to be indentically zero except over the interval $0\leqslant x_i<\infty$, where $f_i^+(x_i)=f_i(x_i)$. Using such a partitioning, one can then express the p.d.f. $h_2(y)$ of the product r.v. $Y=X_1X_2$ in terms of pairs of functions defined over the interval $(0,\infty)$ whose Mellin transforms are well defined, as is now shown. Note that the p.d.f. of the r.v. $Y=X_1X_2$ is the Mellin convolution

$$h_2(y)=\int_{-\infty}^{\infty}\frac{1}{x_1}f_2\left(\frac{y}{x_1}\right)f_1(x_1)dx_1, \tag{4.5.2}$$

which, in view of (4.5.1) is expressible as

$$\begin{aligned}h_2(y)&=\int_{-\infty}^{\infty}\frac{1}{x_1}\left(f_2^+\left(\frac{y}{x_1}\right)+f_2^-\left(\frac{y}{x_1}\right)\right)(f_1^+(x_1)+f_1^-(x_1))\,dx_1\\&=\int_{-\infty}^{\infty}\frac{1}{x_1}f_2^+\left(\frac{y}{x_1}\right)f_1^+(x_1)\,dx_1+\int_{-\infty}^{\infty}\frac{1}{x_1}f_2^+\left(\frac{y}{x_1}\right)f_1^-(x_1)\,dx_1\\&\quad+\int_{-\infty}^{\infty}\frac{1}{x_1}f_2^-\left(\frac{y}{x_1}\right)f_1^+(x_1)\,dx_1+\int_{-\infty}^{\infty}\frac{1}{x_1}f_2^-\left(\frac{y}{x_1}\right)f_1^-(x_1)\,dx_1.\end{aligned} \tag{4.5.3}$$

Note also that when $y<0$ and $-\infty<x_1<\infty$,

$$f_2^+\left(\frac{y}{x_1}\right)f_1^+(x_1)=0 \qquad\text{and}\qquad f_2^-\left(\frac{y}{x_1}\right)f_1^-(x_1)=0, \tag{4.5.4}$$

so that (4.5.3) becomes

$$\begin{aligned}h_2(y)&=\int_{-\infty}^{\infty}\frac{1}{x_1}\left(f_2^-\left(\frac{y}{x_1}\right)f_1^+(x_1)+f_2^+\left(\frac{y}{x_1}\right)f_1^-(x_1)\right)dx_1\\&=\int_0^{\infty}\frac{1}{x_1}\left[f_2^-\left(\frac{y}{x_1}\right)f_1^+(x_1)+f_2^+\left(\frac{y}{-x_1}\right)f_1^-(-x_1)\right]dx_1,\\&\qquad\qquad -\infty<y\leqslant 0.\end{aligned} \tag{4.5.5}$$

Similarly, when $y>0$ and $-\infty<x_1<\infty$,

$$f_2^+\left(\frac{y}{x_1}\right)f_1^-(x_1)=0 \qquad \text{and} \qquad f_2^-\left(\frac{y}{x_1}\right)f_1^+(x_1)=0, \tag{4.5.6}$$

and as a result (4.5.3) reduces to

$$h_2(y)=\int_0^\infty \frac{1}{x_1}\left(f_2^+\left(\frac{y}{x_1}\right)f_1^+(x_1)\right.$$
$$\left.+f_2^-\left(\frac{y}{-x_1}\right)f_1^-(-x_1)\right)dx_1, \qquad 0\leqslant y<\infty. \tag{4.5.7}$$

If one now defines

$$h_2(y)=h_2^-(y)+h_2^+(y), \tag{4.5.8}$$

where

$$h_2^-(y)=\begin{matrix} h_2(y), & \text{if} \quad -\infty<y\leqslant 0 \\ 0; & \text{elsewhere} \end{matrix} \tag{4.5.9}$$

and

$$h_2^+(y)=\begin{matrix} h_2(y), & \text{if} \quad 0\leqslant y<\infty \\ 0; & \text{elsewhere,} \end{matrix} \tag{4.5.10}$$

then for $0\leqslant y<\infty$,

$$h_2^-(-y)=\int_0^\infty \frac{1}{x_1}f_2^+\left(\frac{y}{x_1}\right)f_1^-(-x_1)\,dx_1$$
$$+\int_0^\infty \frac{1}{x_1}f_2^-\left(-\frac{y}{x_1}\right)f_1^+(x_1)\,dx_1 \tag{4.5.11a}$$

and

$$h_2^+(y)=\int_0^\infty \frac{1}{x_1}f_2^+\left(\frac{y}{x_1}\right)f^+(x_1)\,dx_1$$
$$+\int_0^\infty \frac{1}{x_1}f_2^-\left(\frac{-y}{x_1}\right)f_1^-(-x_1)\,dx_1. \tag{4.5.11b}$$

That is, $h_2^+(y)$ and $h_2^-(-y)$ have been expressed in terms of convolutions of pairs of functions defined over the interval $(0, \infty)$, whose Mellin transforms are well defined by (4.2.1) and by

$$M_s(h_2^-(-y)) = M_s(f_2^+(x_2))M_s(f_1^-(-x_1)) + M_s(f_2^-(-x_2))M_s(f_1^+(x_1)), \tag{4.5.12a}$$

$$M_s(h_2^+(y)) = M_s(f_2^+(x_2))M_s(f_1^+(x_1)) + M_s(f_2^-(-x_2))M_s(f_1^-(-x_1)). \tag{4.5.12b}$$

The inversion integral (4.2.3) then yields $h_2^+(y)$ and $h_2^-(-y)$. In turn, $h_2^-(-y)$ defines $h_2^-(y)$.

If $(n-1)$ successive applications of the foregoing procedure are carried out, one obtains the p.d.f. of the r.v.

$$Y = \prod_{i=1}^{n} X_i,$$

namely,

$$h(y) = h_n^-(y) + h_n^+(y), \tag{4.5.13}$$

whose components are defined by inverting the Mellin transforms

$$M_s(h_n^-(-y)) = M_s(f_n^+(y))M_s(h_{n-1}^-(-y)) + M_s(f_n^-(-y))\cdot M_s(h_{n-1}^+(y)) \tag{4.5.14a}$$

and

$$M_s(h_n^+(y)) = M_s(f_n^+(y))M_s(h_{n-1}^+(y)) + M_s(f_n^-(-y))M_s(h_{n-1}^-(-y)). \tag{4.5.14b}$$

The two products on the right in (4.5.14a), when expanded into terms involving $f_i^+(x_i)$ and $f_i^-(-x_i)$, result in 2^{n-1} products, where each product consists of n factors, each factor being either $f_i^+(x_i)$ or $f_i^-(-x_i)$. Also, each product contains an *odd* number of factors of the form $f_i^-(-x_i)$. For example, when $n=3$, $Y = X_1X_2X_3$, and

$$\begin{aligned} M_s(h^-(-y)) = {} & M_s(f_1^-(-x_1))M_s(f_2^+(x_2))M_s(f_3^+(x_3)) \\ & + M_s(f_1^+(x_1))M_s(f_2^-(-x_2))M_s(f_3^+(x_3)) \\ & + M_s(f_1^+(x_1))M_s(f_2^+(x_2))M_s(f_3^-(-x_3)) \\ & + M_s(f_1^-(-x_1))M_s(f_2^-(-x_2))M_s(f_3^-(-x_3)). \end{aligned} \tag{4.5.15a}$$

Similarly, the two products on the right in (4.5.14b), when expanded into terms involving $f_1^+(x_1)$ and $f_i^-(-x_i)$, result in 2^{n-1} products, where each product consists of n factors, each factor being either $f_i^+(x_i)$ or $f_i^-(-x_i)$. Now, however, each product involves an *even* number of factors of the form $f_i^-(-x_i)$. For example, when $n=3$,

$$\begin{aligned} M_s(h^+(y)) = {} & M_s(f_1^+(x_1))M_s(f_2^+(x_2))M_s(f_3^+(x_3)) \\ & + M_s(f_1^+(x_1))M_s(f_2^-(-x_2))M_s(f_3^-(-x_3)) \\ & + M_s(f_1^-(-x_1))M_s(f_2^-(-x_2))M_s(f_3^+(x_3)) \\ & + M_s(f_1^-(-x_1))M_s(f_2^+(x_2))M_s(f_3^-(-x_3)). \end{aligned} \tag{4.5.15b}$$

Thus the p.d.f. $h(y)$ consists of the component p.d.f.'s $h^-(y)$ and $h^+(y)$, which are valid for negative and nonnegative values of Y, respectively, and are obtained by inverting the Mellin transforms $M_s(h^-(-y))$ and $M_s(h^+(y))$:

$$\begin{aligned} h^-(y) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s} M_s(h^-(-y))\,ds, \qquad -\infty < y < 0 \\ &= 0, \qquad \text{otherwise} \end{aligned} \tag{4.5.16a}$$

$$\begin{aligned} h^+(y) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s} M_s(h^+(y))\,ds, \qquad 0 < y < \infty \\ &= 0, \qquad \text{otherwise,} \end{aligned} \tag{4.5.16b}$$

where $M_s(h^-(-y))$ and $M_s^+(y)$ are expanded in the manner indicated. Finally, it should be pointed out that these inversion integrals are usually evaluated by the method of residues.

In the special case where the $f_i(x_i)$ are identical even functions of x, that is,

$$f_i^-(-x_i) = f_i^+(x_i) = f(x), \qquad i = 1, 2, \ldots, n, \tag{4.5.17}$$

the p.d.f. of the product $Y = \prod_{i=1}^n X_i$ is even, so that

$$h_n^+(x) = h_n^-(-x) \tag{4.5.18}$$

and

$$M_s(h^+(x)) = M_s(h^-(x)) = 2^{n-1}\left[M_s(f^+(x))\right]^n. \tag{4.5.19}$$

Equation 4.5.19 supplies a direct relation between the p.d.f. $h(y)$ of the product $Y=\prod_{i=1}^{n}X_i$ and the common p.d.f. $f(x_i)$ of the i.r.v.'s X_i, $-\infty<X_i<\infty$, $i=1,2,\ldots,n$.

The quotient $Y=X_1/X_2$ of two i.r.v.'s X_1 and X_2 may be considered as the product of X_1 and $1/X_2$. As has already been pointed out, if X is a nonnegative r.v. and $W=1/X$ has the p.d.f. $g(w)$, then

$$M_s(g(w))=M_{-s+2}(f(x)). \tag{4.5.20}$$

Partitioning the p.d.f. $h(y)$ of the quotient $Y=X_1/X_2$ of two i.r.v.'s X_i with p.d.f.'s $f_i(x_i)$, $-\infty<X_i<\infty$, $i=1,2$, one has

$$h(y)=h^-(y)+h^+(y), \tag{4.5.21}$$

in which $h^-(y)$ and $h^+(y)$ denote the components of $h(y)$ that are valid over the negative and positive ranges of y, respectively, and where

$$\begin{aligned} M_s(h^-(-y))=&M_s(f_1^+(x_1))M_{2-s}f_2^-(-x_2)\\ &+M_s(f_1^-(-x_1))M_{2-s}(f_2^+(x_2)) \end{aligned} \tag{4.5.22a}$$

and

$$\begin{aligned} M_s(h^+(y))=&M_s(f_1^+(x_1))M_{2-s}(f_2^+(x_2))\\ &+M_s(f_1^-(-x_1))M_{2-s}(f_2^-(-x_2)). \end{aligned} \tag{4.5.22b}$$

Evaluation of the inversion integral (4.2.3) for these two Mellin transforms leads to the p.d.f. $h(y)$ as given by (4.5.21). In particular, if the x_i are identical even i.r.v.'s with p.d.f. $f(x)$, then $h(y)$ is even and

$$h(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}2y^{-s}\left[M_s(f^+(x))M_{2-s}(f^+(x))\right]ds, \qquad -\infty<y<\infty \tag{4.5.23}$$

which was previously obtained by Epstein [92].

4.5.2 The Distribution of the Product of n Independent Normal Random Variables $N(0,\sigma_i)$

In deriving the p.d.f. $h(y)$ of the product $Y=\prod_{i=1}^{n}X_i$ of n i.r.v.'s, each having p.d.f.

$$f(x_i)=\frac{1}{\sigma_i\sqrt{2\pi}}\exp\left(\frac{-x_i^2}{2\sigma_i^2}\right), \qquad -\infty<x_i<\infty, \qquad i=1,2,\ldots,n, \tag{4.5.24}$$

note first [95, p. 312] that

$$M_s(e^{-x_i}) = \Gamma(s), \qquad Re(s) > 0.$$

Using this fact in conjunction with properties 4 and 10, Section 2.8.2, and (4.5.19), one obtains the Mellin transform of $f^+(x_i)$, namely,

$$M_s(f^+(x_i)) = \frac{2^{(s-3)/2}}{\sqrt{\pi}} \sigma_i^{s-1} \Gamma\left(\frac{s}{2}\right), \qquad Re(s) > 0. \tag{4.5.25}$$

The Mellin transform of $h(y)$ is, therefore, then

$$M_s(h^+(y)) = 2^{n-1} \prod_{i=1}^{n} M_s(f^+(x_i)) \tag{4.5.26}$$

and the associated inversion integral is

$$h^+(y) = 2^{n-1}\left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{1-n}(2\pi)^{-n/2}\left(\prod_1^n \sigma_i\right)^{-1}\left(y^2 2^{-n}\prod_{i=1}^n \sigma_i^{-2}\right)^{-s/2} \times \left(\Gamma\left(\frac{s}{2}\right)\right)^n d\left(\frac{s}{2}\right)\right], \qquad 0 < y < \infty. \tag{4.5.27}$$

Also, because of symmetry, $h^-(y) = h^+(y) = h(y)$. Letting $V = 2^{-n}(\prod_{i=1}^n \sigma_i^{-2})y^2$, (4.5.27) becomes equivalent to

$$h^+(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{(2\pi)^{n/2}} \frac{v}{\prod_1^n \sigma_i}^{-s} \Gamma^n(s)\, ds, \tag{4.5.28}$$

It remains to evaluate the inversion integral (4.5.28), which can be accomplished by using the residue theorem, which is applicable because from (6.3.8), Theorem 6.4.1, and Appendix F, $h^+(y)$ is an H-function inversion integral to which Jordan's lemma applies. Thus

$$h^+(v) = \sum_{j=0}^{\infty} R(v,n,j), \tag{4.5.29}$$

where

$$R(v,n,j) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{ds^{n-1}}\left[\frac{(s+j)^n v^{-s}\Gamma^n(s)}{(2\pi)^{n/2}\prod_1^n \sigma_i}\right]\right\}\Bigg|_{s=-j} \tag{4.5.30}$$

is the residue of the integrand of Equation (4.5.28) at the pole of order n at $s=-j$, $j=0,1,2,\ldots$. The evaluation of these residues is considerably simplified by the application of Leibniz's rule for the differentiation of products, which when applied to (4.5.28), yields

$$R(v,n,j)=\frac{v^{-s}}{(2\pi)^{n/2}(\Pi_1^n\sigma_i)^1}\sum_{k=0}^{n-1}\frac{(-\ln v)^{n-1-K}}{(n-1-k)!k!}\left[\frac{d^K}{ds^K}(s+j)^n\Gamma^n(s)\right]\Bigg|_{s=-j}. \tag{4.5.31}$$

To simplify the notation, let

$$U(s)=[(s+j)\Gamma(s)]^n \tag{4.5.32a}$$

$$U^{(K)}(s)=\frac{d^K}{ds^K}U(s), \qquad K\geqslant 1 \tag{4.5.32b}$$

$$U^{(0)}(s)=U(s). \tag{4.5.32c}$$

Then, as we show presently, the required residues (4.5.31) can be obtained recursively from the relationship

$$R(v,n,j)=\lambda v^j\sum_{K=0}^{n-1}\frac{(-\ln v)^{n-1-K}}{(n-1-K)!K!}U^{(K)}(s)\Bigg|_{s=-j} \tag{4.5.33}$$

where $\lambda=(2\pi)^{-n/2}(\Pi_1^n\sigma_i)^{-1}$ or equivalently, in terms of y,

$$R(y,n,j)=\lambda\left(\frac{y^2}{2^n\Pi_1^n\sigma_i^2}\right)^j\sum_{K=0}^{n-1}$$

$$\times\frac{\left(-\ln\dfrac{y^2}{2^n\Pi_1^n\sigma_i^2}\right)^{n-K-1}}{(n-1-K)!K!}U^{(K)}(s)\Bigg|_{s=-j}, \qquad j=0,1,2,\ldots, \tag{4.5.34}$$

where $U^{(K)}(s)$ is obtained from the lower derivatives by the recursion formula

$$U^{(K)}(s)=n\sum_{m=0}^{K-1}(-1)^{K-m}\frac{(K-1)!}{m!(K-1-m)!}\left[\zeta(K-m,1)-\sum_{i=0}^{j-1}\frac{1}{(j-i)^{K-m}}\right]U^{(m)}(s) \tag{4.5.35}$$

and $\zeta(K-m,1)$ is a Riemann zeta function previously defined by (4.4.32). These recursive relationships make the residues as expressed in (4.5.32) particularly amenable to evaluation by an electronic computer.

To evaluate the residues (4.5.33) and (4.5.34) necessary for determining $h(y)$, note first that $U(s)$, defined by (4.5.32a), can be written in the form

$$U(s)=\left(\frac{\Gamma(s+j+1)}{s(s+1)(s+2)\cdots(s+j-1)}\right)^{n}, \tag{4.5.36}$$

in which the denominator is understood to be one when $j=0$. Then

$$\ln U(s)=n\left[\ln\Gamma(s+j+1)-\sum_{i=0}^{j-1}\ln(s+i)\right], \tag{4.5.37}$$

and differentiation of (4.5.37) leads to the result

$$U^{(1)}(s)=nU(s)\left[\psi(s+j+1)-\sum_{i=0}^{j-1}\frac{1}{s+i}\right]. \tag{4.5.38}$$

Application of Leibniz's rule to (4.5.38), making use of (4.4.36), leads to the recursion formula (4.5.35), which involves both the Euler psi function and the Riemann zeta function.

Thus the p.d.f. $h(y)$ of the product of n normal i.r.v.'s $N(0,\sigma_i), i=1,2,\ldots,n$, is given by

$$\begin{aligned} h(y)&=h^{-}(y), \qquad -\infty<y<0\\ &=h^{+}(y), \qquad 0<y<\infty, \end{aligned} \tag{4.5.39}$$

where, because of symmetry, $h^{-}(y)=h^{+}(y)$, and where

$$h^{+}(y)=\lambda\sum_{j=0}^{\infty}\left[\left(\frac{y^2}{2^n\prod_1^n\sigma_i^2}\right)^{-s}\sum_{K=0}^{n-1}\left[-\ln\left(\frac{y^2}{2^n\prod_1^n\sigma_i^2}\right)\right]^{n-1-K}U^{(K)}(s)\right]\Bigg|_{s=-j},$$

$$j=0,1,2,\ldots, \qquad 0<y<\infty. \tag{4.5.40}$$

As has previously been shown (see (4.3.8)), the distribution function $H^{+}(y)$ may be determined directly from a knowledge of the Mellin transform $M_s(h^+(y))$. Specifically,

$$G^+(y) = 1 - H^+(y), \qquad y > 0$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s} M_s(G^+(y))\,ds. \tag{4.5.41}$$

Also, from (4.3.10a, b),

$$M_s(G(y)) = \frac{1}{s} M_{s+1}(h^+(y)),$$

so that (4.5.41) becomes

$$H^+(y) = 1 - \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}\left(\frac{1}{s}\right) M_{s+1}(h^+(y))\,ds$$

$$= 1 - \sum_{j=0}^{\infty} \Phi(y, n, j+1), \tag{4.5.42}$$

where

$$\Phi(y,n,j+1) = \lambda\left(\frac{Y}{2^n/2\Pi_1^n\sigma_i}\right)^{2j+1}$$

$$\times \sum_{K=0}^{n-1} \frac{-\ln\left(\dfrac{y^2}{2^n\Pi_1^n\sigma_i}\right)^{n-1-K}}{(n-1-K)!\,K!}\left[\frac{1}{s}U(s+1)\right]^{(K)}\Bigg|_{s=-j}$$

$$j = 1, 2, \ldots. \tag{4.5.42a}$$

Finally,

$$H^-(y) = 1 - H^+(y), \qquad y < 0$$

Example 4.5.1 Determine the p.d.f. $h(y)$ and d.f. $H(y)$ of the product of six normal i.r.v.'s $N(0, 1)$.

The p.d.f. $h(y)$ is obtained from (4.5.34) by setting $\sigma_i=1, i=1,2,\ldots,6$ and evaluating the required derivatives $U^{(K)}(-j), K=0,1,2,3,4,5$. These derivatives, obtained in a straightforward manner with the aid of the recursion formula (4.5.35), are given below. The symbol $\zeta'(b,a)$ denotes the modified Riemann zeta function, defined as

$$\zeta'(b,a)=\zeta(b,a)+2\sum_{r=1}^{a-1}\frac{1}{r^b} \tag{4.5.43}$$

$$U^{(0)}(-j)=\frac{(-1)^{jn}}{(j!)^n}$$

$$U^{(1)}(-j)=\frac{(-1)^{jn}}{(j!)^n}\left[n\psi(j+1)\right]$$

$$U^{(2)}(-j)=\frac{(-1)^{jn}}{(j!)^n}\left[n^2\psi^2(j+1)+n\zeta'(2,j+1)\right]$$

$$U^{(3)}(-j)=\frac{(-1)^{jn}}{(j!)^n}\left[n^3\psi^3(j+1)+3n^2\psi(j+1)\zeta'(2,j+1)-2n\zeta(3,j+1)\right]$$

$$U^{(4)}(-j)=\frac{(-1)^{jn}}{(j!)^n}\left[n^4\psi^4(j+1)+6n^3\psi^2(j+1)\zeta'(2,j+1)\right.$$

$$\left.-8n^2\psi(j+1)\zeta(3,j+1)+3n^2\zeta'^2(2,j+1)+6n\zeta'(4,j+1)\right]$$

$$U^{(5)}(-j)=\frac{(-1)^{jn}}{(j!)^n}\left[n^5\psi^5(j+1)+10n^4\psi^3(j+1)\zeta'(2,j+1)\right.$$

$$-20n^3\psi^2(j+1)\zeta(3,j+1)+15n^3\psi(j+1)\zeta'^2(2,j+1)$$

$$+30n^2\psi(j+1)\zeta'(4,j+1)$$

$$\left.-20n^2\zeta'(2,j+1)\zeta(3,j+1)-24n\zeta(5,j+1)\right].$$

Substituting these results in (4.5.34) gives the p.d.f. $h(y)$ of the product $y=\Pi_{i=1}^{6}x_i$ of six independent standardized normal r.v.'s, shown below.

$$h(y)=\frac{1}{(2\pi)^3}\sum_{j=0}^{\infty}\frac{(-1)^{6j}\left(\frac{y^2}{2^6}\right)^j}{(j!)^6}\left[\frac{1}{5!}\left(-\ln\left(\frac{y^2}{2^6}\right)\right)^5\right.$$

$$+\frac{1}{4!}\left(-\ln\left(\frac{y^2}{2^6}\right)\right)^4 6\psi(j+1)$$

$$+\frac{1}{3!2!}\left[-\ln\left(\frac{y^2}{2^6}\right)^3\right]\{36\psi^2(j+1)+6\zeta'(2,j+1)\}+\frac{1}{2!3!}\left(-\ln\left(\frac{y^2}{2^6}\right)\right)^2$$

$$\times\{216\psi^3(j+1)+108\psi(j+1)\zeta'(2,j+1)-12\zeta(3,j+1)\}$$

$$+\frac{1}{4!}\left(-\ln\left(\frac{y^2}{2^6}\right)\right)\{1296\psi^4(j+1)+1296\psi^2(j+1)\zeta'(2,j+1)$$

$$\left.-288\psi(j+1)\zeta(3,j+1)+108\zeta'^2(2,j+1)+36\zeta'(4,j+1)\}\right]$$

$$+\frac{1}{5!}\{7776\psi^5(j+1)+12{,}960\psi^3(j+1)\zeta'(2,j+1)$$

$$-4320\psi^2(j+1)\zeta(3,j+1)+3240\psi(j+1)\zeta'^2(2,j+1)$$

$$+1080\psi(j+1)\zeta'(4,j+1)-720\zeta'(2,j+1)\zeta(3,j+1)-144\zeta(5,j+1)\}. \tag{4.5.44}$$

The graph of $h(y)$ is shown in Fig. 4.5.1 and is tablulated elsewhere [353,358], together with the distribution function $H(y)$. The number of residues $R(y,n,j), j=0,1,2,\ldots,$ which must be evaluated, depends on the accuracy required for $h(y)$ or $H(y)$.

Similarly, by utilizing (4.5.42a) and (4.5.42b), one can obtain the distribution function $H(y)$.

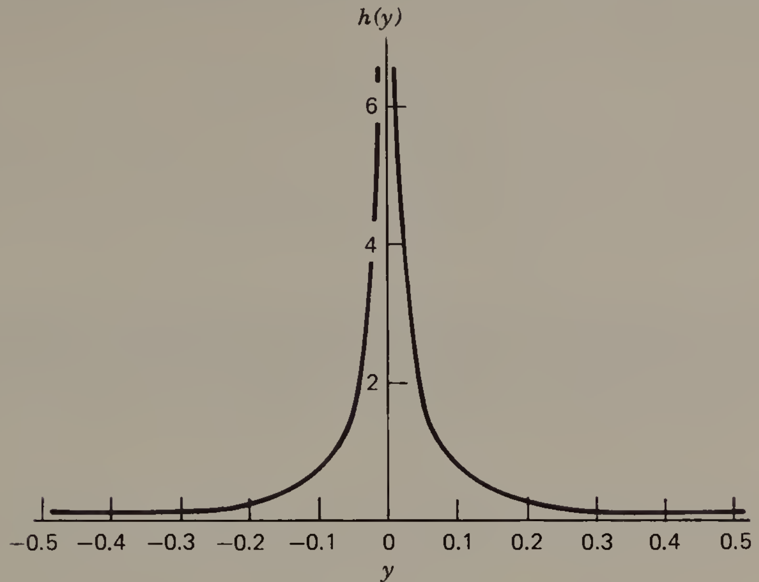


Fig. 4.5.1 P.d.f. of product of six standardized normal r.v.'s.

4.6* THE DISTRIBUTION OF PRODUCTS AND QUOTIENTS OF CONTINUOUS, NONSTANDARDIZED, INDEPENDENT NORMAL RANDOM VARIABLES

As an earlier section pointed out, a change of unit (scale) can be readily accommodated by means of the Mellin transform, but a change of origin cannot. This considerably complicates the derivation of the mathematical form of the distribution of products and quotients of nonstandardized normal i.r.v.'s. Several authors [83, 17, 241] have dealt with products and quotients of nonstandardized normal i.r.v.'s but have left the distribution expressed in integral form (see Exercise 4.27). This section derives the distribution of the product of n nonstandardized normal i.r.v.'s in terms of the parabolic cylinder function (Appendix D.1) and also in series form; the distribution of the quotient is expressed in series form.

4.6.1 Distribution of the Product of Independent, Nonstandardized Normal Random Variables

In general, the Mellin transform of the p.d.f. of products of nonstandardized r.v.'s cannot be *simply* expressed in terms of the Mellin transforms of the corresponding standardized r.v.'s. These facts become painfully evident in the following derivation of the p.d.f. of products of nonstandardized r.v.'s.

Consider, then, the derivation of the product of n nonstandardized normal r.v.'s having p.d.f.'s

$$f_i(x_i)=\frac{1}{\sqrt{2\pi}\,\sigma_i}\exp\left[\frac{-(x_i-\mu_i)^2}{2\sigma_i^2}\right], \qquad -\infty<x_i<\infty, \qquad i=1,2,\ldots,n. \tag{4.6.1}$$

Since each of these p.d.f.'s is of doubly infinite range, it must be partitioned by way of (4.5.1) into the two components $f_i^-(x_i)$ and $f_i^+(x_i)$, with Mellin transforms [95, p. 313, (13)]

$$M_r\big(f_i^{\pm}(\pm x_i)\big)=\frac{e^{-\mu_i^2/4\sigma_i^2}}{\sqrt{2\pi}\,\sigma_i}\sigma_i^r\Gamma(r)D_{-r}\left[\mp\frac{\mu_i}{\sigma_i}\right], \qquad \operatorname{Re}(r)>0, \tag{4.6.2}$$

where $\Gamma(r)$ is the gamma function with complex argument r, and $D_{-r}[\mp\mu_i/\sigma_i]$ is the parabolic cylinder function. This parabolic cylinder function can be expressed in terms of the more manageable gamma function and the confluent hypergeometric function (Appendix D.1); specifically,

$$D_{-r}\left[\mp\frac{\mu_i}{\sigma_i}\right]=2^{-r/2}e^{-\mu_i^2/4\sigma_i^2}$$

$$\times\left[\frac{\Gamma(\frac{1}{2})}{\Gamma[(r+1)/2]}\Phi\left(\frac{r}{2};\frac{1}{2};\frac{\mu_i^2}{2\sigma_i^2}\right)\mp\frac{\mu_i}{\sqrt{2}\,\sigma_i}\frac{\Gamma(-\frac{1}{2})}{\Gamma(r/2)}\Phi\left(\frac{r+1}{2};\frac{3}{2};\frac{\mu_i^2}{2\sigma_i^2}\right)\right]. \tag{4.6.3}$$

By means of the relationship

$$\Gamma(2y)=(2\pi)^{-1/2}2^{2y-1/2}\Gamma(y)\Gamma\left(y+\tfrac{1}{2}\right),$$

(4.6.3) can be reduced to the form

$$D_{-r}\left[\frac{-\mu_i}{\sigma_i}\right]=\frac{2^{r/2}e^{-\mu_i^2/4\sigma_i^2}}{\Gamma(r)}\left[\frac{1}{2}\Gamma\left[\frac{r}{2}\right]\Phi_1(i)\mp\frac{\mu_i}{\sqrt{2}\,\sigma_i}\Gamma\left[\frac{r+1}{2}\right]\Phi_2(i)\right] \tag{4.6.4}$$

where

$$\Phi_1(i)=\Phi\left(\frac{r}{2};\frac{1}{2};\frac{\mu_i^2}{2\sigma_i^2}\right),$$

$$\Phi_2(i)=\Phi\left(\frac{r+1}{2};\frac{3}{2};\frac{\mu_i^2}{2\sigma_i^2}\right),$$

and $\Phi(\cdot)$ denotes the confluent hypergeometric function (Appendix D.1), sometimes denoted by ${}_1F_1(\cdot)$. Then the Mellin transforms of $h^{\pm}(\pm y)$, as given by (4.5.14a, b), respectively, reduce to

$$M_r\left(h_n^{\pm}(\pm y)\right)|=\frac{\exp\left[-\frac{1}{4}\Sigma_i\left(\frac{\mu_i}{\sigma_i}\right)^2\right]}{\left(2\pi\Pi_{i=1}^n\sigma_i^2\right)^{n/2}}$$

$$\times\left[\frac{1}{2^n\Pi_{i=1}^n\sigma_i^2}\right]^{-r/2}2^{n-1}\left[\prod_{i=1}^n A_i\pm\prod_{i=1}^n B_i\right], \quad (4.6.5)$$

where

$$A_i=\frac{1}{2}\Gamma\left(\frac{r}{2}\right)\Phi_1(i),$$

$$B_i=\frac{\mu_i}{\sqrt{2}\,\sigma_i}\Gamma\left(\frac{r+1}{2}\right)\Phi_2(i).$$

To determine the p.d.f. $h(y)$ of the product $Y=\Pi_i^n X_i$ of n noncentral normal variables, it is necessary to evaluate the inversion integral

$$h_n^{\pm}(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}y^{-r}M_r\left(h_n^{\pm}(\pm y)\right)dr, \qquad 0<\mathrm{Re}(r)<1, \quad (4.6.6)$$

which in light of (4.6.5) becomes

$$h_n^{\pm}(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}y^{-r}\frac{\exp\left[-\frac{1}{4}\Sigma_i\left(\frac{\mu_i}{\sigma_i}\right)^2\right]}{(2\pi)^{n/2}\Pi_{i=1}^n\sigma_i}\left[\frac{1}{2^n\Pi_{i=1}^n\sigma_i^2}\right]^{-r/2}$$

$$\times 2^{n-1}\left[\frac{1}{2^n}\Gamma^n\left(\frac{r}{2}\right)\prod_{i=1}^n\Phi_1(i)\pm\frac{\Gamma^n[(r+1)/2]}{2^{n/2}}\prod_{i=1}^n\frac{\mu_i}{\sigma_i}\Phi_2(i)\right]\frac{dr}{2} \quad (4.6.7)$$

Or, equivalently,

$$h^{\pm}(w)=A^*\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} w^{-r/2}\Gamma^n\left(\frac{r}{2}\right)\prod_{i=1}^{n}\Phi_1(i)\frac{dr}{2}\right)$$

$$\pm B^*\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} w^{-r/2}\Gamma^n\left(\frac{r+1}{2}\right)\prod_{i=1}^{n}\Phi_2(i)\frac{dr}{2}\right), \qquad (4.6.8)$$

where

$$A^*=\frac{\exp\left[-\frac{1}{2}\Sigma_i\left(\frac{\mu_i}{\sigma_i}\right)^2\right]}{(2\pi)^{n/2}\Pi_{i=1}^n\sigma_i}, \qquad (4.6.9a)$$

$$B^*=2^{n/2}A^*\prod_{i=1}^{n}\left(\frac{\mu_i}{\sigma_i}\right), \qquad (4.6.9b)$$

$$w=\frac{y^2}{2^n\Pi_i\sigma_i^2}. \qquad (4.6.9c)$$

Letting $s=r/2$, (4.6.8) becomes

$$h^{\pm}(w)=A^*\left[\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} w^{-s}\Gamma^n(s)\prod_{i=1}^{n}\Phi_1(i)\,ds\right]$$

$$\pm B^*\left[\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} w^{-s}\Gamma^n\left(s+\tfrac{1}{2}\right)\prod_{i=1}^{n}\Phi_2(i)\,ds\right], \qquad (4.6.10)$$

where

$$\Phi_1(i)=\Phi_1\left(s;\frac{1}{2};\frac{\mu_i^2}{2\sigma_i^2}\right), \qquad (4.6.11a)$$

$$\Phi_1(i)=\Phi_2\left(s;\frac{3}{2};\frac{\mu_i^2}{2\sigma_i^2}\right). \qquad (4.6.11b)$$

The inversion integrals in (4.6.10) can be evaluated by utilizing the left-hand Bromwich contour $C_L=QKLMPQ$ (Fig. 2.9.1*a*) when $0<|w|\leqslant 1$ and the right-hand Bromwich contour $C_R=QTPQ$ (Fig. 2.9.1*b*), using the transformation $r=-s$ to shift the poles from the LHP to the RHP when $1\leqslant|w|<\infty$. (See Appendix F.2) And since the conditions of Jordan's lemma are satisfied, these integrals evaluated over the Bromwich contours C_L and C_R as $R\to\infty$ are precisely the same as the integrals evaluated over the Bromwich path $(c-i\infty,c+i\infty)$ when $0<|w|\leqslant 1$ and $1\leqslant|w|<\infty$, respectively, In particular,

$$h^{\pm}(w)=\sum_{j=0}^{\infty} R(w;n,j)\pm\sum_{k=0}^{\infty} R'(w;n,k), \tag{4.6.12}$$

where $R(w;n,j)$ denotes the residue of the first integrand in (4.6.10) at the nth order pole $s=-j$, $j=0,1,2,\ldots,$and $R'(w;n,k)$ denotes the residue of the second integrand in (4.6.10) at the nth order pole $s=-k$, $k=\frac{1}{2},\frac{3}{2},\ldots$. More specifically,

$$R(w;n,j)=\frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}\left\{(s+j)^n w^{-s}\Gamma^n(s)\prod_{i=1}^{n}\Phi_1(i)\right\}_{s=-j},$$

$$j=0,1,2,\ldots \tag{4.6.13a}$$

and

$$R'(w;n,k)=\frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}(s+j)^n w^{-s}\Gamma^n\left(\frac{s}{2}+1\right)\prod_{i=1}^{n}\Phi_2(i)\Big|_{s=-k},$$

$$k=\tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2},\ldots. \tag{4.6.13b}$$

Application of Leibniz's rule to (4.6.13a) and (4.6.13b), respectively, yields

$$R(w;n,j)=w^{-s}\sum_{q=0}^{n-1}\frac{1}{(n-1-q)!}\frac{1}{q!}(-\ln w)^{n-1-q}$$

$$\times\frac{d^q}{ds^q}\left[(s+j)^n\Gamma^n(s)\prod_{i=1}^{n}\Phi_1(i)\right]\Big|_{s=-j},$$

$$j=0,1,2,\ldots \tag{4.6.14a}$$

and

$$R'(w;n,k)=w^{-s}\sum_{q=0}^{n-1}\frac{1}{(n-1-q)!}\frac{1}{q!}(-\ln w)^{n-1-q}$$

$$\times\frac{d^q}{ds^q}(s+j)^n\Gamma^n\left(s+\tfrac{1}{2}\right)\prod_{i=1}^{n}\Phi_2(i)\Big|_{s=-k},$$

$$k=\tfrac{1}{2},\tfrac{3}{2},\ldots. \tag{4.6.14b}$$

Through the reapplication of Leibniz's rule, the derivatives

$$\frac{d^q}{ds^q}\left[(s+j)^n\Gamma^m(s)\prod_{i=1}^{n}\Phi_1(i)\right] \tag{4.6.15a}$$

and

$$\frac{d^q}{ds^q}\left(s+j+\tfrac{1}{2}\right)^n\Gamma^n\left(s+\tfrac{1}{2}\right)\prod_{i=1}^{n}\Phi_2(i) \tag{4.6.15b}$$

are expressible recursively in terms of derivatives of lower order, which in turn involves the gamma, digamma, polygamma, and hypergeometric functions. To establish this, it is again helpful to utilize the identity

$$(s+b-1+j)\Gamma(s+b-1)\equiv\frac{\Gamma(s+b+j)}{(s+b-1)(s+b)\cdots(s+b-2+j)} \tag{4.6.16}$$

together with the notation

$$U_b(s,j)=\left(\frac{\Gamma(s+b+j)}{(s+b-1)(s+b)\cdots(s+b-2+j)}\right)^n, \tag{4.6.17}$$

$$U_b^{(q)}(s,j)=\frac{d^q}{ds^q}\left(\frac{\Gamma(s+b+j)}{(s+b-1)(s+b)\cdots(s+b-2+j)}\right)^n. \tag{4.6.18}$$

Then the derivatives (4.6.15a) and (4.6.15b) may be written in the form

$$\frac{d^q}{ds^q}\left[(s+j)^n\Gamma^n(s)\prod_{i=1}^{n}\Phi_1(i)\right]=\sum_{m=0}^{q}\binom{q}{m}U_1^{(m)}(s,j)\left(\prod_{i=1}^{n}\Phi_1(i)\right)^{(q-m)} \tag{4.6.19a}$$

and

$$\frac{d^q}{ds^q}\left[\left(s+j+\tfrac{1}{2}\right)^n\Gamma^n\left(s+\tfrac{1}{2}\right)\prod_{i=1}^{m}\Phi_2(i)\right]$$

$$=\sum_{m=0}^{q}\binom{q}{m}U_{3/2}^{(m)}\left(s+j+\tfrac{1}{2}\right)\left(\prod_{i=1}^{n}\Phi_2(i)\right)^{(q-m)}. \tag{4.6.19b}$$

From Leibniz's rule for the differentiation of the product $U_b(s+b+j)$

$\Pi_{i=1}^{n}\Phi_a(i)$ of two factors, it follows that

$$\frac{d^q}{ds^q}\left[U_b(s,j)\Phi_a(i)\right]=\sum_{m=0}^{q}\binom{q}{m}U_b^{(m)}(s,j)\left(\prod_{i=1}^{n}\Phi_a(i)\right)^{(q-m)}, \qquad a=1,2. \tag{4.6.20}$$

It remains to determine the derivatives $U_b^{(m)}(s,j)$ and $(\Pi_{i=1}^{n}\Phi_a(i))^{(q-m)}$.
Consider first the derivatives $U_b^{(m)}(s,j)$. From (4.6.17),

$$\ln U_b(s,j)=n\ln\Gamma(s+b+j)-\sum_{i=0}^{j-1}\ln(s+b-1+i), \tag{4.6.21}$$

which, on differentiation, yields

$$\begin{aligned}U_b^{(1)}(s,j)&=U_b(s,j)\left(n\Psi(s+b+j)-\sum_{i=0}^{j-1}\frac{1}{s+b-1+i}\right)\\&=U_b(s,j)V_b(s,j),\end{aligned} \tag{4.6.22}$$

where

$$V_b(s,j)=n\Psi(s+b+j)-\sum_{i=0}^{j-1}(s+b-1+i)^{-1}. \tag{4.6.23}$$

Again, application of Leibniz's rule to (4.6.22) gives the general result

$$U_b^{(q+1)}(s)=\sum_{l=0}^{q}\binom{q}{l}U_b^{(q-l)}(s)V_b^{(l)}(s), \qquad q=0,1,\ldots \tag{4.6.24}$$

where

$$\begin{aligned}V_b^{(l)}(s)=n\Psi^{(l)}(s+b+j)+\sum_{i=0}^{j-1}(-1)^{l+1}\\ \times(l+1)!(s+b-1+i)^{-l-1},\end{aligned} \tag{4.6.25}$$

$\Psi^{(1)}(s+b+j)=$digamma function

$\psi^{(l)}(s+b+j)=$polygamma function, $l>1$ (defined by (4.4.36)).

The remaining derivatives that need to be evaluated in (4.6.19a) and (4.6.19b) are $(\Pi_{i=1}^{n}\Phi_{\alpha}(i))^{(r)}$, $r=q-m=0,1,\ldots,q$. The evaluation of these derivatives is accomplished by utilizing Leibniz's rule for differentiating a product of r factors [113], from which it follows that

$$\left[\prod_{i=1}^{n}\Phi_{a}(i)\right]^{(r)}=\sum_{j_1=0}^{r}\sum_{j_2=0}^{j_1}\cdots\sum_{j_n=0}^{j_{n-1}}\binom{r}{j_1}\cdots\binom{j_{n-1}}{j_n}\Phi_{a}^{(r-j_1)}(i)\cdots\Phi_{a}^{(j_{n-1}-j_n)}(n),\tag{4.6.26}$$

where the j_c, $c=1,2,\ldots,n$ must satisfy the constraint $j_1+j_2+\cdots+j_n=r$.

In summary, the residues (4.6.19a) and (4.6.19b) evaluated at the relevant poles are expressible, respectively, as

$$R(w;n,j)=\sum_{q=0}^{n-1}\frac{c_{jq}w^{2j}(-\ln w)^{n-1-q}}{(n-1-q)!q!}\tag{4.6.27}$$

and

$$R(w;n,k)=\sum_{q=0}^{n-1}\frac{d_{kq}w_{q}^{2k}(-\ln w)^{n-1-q}}{(n-1-q)!q!},\tag{4.6.28}$$

where c_{jq} and d_{kq} are obtained by evaluating the derivatives (4.6.19a) and (4.6.19b), respectively, at the values $s=-j$, $j=0,1,2,\ldots,$and $s=-K$, $K=\frac{1}{2},\frac{3}{2},\frac{5}{2}\ldots$. This in turn requires the evaluation of $U_{b}^{(m)}(s,j)$, $b=1,\ \frac{3}{2}$, and $\Pi_{i=1}^{N}\Phi_{a}(i)$, $a=1,2$, by way of (4.6.22) through (4.6.25). The final result in terms of the original variable y (recalling the relationship (4.6.9c)) is

$$h(y)=h^{-}(y),\qquad -\infty<y<0\tag{4.6.29a}$$

$$=h^{+}(y),\qquad 0<y<\infty,\tag{4.6.29b}$$

where the component density functions $h^{-}(y)$ and $h^{+}(y)$ are given by (4.6.7) and are expressible in the exact series forms

$$\begin{aligned}h^{\pm}(y)=A^{*}\sum_{j=0}^{\infty}\sum_{q=0}^{n-1}c_{jq}\left(\frac{y^2}{2^{n}\Pi_{i=1}^{n}\sigma_i^2}\right)^{j}\left[-\ln\left\{\frac{y^2}{2^{n}\Pi_{i=1}^{n}a_i^2}\right\}\right]^{n-1-q}\frac{y}{2^{n-1}\Pi_{i}^{n}\sigma_i}\\ \pm B^{*}\sum_{k=1/2}^{\infty}\sum_{q=0}^{n-1}d_{kq}\left(\frac{y^2}{2^{n}\Pi_{i=1}^{n}\sigma_i^2}\right)^{k}\left(-\ln\frac{y^2}{2^{n}\Pi_{i=1}^{n}\sigma_i^2}\right)^{n-1-q}\frac{y}{2^{n-1}\Pi_{i}^{n}\sigma_i}\end{aligned}\tag{4.6.30}$$

where A^{*} and B^{*} are given by (4.6.9a) and (4.6.9b).

4.6.2 An Alternative Derivation of the Distribution of Products of Independent, Nonstandardized, Normal Random Variables[25]

The form of the p.d.f. $h(y)$ of products of nonstandardized normal r.v.'s as given in the preceding section is obtained from a Mellin inversion integral that involves the rather complex parabolic cylinder function. It is possible to express the Mellin transform of $h(y)$ in terms of gamma functions, as we now show. The evaluation of the resultant inversion integral is still tedious for $n>2$, but it provides a feasible procedure for deriving the distribution of the quotient of nonstandardized normal r.v.'s in analytical form, as the following section demonstrates.

Consider, then, n independent nonstandardized normal r.v.'s $N(\mu_i, \sigma_i)$ with p.d.f.'s

$$f_i(x_i) = \frac{1}{\sqrt{2\pi}\,\sigma_i} \exp\left[-\frac{(x_i-\mu_i)^2}{2\sigma_i^2} \right], \qquad i=1,2,\ldots,n, \qquad -\infty < x_i < \infty. \tag{4.6.31}$$

Without loss of generality (since a scale factor can be absorbed in the Mellin transform), one can utilize the transformed variable $w_i = x_i/\sigma_i$, so that

$$w_i - \mu_i = \frac{x_i - \mu_i}{\sigma_i}, \qquad i=1,2,\ldots,n \tag{4.6.32}$$

and analyze the product

$$w = \prod_{i=1}^{n} w_i. \tag{4.6.33}$$

Then

$$f_i(w_i) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(w_i-\mu_i)^2 \right], \qquad i=1,2,\ldots,n. \tag{4.6.34}$$

Since there is no simple formula relating the Mellin transform of the p.d.f. of a nonstandardized normal r.v. to the Mellin transform of the p.d.f. of a standardized normal r.v., it is natural to reduce (4.6.31) to a form involving

[25] The results in Sections 4.6.2 and 4.6.3 were derived by J. M. Pruett in a doctoral dissertation entitled "The Distribution of Products of Some Independent Nonstandardized Random Variables," University of Arkansas, 1972.

the p.d.f. of a standardized normal variable $N(0,1)$. Writing $f_i(w_i) = (\sqrt{2\pi})^{-1} e^{-\mu_i^2} e^{-w_i^2/2} e^{\mu_i w_i}$ and expanding $\exp[-\frac{1}{2}(w_i-\mu_i)^2]$ in series form, one finds that (4.6.34) reduces to

$$f_i(w_i) = \frac{1}{\sqrt{2\pi}} e^{-\mu_i^2/2} \sum_{j=0}^{\infty} \frac{1}{j!} \mu_i^j w_i^j e^{-w_i^2/2}, \qquad i = 1,2,\dots,n. \quad (4.6.35)$$

Now it is easily shown that

$$M_s(e^{(-w_i)}) = \Gamma(s) \quad (4.6.36)$$

and applying properties 4 and 10, respectively, of Section 2.8.2 to (4.6.36) gives

$$M_s(e^{-w_i^2/2}) = 2^{(s/2)-1}\Gamma\left(\frac{s}{2}\right) \quad (4.6.37)$$

and

$$M_s\left(w_i^j\left[e^{(-w_i^2/2)}\right]\right) = 2^{[(s+j)/2]-1}\Gamma\left(\frac{s+j}{2}\right). \quad (4.6.38)$$

Hence in light of the partitioning procedure defined by (4.5.1),

$$M_s(f_i^+(w_i)) = \frac{1}{\sqrt{2\pi}} 2^{(s/2)-1} e^{-\mu_i^2/2} \sum_{j=0}^{\infty} \frac{1}{j!} (\mu_i\sqrt{2})^j \Gamma\left(\frac{s+j}{2}\right)$$

$$= \frac{1}{\sqrt{2\pi}} 2^{(s/2)-1} e^{-\mu_i^2/2}[E_i + O_i], \quad (4.6.39)$$

where

$$E_i = \sum_{j=0}^{\infty} \frac{1}{(2j)!} (\mu_i\sqrt{2})^{2j} \Gamma\left(\frac{s}{2} + j\right), \quad (4.6.40)$$

$$O_i = \sum_{j=0}^{\infty} \frac{1}{(2j+1)!} (\mu_i\sqrt{2})^{2j+1} \Gamma\left(\frac{s+1}{2} + j\right). \quad (4.6.41)$$

Similarly,

$$M_s(f_i^-(-w_i)) = \frac{1}{\sqrt{2\pi}} 2^{(s/2)-1} e^{-\mu_i^2/2} \sum_{j=0}^{\infty} \frac{1}{j!} (\mu_i\sqrt{2})^j \Gamma\left(\frac{s+j}{2}\right)$$

$$= \frac{1}{\sqrt{2\pi}} 2^{(s/2)-1} e^{\mu_i^2/2}[E_i - O_i]. \quad (4.6.42)$$

Then, with a little algebraic manipulation, it is easily shown by mathematical induction [298, pp. 49–54] that

$$M_s(h^+(w)) = \frac{1}{(2\pi)^{n/2}} 2^{(n/2)(s-2)} \exp\left(\sum_{i=1}^{n} \frac{\mu_i^2}{2}\right)$$

$$\cdot 2^{n-1}(E_1E_2\cdots E_n + O_1O_2\cdots O_n) \tag{4.6.43}$$

$$M_s(h^-(-w)) = \frac{1}{(2\pi)^{n/2}} 2^{(n/2)(s-2)} \exp\left(\sum_{-i=1}^{n} \frac{\mu_i^2}{2}\right)$$

$$\cdot 2^{n-1}(E_1E_2\cdots E_n - O_1O_2\cdots O_n). \tag{4.6.44}$$

Then

$$h^+(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} w^{-s} M_s(h^+(w))\,ds \tag{4.6.45}$$

$$h^-(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} w^{-s} M_s(h^-(-w))\,ds, \tag{4.6.46}$$

which integrals may be evaluated by the residue theorem.

4.6.3 The Distribution of the Quotient of Two Independent, Nonstandardized, Normal Random Variables

One of the important unsolved problems in the field of survey sampling is that of the determination of the distribution of the quotient y/x of two independent, nonstandardized, normal r.v.'s. Perhaps one reason for the elusiveness of the determination of the distribution of this quotient is that none of its absolute moments exists. It is well-known that the ratio of two standard normal i.r.v.'s has a Cauchy distribution, which has the characteristic of having no absolute finite moments. In particular, its even order moments are all infinite, whereas its moments of odd order are zero only because of symmetry. It is not surprising, therefore, that the distribution of the ratio of nonstandardized normal i.r.v.'s also has this property, since the standardized normal r.v. is a special case of a nonstandardized normal r.v.

with mean zero and variance one. That such is in fact the case is now shown. It is also shown that the quotient distribution is in the form of an infinite series involving the parameter ratios μ_1/σ_1 and μ_2/σ_2. Consider, then, the two noncentral normal r.v.'s X_1 and X_2 having the p.d.f.'s.

$$f_i(x_i)=\frac{1}{\sigma_i\sqrt{(2\pi)}}\exp\left[-\frac{1}{2}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2\right],$$

$$-\infty<x_i<\infty,\quad i=1,2, \quad (4.6.47)$$

whose quotient $V=X_1/X_2$ is regarded as the product of the two r.v.'s X_1 and $1/X_2$ with p.d.f.'s $f_1(x_1)$ and $g_2(1/x_2)$, respectively. Then, as previously shown,

$$M_r\left(g_2\left(\frac{1}{x_2}\right)\right)=M_{-r+2}(f_2(x_2)), \quad (4.6.48)$$

where r is a complex variable and the Mellin transform of the p.d.f. $g(v)$ of the quotient $V=X_1/X_2$ is

$$M_r(g(v))=M_r(f_1(x_1))M_{-r+2}(f_2(x_2)). \quad (4.6.49)$$

Evaluation by the method of residues of the inversion integral

$$g(v)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}v^{-r}M_r(g(v))\,dr \quad (4.6.50)$$

yields the desired p.d.f. of V.

To evaluate this inversion integral, one must utilize the partitioning technique defined by (4.5.1) and discussed in Section 4.5.1, since the variables X_i may assume both positive and negative values. Thus

$$M_r[f_i^+(x_i)]=\int_0^\infty x_i^{r-1}f_i^+(x_i)\,dx_i \quad (4.6.51a)$$

$$M_r[f_i^-(-x_i)]=\int_0^\infty x_i^{r-1}f_i^-(-x_i)\,dx_i, \quad (4.6.51b)$$

where, as before, r is a complex variable. Then

$$M_r(f_i^+(x_i)) = \frac{1}{\sqrt{(2\pi)}} \sigma_i^{r-1} 2^{(r/2)-1} e^{-\mu_i^2/2\sigma_i^2} \cdot \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\mu_i \sqrt{2}}{\sigma_i} \right)^j \Gamma\left(\frac{r+j}{2} \right), \tag{4.6.52a}$$

$$M_r(f_i^-(-x_i)) = \frac{1}{\sqrt{(2\pi)}} \sigma_i^{r-1} 2^{(r/2)-1} e^{-\mu_i^2/\sigma_i^2} \cdot \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{-\mu_i \sqrt{2}}{\sigma_i} \right)^j \Gamma\left(\frac{r+j}{2} \right), \tag{4.6.52b}$$

$$M_{-r+2}(f_i^+(x_i)) = \frac{1}{\sqrt{(2\pi)}} \sigma_i^{-r+1} 2^{-r/2} e^{-\mu_i^2/2\sigma_i^2} \cdot \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\mu_i \sqrt{2}}{\sigma_i} \right)^j \Gamma\left(\frac{-r}{2} + \frac{j}{2} + 1 \right), \tag{4.6.52c}$$

and

$$M_{-r+2}(f_i^-(-x_i)) = \frac{1}{\sqrt{(2\pi)}} \sigma_i^{-r+1} 2^{-r/2} e^{-\mu_i^2/2\sigma_i^2} \cdot \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{-\mu_i \sqrt{2}}{\sigma_i} \right)^j \Gamma\left(\frac{-r}{2} + \frac{j}{2} + 1 \right). \tag{4.6.52d}$$

As shown previously (4.5.22a, 4.5.22b) these Mellin transforms have the partitioned form

$$M_r(g^+(v)) = M_r(f_1^+(x_1)) M_{-r+2}(f_2^+(x_2)) + M_r(f_1^-(-x_1)) M_{-r+2}(f_2^-(-x_2)) \tag{4.6.53a}$$

$$M_r(g^-(-v)) = M_r(f_1^+(x_1)) M_{-r+2}(f_2^-(-x_2)) + M_r(f_1^-(-x_1)) M_{-r+2}(f_2^+(x_2)). \tag{4.6.53b}$$

Substituting (4.6.52a–d) for the Mellin transforms in (4.6.53a, b) and utilizing the transformation $s=r/2$, one obtains the results

$$\begin{aligned}
M_s(g^+(v))=&\left[\frac{1}{\sqrt{(2\pi)}}\sigma_1^{2s-1}2^{s-1}e^{-\mu_1^2/2\sigma_1^2}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\mu_1\sqrt{2}}{\sigma_1}\right)\Gamma\left(s+\frac{j}{2}\right)\right]\\
&\cdot\left[\frac{1}{\sqrt{(2\pi)}}\sigma_2^{-2s+1}2^{-s}e^{-\mu_2^2/2\sigma_2^2}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\mu_2\sqrt{2}}{\sigma_2}\right)\Gamma\left(-s+\frac{j}{2}+1\right)\right]\\
&+\left[\frac{1}{\sqrt{(2\pi)}}\sigma_1^{2s-1}2^{s-1}e^{-\mu_1^2/2\sigma_1^2}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{-\mu_1\sqrt{2}}{\sigma_1}\right)\Gamma\left(s+\frac{j}{2}\right)\right]\\
&\cdot\left[\frac{1}{\sqrt{(2\pi)}}\sigma_2^{-2s+1}2^{-s}e^{-\mu_2^2/2\sigma_2^2}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{-\mu_2\sqrt{2}}{\sigma_2}\right)\Gamma\left(-s+\frac{j}{2}+1\right)\right]
\end{aligned}\tag{4.6.54a}$$

and

$$\begin{aligned}
M_s(g^-(-v))=&\left[\frac{1}{\sqrt{(2\pi)}}\sigma_1^{2s-1}2^{s-1}e^{-\mu_1^2/2\sigma_1^2}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\mu_1\sqrt{2}}{\sigma_1}\right)\Gamma\left(s+\frac{j}{2}\right)\right]\\
&\cdot\left[\frac{1}{\sqrt{(2\pi)}}\sigma_2^{-2s+1}2^{-s}e^{-\mu_2^2/2\sigma_2^2}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{-\mu_2\sqrt{2}}{\sigma_2}\right)\Gamma\left(-s+\frac{j}{2}+1\right)\right]\\
&+\left[\frac{1}{\sqrt{(2\pi)}}\sigma_1^{2s-1}2^{s-1}e^{-\mu_1^2/2\sigma_1^2}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{-\mu_1\sqrt{2}}{\sigma_1}\right)\Gamma\left(s+\frac{j}{2}\right)\right]\\
&\cdot\left[\frac{1}{\sqrt{(2\pi)}}\sigma_2^{-2s+1}2^{-s}e^{-\mu_2^2/2\sigma_2^2}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\mu_2\sqrt{2}}{\sigma_2}\right)\Gamma\left(-s+\frac{j}{2}+1\right)\right].
\end{aligned}\tag{4.6.54b}$$

The resultant p.d.f. $g(v)$ is then

$$g(v)=g^+(v)+g^-(v),\tag{4.6.55}$$

where

$$g^{+}(v)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} v^{-s}M_s(g^{+}(v))\,ds, \tag{4.6.56a}$$

$$g^{-}(v)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} v^{-s}M_s(g^{-}(-v))\,ds, \tag{4.6.56b}$$

and $M_s(g^{+}(v))$ and $M_s(g^{-}(v))$ are given by (4.6.54a) and (4.6.54b), respectively. If one now employs the transformation

$$w=\frac{x_1/\sigma_1}{x_2/\sigma_2}=\left(\frac{\sigma_2}{\sigma_1}\right)v, \tag{4.6.57}$$

letting $M_1=\mu_1/\sigma_1$, $M_2=\mu_2/\sigma_2$, then (4.6.56a) and (4.6.56b) become, respectively,

$$\begin{aligned} h^{+}(w)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}&\frac{1}{2\pi}(w^2)^{-s}\exp\left[-\tfrac{1}{2}(M_1^2+M_2^2)\right]\\ &\times\left\{\left[\sum_{j=0}^{\infty}\frac{1}{j!}(M_1\sqrt{2}\,)^j\Gamma\left(s+\frac{j}{2}\right)\right]\right.\\ &\cdot\left[\sum_{j=0}^{\infty}\frac{1}{j!}(M_2\sqrt{2}\,)^j\Gamma\left(-s+\frac{j}{2}+1\right)\right]\\ &+\left[\sum_{j=0}^{\infty}\frac{1}{j!}(-M_1\sqrt{2}\,)^j\Gamma\left(s+\frac{j}{2}\right)\right]\\ &\left.\cdot\left[\sum_{j=0}^{\infty}\frac{1}{j!}(-M_2\sqrt{2}\,)^j\Gamma\left(-s+\frac{j}{2}+1\right)\right]\right\}ds \end{aligned} \tag{4.6.58a}$$

and

$$\begin{aligned} h^{-}(w)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}&\frac{1}{2\pi}(w^2)^{-s}\exp\left[-\tfrac{1}{2}(M_1^2+M_2^2)\right]\\ \cdot\left\{\left[\sum_{j=0}^{\infty}\frac{1}{j!}(M_1\sqrt{2}\,)^j\Gamma\left(s+\frac{j}{2}\right)\right]\cdot\right.&\left[\sum_{j=0}^{\infty}\frac{1}{j!}(-M_2\sqrt{2}\,)^j\Gamma\left(-s+\frac{j}{2}+1\right)\right]\\ +\left[\sum_{j=0}^{\infty}\frac{1}{j!}(-M_1\sqrt{2}\,)^j\Gamma\left(s+\frac{j}{2}\right)\right]\cdot&\left.\left[\sum_{j=0}^{\infty}\frac{1}{j!}(M_2\sqrt{2}\,)^j\Gamma\left(-s+\frac{j}{2}+1\right)\right]\right\}ds. \end{aligned} \tag{4.6.58b}$$

As Section 4.6.2 indicated, one can separate each summation in (4.6.58a, b) into two components, one of which consists of a sum $E_i(s,M_i)$ of even powers of the variables, and the other a sum $O_i(s,M_i)$ of odd powers of the parameters M_i, $i=1,2$, where

$$E_1(s,M_1)=\sum_{j=0}^{\infty}\frac{1}{(2j)!}(M_1\sqrt{2})^{2j}\Gamma(s+j), \tag{4.6.59a}$$

$$O_1(s,M_1)=\sum_{j=0}^{\infty}\frac{1}{(2j+1)!}(M_1\sqrt{2})^{2j+1}\Gamma\left(s+j+\frac{1}{2}\right), \tag{4.6.59b}$$

$$E_2(s,M_2)=\sum_{j=0}^{\infty}\frac{1}{(2j)!}(M_2\sqrt{2})^{2j}\Gamma(-s+j+1), \tag{4.6.59c}$$

$$O_2(s,M_2)=\sum_{j=0}^{\infty}\frac{1}{(2j+1)!}(M_2\sqrt{2})^{2j+1}\Gamma\left(-s+j+\frac{3}{2}\right). \tag{4.6.59d}$$

Note also that

$$E_i(s,M_i)=E_i(s,-M_i),\qquad i=1,2, \tag{4.6.60a}$$

$$O_i(s,M_i)=-O_i(s,-M_i),\qquad i=1,2, \tag{4.6.60b}$$

so that (4.6.58a) and (4.6.58b) are expressible as

$$h^{+}(w)=\frac{\exp\left[-\frac{1}{2}(M_1^2+M_2^2)\right]}{2\pi}\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}w^{-2s}\left[A(s)+B(s)\right]ds\right\} \tag{4.6.61a}$$

and

$$h^{-}(w)=\frac{\exp\left[-\frac{1}{2}(M_1^2+M_2^2)\right]}{\pi}\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}w^{-2s}\left[A(s)-B(s)\right]ds\right\}, \tag{4.6.61b}$$

where

$$A(s)=\left(\sum_{j=0}^{\infty}\frac{1}{(2j)!}(M_1\sqrt{2})^{2j}\Gamma(s+j)\right)\times\left(\sum_{k=0}^{\infty}\frac{1}{(2k)!}(M_2\sqrt{2})^{2k}\Gamma(-s+k+1)\right) \tag{4.6.62}$$

and

$$B(s)=\left(\sum_{j'=0}^{\infty}\frac{1}{(2j'+1)!}\left(M_1\sqrt{2}\right)^{2j'+1}\Gamma\left(s+j'+\frac{1}{2}\right)\right)$$

$$\times\left(\sum_{k'=0}^{\infty}\frac{1}{(2k'+1)!}\left(M_2\sqrt{2}\right)^{2k'+1}\Gamma\left(-s+k'+\frac{3}{2}\right)\right). \qquad (4.6.63)$$

The procedure is to evaluate the residues at the poles of $A(s)$ and $B(s)$ and then use the residue theorem to evaluate the inversion integrals. To show that the residue theorem is applicable, partition the form of $h^+(w)$ into two parts: $h_1^+(w)$, which is valid when $0<w<1$, and $h_2^+(w)$, which is valid when $1<w<\infty$. Similarly, $h^-(w)$ is structured into two components: $h_1^-(w)$, which is applicable when $-1<w<0$, and $h_2^-(w)$, which is applicable when $-\infty<w<-1$.

Note that in view of the identity

$$\begin{aligned} w^{-2s} &\equiv e^{\ln(w^{-2s})} \\ &\equiv e^{-(2\ln w)s}, \end{aligned} \qquad (4.6.64)$$

the integral in (4.6.61a) is expressible in the form

$$\int_{c-i\infty}^{c+i\infty} e^{-(2\ln w)s}\left[A(s)+B(s)\right]ds. \qquad (4.6.65)$$

Thus if one restricts the values of w to the range $0<w<1$, the coefficient of s, namely, $-2\ln w$, is positive so that the conditions of Jordan's lemma are satisfied relative to the closed contour $C_L=QKLMPQ$ (Fig. 2.9.1a), and the equality

$$\int_{C_L} e^{-(2\ln w)s}\left[A(s)+B(s)\right]ds=\int_{c-i\infty}^{c+i\infty} e^{-(2\ln w)s}\left[A(s)+B(s)\right]ds \qquad (4.6.66)$$

holds. Therefore,

$$\begin{aligned} h_1^+(w) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} w^{-2s}\left[A(s)+B(s)\right]ds \\ &= \left(\sum_j R_j+\sum_{j'} R_{j'}\right), \qquad 0<w<1, \end{aligned} \qquad (4.6.67)$$

where R_j denotes the residues at the poles located at $s=-j$, $j=-\frac{1}{2},-\frac{3}{2},\ldots$ and $R_{j'}$ those at the poles located at $s=-(j'+\frac{1}{2}), j'=0,1,2,\ldots$.

For values $1<w<\infty$, the coefficient of s in the kernel $e^{-(2\ln w)s}$ is negative, so that the conditions of Jordan's lemma are satisfied relative to the closed contour $C_R = QTPQ$ (Fig. 2.9.1b), and the equality

$$\int_{C_R} e^{-(2\ln w)s}\left[A(s)+B(s)\right]ds = \int_{c-i\infty}^{c+i\infty} e^{-(2\ln w)s}\left[A(s)+B(s)\right]ds$$

holds. Therefore,

$$h_2^+(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} w^{-2s}\left[A(s)+B(s)\right]ds$$

$$= \left(\sum_k R_k + \sum_{k'} R_{k'}\right), \qquad 1<w<\infty, \tag{4.6.68}$$

where R_k denotes the residues at the poles located at $s=k+1, k=-1, -2, \ldots,$ and $R_{k'}$ those at the poles located at $s=k'+\frac{3}{2}, k'=-\frac{3}{2}, -\frac{5}{2}, \ldots.$ The same reasoning is applicable to $h^-(w)$ as given in Equation (4.6.61b), so that

$$h^-(w) = h_1^-(w) + h_2^-(w), \tag{4.6.69}$$

where

$$h_1^-(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-(2\ln(-w))s}\left[A(s)-B(s)\right]ds$$

$$= \left(\sum_j R_j - \sum_{j'} R_{j'}\right), \qquad -1<w<0 \tag{4.6.70a}$$

$$h_2^-(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-(2\ln(-w))s}\left[A(s)-B(s)\right]ds$$

$$= \left(\sum_k R_k - \sum_{k'} R_{k'}\right), \qquad -\infty<w<-1, \tag{4.6.70b}$$

where the residues R_j, $R_{j'}$, R_k, and $R_{k'}$, have already been defined. Finally, by a systematic arrangement of the residues [298, pp. 66–73] in the sums above (considerably simplified by the fact that only first order poles occur), $h_1^{\mp}(w)$ and $h_2^{\mp}(w)$ are expressible as the following series:

$$h_1^{\mp}(w)=\frac{1}{2\pi}\exp\left[-\frac{1}{2}\left(\frac{\mu_1^2}{\sigma_1^2}+\frac{\mu_2^2}{\sigma_2^2}\right)\right]$$

$$\cdot\left\{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(2j)!}\frac{1}{(2k)!}\left(\frac{\mu_1\sqrt{2}}{\sigma_1}\right)^{2j}\left(\frac{\mu_2\sqrt{2}}{\sigma_2}\right)^{2k}\right.$$

$$\cdot\sum_{l=0}^{\infty}w^{2(j+l)}\frac{\Gamma(j+k+l+1)}{l!(-1)^l}$$

$$\pm\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(2m+1)!}\frac{1}{(2n+1)!}\left(\frac{\mu_1\sqrt{2}}{\sigma_1}\right)^{2m+1}\cdot\left(\frac{\mu_2\sqrt{2}}{\sigma_2}\right)^{2n+1}$$

$$\left.\cdot\sum_{p=0}^{\infty}w^{2m+2p+1}\frac{\Gamma(m+n+p+2)}{p!(-1)^p}\right\},\qquad 0<|w|<1 \tag{4.6.71a}$$

and

$$h_2^{\mp}(w)=\frac{1}{\pi}\exp\left[-\frac{1}{2}\left(\frac{\mu_1^2}{\sigma_1^2}+\frac{\mu_2^2}{\sigma_2^2}\right)\right]$$

$$\times\left\{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(2j)!}\frac{1}{(2k)!}\left(\frac{\mu_1\sqrt{2}}{\sigma_1}\right)^{2j}\left(\frac{\mu_2\sqrt{2}}{\sigma_2}\right)^{2k}\right.$$

$$\times\sum_{l=0}^{\infty}w^{-2(k+l+1)}\frac{\Gamma(j+k+l+1)}{l!(-1)^l}$$

$$\pm\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(2m+1)!}\frac{1}{(2n+1)!}\left(\frac{\mu_1\sqrt{2}}{\sigma_1}\right)^{2m+1}\left(\frac{\mu_2\sqrt{2}}{\sigma_2}\right)^{2n+1}$$

$$\left.\times\sum_{p=0}^{\infty}w^{-2n-2p-3}\frac{\Gamma(m+n+p+2)}{p!(-1)^p}\right\},\qquad 1<|w|<\infty. \tag{4.6.71b}$$

In summary, the p.d.f. $h(w)$ of the quotient $W=(X_1/\sigma_1)/(X_2/\sigma_2)$ of two nonstandardized normal r.v.'s is given by

$$h(w)=h_2^-(w), \qquad -\infty<w<-1 \tag{4.6.72a}$$

$$=h_1^-(w), \qquad -1<w<0 \tag{4.6.72b}$$

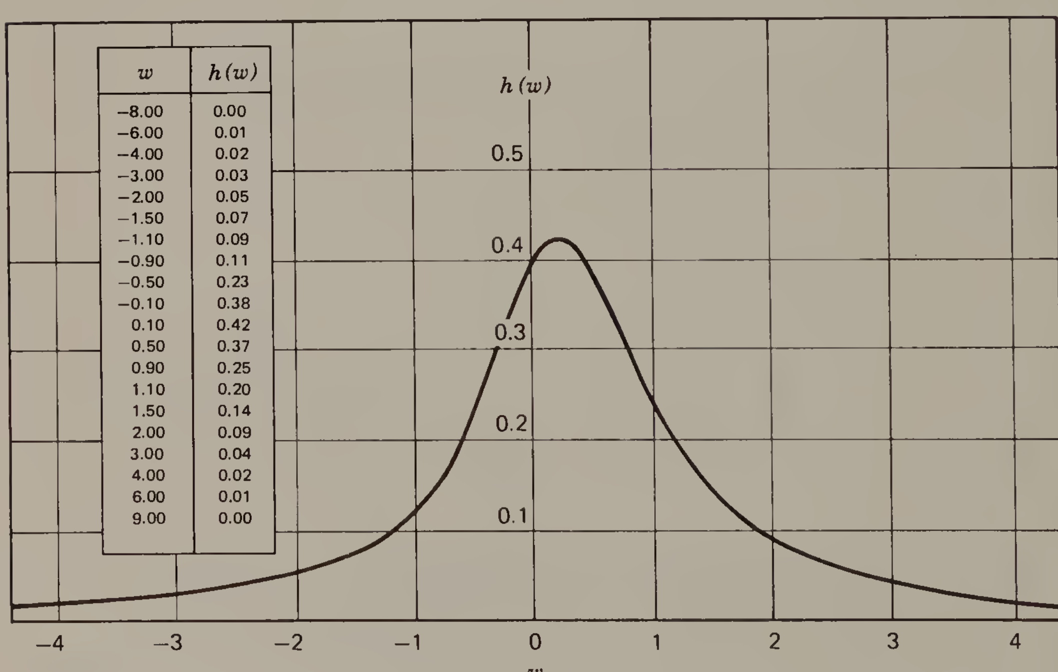
$$=h_1^+(w), \qquad 0<w<1 \tag{4.6.72c}$$

$$=h_2^+(w), \qquad 1<w<\infty \tag{4.6.72d}$$

and $h_1^{\pm}(w)$, $h_2^{\pm}(w)$ are given by (4.6.71a) and (4.6.71b), respectively. A computer program for evaluating these density functions has been written and is operational [298, Appendix C]. There are removable discontinuities at $w=-1,0,1$. The p.d.f. of the quotient $V=X_1/X_2$ may be obtained from these equations through the use of the simple transformation $V=(\sigma_1/\sigma_2)W$.

An example is presented for which the ratio $M_1/M_2=(\mu_1/\sigma_1)/(\mu_2/\sigma_2)$ is 0.50. Figure 4.6.1 plots the p.d.f. with removable discontinuities at $w=-1,0,1$.

Note that the moments of the distributions $g^-(v)$ and $g^+(v)$, as given by the Mellin transforms (4.6.54a, b), do not exist, since $M_s(g^-(-v))$ and $M_s(g^+(v))$ are infinite for $s=-j/2, j=0,1,2,\ldots$ and $s=j/2+1, j=2,4,6,\ldots$.



w	$h(w)$
−8.00	0.00
−6.00	0.01
−4.00	0.02
−3.00	0.03
−2.00	0.05
−1.50	0.07
−1.10	0.09
−0.90	0.11
−0.50	0.23
−0.10	0.38
0.10	0.42
0.50	0.37
0.90	0.25
1.10	0.20
1.50	0.14
2.00	0.09
3.00	0.04
4.00	0.02
6.00	0.01
9.00	0.00

Fig. 4.6.1 P.d.f. of the quotient of two normal i.r.v.'s, $M_1/M_2=(m_1/\sigma_1)/(m_2/\sigma_2)=0.5$.

4.7 THE DISTRIBUTION OF PRODUCTS OF DISCRETE INDEPENDENT RANDOM VARIABLES

In the analysis of discrete r.v.'s the probability mass function (p.m.f.), rather than the p.d.f., is used, in which probabilities rather than probability densities are associated with the possible values of the r.v. Thus if $p(x)$ is the probability mass function for the discrete r.v. x, which may assume the values $x=x_j, j=1,2,\ldots,n$, then $p(x_j)$ gives the probability that the value $x=x_j$ will be chosen if a value of the r.v. is selected at random from the n possible values. If m r.v.'s $x_i, i=1,2,\ldots,m$ are considered, in which the ith r.v. can assume the n_i values $x_{ij}, j=1,2,\ldots,n_i$, then the p.m.f. $p(y)$ of the product r.v. $y=\Pi_{i=1}^{m}x_i$ consists of the probabilities

$$p(y), \qquad y=\{y_k\}, \tag{4.7.1}$$

where $\{y_k\}$ denotes the set of products $x_{1j_1}x_{2j_2}\cdots x_{mj_m}$, each product in the set having the value y_k and being formed by taking one and only one value from each of the sets $\{x_{ij}\}, j=1,2,\ldots,n_i;\ i=1,2,\ldots,m$. The probability distribution function is then

$$P(y_k)=\sum_{s\leqslant k} p\{y_s\}, \tag{4.7.2}$$

where $\{y_s\}$ denotes a possible product $x_1x_2\cdots x_m \leqslant y_s$ for each possible value $y_s \leqslant y_k$ of the product r.v. (y).

Consider, for example, the distribution of the product $Y=X_1X_2$ of two binomial i.r.v.'s with p.m.f.'s

$$p(x_1)=\binom{2}{x_1}p_1^{x_1}(1-p_1)^{2-x_1}, \qquad x_1=0,1,2 \tag{4.7.3a}$$

and

$$p(x_2)=\binom{1}{x_2}p_2^{x_2}(1-p_2)^{1-x_2}, \qquad x_2=0,1. \tag{4.7.3b}$$

The p.m.f. $p(y)$, then, gives the probability that $y=x_1x_2$, where x_1 and x_2, respectively, are the number of successes resulting when two types of Bernoulli trial are carried out, in which the respective probabilities of success in a given trial are p_1 and p_2.

Since x_1 is an element of $\{0,1,2\}$ and x_2 is an element of $\{0,1\}$, then for $y=0$,

$$\begin{aligned} p(y)=p_1(i)\,p_2(j), \quad & i=0, \qquad j=0,1 \\ & i=1,2, \qquad j=0, \end{aligned}$$

where $p_1(i)$ is (4.7.3a) evaluated for $x_1 = i$ and $p_2(j)$ is (4.7.3b) evaluated for $x_2 = j, i = 0, 1, 2; j = 0, 1$. Specifically,

$$p(0) = p_1(0)p_2(0) + p_1(0)p_2(1) + p_1(1)p_2(0) + p_1(2)p_2(0)$$
$$= (1-p_1)^2(1-p_2) + (1-p_1)^2 p_2 + 2p_1(1-p_1)(1-p_2) + p_1^2(1-p_2). \tag{4.7.4a}$$

Similarly,

$$p(1) = p_1(1)p_2(1)$$
$$= 2p_1p_2(1-p_1) \tag{4.7.4b}$$
$$p(2) = p_1(2)p_2(1)$$
$$= p_1^2 p_2. \tag{4.7.4c}$$

The probability distribution for the product $Y = X_1X_2$ is, then,

$$p(y) = (1-p_1)^2(1-p_2) + (1-p_1)^2 p_2 + 2p_1(1-p_1)(1-p_2) + p_1^2(1-p_2),$$
$$y = 0 \tag{4.7.5a}$$
$$= 2p_1p_2(1-p_1), \qquad y = 1 \tag{4.7.5b}$$
$$= p_1^2 p_2, \qquad y = 2. \tag{4.7.5c}$$

As a partial check on the validity of the probability distribution (4.7.5a–c), note that since x_1 and x_2 are independent,

$$E[y] = E[x_1]E[x_2]$$
$$= n_1p_1n_2p_2$$
$$= n_1n_2p_1p_2$$
$$= 2p_1p_2 \tag{4.7.6}$$

$$\sigma^2 = E[y^2] - E[y]^2$$
$$= E[x_1^2]E[x_2^2] - n_1^2n_2^2p_1^2p_2^2$$
$$= 2p_1p_2(1+p_1) - 4p_1^2p_2^2$$
$$= 2p_1p_2 + 2p_1^2p_2 - 4p_1^2p_2^2. \tag{4.7.7}$$

Now, by definition,

$$E[y] = \sum_{\text{All } y} y p(y)$$

$$= 0p(0) + 1p(1) + 2p(2)$$

$$= 2p_1 p_2$$

and

$$\sigma_y^2 = E[y^2] - (E[y])^2$$

$$= 0^2 p(0) + 1^2 p(1) + 2^2 p(2) - (E[y])^2$$

$$= 2p_1 p_2 + 2p_1^2 p_2 - 4p_1^2 p_2^2,$$

which results are identical—as they must be—with (4.7.6) and (4.7.7), respectively.

4.8* THE DISTRIBUTION OF PRODUCTS AND QUOTIENTS OF DEPENDENT RANDOM VARIABLES

Thus far, the analysis of products and quotients by means of Mellin transforms has been limited to independent r.v.'s. Reed [304] and Fox [111] appear to have been among the first to develop the bivariate generalization of Mellin transforms. Later, Subrahmanian [371] combined the earlier results of Springer and Thompson [354] with those of Fox to establish an analogy between the bivariate techniques and the univariate (independent) methods. His results are summarized in this section. Although the use of Mellin transforms, under relevant conditions, may be extended to the general case of products of n dependent r.v.'s, only products (and quotients) of two dependent r.v.'s are considered here.

4.8.1 The Two-Dimensional Mellin Transform

Let (U, V) be a two-dimensional r.v. for which U and V are not independent, and let (U, V) have the joint p.d.f. $f(u,v)$ that is positive in the first quadrant and zero elsewhere. Fox [111] has defined the Mellin transform of $f(u,v)$ as

$$M(s_1, s_2) = \int_0^\infty \int_0^\infty u^{s_1 - 1} v^{s_2 - 1} f(u,v)\, du\, dv \tag{4.8.1}$$

with the inverse

$$f(u,v)=\frac{1}{(2\pi i)^2}\int_{h-i\infty}^{h+i\infty}\int_{k-i\infty}^{k+i\infty}u^{-s_1}v^{-s_2}M(s_1,s_2)\,ds_1\,ds_2. \tag{4.8.2}$$

The conditions under which (4.8.1) and (4.8.2) are valid are stated without proof in the theorems that follow. The proofs are given by Fox [111].

Theorem 4.8.1. If

(i) $M(s_1,s_2)$ is a regular function of both the variables s_1, s_2 in the strips $a<s_1<b$, $c<s_2<d$ for some a,b,c,d.
(ii) in these strips $|M(s_1,s_2)|=O(|s_1|^{-m})O(|s_2|^{-n})$ for some $m>0$, $n>0$, as $|s_1|$ and $|s_2|$ tend to infinity, independently of each other;
(iii) $a<h<b$ and $c<k<d$;
(iv) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|M(s_1,s_2)|\,|ds_1|\,|ds_2|$ exists when taken along any lines parallel to the imaginary axes in the strips defined in (i);
(v) $f(u,v)$ is defined by the equation

$$f(u,v)=\frac{1}{(2\pi i)^2}\int_{h-i\infty}^{h+i\infty}\int_{k-i\infty}^{k+i\infty}u^{-s_1}v^{-s_2}M(s_1,s_2)\,ds_1\,ds_2, \tag{4.8.3}$$

then

$$M(s_1,s_2)=\int_0^\infty\int_0^\infty u^{s_1-1}v^{s_2-1}(f(u,v))\,du\,dv \tag{4.8.4}$$

is true.

Theorem 4.8.2. Let U denote a part of the complex $u=(u_1'+iu_2')$ plane, which is bounded by two lines through the origin and includes the whole of the positive real u-axis from 0 to $+\infty$. Let V denote a similar region in the complex $v=(v_1'+iv_2')$ plane. If with u in U and v in V the following conditions are satisfied:

(i) there exist two real numbers h and k so that $u^hv^kf(u,v)$ is a regular function of both u and v;
(ii) $u^hv^kf(u,v)=O(|\log u|^{-m})O(|\log v|^{-n})$, $m>0$, $n>0$, as u and v tend independently either to zero or to infinity;
(iii) $\int\int|u^hv^kf(u,v)|\,|du|\,|dv|$ exists, when taken along any lines in the U and V regions extending from the origin to infinity;

(iv) $M_{s_1,s_2}(f(u,v))$ is defined by (4.8.4);
then the equation

$$f(u,v)=\frac{1}{(2\pi i)^2}\int_{h-i\infty}^{h+i\infty}\int_{k-i\infty}^{k+i\infty}u^{-s_1}v^{-s_2}M_{s_1,s_2}(f(u,v))\,ds_1 ds_2 \quad (4.8.5)$$

is true.

Only two cases are discussed here.

1. If $Y=UV$, then $h(y)$, the p.d.f. of Y, has the Mellin transform

$$M_{s_1,s_2}(h(y))=M_{s,s}(h(y))=M_{s,s}(f(u,v)). \quad (4.8.6)$$

2. If $W=U/V$, then $g(w)$, the p.d.f. of W, has the Mellin transform

$$M_{s_1,s_2}(g(w))=M_{s;-s+2}(g(w))=M_{s,-s+2}(f(u,v)). \quad (4.8.7)$$

Extensions of (4.8.1) and (4.8.2) to the case where U and V are not everywhere positive can be readily accomplished by the method discussed in Section 4.5. In particular, denote by $M^{++}(s_1,s_2)$, $M^{+-}(s_1,s_2)$, $M^{-+}(s_1,s_2)$, and $M^{--}(s_1,s_2)$ the Mellin transform of $f(u,v)$ in the four quadrants. Then, after assigning the appropriate sign to the negative variable involved, one can write the expressions for the Mellin transform of $h(y)$ and $g(w)$. Specifically, the Mellin transform of $h(y)$ is

$$M_s^+(h(y))=M^{++}(s,s)+M^{--}(s,s), \qquad y>0 \quad (4.8.8a)$$

$$M_s^-(h(-y))=M^{+-}(s,s)+M^{-+}(s,s), \qquad y<0 \quad (4.8.8b)$$

while that for $g(w)$ is

$$M_s^+(g(w))=M^{++}(s,-s+2)+M^{--}(s,-s+2), \qquad w>0 \quad (4.8.9a)$$

$$M_s^-(g(-w))=M^{+-}(s,-s+2)+M^{-+}(s,-s+2), \qquad w<0. \quad (4.8.9b)$$

Inversion of the Mellin transforms (4.8.8b, a) and (4.8.9b, a), which are functions of the single complex variable s along the real line, in the manner discussed in Section 4.2, yields the p.d.f.'s

$$h(y)=h^-(y), \qquad -\infty<y<0 \quad (4.8.10a)$$

$$=h^+(y), \qquad 0<y<\infty \quad (4.8.10b)$$

$$g(w)=g^-(w), \qquad -\infty<w<0 \quad (4.8.11a)$$

$$=g^+(w), \qquad 0<w<\infty. \quad (4.8.11b)$$

4.8.2 The Case of the Symmetrical Distribution

If the joint distribution $f(u,v)$ satisfies the conditions of Theorem 4.8.2 and is symmetrical about both the u and v axes, namely,

$$f(-u,v)=f(-u,-v)=f(u,-v)=f(u,v), \tag{4.8.12}$$

then

$$M^{++}(s_1,s_2)=M^{+-}(s_1,s_2)=M^{-+}(s_1,s_2)=M^{--}(s_1,s_2), \tag{4.8.13}$$

so that

$$M_{s_1,s_2}(f(u,v))=4M^{++}(s_1,s_2), \tag{4.8.13a}$$

If $Y=UV$, inversion of (4.8.13a) yields

$$h(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 4y^{-s}M^{++}(s,s)\,ds. \tag{4.8.14}$$

More generally [111], if the n pairs of bivariate r.v.'s (U_i,V_i) have the joint p.d.f.'s $f_i(u_i,v_i)$, $i=1,2,\ldots,n$, which are all symmetrical about the axes of (u_i) and v_i, the joint p.d.f. of the pair of r.v.'s $u=\prod_1^n u_i, v=\prod_1^n v_i$ is

$$f(u,v)=\frac{1}{(2\pi i)^2}\int_{h-i\infty}^{h+i\infty}\int_{k-i\infty}^{k+i\infty} 4^{n-1}u^{-s_1}v^{-s_2}\prod_{i=1}^{n} M_i^{++}(s_1,s_2)\,ds_1,ds_2, \tag{4.8.15}$$

where

$$M_i^{++}(s_1,s_2)=\int_0^\infty\int_0^\infty u_i^{s_1-1}v_i^{s_2-1}f_i(u_i,v_i)\,du_i\,dv_i \tag{4.8.16}$$

and $M_i^{++}(s_1,s_2)$, $f_i(u_i,v_i)$ satisfy the conditions of Theorems 4.8.1 and 4.8.2, respectively.

The following example, which illustrates how the method of Mellin transforms applies to the derivation of the distribution of products of dependent r.v.'s, is due to Subrahmanian [371].

4.8.3 The Bivariate Normal Distribution

Consider the standardized form of the bivariate normal distribution, namely,

$$f(x_1,x_2)=\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{\frac{-[x_1^2-2\rho x_1x_2+x_2^2]}{2(1-\rho^2)}\right\}, \tag{4.8.17}$$

where ρ denotes the coefficient of correlation between x_1 and x_2. Since (4.8.17) is a symmetrical p.d.f., (4.8.13) holds. Substituting in (4.8.1) and integrating with respect to x_1 (see ref. 95, (13), p. 313), one obtains

$$M^{++}(s_1,s_2)=\frac{\Gamma(s_1)}{2\pi}(1-\rho^2)^{(s_1-1)/2}$$
$$\times\int_0^\infty x_2^{s_2-1}\exp\left\{\frac{-(2-\rho^2)}{4(1-\rho^2)}x_2^2\right\}D_{-s_1}\left[\frac{-\rho x_2}{(1-\rho^2)^{1/2}}\right]dx_2, \tag{4.8.18}$$

where $D_v(y)$ is the parabolic cylinder function (Appendix D.1). Using the integral equation [98, (11), p. 121]

$$\int_0^\infty e^{-zt}t^{-1+\beta/2}D_{-s_1}\left[2(kt)^{1/2}\right]dt=\frac{2^{1-\beta-s_1/2}\pi^{1/2}\Gamma(\beta)}{\Gamma[(s_1+\beta+1)/2]}(z+k)^{-\beta/2}$$
$$\times F\left(\frac{s_1}{2},\frac{\beta}{2};\frac{s_1+\beta+1}{2},\frac{z-k}{z+k}\right),$$
$$\operatorname{Re}(\beta)>0,\qquad \operatorname{Re}\left(\frac{z}{k}\right)>0 \tag{4.8.19}$$

with $x_2=2t^{1/2}$, where $F(a,b,c,z)$ is Gauss's hypergeometric series (Appendix D.1), one can complete the integration in (4.8.18) to obtain

$$M^{++}(s_1,s_2)=\frac{\Gamma(s_1)\Gamma(s_2)(1-\rho^2)^{((s_1+s_2+1)/2)-1}}{2^{(s_1+s_2+2)/2}\Gamma(\frac{1}{2})\Gamma((s_1+s_2+1)/2)}$$
$$\cdot {}_2F_1\left[\frac{s_1}{2},\frac{s_2}{2};\frac{s_1+s_2+1}{2};1-\rho^2\right], \tag{4.8.20}$$

where ${}_2F_1(a,b;c,x)$ is Gauss's hypergeometric series expressed in the notation of the generalized hypergeometric series (see Appendix D.1).

From (4.8.6) and (4.8.20), it follows that the Mellin transform of the p.d.f. $h(y)$ of the product $Y=X_1X_2$ is

$$M_s(h^+(y))=M_s(h^-(y))$$
$$=\frac{[\Gamma(s)]^2(1-\rho^2)^{s-(1/2)}}{2^s\Gamma(\frac{1}{2})\Gamma(s+\frac{1}{2})}{}_2F_1\left[\frac{s}{2},\frac{s+1}{2};s+\tfrac{1}{2};1-\rho^2\right]. \tag{4.8.21}$$

Then, from (4.8.8a, b) and from specializing the transform in formula (29) of ref. 95 (p. 331) for $\nu=0$, $\beta=1$, one obtains

$$h(y)=\frac{1}{\pi(1-\rho^2)^{1/2}}\exp\left[\frac{\rho|y|}{1-\rho^2}\right]K_0\left[\frac{|y|}{1-\rho^2}\right], \qquad -\infty<y<\infty, \quad (4.8.22)$$

where $K_0(x)$ is the modified Bessel function of the second kind of order zero (Appendix D.1).

Similarly, one can obtain the p.d.f. $g(w)$ of the quotient $W=X_1/X_2$ of two dependent r.v.'s having the p.d.f. (4.8.17). Specifically, from (4.8.9a, b) it follows that the Mellin transform of $g(w)$ is (4.8.20) with $s_1=s$, $s_2=-s+2$, so that

$$M_s(g^+(w))=\frac{\Gamma(s)\Gamma(-s+2)(1-\rho^2)^{1/2}}{2\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}\,{}_2F_1\left[\frac{s}{2},\frac{-s}{2}+1;\frac{3}{2};1-\rho^2\right]. \quad (4.8.23)$$

Then, using equations due to Erdelyi [97, (12), p. 101] and [95, (12), p. 309], it follows that

$$g(w)=\frac{(1-\rho^2)^{1/2}}{\pi}\,\frac{1}{w^2-2\rho w+1}, \qquad -\infty<w<\infty, \qquad (4.8.24)$$

as has been obtained with other methods by Fieller [102] and Craig [71].

The distribution of y was obtained with other methods by Craig [69], Aroian [17], and Taneja, Cornwell, and Aroian [373]. Craig and Aroian considered the product $v=x_1x_2/\sigma_1\sigma_2$, where x_1 and x_2 follow a bivariate normal distribution $N(\mu_1,\mu_2;\sigma_1,\sigma_2;\rho)$. They obtained the distribution of v for various values of δ_1, δ_2, and ρ, where $\delta_1=\mu_1/\sigma_1$, $\delta_2=\mu_2/\sigma_2$. Taneja et al. derived the mathematical forms of the distribution of v for five new cases: (1) δ_1,δ_2,ρ; (2) $\delta_1=0,\delta_2,\rho$; (3) $\delta_1=\delta_2=\delta,\rho$; (4) $\delta_1=\delta_2=0,\rho=1$; (5) $\delta_1=\delta_2=\delta,\rho=1$. Special cases include the normal distribution and both the central and noncentral chi-square distributions for one degree of freedom. They also proved that if $\delta_1=\delta_2=\delta$ and δ becomes large, the standardized distribution of the product approaches a Pearson type III distribution. Their results can be used to provide tables for $\rho=0$ (already available) and for $\rho\neq 0$.

EXERCISES

4.1 Show that the p.d.f. $h(y)$ of the product $Y=\Pi_1^n X_j$ of n identically distributed uniform i.r.v.'s with p.d.f.

$$f(x)=\frac{1}{2a}, \qquad -a\leqslant x\leqslant a$$

is

$$h(y)=\frac{a^{-n}}{2}\cdot\frac{1}{(n-1)!}\left(\ln\left|\frac{a^n}{y}\right|\right)^{n-1}, \qquad 0<|y|\leqslant a^n$$

$$=0, \qquad |y|\geqslant a^n.$$

4.2 Let X_1 and X_2 be two uniform i.r.v.'s with means $\mu_1>\mu_2>\frac{1}{2}$ and with p.d.f.'s

$$f_i(x_i)=1, \qquad |x_i-\mu_i|\leqslant\tfrac{1}{2}, \qquad i=1,2$$

$$=0, \qquad \text{otherwise.}$$

Show that the product $Y=X_1X_2$ has p.d.f.

$$g(y)=0, \qquad y>(\mu_1+\tfrac{1}{2})(\mu_2+\tfrac{1}{2})$$

$$=-\ln\frac{y}{(\mu_1+\frac{1}{2})(\mu_2+\frac{1}{2})}, \quad (\mu_1-\tfrac{1}{2})(\mu_2+\tfrac{1}{2})<y<(\mu_1+\tfrac{1}{2})(\mu_2+\tfrac{1}{2})$$

$$=-\ln\frac{\mu_1-\frac{1}{2}}{\mu_1+\frac{1}{2}}, \qquad (\mu_1+\tfrac{1}{2})(\mu_2-\tfrac{1}{2})<y<(\mu_1-\tfrac{1}{2})(\mu_2+\tfrac{1}{2})$$

$$=-\ln\frac{(\mu_1-\frac{1}{2})(\mu_2-\frac{1}{2})}{y}, \quad (\mu_1-\tfrac{1}{2})(\mu_2-\tfrac{1}{2})<y<(\mu_1+\tfrac{1}{2})(\mu_2-\tfrac{1}{2})$$

$$=0, \qquad 0<y<(\mu_1-\tfrac{1}{2})(\mu_2-\tfrac{1}{2}).$$

4.3* Show that the p.d.f. $h(y)$ of the product $Y=X_1X_2$ of a uniform and Cauchy r.v. with p.d.f.'s

$$f_1(x_1)=\frac{1}{2a}, \qquad -a\leqslant x_1\leqslant a$$

$$=0, \qquad |x_1|>a$$

and

$$f_2(x_2)=\frac{c}{\pi(c^2+x_2^2)}, \qquad -\infty<x_2<\infty,$$

respectively, is

$$h(y)=\frac{1}{2\pi ac}\ln\left[\frac{(a^2c^2+y^2)}{y^2}\right], \qquad -\infty<y<\infty.$$

Hint. Note that $M_s(h^+(y))=[-(ac)^{s-1}/2s]\csc(\pi s/2), 0<\mathrm{Re}(s)<2$ or equivalently, (from ref. 95, (11), p. 307)

$$M_s\left(y\frac{dh^+(y)}{dy}\right)=\frac{(ac)}{2}^{s-1}\csc\left(\frac{\pi s}{2}\right).$$

Now observe that this is the Mellin transform of the Cauchy density function $f_2(x_2)$ with c replaced by ac.

4.4* Show that the p.d.f. $h(y)$ of the product $Y=X_1X_2$ of a uniform and gamma r.v. with p.d.f.

$$f_1(x_1)=\frac{1}{2a}, \qquad -a<x_1<a$$

and

$$f_2(x_2)=\frac{x_2^c e^{-x_2}}{\Gamma(c+1)}, \qquad 0\leqslant x_2<\infty,$$

respectively, is

$$h(y)=\frac{1}{2ac}\left[1-\frac{\Gamma(c,|y/a|)}{\Gamma(c)}\right], \qquad -\infty<\frac{y}{a}<\infty,$$

where $\Gamma(c,y/a)$ denotes the incomplete gamma function with parameter c [283].

In Exercises 4.5 to 4.9, the Cauchy i.r.v.'s are identically distributed with p.d.f.

$$f(x)=\frac{a}{\pi(a^2+x^2)}, \qquad -\infty<x<\infty.$$

4.5 Show that the p.d.f. $h(y)$ of the product $Y=X_1X_2$ of two Cauchy i.r.v.'s is

$$h(y)=\frac{a^2}{\pi^2(y^2-a^4)}\ln\left(\frac{y^2}{a^4}\right), \qquad -\infty<y<\infty.$$

4.6 Show that the p.d.f. $h(v)$ of the quotient $V=X_1/X_2$ of two Cauchy i.r.v.'s is identical with the p.d.f. $h(y)$ of the product.

4.7 Show that the p.d.f. $h(y)$ of the product $Y=X_1X_2X_3$ of three identically distributed Cauchy i.r.v.'s is

$$h(y)=\frac{a^3}{2!\pi^3(y^2+a^6)}\left[\left\{\ln\left(\frac{y^2}{a^6}\right)\right\}^2+\pi^2\right],\qquad -\infty<y<\infty.$$

4.8 Show that the p.d.f. $h(y)$ of the product $Y=\Pi_1^5 X_j$ of five Cauchy i.r.v.'s is

$$h(y)=\frac{a^5}{4!\pi^5(y^2+a^{10})}\left[\left\{\ln\left(\frac{y^2}{a^{10}}\right)\right\}^4\right.$$
$$\left.+10\pi^2\left\{\ln\left(\frac{y^2}{a^{10}}\right)\right\}^2+9\pi^4\right],\qquad -\infty<y<\infty.$$

4.9 Show that the p.d.f. $h(y)$ of the product $Y=\Pi_1^{10} X_j$ of 10 independent Cauchy i.r.v.'s is

$$h(y)=\frac{a^{10}}{9!\pi^{10}(y^2-a^{20})}\left[\left\{\ln\left(\frac{y^2}{a^{20}}\right)\right\}^9+120\pi^2\left\{\ln\left(\frac{y^2}{a^{20}}\right)\right\}^7\right.$$
$$+4368\pi^4\left\{\ln\left(\frac{y^2}{a^{20}}\right)\right\}^5+52480\pi^6\left\{\ln\left(\frac{y^2}{a^{20}}\right)\right\}^3$$
$$\left.+147{,}456\pi^8\left\{\ln\left(\frac{y^2}{a^{20}}\right)\right\}\right].$$

4.10 Verify that each of the p.d.f.'s $h(y)$ in Exercises 4.5 to 4.9 does in fact satisfy the relationship

$$\int_{-\infty}^{\infty} h(y)\,dy=1.$$

4.11 Given the three beta i.r.v.'s with p.d.f.

$$f(x_j)=\frac{\Gamma(a_j+b_j)}{\Gamma(a_j)\Gamma(b_j)}x_j^{a_j-1}(1-x_j)^{b_j-1},$$

where $a_1=5, b_1=2, a_2=6, b_2=2, a_3=6, b_3=3$, show that the p.d.f. $h(y)$ of the r.v. $Y=X_1X_2X_3$ is

$$h(y)=17{,}640\left[2y^4-33y^5-18y^5\ln y-6y^5(\ln y)^2\right.$$
$$\left.-12y^6\ln y+30y^6+y^7\right], \qquad 0\leqslant y\leqslant 1.$$

[The crux of the problem is the evaluation of the K_{kj}'s in (4.4.23) and (4.4.24). Once these are obtained, $h(y)$ follows immediately from (4.4.21).] Show also that the distribution function is

$$H(y)=7056y^5-89{,}180y^6+79{,}920y^7+2205y^8$$
$$-47{,}040y^6\ln y-17{,}640y^6(\ln y)^2-30{,}240y^7\ln y.$$

4.12 Use the Mellin transform to prove that $Y=-2(\ln \prod_{j=1}^{n} X_j)$ has a chi-square distribution with $2n$ degrees of freedom, where X_i, $i=1,2,\ldots,n$ are uniform i.r.v.'s each having p.d.f.

$$f(x)=1, \qquad 0\leqslant x\leqslant 1$$
$$=0, \qquad \text{otherwise.}$$

4.13 Show that the p.d.f. $h(y)$ of the product $Y=X_1X_2$ of two normal i.r.v.'s $N(0,1)$ is

$$h(y)=\frac{1}{\pi}K_0(y), \qquad -\infty<y<\infty,$$

where $K_0(y)$ is Bessel's function of the second kind with a purely imaginary argument of zero order (see Appendix D.1).

4.14 Let $V_1, V_2, \ldots, V_m, X_1, X_2, \ldots, X_n$ be $(m+n)$ normal i.r.v.'s each having zero mean and variance σ^2. Define $Y=V/W$ where $V=\Sigma_1^m X_i^2$, $W=\Sigma_{i=1}^n X_i^2$. Prove that Y has the F p.d.f.

$$h(y)=\frac{\Gamma((m+n)/2)}{\Gamma(m/2)\Gamma(n/2)}\frac{y^{m/2-1}}{(y+1)^{(m+n)/2}}.$$

4.15 Let $X_0, X_1, X_2, \ldots, X_n$ be $n+1$ normal i.r.v.'s $N(0,1)$, and define $V=(\Sigma_{i=1}^n X_i^2/n)^{1/2}$. Show that $Y=X_0/V$ has the Student t p.d.f.

with n degrees of freedom:

$$h(y)=\frac{1}{\sqrt{n\pi}}\,\frac{\Gamma[(n+1)/2]}{\Gamma(n/2)}\left(1+\frac{y^2}{n}\right)^{-(n+1)/2}, \qquad -\infty<y<\infty.$$

4.16 Let X_1 and X_2 be i.r.v.'s, each exponentially distributed with parameter $\lambda=\frac{1}{2}$ (see Table D.2). Use the method of integral transforms to show that the p.d.f. $h(y)$ of the quotient $Y=X_1/X_2$ is

$$\begin{aligned} h(y)&=(1+y)^{-2}, && y>0\\ &=0, && \text{otherwise.}\end{aligned}$$

4.17* Let $X_j, j=1,2,3,4$ be i.r.v.'s, each exponentially distributed with parameter λ. Find the p.d.f. $h(y)$ of the r.v. $Y=\dfrac{X_1+X_2}{X_3+X_4}$.

4.18 Consider the power r.v. X with parameter α and having p.d.f.

$$\begin{aligned} f(x)&=(\alpha+1)x^{\alpha}, && 0\leqslant x\leqslant 1, \quad \alpha \text{ real}\\ &=0, && \text{otherwise.}\end{aligned}$$

(This is a particular member of the family of beta p.d.f.'s. (4.4.11) with $a_1=\alpha+1, b_1=1$.) Show that the p.d.f. $h(y)$ of the product $Y=\Pi_{i=1}^{n}X_i$ of n such power r.v.'s is

$$\begin{aligned} h(y)&=\frac{(\alpha+1)^n}{(n-1)!}\,y^{\alpha}\left(\ln\frac{1}{y}\right)^{n-1}, && 0\leqslant y\leqslant 1, \quad y\neq 0\\ &=0, && \text{otherwise.}\end{aligned}$$

4.19 Establish that the quotient of two power r.v.'s with parameter α has p.d.f.

$$\begin{aligned} g(w)&=\left(\frac{\alpha+1}{2}\right)w^{\alpha}, && 0<w\leqslant 1\\ &=\left(\frac{\alpha+1}{2}\right)w^{-\alpha-2}, && 1\leqslant w<\infty\\ &=0; && \text{elsewhere.}\end{aligned}$$

4.20* Let X_1 and X_2 be two independent power r.v.'s with p.d.f.'s

$$f_1(x_1)=\frac{(m+1)x_1}{b_1^{m+1}-a_1^{m+1}}, \qquad \begin{array}{l} a_1 \leqslant x_1 \leqslant b_1 \\ a_1 \geqslant 0 \end{array}$$

$$=0 \qquad \text{otherwise.}$$

$$f_2(x_2)=\frac{(m+1)x_2}{b_2^{m+1}-a_2^{m+1}}, \qquad \begin{array}{l} a_2 \leqslant x_2 \leqslant b_2 \\ a_2 \geqslant 0, \end{array}$$

$$=0 \qquad \text{otherwise.}$$

Also, let $b_1 > b_2 > a_1 > a_2$ and $b_1 b_2 > b_1 a_2 > b_2 a_1 > a_1 a_2$. Show that the product $Y = X_1 X_2$ has p.d.f.

$$\begin{aligned} g(y) &= -A(y)\ln\left(\frac{y}{b_1 b_2}\right), && b_1 a_2 \leqslant y \leqslant b_1 b_2 \\ &= A(y)\left[-\ln\left(\frac{y}{b_1 b_2}\right)+\ln\left(\frac{y}{b_1 a_2}\right)\right], && a_1 b_2 \leqslant y \leqslant b_1 a_2 \\ &= A(y)\left[-\ln\left(\frac{y}{b_1 b_2}\right)+\ln\left(\frac{y}{b_1 a_2}\right)+\ln\left(\frac{y}{b_2 a_1}\right)\right], && a_1 a_2 \leqslant y \leqslant a_1 b_2 \\ &= 0 && \text{elsewhere} \end{aligned}$$

where

$$A(y)=\frac{(m+1)^2 y^m}{\left[(b_1 b_2)^{m+1}-(b_1 a_2)^{m+1}-(b_2 a_1)^{m+1}+(a_1 a_2)^{m+1}\right]}$$

Verify that

$$\int_{\substack{\text{Range} \\ \text{of } y}} g(y)\,dy = 1 \qquad \text{(Pruett, 1972)}$$

4.21* Let X be a power r.v. with p.d.f.

$$f(x)=c\varepsilon^{-c}x^{c-1}, \qquad 0 \leqslant x \leqslant \varepsilon, \qquad \varepsilon>0, \qquad c \geqslant 1.$$

Let $X_{(i)}, X_{(j)}, i<j$, be the ith and jth order statistics in a random sample of size n. Prove that the r.v. $Y = X_{(i)}X_{(j)}$ has the p.m.f.

$$h(y) = \sum_{r=0}^{j-i-1} \sum_{s=0}^{n-j} (-1)^{r+s} \binom{j-i-1}{r} \binom{n-j}{s}$$

$$\times \frac{n!\, c\varepsilon^{-c(j+s+1)+1}}{(i-1)!(j-i-1)!(n-j)!(j-2i-2r+s)!}$$

$$\cdot y^{c(r+i)-1}\left[\varepsilon^{c(j-2i-2r+s)} - y^{c/2(j-2i-2r+s)}\right],$$

where $0<i<j\leqslant n$, $\varepsilon>0$, $c\geqslant 1$, and $0\leqslant y\leqslant \varepsilon$.

(Malik and Trudel 1970)

Hint. First obtain the joint probability mass distribution $p(x_{(i)}, x_{(j)})$ and then evaluate the Mellin (product) convolution $h(y)$.

4.22* Prove that the p.d.f. $g(w)$ of the quotient $W = X_{(i)}/X_{(j)}$, $i<j$, of the power r.v.'s $X_{(i)}, X_{(j)}$ in Exercise 4.21 is

$$g(w) = \frac{c}{B(i,j-i)} w^{(c_i-1)}(1-w^c)^{j-i-1},$$

$$0\leqslant w \leqslant 1, \qquad 0<i<j\leqslant n, \qquad c\geqslant 1.$$

(Malik and Trudel, 1976)

Hint. Proceed from the joint probability mass distribution $g(x_{(i)}, x_{(j)})$ to the Mellin quotient convolution.

4.23* (a) Setting $j=i+1$ in Exercises 4.21 and 4.22, find the probability mass functions of the product $Y = X_{(i)}X_{(i+1)}$ and quotient $W = X_{(i)}/X_{(i+1)}$ of consecutive order statistics.
(b) Setting $i=1, j=n$ in Exercises 4.21 and 4.22, respectively, find the p.m.f. of the product $Y = X_{(i)}X_{(j)}$ and quotient $W = X_{(i)}/X_{(j)}$ of the extreme order statistics.

(Malik and Trudel, 1976)

4.24* Let X_1 denote the uniform r.v. with p.d.f.

$$f_1(x_1) = \frac{1}{2a}, \qquad -a<x_1<a$$

$$=0, \qquad \text{elsewhere}$$

and X_2 the normal r.v.

$$f_2(x_2)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-x_2^2/2\sigma^2}, \qquad -\infty<x_2<\infty,$$

and assume X_1 and X_2 are independent. Show that the r.v. $Y=X_1X_2$ has p.d.f.

$$h(y)=-\frac{1}{2\sqrt{2\pi}}\,Ei\left[-\frac{1}{2}\left(\frac{y}{a\sigma}\right)^2\right], \qquad -\infty<y<\infty,$$

where $-Ei(-x)$ denotes the exponential integral [98, p. 143]; that is,

$$E_1(x)=-Ei(-x)=\int_x^\infty\frac{e^{-t}}{t}\,dt, \qquad x>0.$$

Hint. Note that

$$M_s(h^+(y))=\frac{1}{2\sqrt{\pi}}(2a^2\sigma^2)^{(s-1)/2}\frac{\Gamma(s/2)}{s}$$

Also (see ref. 95, (11), p. 307),

$$M_s\left(y\frac{d}{dy}\{h^+(y)\}\right)=-sM_s(h^+(y)),$$

so that

$$\frac{d}{dy}\{h^+(y)\}=-\frac{1}{\sqrt{2\pi}\,a\sigma}\,\frac{\exp\left[-\frac{1}{2}\left(\frac{y}{a\sigma}\right)^2\right]}{y}$$

Integration now leads to the desired result.

4.25 Show that the quotient $Y=X_1/X_2$ of two independent gamma r.v.'s with p.d.f.

$$f(x_i)=\frac{1}{\Gamma(b_i)}x_i^{b-1}e^{-x_i}, \qquad b_i>0, \qquad 0\leqslant x_i<\infty$$

is a *beta r.v. of the second kind* with p.d.f.

$$h(y)=\frac{y^{b_1-1}}{B(b_1,b_2)(1+y)^{b_1+b_2}}, \qquad 0\leqslant y<\infty.$$

4.26 Let X_1, X_2, X_3, and X_4 be uniform i.r.v.'s with p.d.f.

$$f(x_i)=1, \qquad 0\leqslant x_i\leqslant 1, \qquad i=1,2,3,4.$$

Section 3.1 demonstrated that $U=X_1+X_2$ and $W=X_3+X_4$ each has a triangular distribution. Show that the p.d.f. $g(v)$ of the quotient

$$V=\frac{U}{W}$$

of two such triangular i.r.v.'s is given by

$$g(v)=\begin{cases} \dfrac{7v}{6}, & 0\leqslant v\leqslant \dfrac{1}{2} \\ \dfrac{8}{3}-\dfrac{3v}{2}-\dfrac{2}{3v^2}+\dfrac{1}{6v^3}, & \dfrac{1}{2}\leqslant v\leqslant 1 \\ -\dfrac{2}{3}+\dfrac{v}{6}+\dfrac{8}{3v^2}-\dfrac{3}{2v^3}, & 1\leqslant v\leqslant 2 \\ \dfrac{7}{6v^3}, & 2\leqslant v<\infty. \end{cases}$$

(Locker and Perry, 1962)

4.27* Show[26] that the p.d.f. $h(Y)$ of the product $Y=Y_1Y_2$ of two normal i.r.v.'s $N(0,\sigma)$ and $N(\mu,\sigma)$ is

$$h(y)=\frac{e^{-\mu^2/2\sigma^2}}{\pi\sigma^2}\sum_{k=0}^{\infty}\left(\frac{\mu}{\sigma^2}\right)^k\frac{1}{(2k)!}\left(\frac{y}{\sigma^2}\right)^k K_{k/2}\left(\frac{y}{\sigma^2}\right), \qquad -\infty<y<\infty.$$

($K_{k/2}(y/\sigma^2)$ is a modified Bessel function of the third kind of order $k/2$, defined in Appendix D.1.)

Hint. Since

$$f_1(x_1)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-x_1^2/2\sigma^2}, \qquad \text{it follows that}$$

$$M_s(f_1^+(x_1)=M_s(f_1^-(-x_1))=\frac{2^{(s-3)/2}}{\sqrt{\pi}}\Gamma(\tfrac{s}{2})\sigma^{s-1}. \qquad \text{Also}$$

$$f_2(x_2)=\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[\frac{-(x_2-\mu)^2}{2\sigma^2}\right]$$

[26] I am indebted to Dr. W. E. Thompson for suggesting and solving this problem.

which can be written in the form

$$f_2(x_2)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\mu^2/2\sigma^2}\exp\left(\frac{-x_2^2}{2\sigma^2}+\frac{x_2\mu}{\sigma^2}\right)$$

$$=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\mu^2/2\sigma^2}e^{-x_2^2/2\sigma^2}\sum_{k=0}^{\infty}\left(\frac{\mu}{\sigma^2}\right)^k\frac{x_2^k}{k!}.$$

Then

$$M_s(f_2^+(x_2))=e^{-\mu^2/2\sigma^2}\sum_{k=0}^{\infty}\left(\frac{\mu}{\sigma^2}\right)^k\frac{1}{k!}M_{s+k}(f_1^+(x_1))$$

and

$$M_s(f_2^-(x_2))=e^{-\mu^3/2\sigma^2}\sum_{k=0}^{\infty}\left(\frac{-\mu}{\sigma^2}\right)^k\frac{1}{k!}M_{s+k}(f_1^+(x_1)),$$

from which it follows (as can be shown) that

$$M_s(h^+(y))=M_s(h^-(y))$$

$$=e^{-\mu^2/2\sigma^2}\sum_{k=0}^{\infty}2\left(\frac{\mu}{\sigma^2}\right)^{2k}\frac{1}{(2k)!}M_{s+2k}(f_1^+(x_1))M_s(f_1^+(x_1)).$$

The inversion of $M_s(h^+(y))$ then yields $h(y)$, $-\infty<y<\infty$.

4.28* Let X_1 and X_2 be normally distributed i.r.v.'s with means θ_i, variances $\sigma_i^2(i=1,2)$ and correlation coefficient ρ. Let $W=X_1/X_2$ and denote its p.d.f. and distribution function by $g(w)$ and $G(w)$, respectively.

(*a*) Show that

$$g(w)=\frac{b(w)d(w)}{\sqrt{2\pi}\,\sigma_1\sigma_2a^3(w)}\left[\alpha\left\{\frac{b(w)}{\sqrt{(1-\rho^2)a(w)}}\right\}-\alpha\left\{-\frac{b(w)}{\sqrt{1-\rho^2a(w)}}\right\}\right]$$

$$+\frac{\sqrt{1-\rho^2}}{\pi\sigma_1\sigma_2a^2(w)}\exp\left\{-\frac{c}{2(1-\rho^2)}\right\},$$

where

$$a(w)=\left(\frac{w^2}{\sigma_1^2}-\frac{2\rho w}{\sigma_1\sigma_2}+\frac{1}{\sigma_2^2}\right)^{1/2},$$

$$b(w)=\frac{\theta_1 w}{\sigma_1^2}-\frac{\rho(\theta_1+\theta_2 w)}{\sigma_1\sigma_2}+\frac{\theta_2}{\sigma_2^2},$$

$$c=\frac{\theta_1^2}{\sigma_1^2}-2\rho\frac{\theta_1\theta_2}{\sigma_1\sigma_2}+\frac{\theta_2^2}{\sigma_2^2},$$

$$d(w)=\exp\left\{\frac{b^2(w)-ca^2(w)}{2(1-\rho^2)a^2(w)}\right\},$$

$$\alpha(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y}e^{-u^2/2}\,du.$$

(*b*) Show, by direct calculation, that

$$G(w)=L\left\{\frac{\theta_1-\theta_2 w}{\sigma_1\sigma_2 a(w)},\frac{-\theta_2}{\sigma_2};\frac{\sigma_2 w-\rho\sigma_1}{\sigma_1\sigma_2 a(w)}\right\}$$

$$+L\left\{\frac{\theta_2 w-\theta_1}{\sigma_1\sigma_2 a(w)},\frac{\theta_2}{\sigma_2};\frac{\sigma_2 w-\rho\sigma_1}{\sigma_1\sigma_2 a(w)}\right\},$$

where

$$L(h,k;\gamma)=\frac{1}{2\pi\sqrt{1-\gamma^2}}\int_h^\infty\int_k^\infty \exp\left\{-\frac{(x^2-2\gamma xy+y^2)}{2(1-\gamma^2)}\right\}dx\,dy$$

is the standard bivariate normal integral tabulated by the National Bureau of Standards [270].

(*c*) Show that as $\theta_2/\sigma_2\to\infty$, that is, as

$$\Pr(X_2>0)\to 1,$$

$$G(w)\to\alpha\left\{\frac{\theta_2 w-\theta_1}{\sigma_1\sigma_2 a(w)}\right\}. \qquad \text{(Hinkley, 1969)}$$

4.29* Let $u=y^2/(2B)$ be a gamma-distributed r.v. with parameter p, and $v=x^2/A$ an independently distributed beta r.v. of the first kind with parameters $q,p-q$ (i.e., with p.d.f. (4.4.11) with $a_1=q+1, b_1=p-q+1$), where $0<q<p$ and $A,B,>0$.

(*a*) Prove that the product of these r.v.'s is gamma distributed with parameter q.

(Stuart, 1962)

(*b*) Show that for the special case $q=\frac{1}{2}$, it follows that $(2uv)^{1/2}$ is exactly normally distributed with mean zero and variance 1; or equivalently, that xy is exactly normal with mean zero and variance AB.

(Stuart, 1962)

(*c*) Show that if $q=p-q=1$, one obtains the corollary that the product of a gamma variable (parameter 2) with an independent rectangular variable on (0, 1) is gamma distributed with parameter 1 (i.e., is exponentially distributed).

(Stuart, 1962)

(*d*) By setting $p=(n-1)/2, q=\frac{1}{2}, A=(n-1)^2/n, B=1/(n-1), x=(x_i-\bar{x})/s, y=s'$ in part (*a*), show that $(x_i-\bar{x})/s\cdot s'$ is exactly normally distributed with zero mean and variance $(n-1)/n$.

(Stuart, 1962, and Durbin, 1961, with editorial note by E. S. Pearson)

4.30* (Distribution of the ratio of sums of Laguerre polynomials.) Let $X_{1,1},\ldots,X_{1,n}$, be a random sample from a population with p.d.f.

$$f_1(x)=\Gamma^{-1}(m_1+1)e^{-x}x^{m_1}\sum_{i=0}^{k_1} a_{1,i}L_i^{m_1}(x), \qquad x \geqslant 0,$$

where $m_1>-1, k$, is a positive integer, $a_{1,0}=1, a_{1,1},\ldots,a_1,k_1$ are suitably chosen constants, and $L_i^{m_i}(x)$ is the Laguerre polynomial of degree i and order m that is,

$$L_i^{m_i}(x)=\sum_{j=0}^{i}\binom{i+m_1}{i-j}\frac{(-x)^j}{j!}$$

Similarly, let $X_{2,1},\ldots,x_{2,n_2}$ be a second independent random sample from a population with density function

$$f_2(x)=\Gamma^{-1}(m_2+1)e^{-x}x^{m2}\sum_{i=0}^{k_2} a_{2,i}L_i^{m_2}(x), \qquad x \geqslant 0,$$

with $m_2 > -1, k_2$ a positive integer, $a_{2,0}=1, a_{2,1}, \ldots, a_{2,k_2}$ suitably chosen constants, and $L_i^{m_2}(x)$ the Laguerre polynomial of order m_2 and degree i. Zelen and Dannemiller [417] showed that the r.v.

$$S_i = X_{i,1} + \cdots + X_{i,n_i}, \qquad i=1,2$$

has p.d.f.

$$g_i(s) = \sum_{j=0}^{k_i n_i} \Gamma^{-1}(N_i + j)\alpha_{i,j} j! \exp(-s_s^{N_i-1}) L_j^{N_i-1}(s), \qquad s \geqslant 0,$$

for $i=1,2$, where $N_i = n_i(1+m_i)$ and $\alpha_{i,j}$ is the coefficient of x^j in

$$\left\{1 + \binom{m_i+1}{1}\alpha_{i1}x + \cdots + \binom{m_i+k}{k}\alpha_{ik}x^k\right\}^{n_i},$$

namely,

$$\alpha_{ij} = \sum_{j_1+2j_2+\cdots+k_i j_{k_i}=j} \cdots \sum \frac{n_i!}{j_1! \cdots j_{k_i}!(n_i - j_1 - \cdots - j_{k_i})!}$$

$$\cdot\left[\binom{m_i+1}{1}_{\alpha i,1}\right]^{j_1} \cdots \left[\binom{m_i+k_i}{k_i}_{\alpha i,ki}\right]^{j_{k_i}}.$$

Show that the p.d.f. of the r.v. $U = S_1/S_2$ is

$$h(u) = \sum_{i=0}^{k_1 n_1} \sum_{j=0}^{k_2 n_2} \alpha_{1,i}\alpha_{2,j} \sum_{p=0}^{i} \sum_{q=0}^{j} \binom{i}{p}\binom{j}{q}(-1)^{p+q}$$

$$\cdot B^{-1}(N_1+p, N_2+q)u^{p+N_2-1}(1+u)^{-(p+q+N_1+N_2)}, \qquad u \geqslant 0,$$

where $B(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$,

and that the distribution function is

$$H(u) = \sum_{i=0}^{k_1 n_1} \sum_{j=0}^{k_2 n_2} \alpha_{1,i}\alpha_{2,j} \sum_{p=0}^{i} \sum_{q=0}^{j} \binom{i}{p}\binom{j}{q}(-1)^{p+q}$$

$$\cdot I_{u/(u+1)}(N_1+p, N_2+q), \qquad u \geqslant 0,$$

where $I_x(a,b)$ is the incomplete beta function (4.4.42).

(Basu and Lochner, 1971)

4.31* Let $U_1, U_2, \ldots, U_n, V_1, V_2, \ldots, V_m$ be i.r.v.'s, each uniform over $(0,1)$. Show that

$$P\left[\frac{u_1+u_2+\cdots+u_n}{v_1+v_2+\cdots+v_m}<a\right]=\frac{1}{(n+m)!a^m}\sum_{i=0}^{[ma]}\sum_{j=0}^{[(ma-i)/a]}$$

$$\times(-1)^{i+j}\binom{n}{i}\binom{m}{j}[(m-j)a-i]^{n+m}.$$

Evaluate

$$P\left[\frac{u_1+u_2+\cdots+u_n}{v_1+v_2+\cdots+v_5}<0.9\right].$$

(Marsaglia, 1965)

4.32 Let

$$f(x_1)=\frac{1}{\beta_1}\exp\left(\frac{-x_1}{\beta_1}\right), \qquad 0\leqslant x_1<\infty$$

and

$$f(x_2)=\frac{1}{\beta_2}\exp\left(\frac{-x_2}{\beta_2}\right), \qquad 0\leqslant x_2<\infty.$$

(*a*) Prove that the distribution of $Y=X_1/X_2$ is

$$g(y)=\frac{\alpha}{(y+\alpha)^2},$$

where $\alpha=\dfrac{\beta_1}{\beta_2}$

(cf. Exercise 4.16).

(*b*) Show that no moments of $g(y)$ exist.

(*c*) Show that

$$h(x)=\frac{r\alpha^r}{(x+\alpha)^{r+1}}$$

has moments up to $r-1$.

(Lachenbruch and Brogan, 1971)

4.33 Let X be a discrete r.v. with p.m.f.

$$p(x)=\tfrac{1}{4}, \qquad x=1,2,3,4$$
$$=0, \qquad \text{otherwise.}$$

Find the probability distribution $P(y)$ of the product $Y=X_1X_2$ of two such identically distributed i.r.v.'s. Check your results by showing that

$$E[y]=E[x_1]E[x_2]$$
$$=(E[x])^2$$

and

$$E[y^2]=E[x_1^2]E[x_2^2]$$
$$=(E[x^2])^2.$$

Calculate the variance of the probability distribution $P(y)$.

4.34 Let x be a discrete r.v. having the p.m.f. as given:

x	$p(x)$
1	0.08
2	0.17
3	0.50
4	0.17
5	0.08

Find the probability distribution $P(y)$ of the product $Y=X_1X_2$ of two such identically distributed i.r.v.'s X_1 and X_2. Check your results (partially) by showing that

$$E[y]=E[x_1]E[x_2]$$
$$=(E[x])^2$$
$$E[y^2]=E[x_1^2]E[x_2^2]$$
$$=(E[x^2])^2.$$

4.35 Find the probability distribution $P(y)$ of the product $Y = X_1, X_2, X_3$, each $X_j, j = 1,2,3$ having the binomial p.m.f.

$$p(x) = \binom{3}{x} p^x (1-p)^{3-x}, \qquad x = 0, 1, 2, 3.$$

Show that

$$\begin{aligned} E[y] &= E[x_1]E[x_2]E[x_3] \\ &= (E[x])^3 \end{aligned}$$

and

$$\begin{aligned} E[y^2] &= E[x_1^2]E[x_2^2]E[x_3^2] \\ &= (E[x^2])^3. \end{aligned}$$

Compute the variance of $P(y)$.

CHAPTER 5*

The Distribution of Algebraic Functions of Independent Random Variables

This chapter presents fundamental methods for the derivation of the p.d.f.'s of algebraic functions[27] of i.r.v.'s, as well as the moments of such p.d.f.'s. Two theorems and two corollaries are stated establishing the relationships between Laplace and Mellin transforms, and between Fourier and Mellin transforms. (The proofs of these theorems are given in Appendix C and are due to R. D. Prasad[28] [295].) These theorems and corollaries permit one to determine the Laplace (Fourier) transform of p.d.f. from a knowledge of its Mellin transform, and vice versa, without knowing the p.d.f. explicitly. This capability is particularly convenient in deriving the integral transform of the p.d.f. of an algebraic function of i.r.v.'s, whose inversion will then yield the p.d.f. of that algebraic function. Thus if X_k, $k=1,2,3$ are nonnegative i.r.v.'s, and the p.d.f.'s of the sum $W=X_1+X_2$ and product $Y=X_2X_3$ are $g(w)$ and $h(y)$, respectively, then one can find, directly,

$$L_r(g(w))=L_r(f_1(x_1))L_r(f_2(x_2))$$

and

$$M_s(h(y))=M_s(f_3(x_3))M_s(f_2(x_2)),$$

if all the transforms exist. Now consider, for example, the algebraic

[27]An algebraic function [161] is a function containing only algebraic terms and symbols. Any polynomial is algebraic, but functions such as $\log x$, $\sin x, e^x$ are not. An algebraic function in which the variable (or variables) does not appear as an irreducible radical or with fractional exponents, is called a rational function. Thus $3x^3+x\sqrt{y}+\sqrt{5}$ is rational in x, but neither in y nor in x and y together.

[28]These proofs, communicated to me by Prasad, provide the rigor that is somewhat lacking in the referenced paper [294]. The notation differs from that of Prasad, but the content is the same.

function $x_1 + x_2x_3$. One could find the Laplace transform of $x_1 + x_2x_3$ easily if he had a direct way to get the Laplace transform of x_2x_3 from its Mellin transform. Similarly, if one could go directly from the Laplace to the Mellin transform, he could easily compute the Mellin transform of such functions as $x_1(x_2 + x_3)$. Such direct techniques, and more generally, direct techniques for the mutual conversion of Laplace, Fourier, and Mellin transforms, have recently been developed by Prasad [294] and permit the systematic computation of the transforms of any function of $x_1, x_2, \ldots$. These direct techniques are presented in this chapter, as well as operational techniques recently developed by Stark and Shukla [365] for random scalar and vector products, which are particularly useful in deriving the p.d.f. of sums of products of i.r.v.'s.

This chapter also gives a brief discussion of quadratic forms, a special type of algebraic function of i.r.v.'s that occurs in connection with a number of problems in both theoretical and applied statistics. The distribution of such quadratic forms sometimes plays an important role in the solution of these problems, as examples included in the discussion show. Finally, an example is given to illustrate the procedure for deriving the p.d.f. of an algebraic function of i.r.v.'s.

The relation between Chapters 5 and 6 bears stating. Chapter 5 deals with the distribution of the class of algebraic functions of i.r.v.'s. An important special subclass contained therein is the class of algebraic functions of independent H-function r.v.'s, which includes many of the basic distributions in probability and statistics. Chapter 6 covers the derivation of the distributions of such functions.

5.1 PROBABILITY DISTRIBUTIONS OF ALGEBRAIC FUNCTIONS OF INDEPENDENT RANDOM VARIABLES

To deal with algebraic expressions involving the sum, the product, and the ratio of i.r.v.'s, it is necessary to obtain the conversion of the Laplace transform to the Mellin transform, and vice versa. However care must be exercised when the algebraic function resulting from the addition, subtraction, multiplication, and division of i.r.v.'s—or combinations of these operations—is not limited to nonnegative r.v.'s, since as has already been emphasized, the Laplace and Mellin transforms ordinarily apply only to functions of nonnegative r.v.'s. In such cases, the Fourier or bilateral Laplace transform should be used instead of the Laplace transform, in connection with the extended definition of the Mellin transform over the entire real axis [354]. The following theorems enable one to carry out conversions among transforms; specifically, from Laplace transform to

Mellin transform, and vice versa; from Fourier transform to Mellin transform, and vice versa. These theorems are stated here without proofs. The proofs are given in Appendix C (see note 28).

Theorem 5.1.1. Let $L_r(f(x))$ be the Laplace transform of $f(x)$, $x \geqslant 0$ such that $L_r(f(x))$ is analytic and of order $O(r^{-k})$, where $k>1$ for all r in $\mathrm{Re}(r)>\varepsilon<0$; then the Mellin transform of $f(x)$ is given by

$$M_\alpha(f(x)) = \frac{\Gamma(\alpha)}{2\pi i}\int_{c-i\infty}^{c+i\infty} L_r(f(x))(-r)^{-\alpha}\,dr,$$

$$\mathrm{Re}(\alpha)>0, \qquad \varepsilon<c<0. \tag{5.1.1}$$

In the special case when the singularities of $L_r(f(x))$ are poles in $\mathrm{Re}(r) \leqslant \varepsilon$, and $|L_r(f_{(x)})|$ is bounded in $\mathrm{Re}(r) \leqslant \varepsilon$,

$$M_\alpha(f(x)) = \Gamma(\alpha)\left[\Sigma \operatorname{Res} L_r(f(x))(-r)^{-\alpha} \quad \text{at poles of } L_r(f(x))\right],$$

where Res stands for "residue of." When $L_r(f(x))$ is entire,

$$M_\alpha(f(x)) = \frac{M_{1-\alpha}(L_r(f(x)))}{\Gamma(1-\alpha)}$$

and $M_\alpha(f(x))$ is analytic in $\mathrm{Re}(\alpha)>0$.

Theorem 5.1.2. Let $L_r(f(x))$ and $M_\alpha(f(x))$ be the Laplace and Mellin transforms of $f(x)$, $x \geqslant 0$, respectively. If $f(x)$ is of bounded variation and $x^k f(x) \in L^2$ on $(0,1)$[29] and $x^l f(x) \in L^2$ on $(1,\infty)$ with $k<l$, then

$$L_r(f(x)) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M_\alpha(f(x))\Gamma(1-\alpha)r^{\alpha-1}\,d\alpha,$$

$$k+\tfrac{1}{2}<c<\min \quad \left(1, l+\tfrac{1}{2}\right). \tag{5.1.2}$$

(One says that $x^k f(x) \in L^2$ on (a,b) if $f(x)$ is measurable and $\int_a^b |x^k f(x)|^2 dx < \infty$. See ref. 380, p. 10.)

Corollary 5.1.1. If $M_s(f(x))$ is the Mellin transform of $f(x)$, and $f(x)$ satisfies the conditions of Theorem 5.1.2, then the Fourier transform of

[29]Read "$x^k f(x)$ belongs to, or is, $(0,1)$."

$f(x)$ is given by

$$F_t(f(x)) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M_s(f(x))\Gamma(1-s)(it)^{s-1}\,ds. \tag{5.1.3}$$

Corollary 5.1.2. If $F_t(f(x))$ is the complex Fourier transform of $f(x)$, and is analytic and of order $O((it)^{-k})$, where $k>1$ and $Im(t)\neq 0$, then the Mellin transform of $f(x)$ is given by

$$M_s(f(x)) = \frac{\Gamma(s)}{2\pi}\int_{-\infty}^{\infty}\Big[F_t(f(x))(it)^{-s}\big|_{\mathrm{Im}(t)>0}$$

$$+(-1)^{s-1}F_t(f(x))(-it)^{-s}\big|_{\mathrm{Im}(t)<0}\Big]\,dt \tag{5.1.4}$$

The following examples, due to Prasad [294], illustrate the application of the theorems.

Example 5.1.1. Use Theorem 5.1.1 to find the Mellin transform of $f(x)=e^{-ax}\sin bx$.

The Laplace transform of $f(x)$ is given by

$$L_r(f(x)) = \int_0^\infty e^{-ax}\sin bx\; e^{-rx}\,dx$$

$$= \frac{b}{(r+a)^2+b^2}. \tag{5.1.5}$$

$L_r(f(x))$ has two simple poles in the left half of the complex r-plane, namely, $r=-a+ib$ and $r=-a-ib$. Hence from Theorem 5.1.1, the Mellin transform of $f(x)$ is given by

$$M_\alpha(f(x)) = \Gamma(\alpha)\big[\Sigma \operatorname{Res} L_r(f(x))(-r)^{-\alpha} \quad \text{at poles of } L_r(f(x))\big]$$

$$=\Gamma(\alpha)b\left[-\frac{(a+ib)^{-\alpha}}{2ib}+\frac{(a-ib)^{-\alpha}}{2ib}\right]$$

$$=\frac{\Gamma(\alpha)}{2i}\left[\frac{(a+ib)^{\alpha}-(a-ib)^{\alpha}}{(a^2+b^2)^{\alpha}}\right]. \tag{5.1.6}$$

The expression above for $M_\alpha(f(x))$ can be simplified by using the substitution

$$a = A\cos\theta, \qquad b = A\sin\theta,$$

$$A = \sqrt{a^2+b^2}\,, \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right) \tag{5.1.7}$$

which leads to the result

$$\begin{aligned} M_\alpha(f(x)) &= \frac{\Gamma(\alpha)}{2i}\left[\frac{A^\alpha(\cos\theta + i\sin\theta)^\alpha - A^\alpha(\cos\theta - i\sin\theta)^\alpha}{(a^2+b^2)^\alpha}\right] \\ &= \frac{\Gamma(\alpha)}{2i}\,\frac{2i\sin\alpha\theta}{A^\alpha} \\ &= \frac{\Gamma(\alpha)\sin(\alpha\tan^{-1}(b/a))}{(a^2+b^2)^{\alpha/2}}. \end{aligned} \tag{5.1.8}$$

Example 5.1.2. Use Theorem 5.1.2 to find the Laplace transform of the exponential integral [98, p. 143]

$$f(x) = E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt$$

(cf. Exercise 4.24).

The Mellin transform of $f(x)$ is given by

$$\begin{aligned} M_\alpha(f(x)) &= \int_0^\infty x^{\alpha-1} f(x)\,dx \\ &= \int_0^\infty \left\{\int_x^\infty \frac{e^{-t}}{t}\,dt\right\} x^{\alpha-1}\,dx \\ &= \frac{x^\alpha}{\alpha}\int_x^\infty \frac{e^{-t}}{t}\,dt\,\Big|_0^\infty + \int_0^\infty \frac{x^\alpha}{\alpha}\cdot\frac{e^{-x}}{x}\,dx \\ &= \frac{1}{\alpha}\int_0^\infty x^{\alpha-1}e^{-x}\,dx \\ &= \frac{\Gamma(\alpha)}{\alpha}. \end{aligned} \tag{5.1.9}$$

From Theorem 5.1.2, the Laplace transform of $f(x)$ will be given by

$$L_r(f(x)) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{1}{\alpha}\right)M_\alpha(f(x))\Gamma(1-\alpha)r^{\alpha-1}\,d\alpha$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{1}{\alpha}\right)\Gamma(\alpha)\Gamma(1-\alpha)r^{\alpha-1}\,d\alpha. \qquad (5.1.10a)$$

Since (as can be shown) the conditions of Jordan's lemma are satisfied, this inversion integral may be evaluated by the method of residues by integrating over the Bromwich contours $\lim_{L_k\to\infty} C_{L_k}$ and $\lim_{R_k\to\infty} C_{R_k}$ (Fig. F.1.1), where these Bromwich contours are chosen so as not to pass through any of the poles. (Section F.1 discusses the case in which the integrand contains an infinite number of poles.) Hence α is not an integer, and one may utilize *Euler's* functional equation [18, (4.5), p. 26]

$$\Gamma(\alpha)\Gamma(1-\alpha) = \pi\csc\pi\alpha$$

and express (5.1.10a) in the form

$$L_r(f(x)) = \frac{1}{2\pi i r}\int_{c-i\infty}^{c+i\infty}(\pi\csc\pi\alpha)\frac{r^\alpha}{\alpha}\,d\alpha. \qquad (5.1.10b)$$

The integrand has a double pole at $\alpha = 0$ and simple poles at $\alpha = 1, 2, \ldots,$ in the right half-plane. Thus

$$L_r(f(x)) = \frac{1}{r}\Sigma\mathrm{Res}\left[\frac{\pi(\csc\pi\alpha)r^\alpha}{\alpha}\right], \qquad \text{at } \alpha = 0, 1, 2, \ldots$$

$$= \frac{1}{r}\left(r - \frac{r^2}{2} + \frac{r^3}{3} - \cdots\right)$$

$$= \frac{\log(1+r)}{r}. \qquad (5.1.11)$$

Example 5.1.3. Use Corollary 5.1.1 to find the Fourier transform of the algebraic expression $W = X_1 + X_2X_3$ where the X_j are independent normally distributed r.v.'s with means m_j and standard deviations σ_j, $j = 1, 2, 3$.

The Mellin transform of the p.d.f. $f_j(x_j)$, $j=2,3$, is

$$M_s(f_j(x_j))=\int_{-\infty}^{\infty} x_j^{s-1}f_j(x_j)\,dx_j$$

$$=\int_{-\infty}^{0} x_j^{s-1}f_j(x_j)\,dx_j+\int_{0}^{\infty} x_j^{s-1}f_j(x_j)\,dx_j$$

$$=\frac{e^{i\pi(s-1)}}{\sqrt{2\pi}\,\sigma_j}\int_0^\infty x_j^{s-1}\exp\left[-\frac{(x_j+m_j)^2}{2\sigma_j^2}\right]dx$$

$$+\frac{1}{\sqrt{2\pi}\,\sigma_j}\int_0^\infty x_j^{s-1}\exp\left[-\frac{(x_j-m_j)^2}{2\sigma_j^2}\right]dx$$

$$=\frac{\exp(-m_j/2\sigma_j^2)}{\sqrt{2\pi}}\left[e^{i\pi(s-1)}\int_0^\infty(\sigma_j y)^{s-1}\exp\left(\frac{-y^2}{2}-\frac{m_j y}{\sigma_j}\right)dy\right.$$

$$\left.+\int_0^\infty(\sigma_j y)^{s-1}\exp\left(\frac{-y^2}{2}+\frac{m_j y}{\sigma_j}\right)dy\right]$$

$$=\frac{\exp(-m_j/4\sigma_j^2)}{\sqrt{2\pi}}\sigma_j^{s-1}\Gamma(s)\left[U\left(s-\frac{1}{2},\frac{m_j}{\sigma_j}\right)e^{i\pi(s-1)}\right.$$

$$\left.+U\left(s-\frac{1}{2},\frac{-m_j}{\sigma_j}\right)\right],\qquad j=2,3$$

where

$$U(a,b)=D_{-a-1/2(b)},\tag{5.1.12}$$

where $U(a,x)$ is the parabolic cylinder function [2, (19.3.1), p. 687]. (The notation $D_v(x)$ is defined in Appendix D.1.)

In view of (5.1.12), one notes that since X_2 and X_3 are i.r.v.'s, the Mellin transform of the p.d.f. $g(y)$ of $Y=X_2X_3$ is,

$$M_s(g(y))=\frac{[\Gamma(s)]^2}{2\pi}\prod_{j=2}^{3}\sigma_j^{s-1}\exp\left[\frac{-m_j^2}{4\sigma_j^2}\right]$$

$$\cdot\left[U\left(s-\frac{1}{2},\frac{m_j}{\sigma_j}\right)e^{i\pi(s-1)}+U\left(s-\frac{1}{2},\frac{-m_j}{\sigma_j}\right)\right]. \tag{5.1.13}$$

The Fourier transform of $g(y)$ is, from Corollary 5.1.1

$$F_t(g(y))=\sum_{k=0}^{\infty}\frac{(-it)^k}{k!}M_s(g(y))\bigg|_{s=k+1}$$

$$=\frac{1}{2\pi}\sum_{k=0}^{\infty}(-it)^k\Gamma(k+1)\prod_{j=2}^{3}\sigma_j^k\exp\left[\frac{-m_j^2}{4\sigma_j^2}\right]$$

$$\cdot\left[U\left(k+\frac{1}{2},\frac{m_j}{\sigma_j}\right)e^{ki\pi}+U\left(k+\frac{1}{2},\frac{-m_j}{\sigma_j}\right)\right]. \tag{5.1.14}$$

Also, the Fourier transform of $f_1(x_1)$ is

$$F_t(f_1(x_1))=\exp\left(-im_1t-\tfrac{1}{2}\sigma_1^2t^2\right), \tag{5.1.15}$$

so that the Fourier transform of the p.d.f. $h(w)$ of w is

$$F_t(h(w))=F_t(f_1(x_1))F_t(g(y))$$

$$=\exp\left(-im_1t-\tfrac{1}{2}\sigma_1^2t^2\right)\cdot\frac{1}{2\pi}\sum_{k=0}^{\infty}(-it)^k\Gamma(k+1)\prod_{j=2}^{3}\sigma_j^k\exp\left[\frac{-m_j^2}{4\sigma_j^2}\right]$$

$$\cdot\left[U\left(k+\frac{1}{2},\frac{m_j}{\sigma_j}\right)e^{ki\pi}+U\left(j+\frac{1}{2},\frac{-m_j}{\sigma_j}\right)\right]. \tag{5.1.16}$$

5.2 OPERATIONAL TECHNIQUES FOR RANDOM AND SCALAR PRODUCTS

This section deals with operational techniques for random models based on products of i.r.v.'s as developed by Stark and Shukla [365] and Stark and Nichols [366, pp. 483–487]. In particular, they derive operational expressions for scalar and vector products of vectors having mutually independent random components and state that applications of such operational techniques to mechanics appear promising in view of the basic role of vector multiplication.

The reader is reminded at this point that if X_i has p.d.f. $f_i(x_i)$ and the X_i are mutually i.r.v.'s, $i=1,2,\ldots,n$, then the p.d.f. $g(w)$ of the product variable $Y=\Pi_{i=1}^{n}X_i$ has the Mellin transform

$$M_s(g(y))=\prod_{i=1}^{m} M_s(f_i(x_i)).$$

It is easily verified that the Mellin transform of the p.d.f. $h(y)$ of $Y=X_1^aX_2^b\cdots X_n^r$, where $a,b,\ldots,r$ are constants, is

$$M_s(h(y))=M_{a(s-1)+1}f_1(x_1)M_{b(s-1)+1}f_2(x_2)\cdots M_{r(s-1)+1}f_n(x_n).$$

Note also that if $a,b,\ldots,r$ assume negative values, one obtains a valid counterpart for quotients instead of products.

5.2.1 Compound Random Products

When the number of r.v.'s in the product is also a random variable N, the product is called *compound*. The Mellin transform for the compound product of identically distributed i.r.v.'s

$$Y=X_1X_2\cdots X_N$$

is

$$M_s(h(y))=G_N(M_s(f(x_i))),$$

where $G_N(s)$ is the generating function for the probabilities $q_1,q_2,\ldots,$ corresponding to the number of terms $N=1,2,\ldots,$ in the product and $M_s(f(x_i))$ is the common Mellin transform. That is,

$$\begin{aligned}M_s(h(z))&=q_1M_s(f(x))+q_2[M_s(f(x))]^2+\cdots+q_n[M_s(f(x))]^n\\&=G_n(M_s(f(x))).\end{aligned}$$

These considerations suggest the more general result stated in the following theorem, which is easily verified.

Theorem 5.2.1. Let $Y = X_1X_2\cdots X_n$ be a compound product of independent nonnegative r.v.'s with associated Mellin transforms $M_s(f(x_i))$, $i = 1,2,\ldots$. Let $q_1, q_2, \ldots,$ be the probabilities corresponding to the number of terms $n = 1,2,\ldots,$ in the product. The Mellin transform for Y is

$$M_s(h(y)) = q_1M_s(f(x_1)) + q_2M_s(f(x_1))M_s(f(x_2)) + \cdots$$
$$+ q_nM_s(f(x_1))M_s(f(x_2))\cdots M_s(f(x_n)) + \cdots.$$

5.2.2 Product Sums

It is easy to devise expressions of operational techniques for sums (and differences) of products (and quotients) of independent nonnegative r.v.'s, including quadratic and bilinear forms. Consider, for example, the basic random model

$$W = X_1Y_1 + X_2Y_2 + \cdots + X_nY_n,$$

where W has p.d.f. $h(w)$ and X_j, Y_j are mutually i.r.v.'s for all $j = 1,2,\ldots,n$, with p.d.f.'s $f_j(x_j)$ and $g_j(y_j)$. The characteristic function $F_t(h(w))$ for the density of W can be written in the form

$$\begin{aligned} F_t(h(w)) &= E[e^{itw}] \\ &= \prod_{j=1}^{n} E[e^{itx_jy_j}] \\ &= \prod_{j=1}^{n} \sum_{s=1}^{\infty} \frac{(itx_jy_j)^{s-1}}{(s-1)!} f_j(X_j)\, g_j(y_j) \\ &= \prod_{j=1}^{n} \sum_{s=1}^{\infty} \frac{(it)^{s-1}}{(s-1)!} M_s(f_j(X_j))M_s(g_j(y_j)), \end{aligned}$$

Clearly, this important ability to use operational techniques can easily be extended to more general sums of products considered earlier, including a random number of terms. Of course specific expressions may not always be convenient.

5.2.3 Random Scalar and Vector Products

The basic random model for a sum of random products describes a variety of situations. For example, (1) the total scalar moment arising from a series of forces $Y_1, Y_2, \ldots, Y_n$ acting at respective positions $X_1, X_2, \ldots, X_n$; (2) the total work done by a system of forces; (3) the numbers of vehicles passing a sequence of two traffic signals where X_i and Y_i are zero or one for the ith vehicle at the two intersections according as the vehicle encountered "red" or "green"; (4) the total dollar volume of stocks traded in n transactions where the ith transaction involves X_i shares at price Y_i, and so on.

Actually, for $n=3$ the expression for $F_t(h(w))$ in terms of the Mellin transforms for the p.d.f.'s of X_j and Y_j is the characteristic function for the scalar product $\mathbf{X}\cdot\mathbf{Y}$ of the vectors

$$\mathbf{X} = X_1\boldsymbol{\varepsilon}_1 + X_2\boldsymbol{\varepsilon}_2 + X_3\boldsymbol{\varepsilon}_3 \qquad \text{and} \qquad \mathbf{Y} = Y_1\boldsymbol{\varepsilon}_1 + Y_2\boldsymbol{\varepsilon}_2 + Y_3\boldsymbol{\varepsilon}_3,$$

where $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$, and $\boldsymbol{\varepsilon}_3$ are unit vectors. Furthermore, the characteristic function for the component of the vector product along $\boldsymbol{\varepsilon}_k$ is

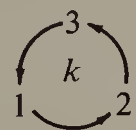
$$F_t(h(w)) = F_t[h(w(k^+))]F_t[h(w(k^-))],$$

where

$$w(k^{\pm}) = X_{k\pm1}Y_{k\pm2},$$

$$F_t[h(w(k^{\pm}))] = \sum_{s=1}^{\infty} \frac{(it)^{s-1}}{(s-1)!} M_s f(x_{k\pm1}) M_s(g(y_{k\pm2}))$$

and k is the cyclic variable



.

5.3 QUADRATIC FORMS

One type of algebraic function of some importance in both theoretical and applied statistics is the quadratic form. A *quadratic form* [32, p. 127] is a homogeneous polynomial of the second degree, namely,

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_i x_j.$$

If it is positive for all real values of the variables $\{x_i\}$, it is called a *positive definite* quadratic form [161, p. 148]; if it is positive or zero, it is called semidefinite. There exists a linear transformation of the form

$$y_i = \sum_{j=1}^{r} d_{ij} x_j, \qquad i = 1,2,\dots,r, \qquad r \leqslant n \tag{5.3.1}$$

such that

$$\sum_{j=1}^{n} d_{ij} d_{kj} = 0, \qquad i \neq k, \qquad d_{r+1}^2 + \cdots + d_n^2 = 0;$$

that is, an orthogonal transformation that reduces the quadratic form Q to a sum of squares Q':

$$Q' = \sum_{i,j=1}^{r} e_{ij} y_j^2,$$

where $r \leqslant n$ is the rank of the matrix of the coefficients in (5.3.1).

Quadratic forms are encountered in various areas of statistical theory, such as the analysis of variance, serial correlation analysis, and multiple regression of time series analysis. Usually the analysis of problems in these areas depends on a knowledge of the distribution of certain quadratic forms. For example, the validity of the F test in the analysis of variance rests on the fact that the components of variance are mutually independent quadratic forms each having a chi-square distribution. The use of quadratic forms in establishing various results in statistical theory is well documented in the literature and is not discussed here. Quadratic forms are introduced in this section because they are a special type of algebraic function and their distribution is usually derived by means of Fourier transforms (characteristic functions), a problem that is clearly relevant to the algebra of r.v.'s. For example, by using characteristic functions it can be readily proven that a necessary and sufficient condition that a quadratic form be distributed as chi-square is that the nonzero roots of its characteristic equation be $+1$.

Quadratic forms are useful in the applied domain as well as in the theoretical. McNolty,[30] for example, derives [253] the distribution of a quadratic form that occurs in certain bombing problems when it is desired to examine the coverage effect on a population of target elements or on a single, randomly located point target where the population or point target

[30]The discussion that follows in this section is essentially that of McNolty, reprinted in part from "Quadratic Form Distributions Associated with Special Functions," by Frank McNolty, *Sankhya, Series B* (1972), Vol. 34, Part I, pp. 21–26, with the permission of the editors of *Sankhya*.

is not distributed uniformly in azimuth and need not be described by identically distributed (x_T, y_T, z_T) components. Frequently, a starting point for the analysis of population coverage or point-target kill is the derivation of the distribution of the positive definite quadratic form

$$Q=(x_T-x_C)^2+(y_T-y_C)^2+(z_T-z_C)^2, \tag{5.3.2}$$

where x_T, y_T, and z_T are the independent target coordinates and $x_{c'}$, $y_{c'}$ and z_c are the independent coordinates of the lethal sphere. Determination of the p.d.f. $h(y)$ of the quadratic form (5.3.2) enables one to evaluate the probability of killing the target. It is here assumed that the independent burst-point coordinates x_c, y_c, z_c have the normal distributions $N(a,\sigma)$. $N(b,\sigma)$, and $N(c,\sigma)$, respectively, while the independent target components are described by (5.3.3) to (5.3.5):

$$f(x;\lambda,\gamma_1,Q_1)=2^{(Q_1/4)-(3/2)}\frac{|x|^{Q/2}}{\gamma_1^{(Q_1/2-1)}}\lambda^{Q_1/2}$$

$$\times \exp\left\{\frac{-\left[\gamma_1^2/(2\lambda)+\lambda x^2\right]}{2}\right\}$$

$$\times I_{(Q_1/2)-1}\frac{\gamma_1|x|}{\sqrt{2}}, \qquad -\infty<x<\infty, \tag{5.3.3}$$

$$f(y)=f(y;\lambda,\gamma_2,Q_2), \qquad -\infty<y<\infty, \tag{5.3.4}$$

and

$$f(z)=f(z;\lambda,\gamma_3,Q_3), \qquad -\infty<z<\infty, \tag{5.3.5}$$

where $I_\nu(\cdot)$ denotes a modified Bessel function of the first kind of order ν (Appendix D.1), and the parameters are subject to the restrictions $\lambda \geqslant 0, \gamma_1, \gamma_2, \gamma_3 \geqslant 0$; $Q_1, Q_2, Q_3 > 0$.

The characteristic function $\phi_w(t)$ of $w=u^2$ is

$$\phi_w(t)=H_1 i\sqrt{\pi}^{\,-i\pi/4}\cdot\frac{1}{\sqrt{t+i/\sigma^2}}\int_{-\infty}^{\infty}|x|^{Q_1/2}$$

$$\times \exp\left[-\left(\frac{\lambda}{2}\right)x^2-\frac{x^2}{(2\sigma^2)}+\frac{ax}{\sigma^2}\right]$$

$$\cdot I_{(Q_1/2)-1}\frac{\gamma_1|x|}{\sqrt{2}}\exp\left\{\frac{i(x-a)^2}{4\sigma^4\left[t+(i/2\sigma^2)\right]}\right\}dx, \tag{5.3.6}$$

where

$$H_1 = \frac{2^{(Q/4)-(3/2)}\lambda^{Q_1/2} e^{-\gamma_1^2/(4\lambda)}}{\sqrt{2\pi}\,\gamma_1^{(Q_1/2)-1}\sigma} e^{-a^2/(2\sigma^2)} \tag{5.3.7}$$

and the characteristic functions $\phi_h(t)$ and $\phi_\rho(t)$ for $h = l^2 = (y_T - y_c)^2$ and $\rho = v^2 = (z_T - z_c)^2$ have the same general form as (5.3.6), so that the characteristic function for $Q = w + h + \rho$ is

$$\phi_Q(t) = \phi_w(t)\phi_h(t)\phi_\rho(t). \tag{5.3.8}$$

Inversion of the characteristic function (5.3.8) yields the p.d.f. $f(u)$ of Q:

$$\begin{aligned} f(u) = {} & H_1 H_2 H_3 \frac{\sqrt{\pi}}{2} i^3 e^{-3i\pi/4} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x|^{Q_1/2} \\ & \cdot |y|^{Q_2/2}|z|^{Q_3/2} \frac{e^{-itu}}{(t + i/(2\sigma^2))^{3/2}} I_{(Q_1/2)-1}\left(\frac{\gamma_1|x|}{\sqrt{2}}\right) I_{(Q_2/2)-1}\left(\frac{\gamma_2|y|}{\sqrt{2}}\right) \\ & \times I_{(Q_3/2)-1}\left(\frac{\gamma_3|z|}{\sqrt{2}}\right) \cdot \exp\left[\frac{-(x^2+y^2+z^2)}{2\sigma^2}(1+\lambda\sigma^2) + \frac{ax+by+cz}{\sigma^2}\right] \\ & \cdot \exp\left[\frac{i\left[(x-a)^2+(y-b)^2+(z-c)^2\right]}{(4\sigma^4)}\left\{t + \left(\frac{i}{2\sigma^2}\right)\right\}^{-1}\right] dx\,dy\,dz\,dt, \end{aligned} \tag{5.3.9}$$

where H_2 and H_3 have the same form as H_1 except that they include the constants (Q_2, γ_2, b) and (Q_3, γ_3, C), respectively, in place of (Q_1, γ_1, a).

To obtain a tractable result, it is desirable to set $a = b = c = 0$ (i.e., assume that the center of the burst is at the origin). Then, if one carries out the indicated integration with respect to t in (5.3.9) and after this introduces the transformation $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, the subsequent integrations with respect to ϕ, θ, and r yield the desired p.d.f. for the quadratic form (5.3.2); namely,

$$\begin{aligned} f(u) = {} & \frac{\lambda^{q/2} e^{-\Gamma^2/(4\lambda)} e^{-u/(2\sigma^2)} u^{1/2}}{\sqrt{2\pi}\,\sigma^{3-q}\Gamma(q/2)} \cdot \sum_{n=0}^{\infty} \frac{2^n u^n \Gamma\left(\frac{q}{2}+n\right)}{(2n+1)!(1+\lambda\sigma^2)^{(q/2)+n}} \\ & \times F_1^1\left[\left(\frac{q}{2}\right)+n; \frac{q}{2}; \frac{\Gamma^2\sigma^2}{\{4(1+\lambda\sigma^2)\}}\right], \qquad u \geqslant 0, \end{aligned} \tag{5.3.10}$$

where $q = Q_1 + Q_2 + Q_3, \Gamma^2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2$ and ${}_1F_1(a;b;x)$ is the confluent hypergeometric function defined in Appendix D.1.

In a subsequent paper, McNolty, Hansen, and Huyhen [255] utilize a quadratic form to represent a mathematical model for the analysis of physical radar target returns. Specifically, the quadratic form involves generalized Bessel and Gaussian components. Its probability distribution is derived and expressed in terms of generalized Bessel and Gaussian components. The use of this mathematical model has a number of interesting practical applications, including target discrimination, the determination of target detection probabilities as a function of signal to noise ratio, and utilization in the design of a radar altimeter in which the terrain-target scattering model is an important part of the radar link analysis.

5.4 SPECIAL FUNCTION DISTRIBUTIONS

McNolty and Tomsky [252] have defined certain special function distributions that had not previously appeared in the literature and discussed some of their properties. They paid particular attention to the presentation of generalized phase and radial density functions associated with seven types of special function distributions. For example, one might wish to reconstruct a phase density function from a given radial (amplitude) density, or perhaps describe bivariate populations directly in terms of phase (azimuth) and radial distributions rather than by means of their Cartesian component distributions. As a typical problem of this type, one might have a random signal voltage $S(t)$ expressed as

$$S(t) = A(t)C(t),$$

in which the amplitude $A(t)$ of the phase-modulated carrier $C(t)$ is related to the x and y components by

$$A^2(t) = X^2(t) + Y^2(t).$$

The authors examine the distribution of various relevant r.v.'s such as $U = X^2/Y^2$ and $V = X^2/(X^2 + Y^2)$, when X and Y have certain specified distributions (e.g., type I Bessel function distributions.) Their results were obtained by employing either characteristic functions or weighted mixture representations.

In a related paper, McNolty [253] goes on to derive the distributions of quadratic forms consisting of the squared differences of the various special function variates and a normal variate. This derivation is accomplished within the framework of bivariate and trivariate situations in which the

(x,y) or (x,y,z) component distributions are not necessarily the same; Kth moments are obtained and a characteristic function is discussed.

5.5 AN EXAMPLE

Before working the exercises at the close of this chapter, the reader may find this example helpful, which is illustrative of the procedure utilized in determining the p.d.f. of a function of several i.r.v.'s.

Let X_i, $i=1,2,3$, be identically distributed i.r.v.'s. with p.d.f.

$$f(x_i)=e^{-x_i}, \qquad i=1,2,3,$$

and let Y be a function of these i.r.v.s, namely,

$$Y=\frac{X_1}{X_2+X_3}.$$

Find the p.d.f. $h(y)$ of the function Y, the latter itself being an r.v.

Let $U=X_2+X_3$ and denote its p.d.f. by $g(u)$. Then the Laplace transform of $g(u)$ is

$$L_r(g(u))=L_r(f_2(x_2))L_r(f_3(x_3))=\frac{1}{(1+r)^2}.$$

From Theorem 5.1.1,

$$\begin{aligned}M_s(g(u))&=\frac{\Gamma(s)}{2\pi i}\int_{c-i\infty}^{c+i\infty}L_r(g(u))(-r)^{-s}dr\\&=\Gamma(s)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(-r)^{-s}}{(1+r)^2}dr\\&=\Gamma(s)\left[-s(-r)^{-s-1}(-1)\right]\Big|_{r=-1}\\&=s\Gamma(s)\\&=\Gamma(s+1).\end{aligned}$$

Since Y is the quotient of the r.v.'s X/U, it follows that

$$M_s(h(y))=M_s(f_1(x_1))M_{-s+2}(g(u))=\Gamma(s)\Gamma(-s+3),$$

so that

$$h(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}y^{-s}\Gamma(s)\Gamma(-s+3)\,ds.$$

Since the conditions of Jordan's lemma are satisfied, one can evaluate the inversion integral above by means of the residue theorem. Specifically,

$$\begin{aligned} h(y) &= R_1, \qquad 0 \leqslant y < 1 \\ &= R_2, \qquad 1 < y < \infty, \end{aligned}$$

where R_1 and R_2 are the residues at the poles $s=0$ and $s=3$, respectively. Specifically,[31]

$$\begin{aligned} R_1 &= \sum_{j=0}^{\infty} y^{-s}(s+j)\Gamma(s)\Gamma(-s+3)\big|_{s=-j} \\ &= \sum_{j=0}^{\infty} y^{-s}\frac{\Gamma(s+j+1)\Gamma(-s+3)}{s(s+1)\cdots(s+j-1)}\bigg|_{s=-j} \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\Gamma(j+3)y^j \\ &= \sum_{j=0}^{\infty} (-1)^j\frac{(j+2)!}{j!}y^j \\ &= \sum_{j=0}^{\infty} (-1)^j(j+2)(j+1)y^j, \qquad 0 \leqslant y < 1. \end{aligned}$$

Similarly,[32]

$$\begin{aligned} R_2 &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}\Gamma(s)\Gamma(-s+3)\,ds, \qquad 1<y<\infty \\ &= \sum_{j=0}^{\infty} y^{-s}(-s+3+j)\Gamma(-s+3)\Gamma(s)\bigg|_{s=-j}, \qquad 1<y<\infty \\ &= \sum_{j=0}^{\infty} y^{-s}\frac{\Gamma(-s+3+j+1)\Gamma(s)}{(-s+3)(-s+4)\ldots(-s+3+j-1)}\bigg|_{s=3+j}, \qquad 1<y<\infty \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}(j+2)!y^{-(j+3)}, \qquad 1<y<\infty \\ &= \sum_{j=0}^{\infty} (-1)^j(j+2)(j+1)y^{-(j+3)}, \qquad 1<y<\infty. \end{aligned}$$

[31]The denominator in the second equation is understood to be 1 when $j=0$.
[32]The denominator in the third equation is understood to be 1 when $j=0$.

EXERCISES

5.1 Let x_1, x_2, x_3, and x_4 be normal i.r.v.'s, each with zero mean and unit variance. Find the p.d.f.'s of

(a) $\dfrac{x_3}{\sqrt{(x_1^2+x_2^2)/2}}$.

(b) $\dfrac{2x_3^2}{x_1^2+x_2^2}$.

(c) $\dfrac{3x_4^2}{x_1^2+x_2^2+x_3^2}$.

(d) $\dfrac{x_1^2+x_2^2}{x_3^2+x_4^2}$.

5.2 The r.v.'s x_1 and x_2 represent, respectively, the amplitudes of sine and cosine waves. Both are independently and uniformly distributed over the interval (0, 1). Let the r.v. R represent the amplitude of their resultant; (i.e., $R^2 = x_1^2 + x_2^2$). Derive the p.d.f. $h(R)$ of R.

$$\begin{aligned} \textit{Ans.}\quad h(R) &= \frac{\pi}{2}, && 0<R<1.\\ h(R) &= \left(2\csc^{-1}R - \frac{\pi}{2}\right), && 1 \leqslant R \leqslant \sqrt{2}\\ &= 0, && \text{elsewhere.} \end{aligned}$$

5.3 Given the following Laplace transforms, use Theorem 5.1.1 to determine the corresponding Mellin transforms $M_s(f(x))$.

(a) $L_r(f(x)) = \left(1-\dfrac{r}{\lambda}\right)^{-1}, \qquad x \geqslant 0.$

(b) $L_r(f(x)) = \dfrac{e^r-1}{r}, \qquad 0 \leqslant x \leqslant 1.$

(c) $L_r(f(x)) = \dfrac{1}{(1-2r)^2}, \qquad x \geqslant 0.$

5.4 Given the following Mellin transforms, use Theorem 5.1.2 to determine the corresponding Laplace transforms.

(a) $M_s(f(x)) = \lambda^{-s}\Gamma(s).$

(b) $M_s(f(x)) = \dfrac{1}{s}, \qquad \text{Re}(s) > 0.$

5.5 Use Corollary 5.1.1 to determine the Fourier transforms of the p.d.f.'s whose Mellin transforms are given in Exercise 5.4.

5.6* Show that $[(a+b)/ab][x/(x+y)]$ is an r.v. having the distribution

$$f(x)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}, \qquad 0\leqslant x\leqslant 1$$

if $x=u_1^{1/2}$, $y=u_2^{1/b}$, where u_1 and u_2 are i.r.v.'s selected from the uniform distribution

$$\begin{aligned} f(u)&=1, \qquad 0\leqslant u\leqslant 1\\ &=0, \qquad \text{otherwise,} \end{aligned}$$

subject to the condition that $x+y\leqslant 1$.

CHAPTER 6*

The Distribution of Algebraic Functions of Independent H-Function Variables[33]

6.1 INTRODUCTION

This chapter presents a new statistical probability distribution introduced by Carter [47] and based on the H-function law; it is (1) the general form of many common distribution or probability laws and (2) easily "transformed" by means of the Mellin integral transform. The H-function distribution is specified by its p.d.f. $f(x)$, which is merely the H-function multiplied by a normalizing constant that makes the integral of the H-function over the relevant range equal to 1. The H-function itself is a transcendental function introduced by Fox [112] in 1961, and its integral over the relevant range is not in general unity.

As Carter has shown, the new H-function distribution includes as special cases many of the more common classical distributions. Therefore it can be considered to be a generalization of these special cases and can serve as a basis for handling algebraic functions of "mixtures" of such variables.

One of the most significant properties of the H-function distribution is that the distribution of products of independent H-function variables is also an H-function distribution—a property not common among the classical distributions. For example, the distribution of the product of normal i.r.v.'s is not normal. On the other hand, since the beta, gamma, Weibull, Maxwell, and various other distributions are special cases of the H-function distribution, products of such identically distributed i.r.v.'s, or

[33]I am indebted to Dr. Bradley D. Carter [47] for the results in Chapter 6 and Sections 8.2 to 8.7, and for his permission to include them in this book.

mixed products of i.r.v.'s having these distributions, will also follow the H-function probability law. Theorems are presented to show that the product of H-function i.r.v.'s is an H-function r.v., the rational power of an H-function r.v. is an H-function r.v., and the quotient or ratio of two H-function i.r.v.'s is an H-function r.v. Also, integral transforms are derived (in the form of integrals) for the p.d.f. of a sum of independent H-function r.v.'s and the p.d.f. of a polynomial in independent H-function r.v.'s. However since it is difficult—if indeed even possible—to evaluate the inversion integrals for these expressions in closed form for the general case, the results are not stated as theorems. Instead, numerical methods are presented in Chapter 8 that are applicable and will lead to numerical solutions in the general case.

In short, this chapter deals with the distribution of the following types of algebraic functions of H-function r.v.'s:

1. A single term consisting of a product or quotient of independent H-function r.v.'s.
2. A single term consisting of a rational power of an H-function r.v.
3. A single term consisting of any combination of products, quotients, or rational power of independent H-function r.v.'s.
4. A sum of independent terms, each of which is an H-function r.v.
5. A polynomial of any number of terms, each term being any combination of products, quotients, or rational powers of independent H-function r.v.'s.

In this section we have simply intoduced the H-function. Section 6.2 defines the H-function and lists simple identities and special cases of this function. Section 6.3 defines the H-function distribution, presents its characteristic function, and identifies many well-known distributions in statistics as special casses of the H-function distribution. Finally, Sections 6.4 to 6.6 deal with the distribution of the types of algebraic function of H-function r.v.'s just listed.

6.2 THE H-FUNCTION

The H-function was introduced by Fox [112] in 1961 as a symmetric Fourier kernel to the Meijer G-function [226, p.139; 97, p. 206]. Soon after, several properties, asymptotic expansions, analytic continuations, and recurrence relations involving the H-function were presented by Braaksma [37] and Gupta [133]. Their contributions formed the basis for much of the work in H-function theory over the following decade.

This period witnessed the development of more expansion formulas for the H-function. These included expansions by Anandani [5,6,9,11] and by Kapoor and Gupta [177] using generalized Legendre functions, Fourier series, and Laplace transforms; expansions by Bessel functions due to Bajpai [19,20]; expansions by hypergeometric functions due to Goyal [127]; and expansions by Gegenbauer (ultraspherical) polynomials due to Soni [351]. Generally these proved unwieldy and applicable only to special cases of the H-function. Also during this period, Dwivedi [87,88] introduced a density function defined in terms of the confluent hypergeometric function and showed that the product of n i.r.v.'s, each having this density function, is an H-function variable.

It was not until 1970 that an integration of the H-function with respect to its parameters was presented by Pendse [284]. His procedure was again rather complex and did not address the more practical problem of evaluating the H-function inversion integral. His techniques were next applied by Taxak [375] to sum certain series of products of two H-functions.

Other contributors to the general theory of H-functions have written papers dealing with numerous and isolated phases of the subject, and a partial list is included in the references. One rather important and significant result was recently derived by Srivastava and Buschman [361], who have shown that an H-function transform of the Mellin convolution of two functions can be expressed in terms of the Mellin convolution of H-function transforms of the functions. Typically, however, the various authors have approached the problems of analyzing the product of an H-function and another function (such as a Bessel [21] or hypergeometric [159]), by differentiating the H-function [8,10] or establishing identities [7]. In 1969, Shaw [327] provided groundwork for application of the Mellin transform and inversion formula toward evaluating the H-function inversion integral. Later, Carter [47,48] developed the results presented in this chapter, with the exception of the exact evaluation of the H-function inversion integral, accomplished by Eldred and Barnes [91] and Lovett [218]. They provided both the theoretical model and the vehicle for application (embodied in the computer program) of the H-function to engineering and related problems involving algebraic functions of independent H-function r.v.'s (Appendix F). In the case of products, quotients, or rational powers (and sometimes sums) of H-function r.v.'s, derivation of the exact distributions is now possible.

6.2.1 Definition of $H(z)$.

Although there are slight variations and generalizations in the definition of the H-function in the literature, this book uses the definition

$$\mathbf{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]=H(z)$$

$$=\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_js)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_js)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_js)}z^s\,ds \qquad (6.2.1)$$

where

$$0\leqslant m\leqslant q,$$

$$0\leqslant n\leqslant p,$$

$$\alpha_j>0 \qquad \text{for} \quad j=1,2,\ldots,p,$$

$$\beta_j>0 \qquad \text{for} \quad j=1,2,\ldots,q,$$

and a_j $(j=1,2,\ldots,p)$ and b_j $(j=1,2,\ldots,q)$ are complex numbers such that no pole of $\Gamma(b_j-\beta_js)$ for $j=1,2,\ldots,m$ coincides with any pole of $\Gamma(1-a_j+\alpha_js)$ for $j=1,2,\ldots,n$. Furthermore, C is a contour in the complex s-plane running from $\omega-i\infty$ .to $\omega+i\infty$ for some real number ω such that the points

$$s=\frac{b_j+k}{\beta_j}$$

for $j=1,2,\ldots,m$ and $k=0,1,\ldots$; and the points

$$s=\frac{a_j-1-k}{\alpha_j}$$

for $j=1,2,\ldots,n$ and $k=0,1,\ldots,$ lie to the right and left of C, respectively. In other words, (6.2.1) is a Mellin-Barnes integral [97, pp. 49–50].

6.2.2 Simple Identities and Special Cases of the H-Function $H(z)$

Variable substitution into (6.2.1) yields the following three identities, which are very useful in manipulating the H-function:

$$\mathbf{H}_{p,q}^{m,n}\left[\frac{1}{z}\middle|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right]$$

$$=\mathbf{H}_{q,p}^{n,m}\left[z\middle|\begin{matrix}(1-b_1,\beta_1),\ldots,(1-b_q,\beta_q)\\(1-a_1,\alpha_1),\ldots,(1-a_p,\alpha_p)\end{matrix}\right] \tag{6.2.2}$$

$$\mathbf{H}_{p,q}^{m,n}\left[z^c\middle|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right]$$

$$=\frac{1}{c}\mathbf{H}_{p,q}^{m,n}\left[z\middle|\begin{matrix}\left(a_1,\frac{\alpha_1}{c}\right),\ldots,\left(a_p,\frac{\alpha_p}{c}\right)\\\left(b_1,\frac{\beta_1}{c}\right),\ldots,\left(b_q,\frac{\beta_q}{c}\right)\end{matrix}\right],\qquad c>0, \tag{6.2.3}$$

and

$$z^c\mathbf{H}_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right]$$

$$=\mathbf{H}_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_1+\alpha_1c,\alpha_1),\ldots,(a_p+\alpha_pc,\alpha_p)\\(b_1+\beta_1c,\beta_1),\ldots,(b_q,+\beta_qc,\beta_q)\end{matrix}\right]. \tag{6.2.4}$$

Many of the so-called special functions are found to be special cases of the H-function, including Gauss's hypergeometric function, the confluent hypergeometric function, Wright's generalized hypergeometric function, MacRobert's E-function, Meijer's G-function; and Bessel functions. Excellent discussions of many of these and other related functions are given by Erdelyi [97] and Luke [226–228]. The relations between the H-function and some of the more important of these special cases are given on pages 197–198.

1. The exponential function

$$\exp(x)=\mathbf{H}_{0,1}^{1,0}[-x|(0,1)], -\infty<x\leqslant 0 \tag{6.2.5}$$

This is easily verified. Specifically, from the definition (6.2.1), one has for $b_1=0, \beta_1=m=q=1, n=p=0$

$$H(-x)=H_{01}^{10}[-x|(0,1)]$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(-x)^s\Gamma(-s)\,ds, \qquad c<0$$

$$=\frac{1}{2\pi i}\left(2\pi i\sum_{j=0}^{\infty}R_j\right),$$

where R_j denotes the residue at the pole $s=j$. That is,

$$R_j=(-s+j)(-x)^s\Gamma(-s)\Big|_{s=j}$$

$$=(-s+j)(-x)^s\frac{\Gamma(-s+j+1)}{-s(-s+1)\cdots(-s+j-1)}\Bigg|_{s=j}, \qquad j=0,1,\ldots,$$

and the denominator is understood to be 1 when $j=0$. Hence

$$R_j=\frac{x^j}{j!}, \qquad j=0,1,2,\ldots,$$

from which it follows that

$$H_{01}^{10}[-x|(0,1)]=\sum_{j=1}^{n}\frac{x^j}{j!}=e^x.$$

2. The generalized, hypergeometric function

$${}_p\mathbf{F}_q\left[\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\middle|x\right]$$

$$=\frac{\Pi_{j=1}^{q}\Gamma(b_j)}{\Pi_{j=1}^{p}\Gamma(a_j)}\mathbf{H}_{p,q+1}^{1,p}\left[-x\middle|\begin{matrix}(1-a_1,1),\ldots,(1-a_p,1)\\ (0,1),(1-b_1,1),\ldots,(1-b_q,1)\end{matrix}\right] \tag{6.2.6}$$

3. Wright's generalized hypergeometric function

$$
{}_p\Psi_q\left[\begin{array}{c|c}(a_1,\alpha_1),\ldots,(a_p,\alpha_p) & \\ (b_1,\beta_1),\ldots,(b_q,\beta_q) & x\end{array}\right]
$$

$$
=\mathbf{H}_{p,q+1}^{1,p}\left[-x\left|\begin{array}{c}(1-a_1,\alpha_1),\ldots,(1-a_p,\alpha_p)\\ (0,1),(1-b_1,\beta_1),\ldots,(1-b_q,\beta_q)\end{array}\right.\right]. \tag{6.2.7}
$$

4. Meijer's G-function

$$
\mathbf{G}_{p,q}^{m,n}\left[x\left|\begin{array}{c}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{array}\right.\right]
$$

$$
=\mathbf{H}_{p,q}^{m,n}\left[x\left|\begin{array}{c}(a_1,1),\ldots,(a_p,1)\\ (b_1,1),\ldots,(b_q,1)\end{array}\right.\right]. \tag{6.2.8}
$$

It should be noted that Luke also gives an extensive list of special cases and identities for the generalized hypergeometric function and for Meijer's G-function, and with the use of (6.2.6) and (6.2.8), these results can be extended to the H-function.

6.2.3 The Mellin and Laplace Transforms of $H(cz)$

Under the preceding definition of the H-function and assuming convergence of the integral in the definition, the Mellin transform [47,48] can be found by interpreting the H-function as the inverse transform of the coefficient on z^{-s} where (6.2.1) is written as

$$
H(cz)=\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}(cz)^{-s}\,ds.
$$

Using the definition of the Mellin transform and denoting the Mellin inversion integral (2.8.10) by $M^{-1}(M_s(f(x))$, one can express $H(cz)$ in the form

$$
H(cz)=M^{-1}\left[\frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}c^{-s}\right],
$$

from which it follows that

$$M_s\left\{\mathbf{H}^{m,n}_{p,q}\left[cz\middle|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right]\right\}=M_s\{H(cz)\}$$

$$=\frac{\Pi_{j=1}^m\Gamma(b_j+\beta_js)\Pi_{j=1}^n\Gamma(1-a_j-\alpha_js)}{\Pi_{j=m+1}^q\Gamma(1-b_j-\beta_js)\Pi_{j=n+1}^p\Gamma(a_j+\alpha_js)}c^{-s} \tag{6.2.9}$$

is the Mellin transform of the H-function with argument cz.

From the definition of the Laplace transform, one has

$$L_r\{H(cz)\}=\int_0^\infty e^{-rz}H(cz)\,dz$$

$$=\int_0^\infty e^{-rz}\frac{1}{2\pi i}\int_C\frac{\Pi_{j=1}^m\Gamma(b_j-\beta_js)\Pi_{j=1}^n\Gamma(1-q_j+\alpha_js)}{\Pi_{j=m+1}^q\Gamma(1-b_j+\beta_js)\Pi_{j=n+1}^p\Gamma(a_j-\alpha_js)}$$

$$\times(cz)^s\,ds\,dz.$$

The contour integral in the s-plane converges absolutely under the conditions given by Erdelyi [97, pp. 49–50] so that when these conditions are satisfied (as they frequently are), the Laplace integral will converge absolutely. Hence the order of integration in the equation above can be changed giving

$$L_r\{H(cz)\}=\frac{1}{2\pi i}\int_C\frac{\Pi_{j=1}^m\Gamma(b_j-\beta_js)\Pi_{j=1}^n\Gamma(1-a_j+\alpha_js)}{\Pi_{j=m+1}^q\Gamma(1-b_j+\beta_js)\Pi_{j=n+1}^p\Gamma(a_j-\alpha_js)}$$

$$\times c^s\left(\int_0^\infty e^{-rs}z^s\,dz\right)ds$$

$$=\frac{1}{2\pi i}\int_C\frac{\Pi_{j=1}^m\Gamma(b_j-\beta_js)\Pi_{j=1}^n\Gamma(1-a_j+\alpha_js)}{\Pi_{j=m+1}^q\Gamma(1-b_j+\beta_js)\Pi_{j=n+1}^p\Gamma(a_j-\alpha_js)}$$

$$c^s\left(\frac{\Gamma(s+1)}{r^{s+1}}\right)ds$$

$$=\frac{1}{r}\mathbf{H}^{m,n+1}_{p+1,q}\left[\frac{c}{r}\middle|\begin{matrix}(0,1),(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right]$$

and from (6.2.4), it follows that

$$L_r\{H(cz)\} = \frac{1}{c}\mathbf{H}_{p,q+1}^{m+1,n}\left[\frac{c}{r}\middle|\begin{matrix}(1,1),(a_1+\alpha_1,\alpha_1),\ldots,(a_p+\alpha_p,\alpha_p)\\(b_1+\beta_1,\beta_1),\ldots,(b_q+\beta_q,\beta_q)\end{matrix}\right].$$

Then from (6.2.2) the Laplace transform is expressible as

$$L_r\left\{\mathbf{H}_{p,q}^{m,n}\left[cz\middle|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right]\right\} = L_r\{H(cz)\}$$

$$= \frac{1}{c}\mathbf{H}_{q,p+1}^{n+1,m}\left[\frac{1}{c}r\middle|\begin{matrix}(1-b_1-\beta_1,\beta_1),\ldots,(1-b_q-\beta_q,\beta_q)\\(0,1),(1-a_1-\alpha_1,\alpha_1),\ldots,(1-a_p-\alpha_p,\alpha_p)\end{matrix}\right].$$

(6.2.10)

6.3 THE H-FUNCTION DISTRIBUTION

6.3.1 Definition

Consider an r.v. X that follows a probability law such that its p.d.f. is given by

$$f(x) = \begin{cases} k\mathbf{H}_{p,q}^{m,n}\left[cx\middle|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right], & x>0 \\ 0, & \text{otherwise,}\end{cases} \qquad (6.3.1)$$

where the symbol $\mathbf{H}$ represents the H-function as defined in (6.2.1) and $k, c, a_j (j=1,\ldots,p)$, $\alpha_j (j=1,\ldots,p)$, $b_j (j=1,\ldots,q)$, and $\beta_j (j=1,\ldots,q)$ are the parameters of the distribution with values such that

$$\int_0^\infty f(x)\,dx = 1$$

with $f(x) \geqslant 0$ for $0 < x < \infty$. Furthermore, the values of a_j $(j=1,\ldots,p)$, α_j $(j=1,\ldots,p)$, b_j $(j=1,\ldots,q)$, and β_j $(j=1,\ldots,q)$ must conform to those restrictions in the definition of the H-function (see (6.2.1)). The r.v. X then is called an H-function r.v., which follows an H-function probability law or H-function distribution.

6.3.2 The Characteristic Function

The characteristic function (or Fourier transform) of $f(x)$ is given as

$$\phi(t)=\int_{-\infty}^{\infty} e^{itx}f(x)\,dx=\int_{0}^{\infty} e^{itx}k\mathbf{H}_{p,q}^{m,n}\left[cx\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]dx$$

$$=L_{-it}\left\{k\mathbf{H}_{p,q}^{m,n}\left[cx\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]\right\}.$$

From (6.2.10), assuming absolute convergence of the integral in the definition of the H-function, the characteristic function of the H-distribution can be given as

$$\phi(t)=\frac{k}{c}\mathbf{H}_{q,p+1}^{n+1,m}\left[-\frac{i}{c}t\left|\begin{matrix}(1-b_1-\beta_1,\beta_1),\ldots,(1-b_q-\beta_q,\beta_q)\\(0,1),(1-a_1,-\alpha_1,\alpha_1),\ldots,(1-a_p-\alpha_p,\alpha_p)\end{matrix}\right.\right].\tag{6.3.2}$$

6.3.3 Moments

Since the derivatives of the H-function exist, the moments of the H-distribution can be found by taking the derivatives of (6.3.2). However, there is a simpler method of finding the general expression for the rth moment about the origin, and it capitalizes on the ease with which the Mellin transform of the p.d.f. may be obtained. In this connection, one recalls that the rth moment about the origin is defined as

$$\mu_r'=E\{x^r\}=\int_{-\infty}^{\infty} x^r f(x)\,dx,\tag{6.3.3a}$$

where E is the expected value operator. From the definition of the Mellin transform, it is clear that $M_s\{f(x)\}=E\{x^{s-1}\}$ for distributions where $\Pr[x<0]=0$, so that the rth moment about the origin may be obtained from the Mellin transform of the relevant p.d.f. Specifically,

$$\mu_r'=M_{r+1}\{f(x)\}$$

$$=M_{r+1}\left\{k\mathbf{H}_{p,q}^{m,n}\left[cx\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_1,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]\right\}.\tag{6.3.3b}$$

Then, from (6.2.9)

$$\mu_r' = k\frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}c^{-s}\Bigg|_{s=r+1}$$

$$=\frac{k}{c^{r+1}}\frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j+\beta_j r)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j-\alpha_j r)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j-\beta_j r)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j+\alpha_j r)}. \tag{6.3.3c}$$

6.3.4 Special Cases of the *H*-Function Distribution

As indicated at the beginning of this chapter, one of the most important assets of the H-function distribution is that many of the classical nonnegative distributions are special cases and can be expressed in the form of (6.3.1). This section gives some of the more common of these special cases; their respective p.d.f.'s appear in the form of (6.3.1). The procedure for converting a p.d.f. into its H-function form is expedited by the use of (6.2.2), (6.2.3), and (6.2.4), as is illustrated in some detail below for the gamma and beta distributions.

The Gamma Distribution

$$f(x)=\frac{x^{\theta-1}\exp(-x/\phi)}{\phi^{\theta}\Gamma(\theta)}, \qquad x>0; \qquad \theta,\phi>0 \tag{6.3.4}$$

$$=\frac{1}{\phi\Gamma(\theta)}\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{\phi}x\middle|(\theta-1,1)\right], \tag{6.3.4a}$$

To establish this result, note that the Mellin transform of $f(x)=e^{-x/\phi}, 0\leqslant x<\infty$, is

$$M_s(f(x))=\phi^{s}\Gamma(s),$$

so that the inversion integral yielding $f(x)$ is, by definition,

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{\phi}\right)^{-s}\Gamma(s)\,ds$$

$$=\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{\phi}x\middle|(0,1)\right],$$

i.e., (6.2.1) with $b_1=0, \beta_1=-1$, $m=q=1, n=p=0$. Then, by (6.2.4),

$$\frac{1}{\phi\Gamma(\theta)}\left(\frac{x}{\phi}\right)^{\theta-1} e^{-x/\phi}=\frac{1}{\phi\Gamma(\theta)}\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{\phi}x\middle|(\theta-1,1)\right],$$

which establishes (6.3.4a).

The Weibull Distribution

$$\begin{aligned} f(x) &= \theta\phi x^{\phi-1}\exp(-\theta x^{\phi}), \qquad x>0 \\ &= \theta\phi x^{\phi-1}\mathbf{H}_{0,1}^{1,0}[\theta x^{\phi}|(0,1)] \\ &= \theta x^{\phi-1}\mathbf{H}_{0,1}^{1,0}\left[\theta^{1/\phi}x\middle|\left(0,\frac{1}{\phi}\right)\right] \\ &= \theta^{1/\phi}\mathbf{H}_{0,1}^{1,0}\left[\theta^{1/\phi}x\middle|\left(1-\frac{1}{\phi},\frac{1}{\phi}\right)\right]. \end{aligned} \tag{6.3.5}$$

The Maxwell Distribution

$$\begin{aligned} f(x) &= \frac{4x^2\exp(-x^2/\theta^2)}{\theta^3\sqrt{\pi}}, \qquad x>0 \\ &= \frac{4}{\theta^3\sqrt{\pi}}x^2\mathbf{H}_{0,1}^{1,0}[(1/\theta^2)x^2|(0,1)] \\ &= \frac{2}{\theta^3\sqrt{\pi}}x^2\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{\theta}x\middle|(0,\tfrac{1}{2})\right] \\ &= \frac{2}{\theta\sqrt{\pi}}\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{\theta}x\middle|(1,\tfrac{1}{2})\right]. \end{aligned} \tag{6.3.6}$$

The Beta Distribution

$$f(x)=\begin{cases}\dfrac{x^{\theta-1}(1-x)^{\phi-1}}{B(\theta,\phi)}, & 0<x\leqslant 1; \qquad \theta,\phi>0\\ 0, & x>1\end{cases}$$

$$=\frac{1}{2\pi i}\int_C M_s\{f(x)\}x^{-s}\,ds$$

$$=\frac{1}{2\pi i}\int_C\left[\int_0^1\frac{1}{B(\theta,\phi)}x^{\theta-1}(1-x)^{\phi-1}x^{s-1}\,dx\right]x^{-s}\,ds$$

$$=\frac{1}{2\pi i}\int_C\frac{\Gamma(\phi)\Gamma(\theta-1+s)}{B(\theta,\phi)\Gamma(\theta+\phi-1+s)}x^{-s}\,ds$$

$$=\frac{\Gamma(\theta+\phi)}{\Gamma(\theta)}\left[\frac{1}{2\pi i}\int_C\frac{\Gamma(\theta-1-s)}{\Gamma(\theta+\phi-1-s)}x^{s}\,ds\right]$$

$$=\frac{\Gamma(\theta+\phi)}{\Gamma(\theta)}\mathbf{H}_{1,1}^{1,0}\left[x\,\middle|\,\begin{matrix}(\theta+\phi-1,1)\\(\theta-1,1)\end{matrix}\right]. \tag{6.3.7}$$

The Half-Normal Distribution

$$f(x)=\frac{2\exp(-x^2/(2\theta^2))}{\theta\sqrt{2\pi}}, \qquad x>0; \qquad \theta>0$$

$$=\frac{2}{\theta\sqrt{2\pi}}\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{2\theta^2}x^2\,\middle|\,(0,1)\right]$$

$$=\frac{1}{\theta\sqrt{2\pi}}\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{\theta\sqrt{2}}x\,\middle|\,(0,\tfrac{1}{2})\right]. \tag{6.3.8}$$

The Exponential Distribution

Let $\theta=1$ in (6.3.4).

$$f(x)=\frac{\exp(-x/\phi)}{\phi}, \qquad x>0; \qquad \phi>0 \tag{6.3.9}$$

$$=\frac{1}{\phi}\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{\phi}x\middle|(0,1)\right]. \tag{6.3.9a}$$

The Chi-Square Distribution

Let $\theta=v/2$ and $\phi=2$ in (6.3.4).

$$f(x)=\frac{x^{v/2-1}\exp(-x/2)}{2^{v/2}\Gamma(v/2)}, \qquad x>0; \qquad v>0$$

$$=\frac{1}{2\Gamma(v/2)}\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{2}x\middle|\left(\frac{v}{2}-1,1\right)\right]. \tag{6.3.10}$$

The Rayleigh Distribution

Let $\theta=1/(2a^2)$ and $\phi=2$ in (6.3.5).

$$f(x)=\frac{x\exp(-x^2/(2a^2))}{a^2}, \qquad x>0$$

$$=\frac{1}{a\sqrt{2}}\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{a\sqrt{2}}x\middle|\left(1/2,\tfrac{1}{2}\right)\right]. \tag{6.3.11}$$

The General Hypergeometric Distribution [97, p. 56]

$$f(x)=\frac{da^{c/d}\Gamma(\beta)\Gamma(r-c/d)}{\Gamma(c/d)\Gamma(r)\Gamma(\beta-c/d)}x^{c-1}{}_1\mathbf{F}_1\left[\begin{matrix}\beta\\ r\end{matrix}\middle|-ax^d\right], \qquad x>0, \tag{6.3.11a}$$

where ${}_1F_1\left[\begin{matrix}\beta\\ r\end{matrix}-ax^d\right]={}_1F_1(\beta,r,-ax^d)$ is the confluent hypergeometric function defined in Appendix D.1.

Then from (6.2.6), the general hypergeometric function $f(x)$ as given by (6.3.11a) becomes

$$f(x)=\frac{da^{c/d}\Gamma(\beta)\Gamma(r-c/d)}{\Gamma(c/d)\Gamma(r)\Gamma(\beta-c/d)}x^{c-1}\left\{\frac{\Gamma(r)}{\Gamma(\beta)}\mathbf{H}_{1,2}^{1,1}\left[ax^d\middle|\begin{matrix}(1-\beta,1)\\(0,1),(1-r,1)\end{matrix}\right]\right\}$$

$$=\frac{a^{c/d}\Gamma(r-c/d)}{\Gamma(c/d)\Gamma(\beta-c/d)}x^{c-1}\mathbf{H}_{1,2}^{1,1}\left[a^{1/d}x\middle|\begin{matrix}\left(1-\beta,\frac{1}{d}\right)\\\left(0,\frac{1}{d}\right),\left(1-r,\frac{1}{d}\right)\end{matrix}\right]$$

$$=\frac{a^{1/d}\Gamma(r-c/d)}{\Gamma(c/d)\Gamma(\beta-c/d)}\mathbf{H}_{1,2}^{1,1}\left[a^{1/d}x\middle|\begin{matrix}\left(1-\beta+\frac{c-1}{d},\frac{1}{d}\right)\\\left(\frac{c-1}{d},\frac{1}{d}\right),\left(1-r+\frac{c-1}{d},\frac{1}{d}\right)\end{matrix}\right].$$

(6.3.12)

The Half-Cauchy Distribution

$$f(x)=\frac{2\theta}{\pi(\theta^2+x^2)},\qquad x>0;\theta>0$$

$$=\frac{1}{2\pi i}\int_C M_s\{f(x)\}x^{-s}\,ds$$

$$=\frac{1}{2\pi i}\int_C\left[\int_0^\infty\frac{2\theta x^{-s}}{\pi(\theta^2+x^2)}dx\right]x^{-s}\,ds$$

$$=\frac{1}{2\pi i}\int_C\frac{\Gamma(\frac{1}{2}s)\Gamma(1-\frac{1}{2}s)\theta^{s-1}}{\pi}x^{-s}\,ds \tag{6.3.13}$$

$$=\frac{1}{\theta\pi}\left[\frac{1}{2\pi i}\int_C\Gamma\left(-\frac{1}{2}s\right)\Gamma\left(1+\frac{1}{2}s\right)\left(\frac{1}{\theta}x\right)^s ds\right]$$

$$=\frac{1}{\theta\pi}\mathbf{H}_{1,1}^{1,1}\left[\frac{1}{\theta}x\middle|\begin{matrix}(0,\frac{1}{2})\\(0,\frac{1}{2})\end{matrix}\right]. \tag{6.3.13a}$$

The Half-Student Distribution

Suppose $X = U/\sqrt{V/(2\theta)}$, where U is a half-normal r.v. with p.d.f.

$$f_1(u) = \frac{2}{\sqrt{2\pi}} e^{-u^2/2}$$

and V is a chi/square r.v. with p.d.f.

$$f_2(v) = \frac{v^{\theta-1} e^{-v/2}}{2^{\theta}\Gamma(\theta)}.$$

Then $f(x)$ has a half-Student distribution with p.d.f.

$$f(x) = \frac{2}{\sqrt{2\theta\pi}} \frac{\Gamma\left(\theta+\frac{1}{2}\right)}{\Gamma(\theta)} \left[\frac{1+x^2}{(2\theta)}\right]^{-[\theta+(1/2)]}, \qquad x>0, \theta>0 \quad (6.3.14)$$

$$= \frac{1}{\sqrt{2\theta\pi}\,\Gamma(\theta)} \mathbf{H}_{1,1}^{1,1}\left[\frac{x}{\sqrt{2\theta}} \middle| \begin{matrix} \left(\frac{1}{2}-\theta, \frac{1}{2}\right) \\ \left(0, \frac{1}{2}\right) \end{matrix}\right]. \quad (6.3.14a)$$

The F Distribution

Suppose $X = (U/\theta_1)/(V/\theta_2)$, where U and V have chi-square distributions with parameters θ_1 and θ_2, respectively. Then $f(x)$ has an F distribution with p.d.f.

$$f(x) = \frac{\Gamma(\theta_1+\theta_2)}{\Gamma(\theta_1)\Gamma(\theta_2)} \left(\frac{\theta_1}{\theta_2}\right)^{\theta_1} \frac{x^{\theta_1-1}}{[1+(\theta_1/\theta_2)x]^{(\theta_1+\theta_2)}} \quad (6.3.15)$$

$$= \frac{\theta_1}{\theta_2\Gamma(\theta_1)\Gamma(\theta_2)} \mathbf{H}_{1,1}^{1,1}\left[\frac{\theta_1}{\theta_2}(x) \middle| \begin{matrix} (-\theta_2, 1) \\ (\theta_1-1, 1) \end{matrix}\right], \qquad x>0, \qquad \theta_1, \theta_2>0.$$

$$(6.3.15a)$$

6.4 PRODUCTS, QUOTIENTS AND POWERS OF H-FUNCTION RANDOM VARIABLES

In this section, theorems are presented to show that the product of independent H-function r.v.'s is an H-function r.v., the rational power of an H-function r.v. is an H-function r.v., and the quotient or ratio of two H-function r.v.'s is an H-function r.v. Also, since the integrand, exclusive of the kernel, is precisely a Mellin transform, the reader is reminded of the various properties of the Mellin transform, which have previously been stated and will not be restated here. Furthermore, when one deals with the complex H-function notation, it will simplify matters to denote the inversion integral by the symbol $M^{-1}(M_s(f(x))$; i.e.,

$$\begin{aligned} f(x) &= M^{-1}M_s(f(x)) \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}M_s(f(x))\,ds. \end{aligned} \tag{6.4.1}$$

6.4.1 The distribution of Products of H-Function Random Variables

In this section, it will be proven that the probability distribution of products of H-function i.r.v.'s is also an H-function distribution. This property permits one to use one mathematical model (or computer program) to obtain the distribution of products of either identically distributed or mixed i.r.v.'s, so long as each r.v. has an H-function distribution.

Theorem 6.4.1 If $X_1, X_2, \ldots, X_n$ are independent H-function r.v.'s with p.d.f.'s $f_1(x_1), f_2(x_2), \ldots, f_n(x_n)$, respectively, where

$$f_j(x_j) = \begin{cases} k_j \mathbf{H}_{p_j, q_j}^{m_j, n_j}\left[c_j x_j \middle| \begin{matrix} (a_{j1}, \alpha_{j1}), \ldots, (a_{jp_j}, \alpha_{jp_j}) \\ (b_{j1}, \beta_{j1}), \ldots, (b_{jq_j}, \beta_{jq_j}) \end{matrix} \right], & x_j > 0 \\ 0, & \text{otherwise}, \end{cases} \tag{6.4.2}$$

for $j = 1, 2, \ldots, n$, then the p.d.f. of the r.v.

$$Y = \prod_{j=1}^{n} X_j$$

is given by

$$h(y) = \begin{cases} \left(\prod_{j=1}^{n} k_j\right) \mathbf{H}_{\sum_{j=1}^{n} p_j,\ \sum_{j=1}^{n} q_j}^{\sum_{j=1}^{n} m_j,\ \sum_{j=1}^{n} n_j}\left[\prod_{j=1}^{n} c_j y \middle| \begin{matrix} (a_{11}, \alpha_{11}), \ldots, (a_{np_n}, \alpha_{np_n}) \\ (b_{11}, \beta_{11}), \ldots, (b_{nq_n}, \beta_{nq_n}) \end{matrix} \right], & y > 0 \\ 0, & \text{otherwise}, \end{cases} \tag{6.4.3}$$

where the sequence of the parameters (a_{jv}, α_{jv}) is

$$v=1,2,\ldots,n_j \qquad \text{for} \quad j=1,2,\ldots,n$$

followed by

$$v=n_j+1,n_j+2,\ldots,p_j \qquad \text{for} \quad j=1,2,\ldots,n,$$

and the sequence of the parameters (b_{jv}, β_{jv}) is

$$v=1,2,\ldots,m_j \qquad \text{for} \quad j=1,2,\ldots,n$$

followed by

$$v=m_j+1,m_j+2,\ldots,q_j \qquad \text{for} \quad j=1,2,\ldots,n.$$

PROOF. From (6.2.9), the Mellin transform of $f_j(x_j)$ is

$$M_s\{f_j(x_j)\}=\frac{k_j}{c_j^s}\frac{\Pi_{v=1}^{m_j}\Gamma(b_{jv}+\beta_{jv}s)\Pi_{v=1}^{n_j}\Gamma(1-a_{jv}-\alpha_{jv}s)}{\Pi_{v=m_j+1}^{q_j}\Gamma(1-b_{jv}-\beta_{jv}s)\Pi_{v=n_j+1}^{p_j}\Gamma(a_{jv}+\alpha_{jv}s)} \tag{6.4.4}$$

and, using (4.3.5), it follows that

$$h^+(y)=M^{-1}\left[\prod_{j=1}^{n} M_s\{f_j(x_j)\}\right], \qquad y>0$$

$$=M^{-1}\left[\prod_{j=1}^{n}\left(\frac{k_j}{c_j^s}\frac{\Pi_{v=1}^{m_j}\Gamma(b_{jv}+\beta_{jv}s)\Pi_{v=1}^{n_j}\Gamma(1-a_{jv}-\alpha_{jv}s)}{\Pi_{v=m_j+1}^{q_j}\Gamma(1-b_{jv}-\beta_{jv}s)\Pi_{v=n_j+1}^{p_j}\Gamma(a_{jv}+\alpha_{jv}s)}\right)\right],$$

$$y>0. \tag{6.4.5}$$

Hence from the definition of the inverse Mellin transform (4.2.2), the equation above can be written as the following integral evaluated over the appropriate Bromwich path $(c-i\infty, c+i\infty)$:

$$h^+(y)=\frac{\Pi_{j=1}^{n}k_j}{2\pi i}\int\frac{\Pi_{j=1}^{n}\Pi_{v=1}^{m_j}\Gamma(b_{jv}+\beta_{jv}s)\Pi_{j=1}^{n}\Pi_{v=1}^{n_j}\Gamma(1-a_{jv}-\alpha_{jv}s)}{\Pi_{j=1}^{n}\Pi_{v=m_j+1}^{q_j}\Gamma(1-b_{jv}-\beta_{jv}s)\Pi_{j=1}^{n}\Pi_{v=n_j+1}^{p_j}\Gamma(a_{jv}+\alpha_{jv}s)}$$

$$\cdot\left(\left[\prod_{j=1}^{n}c_j\right]y\right)^{-s}ds$$

$$=\left(\prod_{j=1}^{n}k_j\right)\mathbf{H}_{\sum_{j=1}^{n}p_j,\ \sum_{j=1}^{n}q_j}^{\sum_{j=1}^{n}m_j,\ \sum_{j=1}^{n}n_j}\left[\left(\prod_{j=1}^{n}c_j\right)y\,\middle|\,\begin{matrix}(a_{11},\alpha_{11}),\ldots,(a_{np_n},\alpha_{np_n})\\(b_{11},\beta_{11}),\ldots,(b_{nq_n},\beta_{nq_n})\end{matrix}\right], \tag{6.4.6}$$

which completes the proof of Theorem 6.4.1.

Example 6.4.1. The Product of n Beta Random Variables. Suppose that in Theorem 6.4.1, $X_1, X_2, \ldots, X_n$ are all beta r.v.'s having the p.d.f. shown in (6.3.7) where, when written in terms of (6.4.6),

$$k_j = \frac{\Gamma(\theta_j + \phi_j)}{\Gamma(\theta_j)},$$

$$a_{j1} = \theta_j + \phi_j - 1,$$

$$b_{j1} = \theta_j - 1,$$

$$\alpha_{j1} = \beta_{j1} = 1,$$

$$c_j = 1,$$

and

$$m_j = 1, \qquad n_j = 0, \qquad p_j = 1, \qquad q_j = 1,$$

for $j = 1, 2, \ldots, n$. Then, substituting into (6.4.6) of Theorem 6.4.1, one has

$$h(y) = \begin{cases} \left[\prod_{j=1}^{n} \frac{\Gamma(\theta_j + \phi_j)}{\Gamma(\theta_j)} \right] \mathbf{H}_{n,n}^{n,0}\left[y \middle| \begin{matrix} (\theta_1 + \phi_1 - 1, 1), \ldots, (\theta_n + \phi_n - 1, 1) \\ (\theta_1 - 1, 1), \ldots, (\theta_n - 1, 1) \end{matrix} \right] \\ 0, \qquad \text{otherwise.} \end{cases} \tag{6.4.7}$$

Application of the identity (6.2.8) now gives

$$h(y) = \begin{cases} \prod_{j=1}^{n} \frac{\Gamma(\theta_j + \phi_j)}{\Gamma(\theta_j)} \mathbf{G}_{n,n}^{n,0}\left[y \middle| \begin{matrix} \theta_1 + \phi_1 - 1, \ldots, \theta_n + \phi_n - 1 \\ \theta_1 - 1, \ldots, \theta_n - 1 \end{matrix} \right], & y > 0 \\ 0, \qquad \text{otherwise.} \end{cases} \tag{6.4.8}$$

That is, the p.d.f. of the product of n independent beta r.v.'s is given by (6.4.8). This result agrees with that of Lomnicki [212, 213] and that of Springer and Thompson [358, p. 731].

Example 6.4.2. The Product of n Gamma Variables. Now suppose that in Theorem 6.4.1, $X_1, X_2, \ldots, X_n$ are all gamma i.r.v.'s having the p.d.f.

given in (6.3.4), where, when written in terms of (6.4.6)

$$k_j = \frac{1}{\phi_j \Gamma(\theta_j)},$$

$$a_{j1} = \theta_j - 1,$$

$$\alpha_{j1} = 1,$$

$$c_{j1} = \phi_j^{-1},$$

and

$$m_j = 1, \qquad n_j = 0, \qquad p_j = 0, \qquad q_j = 1,$$

for $j = 1, 2, \ldots, n$. Then from (6.4.6)

$$h(y) = \begin{cases} \left(\prod_{j=1}^{n} \frac{1}{\phi_j \Gamma(\theta_j)} \right) \mathbf{H}_{0,n}^{n,0}\left[\left(\prod_{j=1}^{n} \phi_j^{-1} \right) y \middle| (\theta_j - 1, 1), \ldots, (\theta_n - 1, 1) \right], \\ 0, \qquad \text{otherwise,} \end{cases}$$

$$y > 0 \qquad (6.4.9)$$

which, on application of (6.2.8), becomes

$$h(y) = \begin{cases} \left(\prod_{j=1}^{n} \frac{1}{\phi_j \Gamma(\theta_j)} \right) \mathbf{G}_{0,n}^{n,0}\left[\left(\prod_{j=1}^{n} \phi_j^{-1} \right) y \middle| \theta_j - 1, \ldots, \theta_n - 1 \right], & y > 0 \\ 0, \qquad \text{otherwise.} \end{cases}$$

$$(6.4.9a)$$

Thus (6.4.9a) expresses the p.d.f. of the product of n independent gamma variables in terms of Meijer G-functions. Equation 6.4.9a agrees with the result obtained by Springer and Thompson [358, p. 722].

6.4.2 The Distribution of Rational Powers of H-Function Random Variables

Another important property of the H-function distribution is that a rational power of an H-function r.v. also follows an H-function distribution, as Theorem 6.4.2 shows.

Theorem 6.4.2. If X is an H-function r.v. with p.d.f.

$$f(x)=\begin{cases} k\mathbf{H}_{p,q}^{m,n}\left[cx\left|\begin{matrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\right.\right], & x>0\\ 0, & \text{otherwise,}\end{cases}$$

then the p.d.f. of the r.v.

$$Y=X^P,$$

where P is a rational number, is given by

$$h(y)=\begin{cases} kc^{P-1}\mathbf{H}_{p,q}^{m,n}\left[c^Py\left|\begin{matrix}(a_1-\alpha_1P+\alpha_1,\alpha_1P),\dots,(a_p-\alpha_pP+\alpha_p,\alpha_pP)\\(b_1-\beta_1P+\beta_1,\beta_1P),\dots,(b_q-\beta_qP+\beta_q,\beta_qP)\end{matrix}\right.\right], \\ 0, \quad \text{otherwise}\end{cases}$$

$$y>0 \qquad (6.4.10)$$

when $P>0$, and

$$h(y)=$$

$$\begin{cases} kc^{P-1}\mathbf{H}_{q,p}^{n,m}\left[c^Py\left|\begin{matrix}(1-b_1+\beta_1P-\beta_1,-\beta_1P),\dots,(1-b_p+\beta_pP-\beta_p,-\beta_pP)\\(1-a_1+\alpha_1P-\alpha_1,-\alpha_1P),\dots,(1-a_p+\alpha_pP-\alpha_p,-\alpha_pP)\end{matrix}\right.\right], \\ 0, \quad \text{otherwise}\end{cases}$$

$$y>0 \qquad (6.4.11)$$

when $P<0$.

PROOF. Equation (4.3.12) shows that $h^+(y)$, the component of the p.d.f. $h(y)$ corresponding to $y>0$, is given by

$$h^+(y)=M^{-1}\left[M_{Ps-P+1}\left(k\cdot\mathbf{H}_{p,q}^{m,n}\left[cx\left|\begin{matrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\right.\right]\right)\right],$$

while from (6.2.9) and (6.4.5)

$$h^+(y)=M^{-1}\left[\frac{k}{c^t}\left.\frac{\prod_{v=1}^m\Gamma(b_v+\beta_vt)\prod_{j=1}^{j}\Gamma(1-a_v-\alpha_vt)}{\prod_{v=m+1}^q\Gamma(1-b_v-\beta_vt)\prod_{v=n+1}^p\Gamma(a_v+\alpha_vt)}\right|_{t=Ps-P+1}\right]$$

$$=M^{-1}\left[\frac{k}{c^{(Ps-P+1)}}\frac{\prod_{v=1}^m\Gamma(b_v-\beta_vP+\beta_v+\beta_vPs)}{\prod_{v=M+1}^q\Gamma(1-b_v+\beta_vP-\beta_v-\beta_vPs)}\right.$$

$$\left.\cdot\frac{\prod_{v=1}^n\Gamma(1-a_v+\alpha_vP-\alpha_v-\alpha_vPs)}{\prod_{v=m+1}^p\Gamma(a_v-\alpha_vP+\alpha_v-\alpha_vPs)}\right].$$

Application of (4.2.2) then yields

$$h^+(y)=\frac{kc^{P-1}}{2\pi i}\int_C \frac{\prod_{v=1}^{m}\Gamma(b_v-\beta_vP+\beta_v+\beta_vPs)}{\prod_{v=M+1}^{q}\Gamma(1-b_v+\beta_vP-\beta_v-\beta_vPs)}$$

$$\cdot\frac{\prod_{v=1}^{n}\Gamma(1-a_v+\alpha_vP-\alpha_v-\alpha_vPs)}{\prod_{v=n+1}^{p}\Gamma(a_v-\alpha_vP+\alpha_v+\alpha_vPs)}(c^Py)^{-s}\,ds,$$

and from the definition of the H-function, it follows that

$$h^+(y)=kc^{P-1}\mathbf{H}_{p,q}^{m,n}\left[c^Py\left|\begin{matrix}(a_1-\alpha_1P+\alpha_1,\alpha_1P),\ldots,(a_p-\alpha_pP+\alpha_p,\alpha_pP)\\(b_1-\beta_1P+\beta_1,\beta_1P),\ldots,(b_q-\beta_qP+\beta_q,\beta_qP)\end{matrix}\right.\right],$$

$$P>0$$

and

$$h^+(y)=kc^{P-1}$$

$$\cdot\mathbf{H}_{q,p}^{n,m}\left[c^Py\left|\begin{matrix}(1-b_1+\beta_1P-\beta_1,-\beta_1P),\ldots,(1-b_q+\beta_qP-\beta_q,-\beta_qP)\\(1-a_1+\alpha_1P-\alpha_1,-\alpha_1P),\ldots,(1-a_p+\alpha_pP-\alpha_q,-\alpha_qP)\end{matrix}\right.\right],$$

$$P<0.$$

Example 6.4.3 The Square of a Standard Half-Normal Variable. Suppose that in Theorem 6.4.2, the r.v. X has a standard half-normal distribution with the p.d.f. given in (6.3.8) with $\theta=1$, where when written in the form of (6.4.10),

$$k=\frac{1}{\sqrt{2\pi}},$$

$$b_1=0,$$

$$\beta_1=\frac{1}{2},$$

$$c=\frac{1}{\sqrt{2}},$$

and

$$m=1,\qquad n=0,\qquad p=0,\qquad q=1.$$

Then from Theorem 6.4.2, the p.d.f. of

$$Y = X^2$$

is given by

$$h(y) = \begin{cases} \dfrac{1}{2\sqrt{\pi}} \mathbf{H}_{0,1}^{1,0}\left[\dfrac{1}{2} y \middle| \left(-\dfrac{1}{2}, 1 \right) \right], & y > 0 \\ 0, & \text{otherwise,} \end{cases} \tag{6.4.12}$$

or, with the use of (6.2.8)

$$h(y) = \begin{cases} \dfrac{1}{2\sqrt{\pi}} \mathbf{G}_{0,1}^{1,0}\left[\dfrac{1}{2} y \middle| -\dfrac{1}{2} \right], & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Since (6.4.12) is equivalent to (6.3.10) with $v = 1$, it follows that $h(y)$ is the p.d.f. for the chi-square distribution with one degree of freedom. This result agrees with the well-known fact that the square of either a standard normal variable or a standard half-normal variable follows a chi-square distribution.

6.4.3 The Distribution of Quotients of H-Function Random Variables

From (4.3.14) and Theorems 6.4.1 and 6.4.2 (with $P = -1$), one obtains yet another important property of the H-function distribution, namely, that quotients of independent nonnegative H-function variables also follow an H-function distribution. This result is stated in the following theorem.

Theorem 6.4.3 If X_1 and X_2 are independent H-function r.v.'s with p.d.f.'s $f_1(x_1)$ and $f_2(x_2)$, respectively, where

$$f_j(x_j) = \begin{cases} k_j \mathbf{H}_{p_j, q_j}^{m_j, n_j}\left[c_j x_j \middle| \begin{matrix} (a_{j1}, \alpha_{j1}), \ldots, (a_{jp_j}, \alpha_{jp_j}) \\ (b_{j1}, \beta_{j1}), \ldots, (b_{jq_j}, \beta_{jq_j}) \end{matrix} \right], & x_j > 0 \\ 0, & \text{otherwise,} \end{cases}$$

for $j = 1, 2$, then the p.d.f. of the r.v.

$$Y = \frac{X_1}{X_2}$$

is given by

$$h(y)=\begin{cases} \dfrac{k_1k_2}{c_2^2}\mathbf{H}^{m_1+n_2,n_1+m_2}_{p_1+q_2,q_1+p_2}\left[\dfrac{c_1}{c_2}y\,\middle|\,\begin{matrix}(a_{11},\alpha_{11}),\ldots\\(b_{11},\beta_{11}),\ldots\end{matrix}\right], & y>0\\ 0, & \text{otherwise,}\end{cases} \tag{6.4.13}$$

where the sequence of the H-function parameters is

$$(a_{11},\alpha_{11}),\ldots,(a_{1n_1},\alpha_{1n_1}),(1-b_{21}-2\beta_{21},\beta_{21}),\ldots,$$

$$(1-b_{2m_2}-2\beta_{2m_2},\beta_{2m_2}),(a_{1,n_1+1},\alpha_{1,n_1+1}),\ldots,(a_{1p_1},\alpha_{1p_1}),$$

$$(1-b_{2,m_2+1}-2\beta_{2,m_2+1},\beta_{2,m_2+1}),\ldots,(1-b_{2q_2}-2\beta_{2q_2},\beta_{2q_2}),$$

and

$$(b_{11},\beta_{11}),\ldots,(b_{1m_1},\beta_{1m_1}),(1-a_{21}-2\alpha_{21},\alpha_{21}),\ldots,$$

$$(1-a_{2n_2}-2\alpha_{2n_2},\alpha_{2n_2}),(b_{1,m_1+1},\beta_{1,m_1+1}),\ldots,(b_{1q_1},\beta_{1q_1}),$$

$$(1-a_{2,n_2+1}-2\alpha_{2,n_2+1},\alpha_{2,n_2+1}),\ldots,(1-a_{2p_2}-2\alpha_{2p_2},\alpha_{2p_2}).$$

PROOF. From (4.3.14), the component of the p.d.f. of Y, which is obtained for nonnegative values of Y, is given by

$$h^+(y)=M^{-1}[M_s\{f_1(x_1)\}M_{2-s}\{f_2(x_2)\}],$$

or, from (6.4.4)

$$h^+(y)=$$

$$M^{-1}\left[\frac{k_1}{c_1^s}\frac{\prod_{v=1}^{m_1}\Gamma(b_{1v}+\beta_{1v}s)\prod_{v=1}^{n_1}\Gamma(1-a_{1v}-\alpha_{1v}s)}{\prod_{v=m_1+1}^{q_1}\Gamma(1-b_{1v}-\beta_{1v}s)\prod_{v=n_1+1}^{p_1}\Gamma(a_{1v}+\alpha_{1v}s)}\right.$$

$$\left.\cdot\frac{k_2}{c_2^{2-s}}\frac{\prod_{v=1}^{m_2}\Gamma(b_{2v}+2\beta_{2v}-\beta_{2v}s)\prod_{v=1}^{n_2}\Gamma(1-a_{2v}-2\alpha_{2v}+\alpha_{2v}s)}{\prod_{v=m_2+1}^{q_2}\Gamma(1-b_{2v}-2\beta_{2v}+\beta_{2v}s)\prod_{v=n_2+1}^{p_2}\Gamma(a_{2v}+2\alpha_{2v}-\alpha_{2v}s)}\right].$$

Rearranging and writing in terms of the Mellin inversion integral (6.4.2), this becomes

$$h^+(y)=\frac{k_1k_2}{c_2^2}\cdot\frac{1}{2\pi i}\int_C\frac{\prod_{v=1}^{n_1}\Gamma(1-a_{1v}-\alpha_{1v}s)\prod_{v=1}^{m_2}\Gamma(b_{2v}+2\beta_{2v}-\beta_{2v}s)}{\prod_{v=n_1+1}^{p_1}\Gamma(a_{1v}+\alpha_{1v}s)\prod_{v=m_2+1}^{q_2}\Gamma(1-b_{2v}-2\beta_{2v}+\beta_{2v}s)}$$

$$\cdot\frac{\prod_{v=1}^{m_1}\Gamma(b_{1v}+\beta_{1v}s)\prod_{v=1}^{n_2}\Gamma(1-a_{2v}-2\alpha_{2v}-\alpha_{2v}s)}{\prod_{v=m_1+1}^{q_1}\Gamma(1-b_{1v}-\beta_{1v}s)\prod_{v=n_2+1}^{p_2}\Gamma(a_{2v}+2\alpha_{2v}-\alpha_{2v}s)}\left(\frac{c_1}{c_2}y\right)^{-s}ds$$

$$=\frac{k_1k_2}{c_2^2}\mathbf{H}_{p_1+q_2,q_1+p_2}^{m_1+n_2,n_1+m_2}\left[\frac{c_2}{c_1}y\,\middle|\,\begin{matrix}(a_{11},\alpha_{11}),\ldots\\(b_{11},\beta_{11}),\ldots\end{matrix}\right],$$

where the sequence of the parameters of the H-function is that given in Theorem 6.4.3.

Example 6.4.4. The Quotient of Two Half-Normal Variables. Suppose that in Theorem 6.4.3, X_1 and X_2 are half-normal r.v.'s having the p.d.f. given in (6.3.8), where when written in the form of (6.4.13),

$$k_j=\frac{1}{\theta_j\sqrt{2\pi}},$$

$$a_{j1}=0,$$

$$\alpha_{j1}=\frac{1}{2},$$

$$c_j=\frac{1}{\theta_j\sqrt{2}},$$

and

$$m_j=1,\qquad n_j=0,\qquad p_j=0,\qquad q_j=1,$$

for $j=1,2$. Then substituting into (6.4.13) of Theorem 6.4.3, one obtains

$$h(y)=\begin{cases}\dfrac{\theta_2}{\theta_1\pi}\cdot\mathbf{H}_{1,1}^{1,1}\left[\dfrac{\theta_2}{\theta_1}y\,\middle|\,\begin{matrix}(0,\frac{1}{2})\\(0,\frac{1}{2})\end{matrix}\right], & y>0\\ 0, & \text{otherwise},\end{cases}\tag{6.4.14}$$

which, when compared to (6.3.13), is recognized to be the p.d.f. of the half-Cauchy distribution

$$f(y)=\frac{2\theta_2/\theta_1}{\pi\left((\theta_2/\theta_1)^2+y^2\right)}, \qquad y>0.$$

6.4.4 Determining the Parameters in the Distribution of Products, Quotients, and Powers of Independent *H*-Function Random Variables

Theorems 6.4.1, 6.4.2, and 6.4.3 show that the exact p.d.f. of a function involving only products, quotients, and powers of H-function r.v.'s can be calculated directly; or more precisely, if the parameters of the p.d.f. of each r.v. are known, the parameters of the probability distribution of the function can be easily calculated, after which the inversion integral can be exactly evaluated by the method of residues, as shown in Chapter 7. Figure 6.4.4 describes, in flow chart form, an algorithm or technique for evaluating the parameters of such a function. The objective of the flow chart is to describe the general logic of a solution procedure, not to serve as an exact description of the computer program.

In examining the flow chart, assume that the function to be evaluated is of the general form

$$Y=h(X_1,X_2,\ldots,X_n),$$

where $X_1,X_2,\ldots,X_n$ and Y are all H-function r.v.'s. Then, according to the symbolism of the flow chart in Fig. 6.4.4, the parameters of the p.d.f. of each X_j, when considered one at a time, are mm, nn, pp, qq, kk, cc, $(aa_j,\alpha\alpha_j,j=1,\ldots,pp)$ and $(bb_j,\beta\beta_j,j=1,\ldots,qq)$, and the respective parameters of the p.d.f. of Y are m, n, p, q, k, c, $(a_j,\ \alpha_j,\ j=1,\ldots,p)$ and $(b_j,\beta_j,j=1,\ldots,q)$ corresponding to the definition of the H-function distribution (6.3.1). Input to the system is an operation type (either multiply or divide) and either a power and the parameters of the probability distribution in the case of an r.v. or simply a numerical value in the case of a constant (deterministic value). Computations can then proceed in accordance with the flow chart until all r.v.'s and constants in the term are accounted for and the parameters of the p.d.f. of Y have been found.

As has just been pointed out, products, powers, and quotients of H-function r.v.'s are themselves H-function r.v.'s whose parameters are expressible in terms of those of the distributions of the original H-function r.v.'s. The p.d.f. of the resultant H-function r.v. is then a Mellin-Barnes type of inversion integral that must be evaluated to obtain the analytical form of the resultant p.d.f. The derivation of the exact (series) solution of

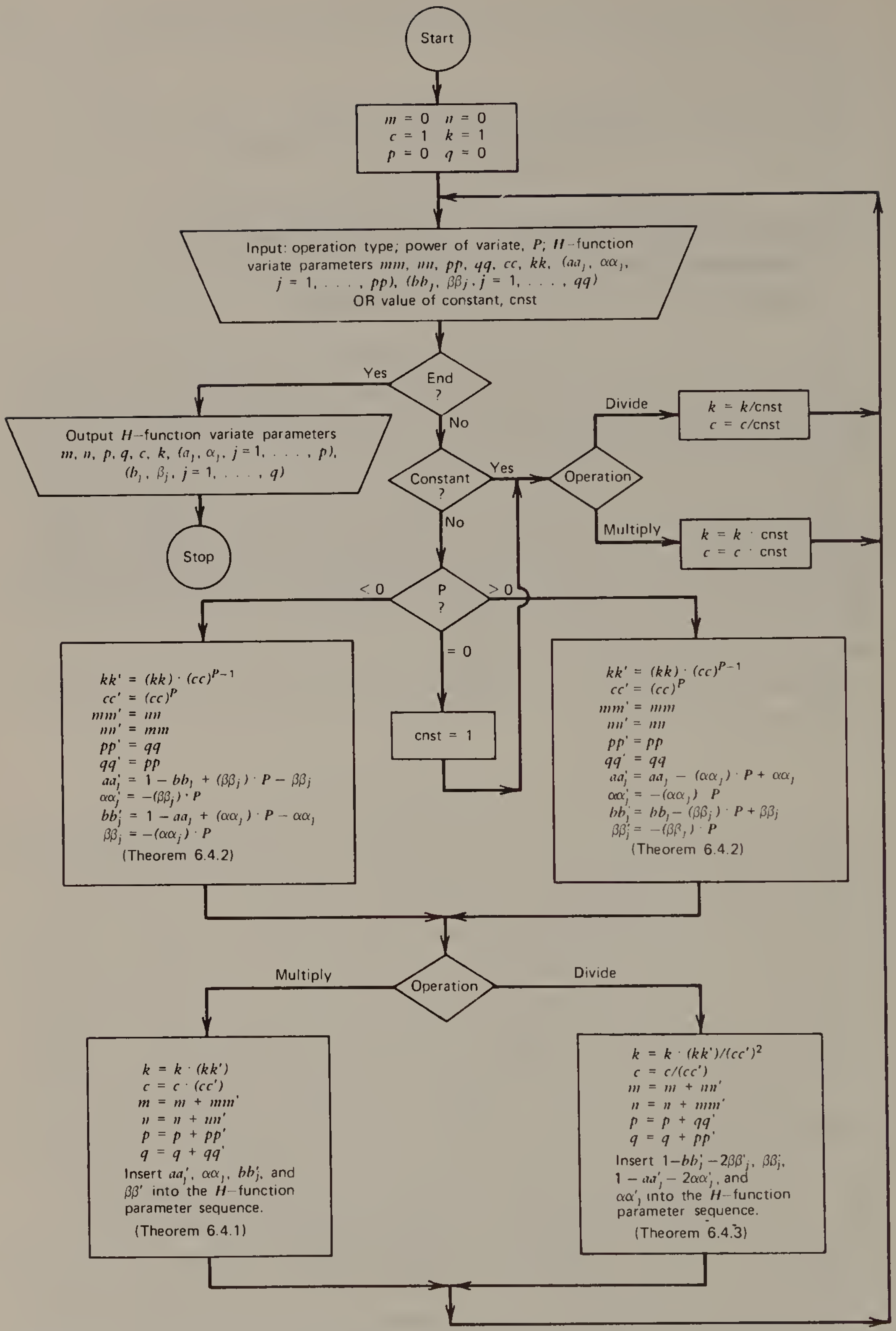


Fig. 6.4.4 Flow chart for determining the distribution parameters of the probability distribution of a function consisting of products, quotients, and powers of H-function random variables.

the H-function inversion integral (6.2.1) over the Bromwich path $(c-i\infty, c+i\infty)$ is due to Eldred and Barnes [91] and Lovett [218] (see Chapter 7). A computer program that evaluates this series solution (i.e., the resultant p.d.f.) and the corresponding cumulative distribution function (c.d.f.) has also been developed by Eldred and Barnes [91].

6.5 THE DISTRIBUTION OF SUMS OF INDEPENDENT H-FUNCTION RANDOM VARIABLES

This section gives expressions for the distribution of sums of independent H-function r.v.'s. Since it is at best difficult to evaluate the inversion integrals for these expressions in closed form in the general case, the results are not stated in theorem form. Nevertheless, the approximation methods presented in Chapter 8 are applicable and will lead to approximate solutions in the general case, for which the accuracy of the approximation can be determined.

The distribution of the sum of n i.r.v.'s is given in (3.2.9) of Theorem 3.2.5 as

$$h(w)=F^{-1}\left[\prod_{j=1}^{n} F_t\{f_j(x_j)\}\right].$$

Substituting the Fourier transform given in (6.3.2), this becomes

$$h(w)=F^{-1}\left\{\prod_{j=1}^{n}\frac{k_j}{c_j}\cdot \mathbf{H}_{q_j,p_j+1}^{n_j+1,m_j}\right.$$

$$\left.\times\left[-\frac{i}{c_j}t\left|\begin{matrix}(1-b_{j1}-\beta_{j1},\beta_{j1}),\ldots,(1-b_{jq_j}-\beta_{jq_j},\beta_{jq_j})\\(0,1),(1-a_{j1}-\alpha_{j1},\alpha_{j1}),\ldots,(1-a_{jp_j}-\alpha_{jp_j},\alpha_{jp_j})\end{matrix}\right.\right]\right\}.\tag{6.5.1}$$

Using the convolution notation, (6.5.1) can be rewritten as

$$h(w)=\frac{k_1}{c_1}\cdot\mathbf{H}_{p_1,q_1}^{m_1,n_1}\left[c_1x_1\left|\begin{matrix}(a_{11},\alpha_{11}),\ldots,(a_{1p_1},\alpha_{1p_1})\\(b_{11},\beta_{11}),\ldots,(b_{1q_1},\beta_{1q_1})\end{matrix}\right.\right]*\cdots$$

$$\times\cdots*\frac{k_n}{c_n}\cdot\mathbf{H}_{p_n,q_n}^{m_n,n_n}\left[c_nx_n\left|\begin{matrix}(a_{n1},\alpha_{n1}),\ldots,(a_{np_n},\alpha_{np_n})\\(b_{n1},\beta_{n1}),\ldots,(b_{nq_n},\beta_{nq_n})\end{matrix}\right.\right].\tag{6.5.2}$$

where the symbol $*$ represents the Fourier exponential convolution operation (and also the Laplace convolution in this case, since $\Pr(X_j \leqslant 0)=0$). If the p.d.f. of $h(y)$ cannot be determined in closed form from either (6.5.1) or (6.5.2), it can be approximated by obtaining approximate solutions to (6.5.1) using the methods of Chapter 8.

6.6 THE DISTRIBUTION OF A POLYNOMIAL IN INDEPENDENT H-FUNCTION VARIABLES

Now consider the equation of the form

$$W = \sum_{h=1}^{M} R_h \prod_{j=1}^{N_h} X_{hj}^{P_{hj}} \tag{6.6.1}$$

where R_h is a constant and the X_{hj}'s are independent H-function r.v.'s. What is the distribution of W? It is the purpose of this section to obtain expressions for:

1. The p.d.f. of W.
2. The rth noncentral moment of the distribution of W when the p.d.f. of X_{hj} is given by

$$f_{hj}(x_{hj}) = \begin{cases} k_{hj} \mathbf{H}_{p_{hj}, q_{hj}}^{m_{hj}, n_{hj}} \left[c_{hj} x_{hj} \left| \begin{matrix} (a_{hj1}, \alpha_{hj1}), \ldots, (a_{hjp_{hj}}, \alpha_{hjp_{hj}}) \\ (b_{hj1}, \beta_{hj1}), \ldots, (b_{hjq_{hj}}, \beta_{hjq_{hj}}) \end{matrix} \right. \right], & x_{hj} > 0 \\ 0, \quad \text{otherwise.} \end{cases} \tag{6.6.2}$$

Assuming that all $P_{hj} > 0$, (6.4.10) of Theorem 6.4.2 shows that the p.d.f. of the distribution of

$$X'_{hj} = X_{hj}^{P_{hj}}$$

is expressible in the form

$$f'_{hj}(x'_{hj}) = \begin{cases} k_{hj} c_{hj}^{P_{hj}-1} \cdot \mathbf{H}_{p_{hj}, q_{hj}}^{m_{hj}, n_{hj}} \left[c_{hj}^{P_{hj}} x'_{hj} \left| \begin{matrix} (a_{hj1} - \alpha_{hj1} P_{hj} + \alpha_{hj1}, \alpha_{hj1} P_{hj}), \ldots, \\ (b_{hj1} - \beta_{hj1} P_{hj} + \beta_{hj1}, \beta_{hj1} P_{hj}), \ldots, \\ (a_{hjp_{nj}} - \alpha_{hjp_{nj}} P_{hj} + \alpha_{hjp_{nj}}, \alpha_{hjp_{nj}} P_{hj}) \\ (b_{hjq_{hj}} - \beta_{hjp_{nj}} P_{hj} + \beta_{hjp_{nj}}, \beta_{hjq_{hj}} P_{hj}) \end{matrix} \right. \right] \\ 0, \quad \text{otherwise.} \end{cases}$$

$$x'_{hj} > 0, \tag{6.6.3}$$

If $P_{hj}<0$, (6.4.11) will be used to find $f'_{hj}(x'_{hj})$ instead of (6.4.10). To simplify the notation as much as possible, only the results of $P_{hj}>0$ will be used in the following expressions.

From (6.4.6) of Theorem 6.4.1, it follows that the p.d.f. of the distribution of

$$Z_h = \prod_{j=1}^{N_h} X'_{hj}$$

is given by

$$g_h(z_h)=\begin{cases} \left(\prod_{j=1}^{N_h} k_{hj} c_{hj}^{P_{hj}-1}\right) \cdot \mathbf{H}_{\sum_{j=1}^{N_h} p_{hj},\ \sum_{j=1}^{N_h} q_{hj}}^{\sum_{j=1}^{N_h} m_{hj},\ \sum_{j=1}^{N_h} n_{hj}} \left[\prod_{j=1}^{N_h} c_{hj}^{P_{hj}} z_h \left| \begin{matrix} (a_{h11}-\alpha_{h11}P_{h1}+\alpha_{h11},\alpha_{h11}P_{h1}),\ldots, \\ (b_{h11}-\beta_{h11}P_{h1}+\beta_{h11},\beta_{h11}P_{h1}),\ldots, \\ \left(a_{hN_{h p_h N_h}}-\alpha_{hN_h p_h N_h}P_{hN_h}+\alpha_{hN_h p_h H_h},\alpha_{hN_h p_h N_h}P_{hN_h}\right) \\ \left(b_{hN_h q_{hH_h}}-\beta_{hN_h q_h N_h}P_{hN_h}+\beta_{hN_h q_h N_h},\beta_{hN_h q_h N_h}P_{hN_h}\right) \end{matrix} \right. \right], \\ 0, \qquad \text{otherwise,} \end{cases}$$

$$z_h>0 \qquad (6.6.4)$$

where the sequence of the parameters in the H-function is that given in (6.4.3).

If $Z'_h = R_h Z_h$, where R_h is a constant, then the p.d.f. the distribution of Z'_h can be found by variable substitution into (6.6.4). Specifically, one obtains

$$q'_h(z'_h)=\begin{cases} \frac{1}{|R_h|}\left(\prod_{j=1}^{N_h} k_{hj} c_{hj}^{P_{hj}-1}\right) \mathbf{H}_{\sum_{j=1}^{N_h} p_{hj},\ \sum_{j=1}^{N_h} q_{hj}}^{\sum_{j=1}^{N_h} m_{hj},\ \sum_{j=1}^{N_h} n_{hj}} \left[\frac{1}{R_h}\left(\prod_{j=1}^{N_h} c_{hj}^{P_{hj}}\right) z_h \left| \begin{matrix} (a_{h11}-\alpha_{h11}P_{h1}+\alpha_{h11},\alpha_{h11}P_{h1}),\ldots, \\ (b_{h11}-\beta_{h11}P_{h1}+\beta_{h11},\beta_{h11 ph1}),\ldots, \\ \left(a_{hN_h p_h N_h}-\alpha_{hN_h p_h N_h}P_{hN_h}+\alpha_{hN_h p_h N_h},\alpha_{hN_h p_h N_h}P_{hN_h}\right) \\ \left(b_{hN_h q_h N_h}-\beta_{hN_h q_h N_h}P_{hN_h}+\beta_{hN_h q_h N_h},\beta_{hN_h q_h N_h}P_{hN_h}\right) \end{matrix} \right. \right], \\ 0, \qquad \text{otherwise,} \end{cases}$$

$$z_h>0 \qquad (6.6.5)$$

where the parameters have the same sequence as in (6.6.4).

Hence the p.d.f. of the distribution of

$$Y=\sum_{h=1}^{M} Z_h'=\sum_{h=1}^{M} R_h Z_h=\sum_{h=1}^{M} R_h \prod_{j=1}^{N_h} X_{hj}'$$

$$=\sum_{h=1}^{M} R_h \prod_{j=1}^{N_h} X_{hj}^{P_{hj}}$$

is given by

$$h(y)=g_1'(z_1')*g_2'(z_2')*\cdots*g_M'(z_M'), \tag{6.6.6}$$

where, as before, the symbol $*$ represents the Fourier convolution operation.

Although the expression above for $h(y)$ is complicated and difficult to evaluate, the moments for the distribution are much easier to find. The p.d.f. of the distribution of Z_h' is given by (6.6.5) and, from (6.4.4), its Mellin transform is

$$M_s\{g_h'(z_h')\}=\frac{R_h^s}{|R_h|}\prod_{j=1}^{N_h}\Bigg\{k_{hj}c_{hj}^{P_{hj}(1-s)-s}$$

$$\cdot\frac{\prod_{v=1}^{M_{hj}}\Gamma(b_{hjv}-\beta_{hjv}P_{hj}+\beta_{hjv}+\beta_{hjv}P_{hj}s)}{\prod_{v=m_{hj}+1}^{q_{hj}}\Gamma(1-b_{hjv}+\beta_{hjv}P_{hj}-\beta_{hjv}-\beta_{hjv}P_{hj}s)}$$

$$\cdot\frac{\prod_{v=1}^{n_{hj}}\Gamma(1-a_{hjv}+\alpha_{hjv}P_{hj}-\alpha_{hjv}-\alpha_{hjv}P_{hj}s)}{\prod_{v=n_{hj}+1}^{p_{hj}}\Gamma(a_{hjv}-\alpha_{hjv}P_{hj}+\alpha_{hjv}+\alpha_{hjv}P_{hj}s)}\Bigg\}. \tag{6.6.7}$$

As (6.3.3b) shows, the rth moment about the origin of the distribution of Z_h' can be expressed as

$$M_{r+1}\{g_h'(z_h')\}=E\{(Z_h')^r\}.$$

Therefore, because of the independence of the terms, the rth moment about the origin for the distribution of Y can be expressed as a function of the lower order moments about the origin for the distributions of the individual terms. In other words, $E\{Y^r\}$ can be expressed as a function of $E\{(Z_h')^v\}$ for $h=1,2,\ldots,M$ and for $v=1,2,\ldots,r$.

EXERCISES

6.1* Using Theorem 6.4.3, show that the central t variable is an H-function r.v., and determine its p.d.f. from (6.4.13).

6.2* Using Theorem 6.4.3, show that the central F variable is an H-function r.v., and determine its p.d.f. from (6.4.13).

6.3 Show that

$$Y = X_1 X_2 + X_3 X_4$$

has the Laplace or double exponential distribution

$$g(y) = \frac{1}{2}\exp(-|y|)$$

if the X_i, $i=1,2,3,4$, are normal i.r.v.'s with mean 0 and variance 1.

(Mantel, 1973)

CHAPTER 7*

Analytical Model for Evaluation of the H-Function Inversion Integral

The reader will recall that when normalized with the proper constant, the H-function inversion integral presented in Chapter 6 encompasses an entire class or family of p.d.f.'s. Because this inversion integral (p.d.f.) includes so many basic distributions in statistics as special cases, it is important to be able to evaluate it in analytical (series) form. The tools necessary for doing this were provided in Chapter 6. The actual derivation of the analytical (series) form of the H-function inversion integral, however, is carried out in the following sections. The derivation employs the Euler psi function ψ and its derivatives $\psi^{(m)}$ (the polygamma functions), together with recursive formulas amenable to computer evaluation. The functions ψ and $\psi^{(m)}$ may be evaluated for positive and negative real arguments (except nonpositive integers) to any desired degree of accuracy by series expansions and recurrence relations [169, 2].

The evaluation of the p.d.f. is accomplished[34] in Sections 7.1.1, 7.1.2, and 7.1.3. The solution is completely general, including the cases of absence of poles in the LHP or RHP, and overlapping or partial overlapping of poles. A computer program has been written by Eldred and Barnes [91] which is operational and which achieves the following tasks:

1. Determines the parameters $a_i, \alpha_i, b_i, \beta_i, M, N, p$, and q in the p.d.f. of an H-function r.v., that is the p.d.f. of the product, quotient, or rational power of input H-function r.v.'s.

[34] I am indebted to Mr. B. S. Eldred and Dr. J. W. Barnes [91] of the Mechanical Engineering Department of the University of Texas at Austin, for developing Sections 7.1.1, 7.1.2, Examples 7.3.1, 7.3.2 and for their permission to include these results in this book. The results in Sections 7.1.3, 7.2, and Appendix F are due to Lovett [218].

2. Evaluates the p.d.f. and c.d.f. of the H-function r.v. at any value of the r.v.
3. Plots the p.d.f. and c.d.f.

The computer program, compiled on an MNF compiler and run on a CDC 6600, is very efficient, and poses no precision problems. Should precision problems arise when the program is run on smaller computers, the problem may be solved by compiling the program under IBM's Extended H-compiler.

7.1 ANALYTICAL FORM OF THE PROBABILITY DENSITY FUNCTION OF AN H-FUNCTION VARIABLE

The H-function inversion integral was presented in Chapter 6 in integral form, but for convenience of the reader it appears again below:[35]

$$h(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\prod_{i=1}^{M}\Gamma(b_i+\beta_i s)\prod_{i=1}^{N}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=M+1}^{q}\Gamma(1-b_i-\beta_i s)\prod_{i=N+1}^{p}\Gamma(a_i+\alpha_i s)}y^{-s}ds, \tag{7.1.1}$$

where y is a real number; $0\leqslant M\leqslant q$; $0\leqslant N\leqslant p; a_i>0$ for $i=1,2,\ldots,p$; $\beta_i>0$ for $i=1,2,\ldots,q$; and $a_i(i=1,2,\ldots,p)$ and $b_i(i=1,2,\ldots,q)$ are real numbers such that no pole of

$$\Gamma(b_i+\beta_i s) \quad \text{for} \quad i=1,2,\ldots,M$$

coincides with any pole of

$$\Gamma(1-a_i-\alpha_i s) \quad \text{for} \quad i=1,2,\ldots,N,$$

and the Bromwich path $(c-i\infty,c+i\infty)$ in the complex plane is such that all poles of

$$\prod_{i=1}^{M}\Gamma(b_i+\beta_i s) \qquad \text{and} \qquad \prod_{i=1}^{N}\Gamma(1-a_i-\alpha_i s)$$

lie to the left and right of $(c-i\infty,c+i\infty)$, respectively. This last restriction follows directly from the definition of the H-function (Definition 6.2.1).

[35]The reader is reminded that in the limits of integration $(c-i\infty,c+i\infty)$, $i=\sqrt{-1}$, whereas when used as a subscript in a_i, b_i, and so on, the letter i denotes a running variable, and also that (7.1.1) is equivalent to the definition (6.2.1).

Poles of these two gamma function factors occur at nonpositive integer values of the arguments $(b_i+\beta_i s), i=1,2,\ldots,M$ and $(1-a_i-\alpha_i s), i=1,2,\ldots,p$. Hence, these poles can be given by

$$s_{ij}=-\frac{b_i+j}{\beta_i}, \qquad j=0,1,2,\ldots$$

and

$$s_{ij}=\frac{1-a_i+j}{\alpha_i}, \qquad j=0,1,2,\ldots$$

for the factors

$$\prod_{i=1}^{M}\Gamma(b_i+\beta_i s) \qquad \text{and} \qquad \prod_{i=1}^{N}\Gamma(1-a_i-\alpha_i s),$$

respectively. Then the restriction that the Bromwich path $(c-i\infty, c+i\infty)$ separate the complex s-plane into an LHP and an RHP such that the points

$$s_{ij}=-\frac{b_i+j}{\beta_i}, \qquad j=0,1,2,\ldots$$

lie to the left of $(c-i\infty, c+i\infty)$, and the points

$$s_{ij}=\frac{1-a_i+j}{\alpha_i}, \qquad j=0,1,2,\ldots$$

lie to the right of $(c-i\infty, c+i\infty)$ can be expressed by the constraint

$$\max\left\{\mathrm{Re}\left(-\frac{b_i}{\beta_i}\right)\right\}_{i=1}^{M} < \min\left\{\mathrm{Re}\left(\frac{1-a_i}{\alpha_i}\right)\right\}_{i=1}^{N}.$$

If the poles are in the LHP, the residue theorem is applicable when the value of the inversion integral over the circular arc C_{L_K} (Fig. F.1.1) approaches zero as the radius of the circle approaches infinity. Sufficient conditions for the vanishing of the integral over this arc are given by Jordan's lemma (Appendix A). Similar statements apply when the poles are in the RHP. These statements are verified in Appendix F.

Consequently, the method of residues must be applied once for poles in the LHP and again for poles in the RHP, to evaluate the H-function inversion integral $h(y)$. Thus

$$h(y)=\begin{cases} h_1(y) & \text{for poles in the LHP} \\ h_2(y) & \text{for poles in the RHP,} \end{cases}$$

where $h_1(y)$ is obtained by evaluating residues at poles in the LHP, and $h_2(y)$ is obtained by evaluating residues at poles in the RHP. In the following development, $h_1(y)$ and $h_2(y)$ represent components of the p.d.f. $h(y)$ of a product, quotient, or rational power of independent H-function r.v.'s.

It should be noted that in what follows $h_1(y)$ and $h_2(y)$ also correspond to the components of the p.d.f. for which $0<y\leqslant 1$ and $1\leqslant y<\infty$, respectively. This fact follows from an examination of Jordan's Lemma (Appendix A), which guarantees the applicability of the residue theorem to the evaluation of the H-function inversion integral, as is shown in Appendix F.

The question naturally arises: How many poles need to be considered in the evaluation of the H-function inversion integral when there is an infinite number of poles in the integrand? The number depends of course on the desired accuracy of the distribution function for the particular H-function involved. For most cases utilization of 10 to 20 poles will probably be more than adequate for 5-place accuracy in the distribution function. However, cases have been identified that require the utilization of 30 poles. Because of the characteristic of nonuniform convergence of the series form of the H-function inversion integral, it is difficult to determine the number of poles that are adequate to achieve a specified accuracy in the distribution function in the general case. This is an area for further research.

7.1.1 Evaluation of the Probability Density Function for Poles in the Left-Half Plane

First consider the evaluation of the p.d.f. $h(y)$ at poles in the LHP. In the H-function inversion integral, these poles occur in the factor

$$\prod_{i=1}^{M}\Gamma(b_i+\beta_i s)$$

and are given by

$$s_{ij}=-\frac{b_i+j}{\beta_i},\qquad \begin{aligned} &j=0,1,2,3,\ldots\\ &i=1,2,3,\ldots,M. \end{aligned}$$

Without loss of generality assume that the poles

$$\{s_{ij}\}_{j=0,i=1}^{\infty,M}$$

are ordered algebraically from largest (most positive) to smallest (most negative) and denoted by

$$\{s_k\}_{k=1}^{\infty}$$

Then, by the theory of residues, the portion of $h(y)$ evaluated at poles in the LHP may be given as

$$h_1(y)=\sum_{k=1}^{\infty}(\text{residue evaluted at } s_k)$$

$$=\sum_{k=1}^{\infty}\frac{1}{(\delta_k-1)!}\frac{d^{\delta_k-1}}{ds^{\delta_k-1}}$$

$$\cdot\left.\left\{(s-s_k)^{\delta_k}\frac{\prod_{i=1}^{M}\Gamma(b_i+\beta_i s)\prod_{i=1}^{N}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=M+1}^{q}\Gamma(1-b_i-\beta_i s)\prod_{i=N+1}^{p}\Gamma(a_i+\alpha_k s)}y^{-s}\right\}\right|_{s=s_k}$$

where δ_k is the order of the pole s_k in the integrand of the H-function inversion integral, after any cancellation of terms.

To arrive at a value for δ_k, it is necessary to know the order of the pole s_k in each of the four product terms of the integrand. Hence, define δ_{kM}, δ_{kN}, δ_{kq}, and δ_{kp} to be the orders of the pole in the four terms

$$\prod_{i=1}^{M}\Gamma(b_i+\beta_i s),\qquad \prod_{i=1}^{N}\Gamma(1-a_i-\alpha_i s),\qquad \prod_{i=M+1}^{q}\Gamma(1-b_i-\beta_i s),$$

and $\displaystyle\prod_{i=N+1}^{p}\Gamma(a_i+\alpha_i s)$, respectively.

For example, δ_{kM} is the number of factors in the term

$$\prod_{i=1}^{M}\Gamma(b_i+\beta_i s)$$

such that

$$b_i + \beta_i s_k = j_{ki}$$

where j_{ki} is a nonpositive integer, possibly different for each relevant i. This definition yields

$$\delta_k = \delta_{kM} + \delta_{kN} - \delta_{kq} - \delta_{kp}.$$

However, due to the definition of the H-function, the poles of the two terms

$$\prod_{i=1}^{M} \Gamma(b_i + \beta_i s) \qquad \text{and} \qquad \prod_{i=1}^{N} \Gamma(1 - a_i - \alpha_i s)$$

are separated by the contour of integration. The evaluation here is at the poles of the LHP; therefore δ_{kN} must necessarily be zero. Note also that if $\delta_k < 1$ then there is no residue at s_k.

Given then that

$$\delta_k = \delta_{kM} - \delta_{kq} - \delta_{kp} \geqslant 1,$$

$h_1(y)$ can be expressed as follows:

$$h_1(y) = \sum_{k=1}^{\infty} \frac{1}{(\delta_k - 1)!} \frac{d^{\delta_k - 1}}{ds^{\delta_k - 1}}$$

$$\cdot \left\{ \frac{(s - s_k)^{\delta_{kM}} \prod_{i=1}^{M} \Gamma(b_i + \beta_i s) \prod_{i=1}^{N} \Gamma(1 - a_i - \alpha_i s) y^{-s}}{(s - s_k)^{\delta_{kq}} \prod_{i=M+1}^{q} \Gamma(1 - b_i - \beta_i s)(s - s_k)^{\delta_{kp}} \prod_{i=N+1}^{p} \Gamma(a_i + \alpha_i s)} \right\} \Bigg|_{s = s_k}$$

$$= \sum_{k=1}^{\infty} \frac{1}{(\delta_k - 1)!} \frac{d^{\delta_k - 1}}{ds^{\delta_k - 1}} \{ C_k^{(0)}(s) U_k^{(0)}(s) y^{-s} \} |_{s = s_k} \tag{7.1.2}$$

where

$$C_k^{(0)}(s) = \frac{\prod_{i=1}^{N} \Gamma(1 - a_i - \beta_i s)}{(s - s_k)^{\delta_{kq}} \prod_{i=M+1}^{q} \Gamma(1 - b_i - \beta_i s)(s - s_k)^{\delta_{kp}} \prod_{i=N+1}^{p} \Gamma(a_i + \alpha_i s)}$$

and

$$U_k^{(0)}(s) = (s - s_k)^{\delta_{kM}} \prod_{i=1}^{M} \Gamma(b_i + \beta_i s).$$

Applying Leibnitz's rule for the differentiation of products to the last formulation of $h_1(y)$ yields

$$h_1(y)=\sum_{k=1}^{\infty}\frac{1}{(\delta_k-1)!}\sum_{w=0}^{\delta_k-1}\binom{\delta_k-1}{w}\left[\frac{d^{\delta_k-w-1}}{ds^{\delta_k-w-1}}C_k^{(0)}(s)\right]$$

$$\left[\frac{d^w}{ds^w}\left[U_k^{(0)}(s)y^{-s}\right]\right]\Bigg|_{s=s_k}$$

$$=\sum_{k=1}^{\infty}\frac{1}{(\delta_k-1)!}\sum_{w=0}^{\delta_k-1}\binom{\delta_k-1}{w}\left[\frac{d^{\delta_k-w-1}}{ds^{\delta_k-w-1}}C_k^{(0)}(s)\right]$$

$$\cdot\left[\sum_{v=0}^{w}\binom{w}{v}\left\{\frac{d^{w-v}}{ds^{w-v}}y^{-s}\right\}\left\{\frac{d^v}{ds^v}U_k^{(0)}(s)\right\}\right]\Bigg|_{s=s_k}.$$

From the preceding discussion it is evident that, to calculate $h_1(y)$ efficiently, it is necessary to be able to derive simple forms for the derivatives of $C_k^{(0)}(s)$, $U_k^{(0)}(s)$, and y^{-s}. For this purpose note that

$$\Gamma(x)=\frac{\Gamma(1+x)}{x}.$$

With this in mind, $U_k^{(0)}(s)$ and $C_k^{(0)}(s)$ may be written as follows:

$$U_k^{(0)}(s)=\prod_{\substack{i=1\\ b_i+\beta_i s_k\neq -J}}^{M}\Gamma(b_i+\beta_i s)\;.\prod_{\substack{i=1\\ b_i+\beta_i s_k=-J}}^{M}$$

$$\cdot\frac{(s-s_k)\Gamma(J+1+b_i+\beta_i s)}{(b_i+\beta_i s)(1+b_i+\beta_i s)\dots(J-1+b_i+\beta_i s)(J+b_i+\beta_i s)}$$

$$=\prod_{\substack{i=1\\ b_i+\beta_i s_k\neq -J}}^{M}\Gamma(b_i+\beta_i s)\prod_{\substack{i=1\\ b_i+\beta_i s_k=-J}}^{M}$$

$$\cdot\frac{\Gamma(J+1+b_i+\beta_i s)}{(b_i+\beta_i s)(1+b_i+\beta_i s)\dots(J-1+b_i+\beta_i s)\beta_i}$$

$$C_k^{(0)}(s) = \frac{\prod_{i=1}^{N} \Gamma(1-a_i-\alpha_i s)}{\prod_{\substack{i=M+1 \\ 1-b_i-\beta_i s_k \neq -J}}^{q} \Gamma(1-b_i-\beta_i s) \prod_{\substack{i=M+1 \\ 1-b_i-\beta_i s_k = -J}}^{q}}$$

$$\cdot \frac{1}{\dfrac{(s-s_k)\Gamma(J+2-b_i-\beta_i s)}{(1-b_i-\beta_i s)(2-b_i-\beta_i s)\dots(J-b_i-\beta_i s)(J+1-b_i-\beta_i s)}}$$

$$\cdot \frac{1}{\prod_{\substack{i=N+1 \\ a_i+\alpha_i s_k \neq -J}}^{p} \Gamma(a_i+\alpha_i s) \prod_{\substack{i=N+1 \\ a_i+\alpha_i s_k = -J}}^{p}}$$

$$\cdot \frac{1}{\dfrac{(s-s_k)\Gamma(J+1+a_i+\alpha_i s)}{(a_i+\alpha_i s)(1+a_i+\alpha_i s)\dots(J-1+a_i+\alpha_i s)(J+a_i+\alpha_i s)}}$$

$$= \prod_{i=1}^{N} \Gamma(1-a_i-\alpha_i s) \prod_{\substack{i=M+1 \\ 1-b_i-\beta_i s_k \neq -J}}^{q} [\Gamma(1-b_i-\beta_i s)]^{-1}$$

$$\cdot \prod_{\substack{i=M+1 \\ 1-b_i-\beta_i s_k = -J}}^{q} \left[\frac{\Gamma(J+2-b_i-\beta_i s)}{(1-b_i-\beta_i s)(2-b_i-\beta_i s)\dots(J-b_i-\beta_i s)(-\beta_i)}\right]^{-1}$$

$$\cdot \prod_{\substack{i=N+1 \\ a_i+\alpha_i s_k \neq -J}}^{p} [\Gamma(a_i+\alpha_i s)]^{-1}$$

$$\cdot \prod_{\substack{i=N+1 \\ a_i+\alpha_i s_k = -J}}^{p} \left[\frac{\Gamma(J+1+a_i+\alpha_i s)}{(a_i+\alpha_i s)(1+a_i+\alpha_i s)\dots(J-1+a_i+\alpha_i s)(\alpha_i)}\right]^{-1}$$

where the conditional notations, beneath the product signs, of the form $x(i,k) \neq -J$ and $x(i,k) = -J$, are to be read "not equal to any negative integer" and "equal to some negative integer $-J$ (not necessarily constant with respect to i and k)," respectively. The conditional notations are to delineate between those gamma functions that have poles at s_k and those that do not.

To calculate the first derivatives of $C_k^{(0)}(s)$ and $U_k^{(0)}(s)$ utilize the simple product rule for products involving an arbitrary number of factors. That is

$$\frac{d}{dx}\prod_{i=1}^{L} f_i(x) = \sum_{i=1}^{L} f_i^{(1)}(x)\prod_{\substack{j=1\\ j\neq i}}^{L} f_j(x).$$

Now note that each term of $C_k^{(0)}(s)$ and $U_k^{(0)}(s)$ is of one of the four forms

$$\Gamma(x), \qquad \frac{1}{c+dx}, \qquad [\Gamma(x)]^{-1}, \qquad \text{or} \qquad \left(\frac{1}{c+dx}\right)^{-1}.$$

These forms have the following derivatives:

$$\frac{d}{dx}\Gamma(x)=\Gamma(x)\psi(x) \qquad \text{where} \quad \psi(x)=\frac{d}{dx}\ln\Gamma(x)$$

$$\frac{d}{dx}(c+dx)^{-1}=(c+dx)^{-1}\left(\frac{-d}{c+dx}\right)$$

$$\frac{d}{dx}[\Gamma(x)]^{-1}=-[\Gamma(x)]^{-2}\frac{d}{dx}\Gamma(x)=[\Gamma(x)]^{-1}(-\psi(x))$$

$$\frac{d}{dx}\left(\frac{1}{c+dx}\right)^{-1}=d=\left(\frac{1}{c+dx}\right)^{-1}\left(\frac{d}{c+dx}\right).$$

A common property, then, of these four forms is that for each form $f_i(x)$, it happens that there exists a function $g_i(x)$ such that $f_i^{(1)}(x)=f_i(x)g_i(x)$. Then the simple product rule above takes the form

$$\frac{d}{dx}\left[\prod_{i=1}^{L} f_i(x)\right]=\sum_{i=1}^{L} g_i(x)\prod_{j=1}^{L} f_j(x).$$

Applying the facts of the preceding discussion to $U_k^{(0)}(s)$ and $C_k^{(0)}(s)$ yields

$$U_k^{(1)}(s)=U_k^{(0)}(s)\Bigg\{\sum_{\substack{i=1\\ b_i+\beta_i s_k\neq -J}}^{M}(\beta_i)\psi(b_i+\beta_i s)$$

$$+\sum_{\substack{i=1\\ b_i+\beta_i s_k=-J}}^{M}\left[\beta_i\psi(J+1+b_i+\beta_i s)+\sum_{l=0}^{J-1}\frac{(-\beta_i)}{(l+b_i+\beta_i s)}\right]\Bigg\}$$

$$=U_k^{(0)}(s)\,V_k^{(1)}(s)$$

and

$$
\begin{aligned}
C_k^{(1)}(s) = C_k^{(0)}(s)\Bigg\{ & \sum_{i=1}^{N} (-\alpha_i)\psi(1-a_i-\alpha_i s) \\
& - \sum_{\substack{i=M+1 \\ 1-b_i-\beta_i s_k \neq -J}}^{q} (-\beta_i)\psi(1-b_i-\beta_i s) \\
& - \sum_{\substack{i=M+1 \\ 1-b_i-\beta_i s_k = -J}}^{q} \left[(-\beta_i)\psi(J+2-b_i-\beta_i s) + \sum_{l=1}^{J} \frac{\beta_i}{(l-b_i-\beta_i s)}\right] \\
& - \sum_{\substack{i=N+1 \\ a_i+\alpha_i s_k \neq -J}}^{p} (\alpha_i)\psi(a_i+\alpha_i s) \\
& - \sum_{\substack{i=N+1 \\ a_i+\alpha_i s_k = -J}}^{p} \left[(\alpha_i)\psi(J+1+a_i+\alpha_i s) + \sum_{l=0}^{J-1} \frac{-\alpha_i}{(l+a_i+\alpha_i s)}\right]\Bigg\} \\
= C_k^{(0)}(s)\chi_k^{(1)}(s).
\end{aligned}
$$

From this point it is a straightforward procedure to calculate the higher order derivatives of $C_k^{(0)}(s)$ and $U_k^{(0)}(s)$. That is, simply apply Leibnitz's rule for the differentiation of products in a recursive fashion. Then

$$
C_k^{(r)}(s) = \sum_{l=0}^{r-1} \binom{r-1}{l} C_k^{(r-l-1)}(s)\chi_k^{(l+1)}(s)
$$

and

$$
U_k^{(r)}(s) = \sum_{l=0}^{r-1} \binom{r-1}{l} U_k^{(r-l-1)}(s) V_k^{(l+1)}(s)
$$

where, by simple differentiation of sums,

$$\chi_k^{(t)}(s)=\sum_{i=1}^{N}(-\alpha_i)^t\psi^{(t-1)}(1-a_i-\alpha_i s)$$

$$-\sum_{\substack{i=M+1\\ 1-b_i-\beta_i s_k\neq -J}}^{q}(-\beta_i)^t\psi^{(t-1)}(1-b_i-\beta_i s)$$

$$-\sum_{\substack{i=M+1\\ 1-b_i-\beta_i s_k=-J}}^{q}$$

$$\cdot\left[(-\beta_i)^t\psi^{(t-1)}(J+2-b_i-\beta_i s)+\sum_{l=1}^{J}\frac{(\beta_i)^t(t-1)!}{(l-b_i-\beta_i s)^t}\right]$$

$$-\sum_{\substack{i=N+1\\ a_i+\alpha_i s_k\neq -J}}^{p}(\alpha_i)^t\psi^{(t-1)}(a_i+\alpha_i s)$$

$$-\sum_{\substack{i=N+1\\ a_i+\alpha_i s_k=-J}}^{p}\left[(\alpha_i)^t\psi^{(t-1)}(J+1+a_i+\alpha_i s)+\sum_{l=0}^{J-1}\frac{(-\alpha_i)^t(t-1)!}{(l+a_i+\alpha_i s)^t}\right]$$

and

$$V_k^{(t)}(s)=\sum_{\substack{i=1\\ b_i+\beta_i s_k\neq -J}}^{M}(\beta_i)^t\psi^{(t-1)}(b_i+\beta_i s)$$

$$+\sum_{\substack{i=1\\ b_i+\beta_i s_k=-J}}^{M}\left[(\beta_i)^t\psi^{(t-1)}(J+1+b_i+\beta_i s)+\sum_{l=0}^{J-1}\frac{(-\beta_i)^t(t-1)!}{(l+b_i+\beta_i s)^t}\right].$$

All that remains to complete the derivation of $h_1(y)$ is to derive an expression for the derivatives of y^{-s}. However these derivatives are simply given by

$$\frac{d^r}{ds^r}(y^{-s})=(-\ln y)^r y^{-s}.$$

In summary

$$h_1(y)=\sum_{k=0}^{\infty}\frac{1}{(\delta_k-1)!}\sum_{w=0}^{\delta_k-1}\left\{\binom{\delta_k-1}{w}\left[\frac{d^{\delta_k-w-1}}{ds^{\delta_k-w-1}}C_k^{(0)}(s)\right]\right.$$

$$\left.\cdot\sum_{v=0}^{w}\binom{w}{v}\left[\frac{d^{w-v}}{ds^{w-v}}(y^{-s})\right]\left[\frac{d^v}{ds^v}U_k^0(s)\right]\right\}\Bigg|_{s=s_k}$$

where

$$C_k^{(0)}(s)|_{s=s_k}=\prod_{i=1}^{N}\Gamma(1-a_i-\alpha_i s_k)\prod_{\substack{i=M+1\\1-b_i-\beta_i s_k\neq -J}}^{q}[\Gamma(1-b_i-\beta_i s_k)]^{-1}$$

$$\cdot\prod_{\substack{i=M+1\\1-b_i-\beta_i s_k=-J}}^{q}(-1)^J J!(-\beta_i)$$

$$\cdot\prod_{\substack{i=N+1\\a_i+\alpha_i s_k\neq -J}}^{p}[\Gamma(a_i+\alpha_i s_k)]^{-1}\prod_{\substack{i=N+1\\a_i+\alpha_i s_k=-J}}^{p}(-1)^J J!(\alpha_i)$$

$$U_k^{(0)}(s)|_{s=s_k}=\prod_{\substack{i=1\\b_i+\beta_i s_k\neq -J}}^{M}(b_i+\beta_i s_k)\prod_{\substack{i=1\\b_i+\beta_i s_k=-J}}^{M}\frac{(-1)^J}{J!(\beta_i)}$$

$$\frac{d^r}{ds^r}(C_k^{(0)}(s))\bigg|_{s=s_k}=C_k^{(r)}(s)\bigg|_{s=s_k}=\sum_{t=0}^{r-1}\binom{r-1}{t}C_k^{(r-t-1)}(s)\chi_k^{(t)}(s)\bigg|_{s=s_k}$$

$$\frac{d^r}{ds^r}(U_k^{(0)}(s))|_{s=s_k}=U_k^{(r)}(s)|_{s=s_k}$$

$$=\sum_{t=0}^{r-1}\binom{r-1}{t}U_k^{(r-t-1)}(s)V_k^{(t)}(s)|_{s=s_k}$$

$$\frac{d^r}{dx^r}(y^{-s})=(-\ln y)^r y^{-s}$$

and

$$\chi_k^{(t)}(s)\big|_{s=s_k} = \sum_{i=1}^{N} (-\alpha_i)^t \psi^{(t-1)}(1-a_i-\alpha_i s_k)$$
$$-\sum_{\substack{i=M+1\\ 1-b_i-\beta_i s_k \neq -J}}^{q} (-\beta_i)^t \psi^{(t-1)}(1-b_i-\beta_i s_k)$$
$$-\sum_{\substack{i=M+1\\ 1-b_i-\beta_i s_k = -J}}^{q} \left[(-\beta_i)^t \psi^{(t-1)}(1) + \sum_{l=0}^{J-1} (\beta_i)^t (t-1)!(l-J)^{-t}\right]$$
$$-\sum_{\substack{i=N+1\\ a_i+\alpha_i s_k \neq -J}}^{p} (\alpha_i)^t \psi^{(t-1)}(a_i+\alpha_i s_k)$$
$$-\sum_{\substack{i=N+1\\ a_i+\alpha_i s_k = -J}}^{p} \left[(\alpha_i)^t \psi^{(t-1)}(1) + \sum_{l=0}^{J-1} (-\alpha_i)^t (t-1)!(l-J)^{-t}\right]$$

$$V_k^{(t)}(s)\big|_{s=s_k} = \sum_{\substack{i=1\\ b_i+\beta_i s_k \neq -J}}^{M} (\beta_i)^t \psi^{(t-1)}(b_i+\beta_i s_k)$$
$$+\sum_{\substack{i=1\\ b_i+\beta_i s_k = -J}}^{M} \left[(\beta_i)^t \psi^{(t-1)}(1) + \sum_{l=0}^{J-1} (-\beta_i)^t (t-1)!(l-J)^{-t}\right].$$

This represents that portion of the p.d.f. $h(y)$ for which poles are in the LHP.

7.1.2 Evaluation of the Probability Density Function for Poles in the Right-Half Plane

To complete the evaluation of $h(y)$ poles in the RHP must now be considered. These poles occur in the factor

$$\prod_{i=1}^{N} (1-a_i-\alpha_i s)$$

in the numerator of the inversion integral and are given by

$$s_{ij} = \frac{1-a_i+j}{\alpha_i}, \qquad j=0,1,2,3,\ldots \quad i=1,2,\ldots,N.$$

Without loss of generality assume that the poles

$$\{s_{ij}\}_{j=0,i=1}^{\infty,N}$$

are ordered algebraically from smallest to largest and denoted by

$$\{s_k\}_{k=1}^{\infty}.$$

Again the theory of residues is applicable and the inversion integral can be evaluated over the Bromwich path in a fashion analagous to that used for poles in the LHP. Then the portion of the p.d.f. $h(y)$ evaluated at poles in the RHP may be given as

$$h_2(y)=\sum_{k=1}^{\infty}(-\text{residue evaluated at } s_k)$$

$$=\sum_{k=1}^{\infty}\frac{-1}{(\delta_k-1)!}\frac{d^{\delta_k-1}}{ds^{\delta_k-1}}$$

$$\cdot\left\{(s-s_k)^{\delta_k}\frac{\prod_{i=1}^{M}\Gamma(b_i+\beta_i s)\prod_{i=1}^{N}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=M+1}^{q}\Gamma(1-b_i-\beta_i s)\prod_{i=N+1}^{p}\Gamma(a_i+\alpha_i s)}y^{-s}\right\}\Bigg|_{s=s_k}$$

where δ_k is the order of the pole s_k in the integrand of the H-function inversion integral, after any cancellation of terms. The minus sign in the coefficient

$$\frac{-1}{(\delta_k-1)!}$$

arises from the fact that the integration of the inversion integral is in opposite orientation to that used for the theory of residues, when operating in the RHP.

Then, following the same procedure used for poles in the LHP, we define $\delta_k=\delta_{kN}-\delta_{kq}-\delta_{kp}>0$ (note that this time $\delta_{kM}=0$) and obtain

$$h_2(y)=\sum_{k=1}^{\infty}\frac{-1}{(\delta_k-1)!}\frac{d^{\delta_k-1}}{ds^{\delta_k-1}}\{C_k^{(0)}(s)U_k^{(0)}(s)y^{-s}\}|_{s=s_k}$$

where now

$$C_k^{(0)}(s)=\frac{\prod_{i=1}^{M}\Gamma(b_i+\beta_i s)}{(s-s_k)^{\delta_{kq}}\prod_{i=M+1}^{q}\Gamma(1-b_i-\beta_i s)(s-s_k)^{\delta_{kp}}\prod_{i=N+1}^{p}\Gamma(a_i+\alpha_i s)}$$

and

$$U_k^{(0)}(s)=(s-s_k)^{\delta_{kN}}\Pi_{i=1}^{N}\Gamma(1-a_i-\alpha_i s).$$

Hence, if we apply Leibnitz's rule to $h_2(y)$ and perform the cancellations in $C_k^{(0)}(s)$ and $U_k^{(0)}(s)$ we find that

$$h_2(y)=\sum_{k=1}^{\infty}\frac{-1}{(\delta_k-1)!}\sum_{w=0}^{\delta_k-1}$$

$$\cdot\left\{\binom{\delta_k-1}{w}\frac{d^{\delta_k-w-1}}{ds^{\delta_k-w-1}}C_k^{(0)}(s)\sum_{v=0}^{W}\left[\binom{w}{v}\frac{d^{w-v}}{ds^{w-v}}y^{-s}\frac{d^v}{ds^v}U_k^{(0)}(s)\right]\right\}\Bigg|_{s=s_k}$$

$$C_k^{(0)}(s)=\prod_{i=1}^{M}\Gamma(b_i+\beta_i s)\prod_{\substack{i=M+1\\1-b_i-\beta_i s_k\neq -J}}^{q}[\Gamma(1-b_i-\beta_i s)]^{-1}$$

$$\times\prod_{\substack{i=M+1\\1-b_i-\beta_i s_k=-J}}^{q}\left[\frac{\Gamma(J+2-b_i-\beta_i s)}{(1-b_i-\beta_i s)(2-b_i-\beta_i s)\cdots(J-b_i-\beta_i s)(-\beta_i)}\right]^{-1}$$

$$\times\prod_{\substack{i=N+1\\a_i+\alpha_i s_k\neq -J}}^{p}[\Gamma(a_i+\alpha_i s)]^{-1}$$

$$\times\prod_{\substack{i=N+1\\a_i+\alpha_i s_k=-J}}^{p}\left[\frac{\Gamma(J+1+a_i+\alpha_i s)}{(a_i+\alpha_i s)(1+a_i+\alpha_i s)\cdots(J-1+a_i+\alpha_i s)(\alpha_i)}\right]^{-1}$$

$$U_k^{(0)}(s)=\prod_{\substack{i=1\\i-a_i-\alpha_i s_k\neq -J}}^{N}\Gamma(1-a_i-\alpha_i s)\prod_{\substack{i=1\\1-a_i-\alpha_i s_k=-J}}^{N}$$

$$\cdot\frac{(J+2-a_i-\alpha_i s)}{(1-a_i-\alpha_i s)(1-a_i-\alpha_i s)\cdots(J-a_i-\alpha_i s)(-\alpha_i)}.$$

To find the derivatives of $C_k^{(0)}(s)$ and $U_k^{(0)}(s)$ again use the modified product role:

$$\frac{d}{ds}\prod_{i=1}^{L}f_i(x)=\sum_{i=1}^{L}g_i(x)\prod_{j=1}^{L}f_i(x)$$

where

$$f_i^{(1)}(x)=g_i(x)f_i(x)$$

as developed for the forms considered in the LHP. The same forms appear here composing $C_k^{(0)}(s)$ and $U_k^{(0)}(s)$. Then we find, in a manner completely analogous to that used in the LHP, that

$$C_k^{(r)}(s)=\sum_{t=0}^{r-1}\binom{r-1}{t}C_k^{(r-t-1)}(s)\chi_k^{(t+1)}(s)$$

and

$$U_k^{(r)}(s)=\sum_{t=0}^{r-1}\binom{r-1}{t}U_k^{(r-t-1)}(s)V_k^{(t+1)}(s)$$

where

$$\begin{aligned}
\chi_k^{(t)}(s)=&\sum_{i=1}^{M}(\beta_i)^t\psi^{(t-1)}(b_i+\beta_i s)\\
&-\sum_{\substack{i=M+1\\ 1-b_i-\beta_i s_k\neq -J}}^{q}(-\beta_i)^t\psi^{(t-1)}(1-b_i-\beta_i s)\\
&-\sum_{\substack{i=M+1\\ 1-b_i-\beta_i s_k=-J}}^{q}\\
&\cdot\left[(-\beta_i)^t\psi^{(t-1)}(J+2-b_i-\beta_i s)+\sum_{l=1}^{J}\frac{(\beta_i)^t(t-1)!}{(l-b_i-\beta_i s)^t}\right]\\
&-\sum_{\substack{i=N+1\\ a_i+\alpha_i s_k\neq -J}}^{p}(\alpha_i)^t\psi^{(t-1)}(a_i+\alpha_i s)\\
&-\sum_{\substack{i=N+1\\ a_i+\alpha_i s_k=-J}}^{p}\left[(\alpha_i)^t\psi^{(t-1)}(J+1+a_i+\alpha_i s)+\sum_{l=0}^{J-1}\frac{(-\alpha_i)^t(t-1)!}{(l+a_i+\alpha_i s)^t}\right]
\end{aligned}$$

and

$$\begin{aligned}
V_k^{(t)}(s)=&\sum_{\substack{i=1\\ 1-a_i-\alpha_i s_k\neq -J}}^{N}(-\alpha_i)^t\psi^{(t-1)}(1-a_i-\alpha_i s)\\
&+\sum_{\substack{i=1\\ 1-a_i-\alpha_i s_k=-J}}^{N}\\
&\cdot\left[(-\alpha_i)^t\psi^{(t-1)}(J+2-a_i-\alpha_i s)+\sum_{l=1}^{J}\frac{(\alpha_i)^t(t-1)!}{(l-a_i-\alpha_i s)^t}\right]
\end{aligned}$$

As in the LHP, the derivatives of y^{-s} are given by

$$\frac{d^r}{ds^r}=(-\ln y)^r y^{-s}.$$

In summary

$$h_2(y)=\sum_{k=1}^{\infty}\frac{-1}{(\delta_k-1)!}\sum_{w=0}^{\delta_k-1}$$

$$\cdot\left\{\binom{\delta_k-1}{w}\frac{d^{\delta_k-w-1}}{ds^{\delta_k-w-1}}C_k^{(0)}(s)\sum_{v=0}^{w}\left[\binom{w}{v}\frac{d^{w-v}}{ds^{w-v}}y^{-s}\frac{d^v}{ds^v}U_k^{(0)}(s)\right]\right\}\Bigg|_{s=s_k}$$

where

$$C_k^{(0)}(s)\big|_{s=s_k}=\prod_{i=1}^{M}\Gamma(b_i+\beta_i s_k)\cdot\prod_{\substack{i=M+1\\1-b_i-\beta_i s_k\neq -J}}^{q}$$

$$\cdot\ [\Gamma(1-b_i-\beta_i s_k)]^{-1}\prod_{\substack{i=M+1\\1-b_i-\beta_i s_k=-J}}^{q}(-1)^J J!(-\beta_i)$$

$$\cdot\prod_{\substack{i=N+1\\a_i+\alpha_i s_k\neq -J}}^{p}[\Gamma(a_i+\alpha_i s_k)]^{-1}\prod_{\substack{i=N+1\\a_i+\alpha_i s_k=-J}}^{p}(-1)^J J!(\alpha_i)$$

$$U_k^{(0)}(s)\big|_{s=s_k}=\prod_{\substack{i=1\\1-a_i-\alpha_i s_k\neq -J}}^{N}\Gamma(1-a_i-\alpha_i s_k)\prod_{\substack{i=1\\1-a_i-\alpha_i s_k=-J}}^{N}\frac{(-1)^J}{J!(\alpha)}$$

$$\frac{d^r}{ds^r}C_k^{(0)}(s)\bigg|_{s=s_k}=C_k^{(r)}(s)\bigg|_{s_k}=\sum_{t=0}^{r-1}\binom{r-1}{t}C_k^{(r-t-1)}(s)\chi_k^{(t+1)}(s)\bigg|_{s=s_k}$$

$$\frac{d^r}{ds^r}U_k^{(0)}(s)\bigg|_{s=s_k}=U_k^{(r)}(s)\bigg|_{s=s_k}=\sum_{t=0}^{r-1}\binom{r-1}{t}U_k^{(r-t-1)}(s)V_k^{(t+1)}(s)\bigg|_{s=s_k}$$

$$\frac{d^r}{ds^r}y^{-s}\bigg|_{s=s_k}=(-\ln y)^r y^{-s_k}$$

and

$$\chi_k^{(t)}(s)\big|_{s=s_k} = \sum_{i=1}^{M} (\beta_i)^t \psi^{(t-1)}(b_i+\beta_i s_k) - \sum_{\substack{i=M+1\\ 1-b_i-\beta_i s_k \neq -J}}^{q}$$

$$\cdot (-\beta_i)^t \psi^{(t-1)}(1-b_i-\beta_i s_k)$$

$$- \sum_{\substack{i=M+1\\ 1-b_i-\beta_i s_k = -J}}^{q} \left[(-\beta_i)^t \psi^{(t-1)}(1) + \sum_{l=0}^{J-1} (\beta_i)^t (t-1)!(l-J)^{-t} \right]$$

$$- \sum_{\substack{i=N+1\\ a_i+\alpha_i s_k \neq -J}}^{p} (\alpha_i)^t \psi^{(t-1)}(a_i+\alpha_i s_k)$$

$$- \sum_{\substack{i=N+1\\ a_i+\alpha_i s_k = -J}}^{p} \left[(\alpha_i)^t \psi^{(t-1)}(1) + \sum_{l=0}^{J-1} (-\alpha_i)^t (t-1)!(l-J)^{-t} \right]$$

$$V_k^{(t)}(s)\big|_{s=s_k} = \prod_{\substack{i=1\\ 1-a_i-\alpha_i s_k \neq -J}}^{N} (-\alpha_i)^t \psi^{(t-1)}(1-a_i-\alpha_i s_k) \cdot \prod_{\substack{i=1\\ 1-a_i-\alpha_i s_k = -J}}^{N}$$

$$\cdot \Bigg[(-\alpha_i)^t \psi^{(t-1)}(1)$$

$$+ \sum_{l=0}^{J-1} (\alpha_i)^t (t-1)!(l-J)^{-t} \Bigg].$$

This represents that portion of the p.d.f. $h(y)$ for which poles are in the RHP.

7.1.3 Evaluation of the Probability Density Function Given the Total Absence of Poles in One Half Plane

Often in practice one of the two product terms in the numerator of the integrand of the H-function inversion integral is absent. When this happens there will be no poles in the corresponding half plane. This does not present any difficulties with the inversion technique as the theory of residues may be applied as shown below.

Suppose, as happens in the majority of such cases, that the factor

$$\prod_{i=1}^{N} \Gamma(1-a_i-\alpha_i s)$$

is not present. Then there are no poles in the RHP. The residue theorem may be applied, as before, to the poles of the LHP. However, to apply the residue theorem to the RHP the poles must first be shifted from the LHP to the RHP by the transformation $r=-s$ in the integrand of the H-function inversion integral. Under this transformation the inversion integral becomes

$$h(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\prod_{i=1}^{M}\Gamma(b_i-\beta_i r)\prod_{i=1}^{N}\Gamma(1-a_i+\alpha_i r)}{\prod_{i=M+1}^{Q}\Gamma(1-b_i+\beta_i r)\prod_{i=N+1}^{P}\Gamma(a_i-\alpha_i r)}y^r(-dr).$$

Under the assumption that the factor

$$\prod_{i=1}^{N}\Gamma(1-a_i+\alpha_i r)$$

is absent, the only poles occur in the factor

$$\prod_{i=1}^{M}\Gamma(b_i-\beta_i r)$$

and are given by

$$r=\frac{b_i+j}{\beta_i}, \qquad j=0,1,2,\ldots.$$

Then each r value is the negative of the corresponding s value for the evaluation of poles in the LHP of the original H-function inversion integral. The residues at these poles of the transformed inversion integral may be evaluated by a procedure analogous to that used previously. A similar method applies in the case that the factor

$$\prod_{i=1}^{M}\Gamma(b_i+\beta_i s)$$

is absent ($M<1$) in the original inversion integral.

Although the above describes the correct mathematical procedure for evaluating the inversion integral in the case of absence of poles in either the RHP or the LHP it is not necessary in practice. This method again yields $h_1(y)$ for $0<y\leqslant 1$ and $h_2(y)$ for $1\leqslant y<\infty$. However, in such a situation, $h_1(y)$ and $h_2(y)$ are identical functions. Hence, it suffices to derive *either* $h_1(y)$ or $h_2(y)$, when there is an absence of poles in either half plane, to obtain the p.d.f. $h(y)$ valid for all y on the interval $(0,\infty)$.

7.2 CUMULATIVE DISTRIBUTION FUNCTION

The cumulative distribution function may be found by a procedure analogous to that for the p.d.f. The c.d.f. $H(y)$, defined by

$$H(y)=\int_0^y h(y)\,dy,$$

can be obtained by direct integration or through use of the Mellin transform of $h(y)$. The latter method is preferable, since it avoids the necessity of evaluating $h(y)$ to derive $H(y)$. Specifically (see (4.3.7)),

$$M_s(1-H(y))=s^{-1}M_{s+1}(h(y)).$$

Inverting, one obtains

$$1-H(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}s^{-1}M_{s+1}(h(y))\,ds$$

and

$$H(y)=1-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}s^{-1}M_{s+1}(h(y))\,ds.$$

Writing $H(y)$ in the form of an H-function inversion integral, one has

$$H(y)=1-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\frac{\prod_{i=1}^{M}\Gamma[b_i+\beta_i(s+1)]}{s\prod_{i=M+1}^{q}\Gamma[1-b_i-\beta_i(s+1)]}\right.$$

$$\left.\cdot\frac{\prod_{i=1}^{N}\Gamma[1-a_i-\alpha_i(s+1)]}{\prod_{i=N+1}^{p}\Gamma[a_i+\alpha_i(s+1)]}y^{-s}\,ds\right]$$

$$=1-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}$$

$$\times\frac{\prod_{i=1}^{M}\Gamma(b_i+\beta_i+\beta_i s)\prod_{i=1}^{N}\Gamma(1-a_i-\alpha_i-\alpha_i s)}{s\prod_{i=M+1}^{q}\Gamma(1-b_i-\beta_i-\beta_i s)\prod_{i=N+1}^{p}\Gamma(a_i+\alpha_i+\alpha_i s)}y^{-s}\,ds.$$

If new parameters are defined by

$$a_i' = a_i + \alpha_i, \qquad b_i' = b_i + \beta_i,$$

then the inversion integral above, with the exclusion of the s term in the denominator, takes the form of a typical H-function inversion integral. Hence, it would seem that the general inversion technique introduced here should be applicable with minor alterations to accommodate the s term in the denominator. The modifications are as presented below.

The introduction of the s term in the denominator of the integrand introduces an additional pole at $s=0$. If there is no pole at $s=0$ in the numerator the general model may be applied by substituting a_i' and b_i' as noted, introducing

$$\hat{C}_k^{(0)}(s) = \frac{C_k^{(0)}(s)}{s}$$

$$\hat{\chi}_k^{(t)}(s) = \chi_k^{(t)}(s) - s^{-t}$$

and adding the residue at $s=0$, as given by

$$\operatorname{Res}(s=0) = \frac{\prod_{i=1}^{M}\Gamma(b_i')\prod_{i=1}^{N}\Gamma(1-a_i')}{\prod_{i=M+1}^{q}\Gamma(1-b_i')\prod_{i=N+1}^{p}\Gamma(a_i')},$$

to the sum of the residues for the portion of $H(y)$ which includes the pole at $s=0$. Usually the residue at $s=0$ will be added to $H_1(y)$. It should be noted that, in this case, if any one of the terms in the denominator of the residue formula above has a pole at $s=0$, that is $1-b_i'$ or a_i' is a nonpositive integer, then there is no residue at $s=0$ and the preceding modifications will suffice.

If, on the other hand, a pole occurs in the numerator at $s=0$ then the introduction of the s term in the denominator effectively decreases the order of the pole by 1. In this case the residue at $s=0$ is evaluated as in the general H-function inversion model with the substitution of a_i' and b_i' for a_i and b_i, respectively. The residues at poles other than $s=0$ are handled with the additional substitutions of $\hat{C}_k^0(s)$ and $\hat{\chi}_k^{(t)}(s)$, as defined above, for $C_k^{(0)}(s)$ and $\chi_k^{(t)}(s)$, respectively.

In the manner outlined above the c.d.f., $H(y)$, associated with an H-function p.d.f., $h(y)$, can be evaluated without first obtaining $h(y)$, if it is so desired.

7.3 EXAMPLES

Example 7.3.1 Suppose X_1 and X_2 are random variables with the following Weibull p.d.f.'s:

$$f_1(x_1)=2x_1e^{-x_1^2}, \qquad x_1>0$$

$$f_2(x_2)=3x_2^2e^{-x_2^3}, \qquad x_2>0$$

where $\theta_1=1.0, \phi_1=2.0, \theta_2=1.0$, and $\phi_2=3.0$. To derive the p.d.f. of the product r.v.,

$$Y=X_1X_2,$$

it is necessary to identify the poles of the H-function representing the distribution of Y. By applying Theorem 6.4.1 to the H-function representations of the densities $f_1(x_1)$ and $f_2(x_2)$ (6.3.5), the p.d.f. of Y may be written

$$h(y)=\begin{cases} H_{0,2}^{2,0}\left[y\middle|\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{2}{3},\frac{1}{3}\right)\right], & y>0 \\ 0, & \text{otherwise.}\end{cases}$$

Then the H-function parameters are:

$$M=2, \quad N=0, \quad p=0, \quad q=2, \quad b_1=\tfrac{1}{2}, \quad \beta_1=\tfrac{1}{2}, \quad b_2=\tfrac{2}{3}, \quad \beta_2=\tfrac{1}{3}.$$

The nonzero portion of the p.d.f. may be written in the form of an H-function inversion integral as

$$h(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}s\right)\Gamma\left(\tfrac{2}{3}+\tfrac{1}{3}s\right)y^{-s}\,ds.$$

Note that the factor $\Pi_{i=1}^{N}\Gamma(1-a_i-\alpha_i s)$ is absent. In Section 7.1.3 it was indicated that the p.d.f. $h(y)$ may consequently be obtained by evaluating only the residues at poles in the LHP. However, to illustrate the shifting of poles to the RHP, $h(y)$ may be evaluated in two components as in the general case when there are poles in both half planes.

Poles occurring in the LHP are given, as in the general case, by

$$s_{1j} = -\frac{b_1+j}{\beta_1} = -(1+2j), \qquad j=0,1,2,\ldots$$

$$= -1, -3, -5, -7, \ldots, -(1+2j), \ldots$$

$$s_{2j} = -\frac{b_2+j}{\beta_2} = -(2+3j), \qquad j=0,1,2,\ldots$$

$$= -2, -5, -8, -11, \ldots, -(2+3j), \ldots.$$

Ordering the poles algebraically from largest to smallest yields

$$\{s_k\}_{k=1}^{\infty} = \{-1, -2, -3, -5, -7, -8, -9, -11, -13, -14, -15, \ldots\}.$$

Poles of order 1 occur for $s = -1, -2, -3, -7, -8, -9, -13, -14, -15, \ldots$ while poles of order 2 occur for $s = -5, -11, -17, \ldots$.

Evaluation of the residues at these poles will produce $h_1(y)$, that component of the desired p.d.f. for which $0 < y \leqslant 1$. In particular, at $s = -5$ the evaluation of the residue follows below with $\delta = 2$ and $k = 4$.

$$C_4^{(0)}(s)\big|_{s=-5} = 1.0$$

and

$$C_4^{(1)}(s)\big|_{s=-5} = C_4^{(0)}(s)\chi_4^{(0)}(s)\big|_{s=-5} = 0.0,$$

since $N=0$, $p=0$, and $q<M+1$.

$$U_4^{(0)}(s)\big|_{s=-5} = \frac{(-1)^2}{2!\left(\frac{1}{2}\right)}\,\frac{(-1)^1}{1!\left(\frac{1}{3}\right)} = -3$$

and

$$U_4^{(1)}(s)\big|_{s=-5} = U_4^{(0)}(s)\,V_4^{(1)}(s)\big|_{s=-5}$$

$$= -3\Big[\Big\{\tfrac{1}{2}\psi^{(0)}(1) + \left(-\tfrac{1}{2}\right)(1!)\left[(-2)^{-1} + (-1)^{-1}\right]\Big\}$$

$$+ \Big\{\tfrac{1}{3}\psi^{(0)}(1) + \left(-\tfrac{1}{3}\right)(0!)\left[(-1)^{-1}\right]\Big\}\Big]$$

$$= -1.80696.$$

Hence the residue at $s=-5$ is given by

$$\frac{1}{1!}\left[C_4^{(1)}(s)y^{-s}U_4^{(0)}(s)+C_4^{(0)}(s)\left[(-\ln y)y^{-s}U_4^{(0)}(s)+y^{-s}U_4^{(1)}(s)\right]\right]\Big|_{s=-5}$$

which gives

$$3(\ln y)y^5-1.80696y^5.$$

In order to evaluate that component of the p.d.f. for which $y>1$, poles may be shifted to the RHP, as noted previously. To do this let $r=-s$ in the H-function inversion integral, then

$$h(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\prod_{i=1}^{2}\Gamma(b_i-\beta_i r)y^r(-dr).$$

This transformation shifts the poles to the RHP, where the corresponding r values become positive, so that the residues may be evaluated for $r=1,2,3,5,7,8,9,\ldots$. This produces $h_2(y)$, that component of the desired p.d.f. for which $y\geqslant 1$. As stated in Section 7.1.3 and again in this section, $h_2(y)$ is equivalent to $h_1(y)$.

Example 7.3.2 Suppose X_1, X_2, and X_3 are beta r.v.'s having p.d.f.'s

$$f_1(x_1)=\frac{1}{\beta(9,3)}x_1^8(1-x_1)^2,\qquad 0<x_1\leqslant 1$$

$$f_2(x_2)=\frac{1}{\beta(8,3)}x_2^7(1-x_2)^2,\qquad 0<x_2\leqslant 1$$

and

$$f_3(x_3)=\frac{1}{\beta(4,2)}x_3^3(1-x_3),\qquad 0<x_3\leqslant 1$$

where $\theta_1=9,\phi_1=3,\theta_2=8,\phi_2=3,\theta_3=4,\phi_3=2$. The p.d.f. of the r.v.

$$Y=\prod_{i=1}^{3}X_i$$

has been shown to be [47, 358]

$$h(y)=\begin{cases} \dfrac{\Gamma(12)\Gamma(11)\Gamma(6)}{\Gamma(9)\Gamma(8)\Gamma(4)} \quad H_{3,3}^{3,0}\left[y \middle| \begin{matrix} (11,1),(10,1),(5,1) \\ (8,1),(7,1),(3,1) \end{matrix} \right], & 0<y\leqslant 1 \\ 0, & \text{otherwise.} \end{cases}$$

When written in the form of an H-function inversion integral, the nonzero portion of the p.d.f. becomes

$$h(y)=\frac{\Gamma(12)\Gamma(11)\Gamma(6)}{\Gamma(9)\Gamma(8)\Gamma(4)}\left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\prod_{i=1}^{3}\Gamma(b_i+\beta_i s)}{\prod_{i=1}^{3}\Gamma(a_i+\alpha_i s)}y^{-s}\,ds\right]$$

$$=\frac{\Gamma(12)\Gamma(11)\Gamma(6)}{\Gamma(9)\Gamma(8)\Gamma(4)}\left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(8+s)\Gamma(7+s)\Gamma(3+s)}{\Gamma(11+s)\Gamma(10+s)\Gamma(5+s)}y^{-s}\,ds\right]. \tag{7.1.3}$$

Note that poles occur in the numerator of the integrand for $s=-3,-4,-5,\ldots,$ and poles occur in the denominator for $s=-5,-6,-7,\ldots.$ Thus, an "overlapping" of poles occurs for $s=-5,-6,-7,\ldots.$ Poles in the denominator corresponding to those in the numerator reduce the order of the poles in the numerator. In fact, in this particular example, the orders are reduced to the extent that poles for $s=-5,-6,-11,-12,-13,\ldots$ are not present. This example therefore has only six poles—four of order 1 (for $s=-3,-4,-7,-10$) and two of order 2 (for $s=-8,-9$). This may be shown by writing

$$\frac{\Gamma(8+s)\Gamma(7+s)\Gamma(3+s)}{\Gamma(11+s)\Gamma(10+s)\Gamma(5+s)}=$$

$$\frac{\Gamma(8+s)\Gamma(7+s)\Gamma(3+s)}{(10+s)(9+s)(8+s)\Gamma(8+s)(9+s)(8+s)(7+s)\Gamma(7+s)}$$

$$\cdot\frac{1}{(4+s)(3+s)\Gamma(3+s)}$$

$$=\frac{1}{(10+s)(9+s)^2(8+s)^2(7+s)(4+s)(3+s)}.$$

The evaluation follows directly from here by calculation of the residues. Observe that $h_2(y)=0$ for $y>1$ in this example.

CHAPTER 8

Approximating the Distribution of an Algebraic Function of Independent Random Variables

Chapter 6 proved that products, quotients, and powers of independent H-function r.v.'s are also H-function r.v.'s whose p.d.f.'s may be determined by the method of residues. However the sum of H-function r.v.'s is not *in general* an H-function r.v., and its p.d.f. cannot in general be obtained in exact form. Similarly, it may be difficult, infeasible, or perhaps even impossible to obtain, in exact form, the p.d.f. of algebraic functions of r.v.'s of other types. (An algebraic function of an r.v. would, of course, include the r.v. itself as a simple special case.) In such cases, approximating p.d.f.'s may often be used to advantage, and this chapter suggests some of these p.d.f.'s. A method developed by Posten and Woods [293] for evaluating the accuracy of the approximation, based on the moments of both the desired and approximating p.d.f.'s, is presented, as is also the well-known Connish-Fisher expansion. Revelant papers dealing with the numerical inversion of integral transforms are cited. However, since no procedure has yet been developed that is generally adaptable and not tailored to specific situations, the latter approach is discussed only briefly.

8.1 APPROXIMATING THE DISTRIBUTION OF PRODUCTS, QUOTIENTS, AND POWERS OF INDEPENDENT *H*-FUNCTION RANDOM VARIABLES

Even though products, quotients, and powers of independent H-function r.v.'s are themselves H-function r.v.'s whose p.d.f.'s can be determined (exactly) by the method of residues, it is nevertheless true that in some cases a suitable approximation to the relevant p.d.f. may be more easily obtained. In particular, an approximating p.d.f. with a specified degree of accuracy may be obtained from a knowledge of the moments of the exact (unknown) distribution, which moments are readily obtained from the Mellin transforms of the original H-function r.v.'s (Section 6.3.3). Some feasible approximating p.d.f.'s are cited later in this chapter, together with methods for determining the accuracy of the approximation. Although these methods are discussed primarily from the standpoint of their application to algebraic functions of sums and differences of independent H-function r.v.'s, they are equally valid for approximating the p.d.f. of any r.v. or the algebraic function of any i.r.v.'s, as long as the moments of the desired p.d.f. are known.

8.2 CALCULATING THE MOMENTS OF THE PROBABILITY DENSITY FUNCTION OF AN ALGEBRAIC FUNCTION CONSISTING OF SUMS AND DIFFERENCES OF H-FUNCTION INDEPENDENT RANDOM VARIABLES

Some algebraic functions of H-function r.v.'s, such as the polynomial (6.6.1), can be reduced to a mixture of sums and differences of H-function r.v.'s. As Chapter 6 showed, the general form for the p.d.f. associated with such a function can be expressed only as an n-fold Laplace convolution of H-functions and is not readily evaluated. However finding the moments of such a function is much less difficult.

Figure 8.2.1 is a flow chart describing the logic of a procedure to find the moments of the p.d.f. of an algebraic function of H-function i.r.v.'s where the function can be expressed as sums and products of terms involving only products, quotients, and powers of the variables. The calculated moments can then be used for approximating the p.d.f. and/or cumulative distribution function of the algebraic function.

Carter [47] has written a computer program in FORTRAN language, named STOFAN (stochastic function analyzer), which includes procedures to achieve the following tasks:

1. Input an algebraic function of H-function r.v.'s and constants (when the function can be expressed as sums and differences of H-function i.r.v.'s).
2. Find the parameters of the distribution of products, quotients, and rational powers of H-function i.r.v.'s.
3. Find the moments of the p.d.f. of an algebraic function of H-function i.r.v.'s.

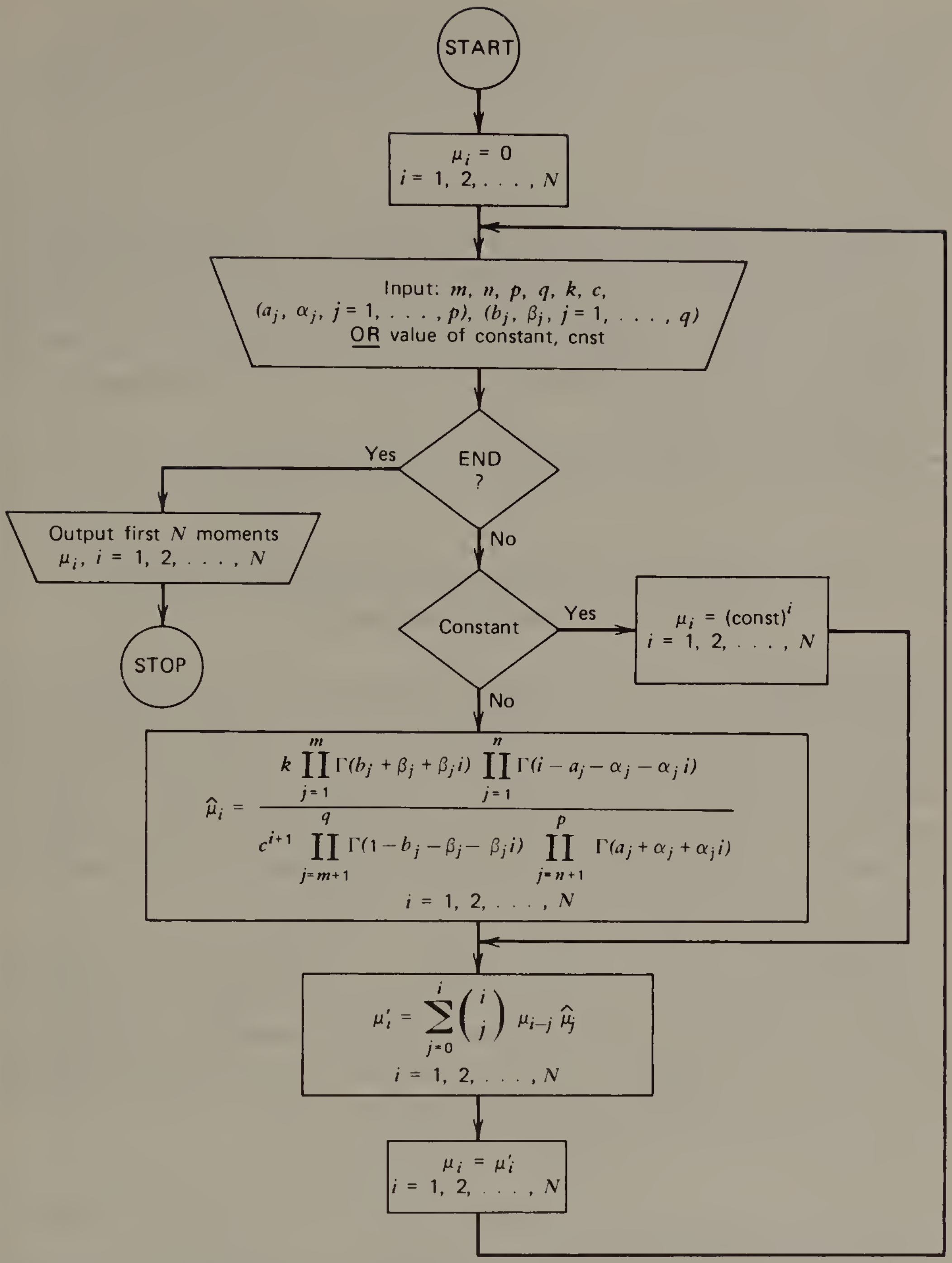


Fig. 8.2.1 Flow chart for determining the first N movements of the p.d.f. of a function consisting of sums (and differences) of H-function i.r.v.'s.

4. Approximate the p.d.f. and the c.d.f. from the moments. The approximation procedure is that given by Hill [148] and briefly described in Sections 8.3 and 8.4.

Program STOFAN consists of a main program and several subprograms all coded in FORTRAN. Although the programs have been run only on the Univac 1106 using the standard Univac 1106/1108 FORTRAN compiler, they should be compatible with any other computer having ANSI (American National Standards Institute) FORTRAN capabilities. Minor modifications should be relatively simple to make, since the programs are not "machine dependent."

The STOFAN program statements are not listed here. The reader interested in the program is referred to Carter's thesis [47].

8.3 APPROXIMATING THE PROBABILITY DENSITY FUNCTION FROM THE MOMENTS

Many techniques have been presented over the years to solve the so-called reduced or finite problem of moments, that is, the problem of determining or approximating a probability distribution from a finite number of its moments. A recent dissertation by Hill [148] is quite complete in evaluating the more common of the existing methods and in presenting an explicit procedure for utilizing them numerically. This section describes Hill's method and the approximation techniques involved.

Hill's procedure includes cases concerning both discrete and continuous distributions and assumes that in addition to a finite number of known moments, the moment-generating function of the probability distribution is known. Only the part of the procedure related to approximating the p.d.f. of a continuous function from a finite number of its moments is discussed here, since sums and differences of H-function i.r.v.'s are always continuous. The procedure then reduces to three basic steps:

1. Determine whether a series approximation of the p.d.f. can be made from the known moments.
2. Make a series approximation (if possible) using either the Gram-Charlier type A (Hermite polynomial) series or the Laguerre polynomial series.
3. Fit the first four moments to the Pearson system of probability distributions.

Figure 8.3.1 is a flow chart showing the overall procedure as described by

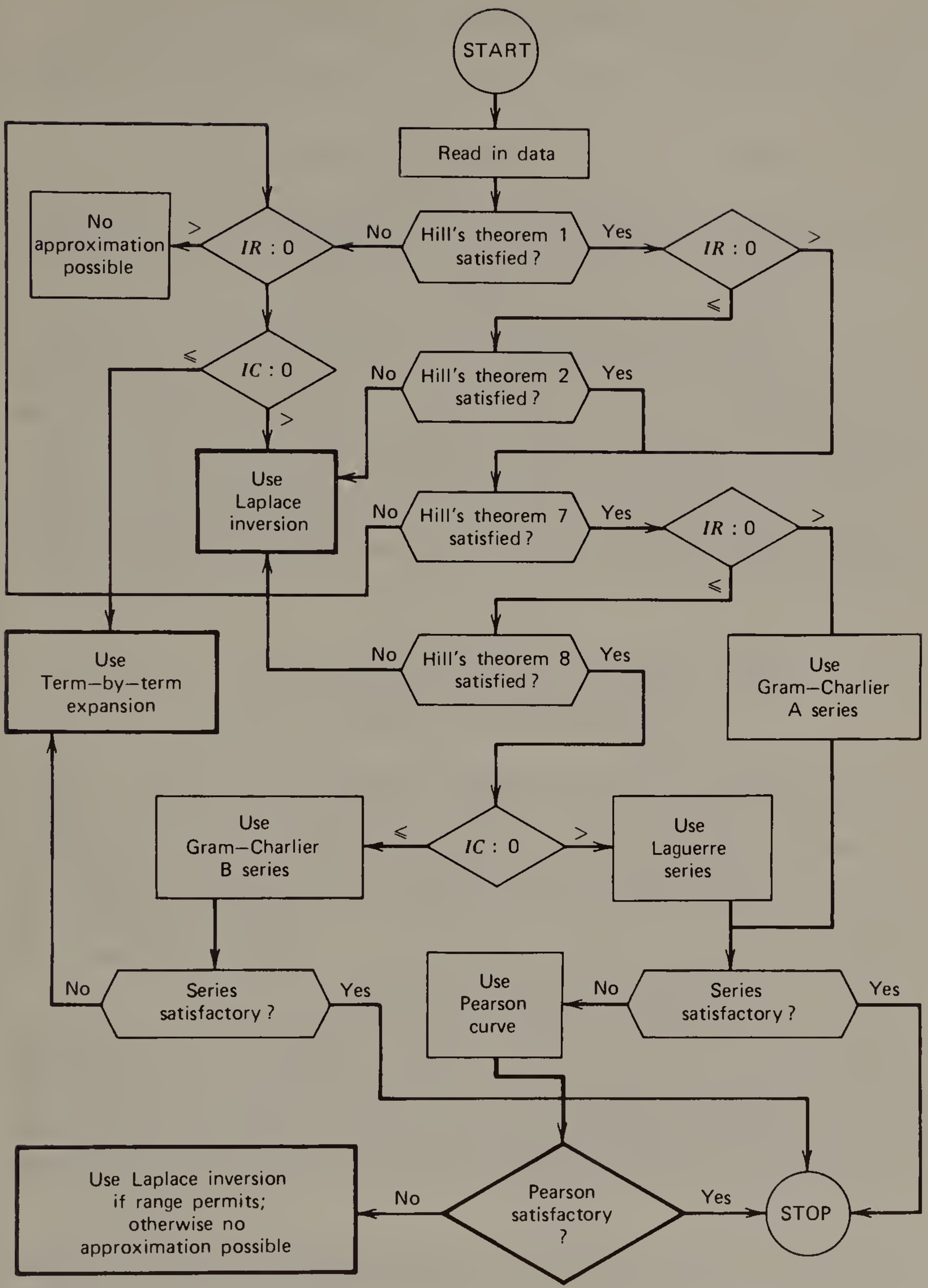


Fig. 8.3.1 Master flow diagram for Hill's procedure for approximating a distribution from its moments.

Hill. The variables IC and IR have the following definitions in the flow chart:

$IR=0$ if the range of the distribution is $(0,\infty)$

$IR=1$ if the range of the distribution is $(-\infty,\infty)$

$IC=0$ if the distribution is discrete

$IC=1$ if the distribution is continuous

Hill's theorems 1 and 2 are explained in the following sections, as are the Gram-Charlier A series, the Laguerre series, and Pearson curves. (Hill uses the Gram-Charlier B series and the term-by-term expansion when dealing with discrete distributions; the Laplace inversion technique is used only when the m.g.f. is known.) Hill also gives a FORTRAN computer program (named TEST) that makes most of the decisions shown in the flow chart of Fig. 8.3.1.

8.4 THEOREMS RELATED TO THE EXISTENCE OF A SERIES SOLUTION

Hill establishes four theorems related to the existence of a series solution for approximating the p.d.f. of a distribution from a specified number of its moments. Theorems 8.4.1 and 8.4.2 (theorems 1 and 2, in Hill's notation) give conditions for the existence of any series solution for distributions of doubly infinite and singly infinite range, respectively, and Hill's theorems 7 and 8 give conditions for a unimodal solution. Only theorems 1 and 2 are given here, without proof.

In some cases it may be easier to establish the existence of the p.d.f. on the basis of the Laplace, Fourier, or Mellin transforms, since the analyticity of the relevant transform in a given strip is a sufficient condition for the uniqueness (hence the existence) of the corresponding p.d.f. However the following theorems give conditions that are both necessary and sufficient for the existence of the p.d.f.[36]

Theorem 8.4.1 (Hill's theorem 1) A necessary and sufficient condition that there should exist at least one nondecreasing function $F(t)$ such that

$$v_j=\int_{-\infty}^{\infty} t^j\, dF(t), \qquad j=0,1,2,\ldots,2n-1$$

[36]Theorems 8.4.1 and 8.4.2 are Hill's version of equivalent theorems previously stated by other authors (see, e.g., refs 3, 335)

is that the quadratic form

$$\sum_{i,j=0}^{n} v_{i+j} x_i x_j$$

be positive (definite or semidefinite), or equivalently, that the sequence $v_0, \ldots, v_{2n-1}$ be positive [148, pp. 24,25].

Theorem 8.4.2 (Hill's theorem 2) A necessary and sufficient condition that there should exist a nondecreasing function $F(t)$ such that

$$v_j = \int_0^\infty t^j \, dF(t); \qquad j = 0, 1, \ldots, n$$

is that the quadratic forms

$$\sum_{i=0}^{[n/2]} \sum_{j=0}^{[n/2]} v_{i+j} x_i x_j$$

and

$$\sum_{i=0}^{[(n+1)/2]} \sum_{j=0}^{[(n+1)/2]} v_{i+j+1} x_i x_j$$

should be positive (definite or semidefinite) [32, p. 150].

8.5 THE PEARSON DISTRIBUTIONS

The Pearson system, developed by Karl Pearson in the late 1880s, consists of a family of 12 types of curves and a set of rules for determining which curve best fits the p.d.f. described by the first four moments of the distribution. The family of curves is generated by solutions to the differential equation

$$\frac{df(x)}{dx} = \frac{dy}{dx} = \frac{(x+a)y}{b_0 + b_1 x + b_2 x^2}.$$

A complete development of the curves and the associated rules is given by Elderton and Johnson [90], Craig [68], and Kendall and Stuart [178, pp. 148–154].

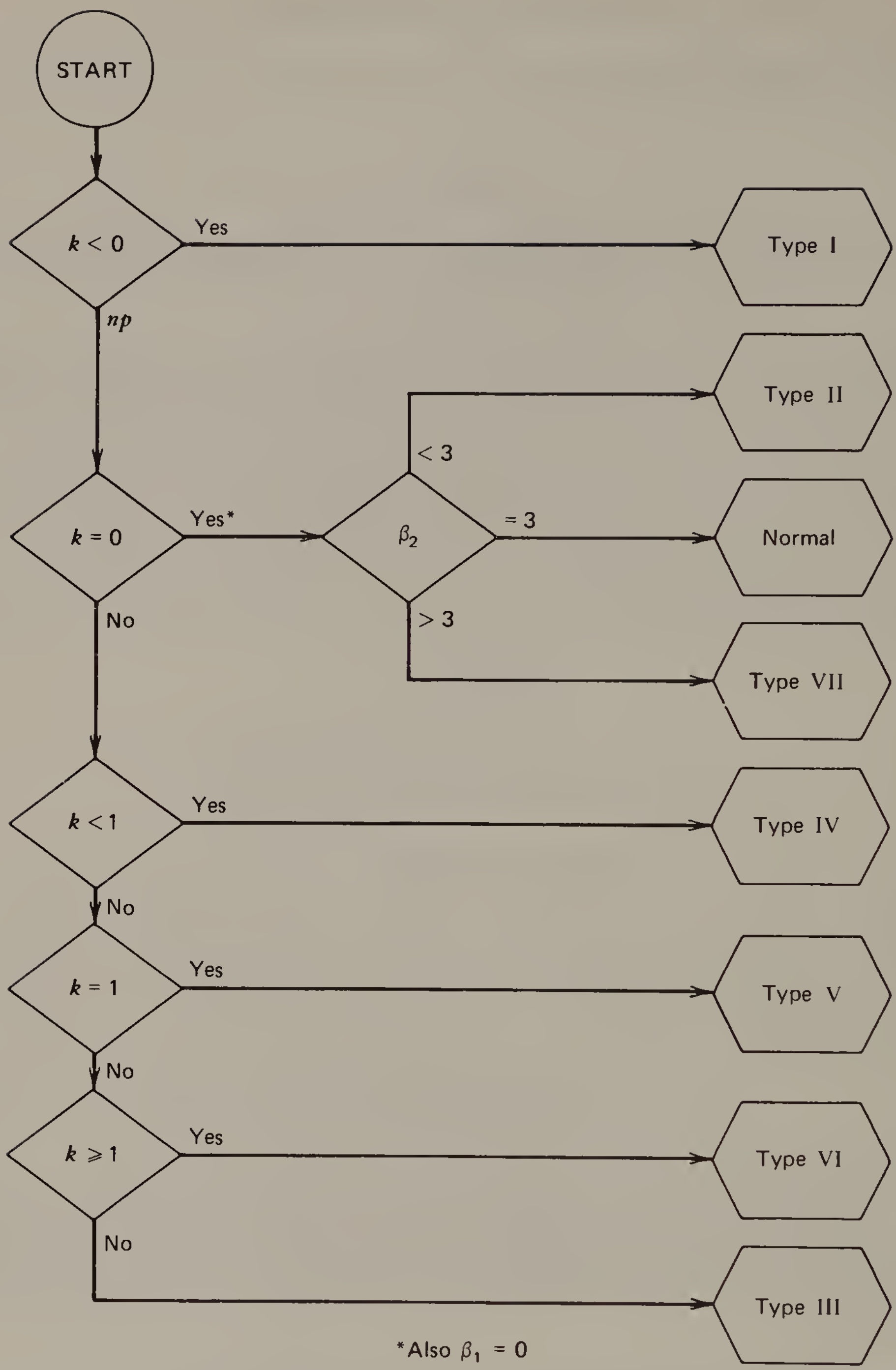


Fig. 8.5.1 Selection criteria for Pearson curves; asterisk indicates "also $\beta_1 = 0$".

If μ_1, μ_2, μ_3, and μ_4 represent the first four moments (about the mean) of the distribution to be approximated, the selection of a particular Pearson curve type is based in the moment ratios

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}, \tag{8.5.1}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2}, \tag{8.5.2}$$

and

$$k = \frac{\beta_1(\beta_2+3)^2}{4(2\beta_2-3\beta_1-6)(4\beta_2-3\beta_1)}. \tag{8.5.3}$$

These three parameters give a basis for selecting one of the 12 curve types.

Figure 8.5.1 shows the criteria for selecting types I through VII. Types VIII through XII are special cases of the other types and are not included here.

8.6 THE GRAM–CHARLIER TYPE A SERIES

The Gram–Charlier type A series which is based on the normal distribution and its derivatives, is well known for its use in approximating the p.d.f. of a distribution whose range is doubly infinite (i.e., a p.d.f. $f(x) \geqslant 0$ for $-\infty < x < \infty$). The general form of the series expansion of the p.d.f. of a distribution in terms of the standardized variable $z = (x-\mu)/\sigma$ is given by

$$f(z) = \sum_{j=0}^{\infty} c_j H_j(z)\phi(z), \qquad -\infty < z < \infty \tag{8.6.1}$$

where

$$c_n = \frac{1}{n!} \sum_{k=0}^{[n/2]} \left(-\frac{1}{2}\right)^k \frac{n!\,v_{n-2k}}{k!(n-2k)!} \tag{8.6.1a}$$

(v_j is defined in Theorem 8.4.1),

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right), \tag{8.6.1b}$$

and $H_n(z)$ is known as a Hermite polynomial such that

$$H_0(z)=1,$$

$$H_1(z)=z,$$

$$H_2(z)=z^2-1, \tag{8.6.1c}$$

and, more generally,

$$H_n(z)=zH_{n-1}(z)-(n-1)H_{n-2}(z). \tag{8.6.1d}$$

Replacing z with $(x-\mu)/\sigma$, where μ and σ are the mean and standard deviation of the distribution of X, one can express the Gram–Charlier type A series in the form

$$f(x)=\sum_{j=0}^{\infty} c_j H_j\left(\frac{x-\mu}{\sigma}\right)\phi(x), \qquad -\infty<x<\infty, \tag{8.6.2}$$

where

$$c_n=\frac{1}{n!}\sum_{k=0}^{[n/2]}\left(\frac{-1}{2}\right)^k \frac{n!}{k!(n-2k)!}\left(\frac{v_{n-2k}}{\sigma^{n-2k}}\right) \tag{8.6.2a}$$

and

$$\phi(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right). \tag{8.6.2b}$$

Equation 8.6.2 is the series expansion of the p.d.f. of a distribution with mean μ.

A complete derivation of the Gram–Charlier type A series is given by Kendall [178, pp. 154–160] and Hill [148, pp. 46–60].

Example 8.6.1 Find the Gram–Charlier type A approximation to the p.d.f. of the r.v.

$$Y=X_1+X_2X_3-5X_4, \tag{8.6.3}$$

where X_1 is a half-normal r.v. with mean zero and standard deviation $\theta=1.5$ (see 6.3.8); X_2 is a gamma r.v. with parameters $\theta=2$, $\phi=1$ (see (6.3.4)); X_3 is an exponential r.v. with parameter $\phi=0.4$ (see (6.3.9)); and X_4 is a half-normal r.v. with mean zero and standard deviation $\theta=2$.

Table 8.6.1 Moments of the Function from STOFAN[a]

1	−.67020188+001	11	−.20708208+015
2	.82143092+002	12	.69242015+016
3	−.12665960+004	13	−.24152253+018
4	.23176458+005	14	.87604507+019
5	−.48128349+006	15	−.32943768+021
6	.11065422+008	16	.12828700+023
7	−.27692034+009	17	−.51439105+024
8	.74526766+010	18	.21320885+026
9	−.21373536+012	19	−.90148586+027
10	.64860079+013	20	.40439913+029

BEGIN PROGRAM "TEST"[b]
FOR APPROXIMATING A PROBABILITY DISTRIBUTION
FROM ITS MOMENTS

THE RANGE IS (—INFINITY, INFINITY)

A SOLUTION EXISTS
FOR SPECIFIED MOMENTS

A UNIMODAL DISTRIBUTION EXISTS HAVING
THE SPECIFIED MOMENTS

[a]The numerical values of computer output (e.g., these and subsequent moments) are stated in computer terminology. For example, −.67020188+001 denotes the number −6.7020188.
[b]Program developed by T. W. Hill [148].

Since all the variables are H-function r.v.'s, the required calculations can be carried out by the computer program STOFAN. The appropriate input is

```
START    EXAMPLE 8.6.1
ADD      HALF-NORMAL      1,1.5
ADD      GAMMA            1,.2,1
MULT     EXPONENTIAL      1,.4
SUB      CONSTANT         5
MULT     HALF-NORMAL      1,2
END
```

The final STOFAN output appears in Tables 8.6.1 and 8.6.2.

On substitution of the values of c_j given in Table 8.6.2, $y=0,1,\ldots,5$, into (8.6.1) gives the Gram–Charlier approximation to the true p.d.f. The Gram–Charlier p.d.f. and c.d.f. are tabulated in Table 8.6.3 and graphed in

Table 8.6.2 Hermite Approximation of the Probability Density Function from STOFAN

j	jth Moment (v_j)[a]	Standardized Moment	jth Series Coefficient (c_j)
0	1		1
1	−.67020188+001	.00000000	.00000000
2	.82143092+002	.10000000+001	.00000000
3	−.12665960+004	−.95581576+000	−.15930263+000
4	.23176458+005	.38293196+001	.34554985−001
5	−.48128349+006	−.95046978+001	.44549839−003

[a]The numerical values of computer output (e.g., these and subsequent moments) are stated in computer terminology. For example, −.67020188+001 denotes the number −6.7020188.

Table 8.6.3 Gram–Charlier Type A Approximation to the True Probability Density Function and Cumulative Distribution Function in Example 8.6.1

x	p.d.f.	c.d.f.
−30.900	0.000358	0.000101
−30.100	0.000525	0.000450
−24.700	0.003871	0.010282
−19.100	0.010043	0.049876
−14.900	0.015541	0.101099
−12.500	0.025989	0.149221
−10.090	0.037832	0.199808
−9.700	0.048500	0.251521
−8.700	0.057566	0.304582
−7.900	0.064254	0.353361
−7.100	0.069894	0.407101
−6.500	0.073158	0.450058
−5.900	0.075417	0.494678
−5.100	0.076677	0.555646
−4.500	0.076239	0.601574
−3.900	0.074622	0.646883
−3.100	0.070750	0.705148
−2.500	0.066716	0.746427
−1.700	0.060140	0.797243
−0.700	0.050647	0.852717
0.300	0.040615	0.898355
1.900	0.025530	0.950973
4.300	0.009224	0.990886
4.900	0.006566	0.995599
5.900	0.003275	1.000412
10.100	0.000000	1.003087
17.700	0.000000	1.003087

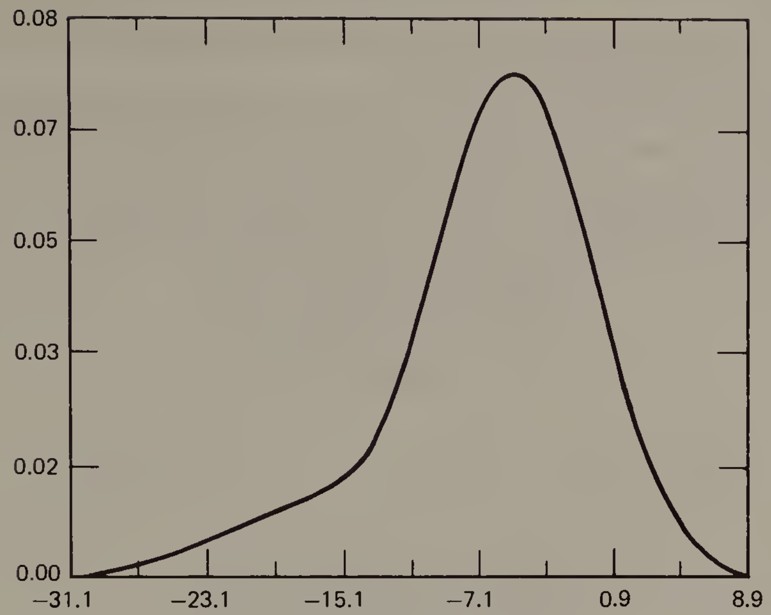


Fig. 8.6.1 Graph of Gram-Charlier density function for Example 8.6.1.

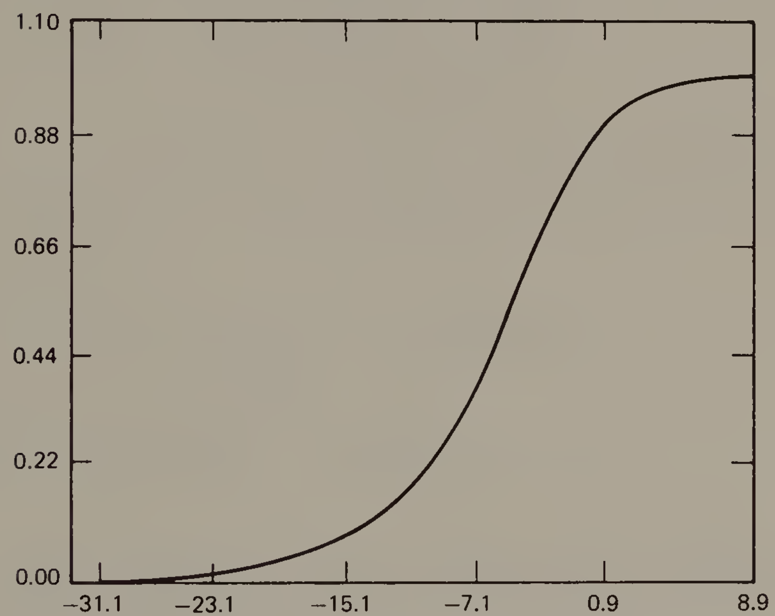


Fig. 8.6.2 Graph of Gram-Charlier c.d.f, for Example 8.6.1.

Figs. 8.6.1 and 8.6.2. Carter tabulates the p.d.f. and c.d.f. at intervals of 0.001.

The accuracy of this Gram–Charlier approximation can be determined by utilizing the method of Posten and Woods involving the Fourier sine series to evaluate the error of the approximation to the distribution function at any value of y. (See Sections 8.12.2 and 8.12.3). One could, of course, use integral transforms to determine the exact p.d.f. $h(y)$ [in series form] of the r.v. y defined by (8.6.3).

STOFAN also selects the Pearson type that best approximates the true distribution in Example 8.6.1. The output is given below.

PEARSON CURVES APPROXIMATION[37]

BETA(1) = .91358378 + 000
BETA(2) = .38293196 + 001
$K = -.10000000 + 002$

USE PEARSON TYPE 1 WITH PARAMETERS

$$R \doteq 6(\beta_2 - \beta_1 - 1)/(6 + 3\beta_1 - 2\beta_2) = .10622204 + 002$$

$M(1) = -.16121036 + 001$
$M(2) = .10234308 + 002$
$A(1) = -.10385197 + 002$
$A(2) = .65929568 + 002$
$Y(0) = -.43009267 + 001$
$U(3) = -.21709228 + 003$

The appropriate Pearson type I (beta) p.d.f. is then [178]

$$h(y) = y_0\left(1 + \frac{y}{a_1}\right)^{m_1}\left(1 - \frac{y}{a_2}\right)^{m_2}, \qquad -a_1 < y < a_2, \qquad a_1 < 0$$

$$= -4.3009267\left(1 + \frac{y}{-10.385197}\right)^{-1.6121036}$$

$$\times\left(1 - \frac{y}{65.929568}\right)^{10.234308} \quad 10.385197 < y < 65.929568,$$

where the origin is at the mode. If the origin is taken at the start of the curve, one obtains the usual form of the beta distribution, namely,

$$h(y) = \frac{1}{B(m_1 + 1, m_2 + 1)} y^{m_1}(1 - y)^{m_2}.$$

Again, the accuracy of this approximation to the true distribution can be evaluated by the method of Posten and Woods (Sections 8.12.2 and 8.12.3).

[37] See note a, Table 8.6.1.

8.7 THE LAGUERRE POLYNOMIAL SERIES

When a probability distribution has nonzero values for its p.d.f. only in the range $0<x<\infty$, then a series developed from the gamma distribution is widely used for approximating the p.d.f. from the moments of the distribution. Using a derivation similar to that used for the Gram–Charlier type A series, one can obtain the following general form of the series expansion of the p.d.f. $f(x)$:

$$f(x)=\sum_{j=0}^{\infty} d_j L_j^{(r)}(x)\psi(x), \qquad 0<x<\infty, \tag{8.7.1}$$

where

$$d_n=\frac{(-1)^n}{n!(1+r)_n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{(1+r)_n}{(1+r)_k}v_k$$

(where v_k is defined in Theorem 8.4.1)

$$\psi(x)=\frac{x^r\exp(-x)}{\Gamma(r+1)},$$

and $L_n^{(r)}$, known as a Laguerre polynomial, satisfies the relations

$$L_0^{(r)}(x)=1$$

$$L_1^{(r)}(x)=x-r-1$$

$$L_2^{(r)}(x)=x^2-2(r+2)x+(r+1)(r+2)$$

$$\vdots$$

$$L_n^{(r)}=(x-r-2n+1)L_{n-1}^{(r)}(x)-(n-1)(n+r-1)L_{n-2}^{(r)}(x).$$

The notation $(n)_m$ above is defined as $(n)(n+1)(n+2)\cdots(n+m-1)$.

Before the approximation above can be used, however, some value must be given to the parameter r. Hill [148, p. 69] shows that the value of r can be selected from a knowledge of the p.d.f. at $x=0$. If $f(0)=0$, then he suggests that $r=1$. For $f(0)\neq 0$, r should be assigned a value of zero. In approximating mixtures of H-function r.v.'s, some of which have $f(0)=0$ and others having $f(0)\neq 0$, a value of 1 is assigned to r.

A complete derivation of the Laguerre series is given by Hill [148, pp. 60–72].

Example 8.7.1. Find the Laguerre polynomial series approximation to the p.d.f. of the r.v.

$$y=0.25x_1x_2+x_3+7.21,$$

where x_1 is a gamma r.v. with distribution parameters $\theta=2$ and $\phi=4$ (see (6.3.4)), x_2 is a beta r.v. with parameters $\theta=2$ and $\phi=0.5$ (see (6.3.7)), and x_3 is an exponential r.v. with parameter $\phi=0.4$ (see (6.3.9)).

Again, since all the variables are H-function r.v.'s, STOFAN can be used with the input:

```
START     EXAMPLE 8.7.1
ADD       CONSTANT          25
MULT      GAMMA             1,2,.4
MULT      BETA              1,2,.5
ADD       EXPONENTIAL       1,.4
ADD       CONSTANT          7.21
END
```

The final STOFAN output appears in Tables 8.7.1 and 8.7.2.

Table 8.7.1 Moments of the Function from STOFAN[a]

1	.77700000+001	11	.77236362+010
2	.60548443+002	12	.63290641+011
3	.47332041+003	13	.52310513+012
4	.37127959+004	14	.43654145+013
5	.29233442+005	15	.36826923+014
6	.23112662+006	16	.31447846+015
7	.18356542+007	17	.27224017+016
8	.14652356+008	18	.23931349+017
9	.11760710+009	19	.21400109+018
10	.94980927+009	20	.18027778+019

[a]The numerical values of computer output (e.g., these and subsequent moments) are stated in computer terminology. For example, −0.6702188+001 denotes the number −6.7020188.

BEGIN PROGRAM "TEST"
FOR APPROXIMATING A PROBABILITY DISTRIBUTION
FROM ITS MOMENTS[b]

THE RANGE IS (0, INFINITY)

A SOLUTION IS POSSIBLE USING
ONLY THE MOMENTS UP TO 18, BECAUSE

THE VALUE OF THE DETERMINANT
OF ORDER 11 IS −.26771528+025

A SOLUTION EXISTS
FOR SPECIFIED MOMENTS

A UNIMODAL DISTRIBUTION EXISTS HAVING
THE SPECIFIED MOMENTS

Table 8.7.2 Laguerre Approximation of the ProbabilityDensity Function from STOFAN[a]

j	jth Moment[b]	Standardized Moment	jth Series Coefficient (d_j)
0	1		1
1	.77700000+001	.56000000+000	−.72000000+000
2	.60548443+002	.48914286+000	.26076190+000
3	.47332041+003	.60160000+000	−.63250794−001
4	.37127959+004	.96920935+000	.11539706−001
5	.29233442+005	.19421014+001	−.16854816−002
6	.23112662+006	.46634494+001	.20474457−003
7	.18356542+007	.13059470+002	−.21202623−004
8	.14652356+008	.41791849+002	.19019141−005
9	.11760710+009	.15045213+003	−.14913518−006
10	.94980927+009	.60181005+003	.10245318−007
11	.77236362+010	.26479660+004	−.61210760−009
12	.63290641+011	.12710239+005	.30979781−010
13	.52310513+012	.66093247+005	−.12244554−011

[a]The numerical values of computer output (e.g., these and subsequent moments) are stated in computer terminology. For example, −0.6702188+001 denotes the number −6.702188.
[b]Program developed by T. W. Hill [148].

Substitution of the values in Table 8.7.2 of d_j into (8.7.1) gives the

Laguerre approximation to the true p.d.f. The Laguerre p.d.f. and c.d.f. are briefly tabulated in Table 8.7.3. (It is feasible, of course, to tabulate them in much greater detail, as does Carter [47].) Their graphs are given in Figs. 8.7.1 and 8.7.2.

Table 8.7.3 Laguerre Approximation to the True Probability Density Function and Cumulative Distribution Function

x	p.d.f.	c.d.f.
7.210	0.000000	0.000000
7.240	0.333317	0.005147
7.250	0.430237	0.008965
7.270	0.604680	0.019345
7.310	0.883967	0.049390
7.360	1.123577	0.1000015
7.400	1.244822	0.147556
7.440	1.316309	0.198917
7.480	1.348003	0.252314
7.520	1.348316	0.306326
7.550	1.332256	0.346562
7.590	1.293964	0.399140
7.630	1.241041	0.449879
7.670	1.177738	0.498281
7.720	1.089258	0.554987
7.760	1.014353	0.597066
7.820	0.900096	0.654500
7.870	0.806516	0.697154
7.940	0.682293	0.749206
8.020	0.554140	0.798556
8.130	0.406165	0.851068
8.280	0.256932	0.900681
8.590	0.097473	0.950829
9.410	0.027438	0.989863
9.930	0.011174	0.999962
10.100	0.006262	1.001434
10.380	0.000109	1.002262

Again, the accuracy of this Laguerre approximation can be evaluated by the method of Posten and Woods (Sections 8.12.2 and 8.12.3).

PEARSON CURVES APPROXIMATION FROM STOFAN[38]

BETA (1) = .31759269 + 001
BETA (2) = .80142563 + 001
K = .85381975 + 001

[38]See note *a* Table 8.7.2

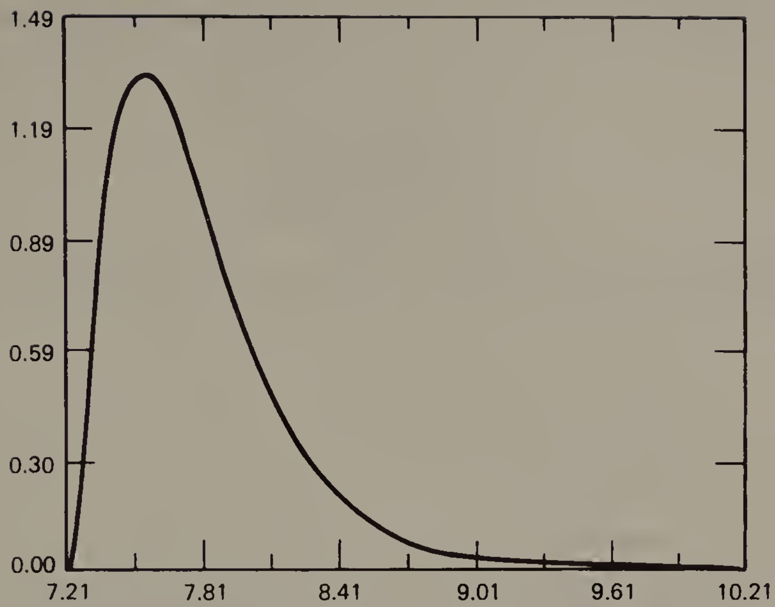


Fig. 8.7.1 Graph of Laguerre density function for Example 8.7.1.

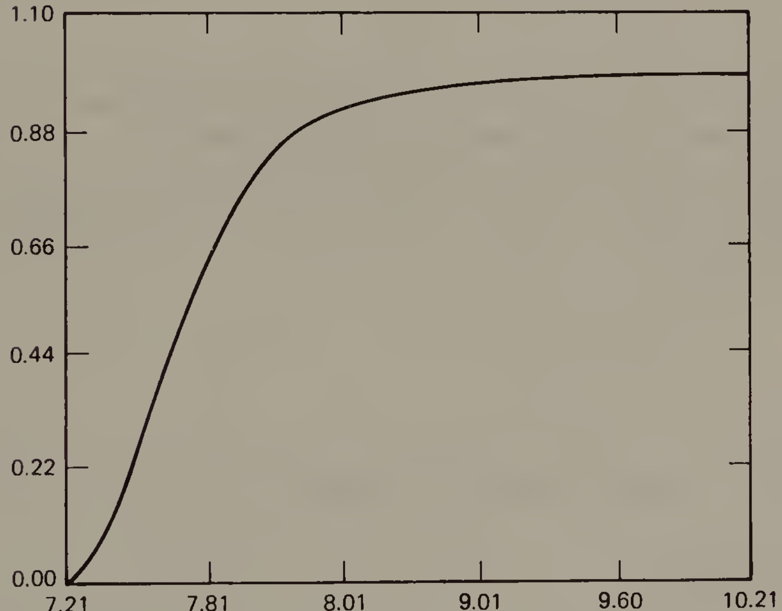


Fig. 8.7.2 Graph of Laguerre cumulative distribution (c.d.f.) for Example 8.7.1

USE PEARSON TYPE III (GAMMA) DISTRIBUTION
WITH PARAMETERS

$D = .26785714+001 \ (= d_n)$
$P = .25947522+000$
$A = .96870748-001$
$Y(0) = .16098245+001$

The Pearson type III approximation to the true p.d.f. is [90, 178]

$$h(y) = y_0\left(1+\frac{y}{a}\right)^{wa} e^{-wy}, \qquad -a<y<\infty$$

where

$$w = \frac{2\mu_2}{\mu_3} \qquad a = \frac{2\mu_2^2}{\mu_3} - \frac{\mu_3}{2\mu_2}$$

$$p = wa = \frac{4}{\beta_1} - 1 \qquad y_0 = \frac{p^{p+1}}{ae^p\Gamma(p+1)}$$

$$\text{mode} = \text{mean} - \frac{\mu_3}{2\mu_2}$$

and the origin is at the mode. Substitution of the foregoing values of these parameters gives the specific Pearson type III p.d.f.
If the origin is shifted to the start of the distribution and a convenient scale is chosen, the density function assumes the familiar form of the gamma distribution, namely,

$$h(y) = \frac{1}{\Gamma(\delta)} y^{\delta-1} e^{-y}, \qquad \delta>0, 0\leqslant y<\infty.$$

8.8 THE BETA APPROXIMATION

The beta distribution

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, \qquad 0\leqslant x\leqslant 1 \tag{8.8.1}$$

is a highly flexible distribution whose approximations to unimodal distributions with varying degrees of asymmetry is often amazingly good. The beta distribution is determined by its first two moments, the parameters a

and b being expressible in terms of the first and second moments by the equations

$$b=\frac{\mu_1'-\mu_2\left(\dfrac{1}{1-\mu_1'}\right)}{\mu_2\left(\dfrac{1}{1-\mu_1'}\right)^2}$$

$$a=b\left(\frac{\mu_1'}{1-\mu_1'}\right), \tag{8.8.2}$$

where μ_r' and μ_r denote, respectively, the rth moments about the origin and the mean.

The actual error can be evaluated to any specified degree of accuracy by a method developed by Posten and Woods [293] and discussed in Sections 8.12.2 and 8.12.3 Example 8.13.1 illustrates the approximation of the distribution of a product of three beta variables with a beta distribution. The results, including the accuracy of the approximation, are given in Table 8.13.1.

8.9 THE VON MISES STEP FUNCTION APPROXIMATION

The method of Von Mises is one of successive approximation of a distribution function $F(x)$ given the sequence of integer moments $M_0, M_1, M_2, \ldots, M_{2m-1}$; it is based on the following theorem of Von Mises [388].

Theorem 8.9.1 If $M_0, M_1, \ldots, M_{2m-1}$ are the moments of a distribution $F(x)$, which increase at m points at least, then there is a unique m-step distribution $V_m(x)$ that has these moments. The m steps are in the interior of the smallest interval that contains all points of increase of $F(x)$. Either $F(x)=V_m(x)$ or $F(x)$ crosses each step of $V_m(x)$.

It should be pointed out that M_k is the kth integer moment of the p.d.f. $f(x)=dF/dx$; that is,

$$M_k=\int_{\substack{\text{Range}\\ \text{of } x}} x^k f(x)\,dx,$$

referred to by Von Mises as the kth moment of the distribution $F(x)$.

When the conditions of Theorem 8.9.1 are satisfied, one may proceed to find the desired step function, which he now knows exists. This step function $V(x)$ has the same first $2m$ moments $M_0, M_1, \ldots, M_{2m-1}$ as the distribution $F(x)$. Furthermore, the step function $V_m(x)$ is such that at m abscissa values $A_1, A_2, \ldots, A_m$ the value of $F(x)$ is contained in step intervals of length $S_1, S_2, \ldots, S_m$, respectively. The derivation of the algorithm for computing the abscissa values and the step sizes of the step function is given by Von Mises [388] and is not repeated here. However the explicit procedure for applying the algorithm is outlined below.

The first step in the procedure is to determine the moments $M_k, k = 0, 1, \ldots, 2m-1$ of the distribution $F(x)$ which is being approximated. Then one solves the following system of linear equations simultaneously for $c_k, k=0, 1, \ldots, 2m-1$:

$$
\begin{aligned}
c_0M_0 + c_1M_1 + \cdots + c_{m-1}M_{m-1} &= -M_m,\\
c_0M_1 + c_1M_2 + \cdots + c_{m-1}M_m &= -M_{m+1},\\
&\cdots\cdots\\
c_0M_1 + c_1M_2 + \cdots + c_{m-1}M_m &= -M_{2m-1}.
\end{aligned}
\tag{8.9.1}
$$

Having obtained the values of $c_k, k=0,1,\ldots,2m-1$ one next obtains the abscissa $A_1, A_2, \ldots, A_{2m-1}$ as the roots of the equation

$$x^m + c_{m-1}x^{m-1} + c_{m-2}x^{m-2} + \cdots + c_1x + c_0 = 0. \tag{8.9.2}$$

After the abscissas $A_1, A_2, \ldots, A_m$ have been found, the step sizes $S_1, S_2, \ldots, S_m$ are computed from the m equations:

$$\sum_{j=1}^{m} A_j^k S_j = M_k, \qquad k=0,1,\ldots,m-1. \tag{8.9.3}$$

The computation is well suited to a digital computer using library programs for the solution of the polynomial equation (8.9.2) and the systems of simultaneous equations (8.9.1) and (8.9.3). The distribution function $F(x)$ is approximated with increasing accuracy as m is increased.

Thompson and Palicio [376] give an interesting and practical application[39] of Von Mises method in which they use the step function $V_m(a)$ to approximate the posterior distribution of system availability a. In this

[39]Copyright © 1975 by The Institute of Electrical and Electronic Engineers, Inc. Reprinted in part, with permission, from "Bayesian Confidence Limits for the Availability of Systems," by W. E. Thompson and P. A. Palicio, *IEEE Transactions on Reliability* (1975), Vol. R-24, pp. 118–120.

problem, they consider a system consisting of N independent subsystems arranged in series (i.e., the system fails if any subsystem fails). This system is characterized by N mutually statistically independent two-state renewal processes [64]. The times to failure u_i and times to repair d_i of subsystem i are statistically independent r.v.'s with exponential p.d.f.'s

$$f_i(u_i)=\exp(-\lambda_i u_i), \qquad u_i \geqslant 0$$

$$f_i(d_i)=\exp(-\mu_i d_i), \qquad d_i \geqslant 0.$$

The probability of finding subsystem i in the up state at an arbitrary time is the subsystem availability, defined as

$$\begin{aligned} a_i &= \frac{E(u_i)}{E(u_i)+E(d_i)} \\ &= \frac{\lambda_i}{\lambda_i+\mu_i}. \end{aligned}$$

Gamma p.d.f.'s are natural conjugate [300] prior p.d.f.'s for λ_i and μ_i; namely, for λ_i,

$$f_i(\lambda_i)=[\Gamma(d_i)]^{-1}\xi_i^{di}\lambda_i^{di-1}\exp(-\lambda_i\xi_i) \qquad \lambda_i \geqslant 0.$$

The posterior p.d.f. of subsystem availability resulting from the gamma prior p.d.f.'s is then the Euler p.d.f. [376], [38],[344, p.3]

$$f_i(a_i)=K_i\frac{a_i^{w_i-1}(1-a_i)^{r_i-1}}{[1-\delta_i a_i]^{r_i+w_i}}, \qquad r_i>0, w_i>0, \qquad |\delta_i|<1, 0\leqslant a_i \leqslant 1, \quad (8.9.4)$$

where

$$r_i \equiv A_i+\alpha_i, \qquad w_i \equiv B_i+\beta_i,$$

$$u_i \equiv V_i+\xi_i, \qquad d_i \equiv D_i+\eta_i,$$

$$U_i \equiv \sum_{j=1}^{A_i} U_{ij}, \qquad D_i \equiv \sum_{j=1}^{B_i} D_{ij},$$

$$\delta_i \equiv 1-\frac{d_i}{u_i}, \qquad K_i \equiv \frac{(1-\delta_i)^{w_i}}{B(r_i, w_i)}. \qquad (8.9.5)$$

Here A_i and D_i denote the number of observations of time to failure and time to repair, respectively, for the ith subsystem, and U_{ij} and D_{ij} denote, respectively, observation j of time to failure and observation j of time to repair for the ith subsystem.

The required moments $M_k, k=0,1,\ldots,2m-1$ of the posterior p.d.f. $h(a)$ are obtained from the moments of the posterior p.d.f.'s of the subsystem availabilities. These moments are precisely the Mellin transforms of the Euler p.d.f.'s given by (8.9.4). Specifically, if there are N subsystems, then

$$M_k = \prod_{i=1}^{N} {}_iM_k, \tag{8.9.6}$$

where

$${}_iM_k = (1-\delta_i)^{w_i} \frac{B(r_i, w_i+r)}{B(r_i, w_i)} {}_2F_1(w_i+r_i, w_i+r;\ \ w_i+r_i+r;\ \ \delta_i) \tag{8.9.7}$$

and ${}_2F_1(a,b;\ c;\ z)$ is the Gauss's hypergeometric function (Appendix D.1).

The example below, by Thompson and Palicio [376], is for a system consisting of two independent subsystems.

Example 8.9.1 Use the Von Mises step function based on 36 moments to approximate the posterior distribution (c.d.f. $H(a)$) of system availabil-

Table 8.9.1 Moments of the Probability Density Function $f(a)$

$M_0=1$	$M_{18}=0.0000323295220$
$M_1=0.3959960052346$	$M_{19}=0.0000221647900$
$M_2=0.1701086745762$	$M_{20}=0.0000153581097$
$M_3=0.6708684832154$	$M_{21}=0.0000107481631$
$M_4=0.0378530414098$	$M_{22}=0.0000075926454$
$M_5=0.0192286501389$	$M_{23}=0.0000054110158$
$M_6=0.0101668257596$	$M_{24}=0.0000038884404$
$M_7=0.0055662340123$	$M_{25}=0.0000028163576$
$M_8=0.0031423813354$	$M_{26}=0.0000020551131$
$M_9=0.0018229907769$	$M_{27}=0.0000015102679$
$M_{10}=0.0010836652823$	$M_{28}=0.0000011173531$
$M_{11}=0.0006584819140$	$M_{29}=0.0000008319640$
$M_{12}=0.0004081683537$	$M_{30}=0.0000006232556$
$M_{13}=0.0002576416534$	$M_{31}=0.0000004696286$
$M_{14}=0.0001653526462$	$M_{32}=0.0000003558428$
$M_{15}=0.0001077571769$	$M_{33}=0.0000002710641$
$M_{16}=0.0000712218678$	$M_{34}=0.0000002075385$
$M_{17}=0.0000476940182$	$M_{35}=0.0000001596788$

Table 8.9.2 Von Mises Step Function and BetaApproximation for Example 8.9.1

Abscissa Values A_j	Step Sizes S_j	Sum of Steps $\frac{\sigma}{j-1} S_j$	Ordinate Plot $(\frac{\sigma S_j}{j-1}) + \frac{1}{2} S_j$	Beta Approximation
0.030604	0.000008	0.000000	0.000004	0.000005
0.059804	0.000225	0.000008	0.000121	0.000039
0.095634	0.002130	0.000233	0.001298	0.000668
0.137443	0.010722	0.002363	0.007724	0.005350
0.184663	0.034895	0.013085	0.030533	0.024721
0.236513	0.080715	0.047980	0.088338	0.090317
0.292141	0.140219	0.128695	0.198805	0.192776
0.350614	0.188104	0.268914	0.362966	0.363533
0.411006	0.198051	0.457018	0.556044	0.563061
0.472337	0.163446	0.655069	0.736792	0.744903
0.533657	0.105386	0.818515	0.871208	0.875252
0.594162	0.051677	0.923901	0.949740	0.951109
0.652768	0.018781	0.975578	0.984969	0.984422
0.709209	0.004785	0.994359	0.996752	0.996199
0.762326	0.000778	0.999144	0.999533	0.999313
0.812294	0.000074	0.999922	0.999959	0.999915
0.858734	0.000003	0.999996	0.999998	0.999993
0.902758	0.000000(04)	0.999999	0.999999	0.999998
1.000000		1.000000	1.000000	1.000000

ity for a system consisting of two subsystems in series, for which both time to failure and time to repair are reported for each failure, and the prior p.d.f.'s are gamma. Assume further that six fail-repair cycles are observed on subsystem 1 and 12 on subsystem 2, with the following results:

$$w_1 = r_1 = 6 \qquad \delta_1 = 0.000 \qquad \hat{a}_1 = 0.500$$

$$w_2 = r_2 = 12 \qquad \delta_2 = 0.750 \qquad \hat{a}_2 = 0.800.$$

SOLUTION. First, the moments M_k of the p.d.f. $h(a)$ are calculated with the combined use of (8.9.6) and (8.9.7) and are found to have the values shown in Table 8.9.1. (All calculations were made using double precision (17 digits) by Thompson and Palicio, but they are here given to somewhat lesser accuracy in the interest of brevity.) These values of M_k are then substituted into the set of equations (8.9.1), which are solved simultaneously for the coefficients $c_k, k=0,1,\ldots,m-1$. One then substitutes these coefficients into (8.9.2) and solves this equation for its roots $A_1, A_2, \ldots, A_m$. Substitution of these A_j into (8.9.3) yields the set of m linear equations in m

unknowns $S_j, j=1,2,\ldots,m$, which are readily solved simultaneously to obtain the step sizes S_j. One now has all the necessary information from which to determine 18 values of the c.d.f. The results appear in Table 8.9.2. When the first m moments of the approximating and exact p.d.f.'s are identical, the error of the approximation can be determined by means of Theorem 8.12.3.

8.10 THE CORNISH–FISHER EXPANSION

Another approach to the determination of an approximate distribution function that approaches the true but unknown distribution function $F(x)$ is to derive a new variable

$$w = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots, \tag{8.10.1}$$

whose distribution function $G(w)$ is approximately that of a standardized normal variable and can be used to obtain a satisfactory approximation to $F(x)$, specifically, an approximation whose accuracy will be of a specified order. That is, the method consists of the "normalization" of the approximating distribution. This is accomplished by beginning with *Edgeworth's form* of the Gram-Charlier type A series as the approximating density function for $f(x)$, then choosing the b's in the variable transformation (8.10.1) in such a way that the standardized normal distribution function $G(w)$ will approximate the distribution function $F(x)$ for corresponding values of w and x. The series expansion (8.10.1), called the Cornish-Fisher expansion, permits the evaluation of $f(x)$ and $F(x)$ by using only a table of ordinates and areas for a standardized normal distribution. This method utilizes cumulants, which have certain properties that make them more useful than moments from the theoretical standpoint [178, pp. 67–71].

The Cornish-Fisher expansion is appropriate and useful as an approximation to an unknown distribution, whose moments are known, when that distribution (specifically, the p.d.f.) depends on a parameter λ in such a way that as λ tends to infinity, the distribution tends to normality. For example, the distribution of the means of n identically distributed i.r.v.'s depends on the parameter $\lambda = n =$ sample size in such a way that as n tends to ∞, the distribution approaches normality, provided the variance is finite. By use of the Cornish-Fisher expansion, one obtains a new variable whose distribution function approaches normality much faster than that of the original variable. One can then utilize the one-to-one correspondence between the two distribution functions to obtain a much better approximation to the original distribution from the standardized normal distribution than would otherwise be possible. Since an adequate presentation of the method would be quite lengthy, and since Kendall and Stuart, for example, give an excellent discussion of the methods in their book [178, pp. 163–166], the interested reader is referred to it for details.

8.11 NUMERICAL INVERSION OF INTEGRAL TRANSFORMS

Another possible approach to the determination of the p.d.f. (or c.d.f., as the case may be) when the inversion integral is mathematically intractable consists of the numerical evaluation of the inversion integral, frequently referred to as the numerical inversion of integral transforms. Thus the determination of the p.d.f. (or c.d.f.) of a sum of i.r.v.'s by this method would involve the numerical inversion of a Laplace or Fourier transform. On the other hand, the direct determination of the p.d.f. $h(y)$ or the c.d.f. $H(y)$ of a product $Y=\Pi_1^n X_i$ of i.r.v.'s would involve the numerical inversion of the relevant Mellin transform. However in this case there is the alternative possibility of first determining the p.d.f. $g(w)$ (or the c.d.f. $G(w)$) of $W=\log Y$ by numerical inversion of the relevant Fourier or Laplace transform, from which the p.d.f. or c.d.f. could be obtained. That is, a tabulation of $G(w)$ versus w could be easily transformed into an equivalent tabulation of $H(y)$ versus y by identifying each w with the corresponding value $y=\text{antilog}\, w$. Thus methods for numerical inversion of Laplace or Fourier transforms could also be used to tabulate $h(y)$ or $H(y)$ versus y; this fact is of some importance, since it appears that thus far no one has constructed a method (much less a computer program) for the direct numerical inversion of the *Mellin* transform.

The various techniques available for the numerical inversion of the transforms have met with limited success insofar as their general application is concerned. Each has some attribute to recommend it over the others. Among all the available techniques, the one that appears easiest to apply is that of Dubner and Abate [85]. Their technique also has the advantage that the order of magnitude of the error is easily preassigned by selection of one parameter in the computation. The computer program needed to realize this technique requires fewer than 20 cards—*a rather pleasant feature*.

Nevertheless, a single method for numerically inverting the Laplace transform that works equally well for all types of problem encountered is lacking [148]. In some instances, the accuracy of the results depends on a judicious choice of certain initial estimates, but so far in such a situation a definite procedure for making the initial estimates in an objective manner in the general case has not been provided (with the exception of Dubner and Abate's method). Inversion of the Fourier transform by means of the fast Fourier transform (FFT) method [119] suffers from an inherent accuracy problem because of the finite word length used in digital computers [176]. Using the FFT, McKenzie [248] encountered some difficulties in its application in the numerical inversion of the Fourier transform in the determination of the p.d.f. of products of i.r.v.'s defined on a finite range. Until these limitations are removed and a single satisfactory method for numerical inversion of a Laplace (or Fourier) transform is provided, the approximation of the closed form inversion of transforms by means of a numerical inversion method is, in my opinion, less satisfactory than the use of the aforementioned approximating distributions (the beta p.d.f., the Laguerre series, the Cornish-Fisher expansion, etc.). For this reason, and because of the rather large amount of space that would be required, the various methods for the numerical inversion of transforms are not discussed in this book. The interested reader is referred to a number of relevant papers (e.g., refs. 156; 148, p. 92; 257, 276, 319, 94).

8.12 EVALUATING THE ACCURACY OF AN APPROXIMATING DISTRIBUTION

When the determination of the exact p.d.f. by evaluating the inversion integral is tedious, difficult, or impossible to obtain, some approximation method must be used to evaluate the associated distribution function, which is ultimately the function of interest. The magnitude of the problem is appreciably reduced inasmuch as the moments of both the exact (unknown) and approximate p.d.f.'s are known (or at least can be obtained).

Several approaches to the problem of determining such approximations may be taken. The earliest approach was that of bracketing the proportion of a population contained within the interval $\mu \pm k\sigma$, by using Chebyshev's inequality [277, p. 226] or modifications thereof that strengthen the results, such as the Gauss and Camp-Meidell inequalities [366, pp. 297–298]. However, even the best of the methods utilizing this approach are usually intolerably weak, and for this reason they are not discussed further.

A second approach to the approximation problem is that of evaluating the relevant inversion integral by numerical methods (e.g., utilizing the FFT). Some of these numerical methods are briefly discussed in a subsequent section.

The most powerful approach to the problem is to use an approximating p.d.f. $g(x)$, then to evaluate the accuracy with which the associated distribution function $G(x)$ approximates the true but unknown distribution function $F(x)$. Of course various p.d.f.'s have long been suggested for such approximations—for example, the Gram-Charlier series, the Laguerre series, the Pearson distributions, the Von Mises step function, and the Cornish-Fisher expansion—all of which have been discussed in this connection. However feasible and powerful methods for obtaining an accurate estimate of the magnitude of the error incurred when using such an approximate p.d.f. have been largely lacking. Recently Posten and Woods [293] have derived a method for determining the accuracy of the approximation on the basis of a knowledge of the moments of both the approximating and exact p.d.f.'s. The accuracy, as expected, will improve as the number of moments having identical values for both distributions increases. The question is, of course, the magnitude of the improvement. Nevertheless, the accuracy of the approximation can be determined—no matter what the approximating function is—as long as the moments of both the approximate and exact p.d.f.'s are known. The method for accomplishing the evaluation of this accuracy, as developed by Posten and Woods, is here presented with their permission.

8.12.1 The Fourier Series

On December 21, 1807, Joseph Fourier (1768–1830) first asserted to the Paris Academy that any arbitrary function defined on a finite interval can be represented by an infinite sum of sine and cosine functions. Although not the first to introduce trigonometric expansions—Clairant, D'Alembert, and D. Bernoulli had used them in their investigations—Fourier did prove that the expansion was valid for certain simple functions. Although his theorem could not be proved for all arbitrary functions, his method was correct. Moreover, his rather sweeping claim that any arbitrary function could be expanded in such a series stimulated their use in mathematical physics, as a result of which they became known as Fourier series, characterized by the following definition.

Definition 8.12.1 Let $g(x)$ be an arbitrary function defined on the interval $(0,\pi)$. The infinite series

$$\sum_{n=1}^{\infty} b_n \sin(nx) \tag{8.12.1}$$

is the Fourier sine series of $g(x)$ if

$$b_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin(nx)\,dx. \tag{8.12.2}$$

The coefficients b_n are called the Fourier sine coefficients of $g(x)$.

Hereafter, the shorter terms "Fourier series" and "Fourier coefficients" are used instead of "Fourier sine series" and "Fourier sine coefficients," respectively. This should cause no confusion, since the Fourier sine series is the only type of Fourier series used in this chapter.

Note that in Definition 8.12.1, no mention of convergence has been made. That is, a Fourier series is simply a series of the form given in the definition. Because of this, the following theorem is particularly significant.

Theorem 8.12.1 (Jordan's theorem[40]) If the function $g(x)$ is of bounded variation on the interval $(0,\pi)$, then its Fourier series

$$\sum_{n=1}^{\infty} b_n \sin(nx)$$

converges at every point x in the interval $(0,\pi)$ to the value

$$\frac{[g(x^+)+g(x^-)]}{2}. \tag{8.12.3}$$

If $g(x)$ is continuous on the interval $I=(a,b)$, the Fourier series is uniformly convergent in I and converges to the value $g(x^+)=g(x^-)=g(x)$.
If the series converges in the sense of Theorem 8.12.1, it will be written in the customary form

$$g(x)=\sum_{n=1}^{\infty} b_n \sin(nx),$$

remembering, of course, that the Fourier series converges to the value (8.12.3) at a point of discontinuity.

Fourier's results provided the stimulus for many fundamental investigations in mathematics. Among the earliest of these was Dirichlet's derivation in 1829 of sufficient conditions for the convergence of a Fourier series. It should be emphasized that these conditions are sufficient. In fact, one of the unsolved problems of Fourier series is whether necessary and sufficient conditions for convergence do exist. For our purposes, it is sufficient to know that although a Fourier series is uniformly convergent in any interval not containing a point of discontinuity, the number of terms needed for an accuracy ε becomes arbitrarily large as one approaches the point of discontinuity.

[40]Not to be confused with Jordan's lemma.

The application of Fourier series has been mostly in mathematics, physics, and engineering. But though Fourier series have been used in mathematical physics with remarkable success, their role in statistical distribution theory has been neglected. Posten and Woods [293] attribute this neglect to the following conditions.

1. An early belief that approximations should be based on the first four moments of the distribution. This belief, one of the main deterrents to the use of Fourier series in statistical distribution theory, probably stemmed from the fact that both the Pearson and Charlier systems were devised to fit theoretical functions to empirical data. Since the sample moments M_k are very unreliable for $k>4$, any attempt to fit a theoretical curve to sample data was necessarily restricted to the methods that produced a satisfactory approximation using no more than the first four sample moments. The problem of finding a satisfactory approximating distribution utilizing the known moments of both the approximating and exact distributions, eliminates any dependence on sample moments and their inherent unreliability.

2. Inapplicability of Fourier series over infinite ranges, the range of many density functions. This appears to be a serious restriction, since it implies that a valid Fourier representation is restricted to the distribution of r.v.'s having a finite range. In many cases, however, it will be both possible and feasible to work with r.v.'s of singly or doubly infinite range by utilizing some type of transformation. For example, if $0 \leqslant X < \infty$, the transformation $Y = e^{-X}$ yields a variable Y confined to the interval $(0,1)$. Similarly, if $-\infty < X < \infty$, the transformed variable

$$\begin{aligned} Y &= -e^{X}, \qquad -\infty < X \leqslant 0 \\ &= e^{-X}, \qquad 0 \leqslant X < \infty \end{aligned}$$

is restricted to the range $(-1,1)$.

3. Lack of reliability of Fourier series in the tails of distributions, usually the most important area for statisticians concerned with hypothesis testing.

4. Slow rate of convergence for Fourier series. The speed of convergence and the reliability of a Fourier series representation are both dependent on the smoothness of the function. In particular, if the function has points of discontinuity, the effect of the Gibbs phenomenon [293, pp. 12–15; 31] will result in a slowly converging series with regions of unreliability around the points of discontinuity. For this reason a Fourier series is not very useful as an approximation to any distribution with jump points.

5. Certain computational difficulties: (*a*) the problem of calculating the Fourier coefficients, (*b*) the evaluation of the series, and (*c*) the lack of efficient computational facilities.

Before electronic computers were available, the evaluation of a Fourier series was painstaking and expensive in terms of time and effort. But with the modern electronic calculators—and even present-day desk calculators—the evaluation of a Fourier series is quite feasible. Nevertheless, to produce an approximating function with the reliability and speed of convergence necessary to determine the numerical accuracy of this approximation to the true distribution function, certain modifications are necessary. It turns out [293] that these modifications can be accomplished in three ways, namely, in the choice of the function to be expanded, the interval of periodicity, and the transformations to be used.

First of all, the convergence of the Fourier series can be substantially improved by employing Kummer's technique, which consists of subtracting from a "suitable" known distribution function $G(x)$ the desired distribution function $F(x)$ and expanding the difference $d(x)=G(x)-F(x)$ in a Fourier series. The choice of the function $G(x)$ is important.

Woods and Posten point out that in general, a relatively small number of terms of the Fourier expansion will suffice to approximate a function with a high degree of accuracy, if its periodic extension is not only differentiable but differentiable to a sufficiently high order. Then if the function $d(x)=G(x)-F(x)$ is sufficiently smooth on the interval $(0,1)$, the order of differentiability will be determined by the behavior of the extended function at the points $x=0,1$. By choosing $G(x)$ such that $d(0)=d(1)=0$, the continuity of any extension of $G(x)$ is ensured. Furthermore, $G(x)$ may be chosen so that its form is as close to the form of a general distribution function as possible. This will generally flatten out $d(x)$ and, as a rule, will make the derivatives of $d(x)$ close to zero at the end points.

Although the standard method of scaling the interval $(0,\pi)$ is to use the linear transformation $\theta=\pi x$, the nonlinear transformation $\theta=\operatorname{Cos}^{-1}(2x-1)$ is more appropriate inasmuch as it simplifies the computational aspects of the representation theorems derived by Posten and Woods.

Using the aforementioned modifications, Posten and Woods derive the representation theorems for the purpose of evaluating the accuracy with which an approximating function represents the true but unknown distribution function, when the moments of both functions are known or obtainable. Some applications of these theorems to specific distribution functions are given in Section 8.13.

8.12.2 Approximation Theorems Based on the Fourier Sine Series

The use of the Fourier sine series to evaluate the error of an approximating distribution is based on the following theorems by Posten and Woods [293].

Theorem 8.12.2 (the first generalized Fourier representation theorem). Let $F(x)$ and $G(x)$ be distribution functions over the interval $(0,1)$. Then

$$F(x)=\begin{cases} G(x)-\sum_{n=1}^{\infty} d_n^* \sin(n\theta), & 0 \leqslant x \leqslant 1 \qquad (8.12.4a) \\ G(x), & \text{elsewhere}, \qquad (8.12.4b) \end{cases}$$

where

$$\theta=\operatorname{Cos}^{-1}(2x-1), \tag{8.12.4c}$$

$$d_n^* = \frac{2}{n\pi}\left[E_F(T_n^*(x)) - E_G(T_n^*(x))\right], \tag{8.12.4d}$$

$$E_F(h(x)) = \int h(x)\,dF(x), \tag{8.12.4e}$$

and $T_n^*(x)$ are the Chebyshev polynomials defined on the interval $(0,1)$ as

$$T_n^*(x)=\cos\left[n\operatorname{Cos}^{-1}(2x-1)\right], \tag{8.12.4f}$$

where n is a nonnegative integer.

Theorem 8.12.3 (the second generalized Fourier representation theorem). Let $F(x)$ and $G(x)$ be distribution functions over the interval $(-1,1)$. Then

$$F(x)=\begin{cases} G(x)-\sum_{n=1}^{\infty} d_n \sin n\theta, & -1 \leqslant x \leqslant 1 \qquad (8.12.5a) \\ G(x), & \text{elsewhere}, \qquad (8.12.5b) \end{cases}$$

where

$$\theta=\operatorname{Cos}^{-1}(x), \tag{8.12.5c}$$

$$d_n = \frac{2}{n\pi}\left[E_F(T_n(x)) - E_G(T_n(x))\right], \tag{8.12.5d}$$

$$E_F(h(x)) = \int h(x)\,dF(x), \tag{8.12.5e}$$

and $T_n(x)$ are the Chebyshev polynomials defined over the interval $(-1,1)$ by

$$T_n(x)=\cos\left[n\operatorname{Cos}^{-1}(x)\right], \tag{8.12.5f}$$

n being a nonnegative integer.

The theorems above are restricted to the r.v.'s defined over the intervals $(0,1)$ or $(-1,1)$. As has been pointed out, however, these Fourier representation theorems may be used to evaluate the distribution of any random variable X as long as there exists a transformation $y=G(x)$ such that $Y=G(X)$ ranges over either of the intervals $(0,1)$ or $(-1,1)$ and the moments of Y are known. The requirement that the moments be known is essential and is the basis for the evaluation of the error incurred when $G(x)$ is used to approximate $F(x)$.

The following theorems are relevant. The proofs are straightforward and are not given here.

Theorem 8.12.4. If X is an r.v. over the interval $(0,\infty)$ with m.g.f. $M(t)$, then $Y=e^{-X}$ is an r.v. over the interval $(0,1)$ with moments $E[Y^K]=M(-K)$.

Theorem 8.12.5. If X is an r.v. over the interval $(-\infty,\infty)$ with p.d.f. $f(x)$, and Y is defined as

$$\begin{aligned} Y &= -e^{X}, \qquad -\infty<X<0 \\ &= e^{-X}, \qquad 0\leqslant X<\infty, \end{aligned}$$

then Y is an r.v. over the interval $(-1,1)$ with p.d.f.

$$h(y)=\begin{matrix} h^{-}(y), & -1<y<0 \\ h^{+}(y), & 0\leqslant y<1 \end{matrix}$$

where

$$h^{-}(y)=-\left(\frac{1}{y}\right)f\left(\ln\frac{1}{|y|}\right), \qquad -1\leqslant y<0,$$

$$h^{+}(y)=\left(\frac{1}{y}\right)f\left(\ln\frac{1}{y}\right), \qquad 0<y\leqslant 1.$$

The m.g.f. of $h(y)$ is

$$\begin{aligned} M(t) &= M^{-}(t), \qquad -1\leqslant y<0 \\ &= M^{+}(t), \qquad 0<y\leqslant 1, \end{aligned}$$

where

$$M^{-}(t)=\int_{-\infty}^{0} e^{-ty}h^{-}(y)\,dy,$$

$$M^{+}(t)=\int_{0}^{\infty} e^{ty}h^{+}(y)\,dy.$$

Theorem 8.12.6. If X is an r.v. over the interval (a,∞) with m.g.f. $M(t)$, then $Y=X-a$ is an r.v. over the interval $(0,\infty)$ with m.g.f. $e^{-at}M(t)$.

Theorem 8.12.7. If X is an r.v. over the interval $(-\infty,a)$ with m.g.f. $M(t)$, then $Y=a-X$ is an r.v. over the interval $(0,\infty)$ with m.g.f. $e^{at}M(-t)$.

Theorem 8.12.8. If X is an r.v. over the interval (a,b) with moments $E[X^K]=\mu_K'$, then $Y=(X-a)/(b-a)$ is an r.v. over the interval $(0,1)$ with moments

$$E[Y^n]=(b-a)^{-n}\sum_{K=0}^{n}\binom{n}{K}(-a)^{n-K}\mu_K'.$$

Theorem 8.12.9. If X is an r.v. over the interval (a,b) with moments $E[X^K]=\mu_K'$, then $Y=(X-b)/(a-b)$ is an r.v. over the interval $(0,1)$ with moments

$$E[Y^n]=(a-b)^{-n}\sum_{K=0}^{n}\binom{n}{K}(-b)^{n-k}\mu_K'.$$

Theorem 8.12.10. If X is an r.v. over the interval (a,b) with moments $E[X^K]=\mu_K'$, then $Y=[2X-(a+b)]/(b-a)$ is an r.v. over the interval $(-1,1)$ with moments

$$E[Y^n]=(b-a)^{-n}\sum_{K=0}^{n}\binom{n}{K}(a+b)^{n-K}\mu_K'.$$

Posten and Woods [293] point out that in some cases one may find the simple approximating p.d.f.

$$G(x)=1-\frac{\theta}{\pi}$$

to be satisfactory, in which case the accuracy of the approximation to $F(x)$ may be determined from the following two theorems, which they call the first and second Fourier representation theorems.

Theorem 8.12.11. (the first Fourier representation theorem). If X is an r.v. over the interval $(0,1)$ with distribution function $F(x)$, then

$$F(x)=\begin{cases} 0, & x<0 & (8.12.6a)\\ 1-\dfrac{\theta}{\pi}-\displaystyle\sum_{n=1}^{\infty} b_n\sin(n\theta), & 0\leqslant x\leqslant 1 & (8.12.6b)\\ 1, & x>1, & (8.12.6c)\end{cases}$$

where $\theta=\cos^{-1}(2x-1)$, $b_n=(2/n\pi)E(T_n^*(X))$, and $E(T_n^*(X))$ is the expected value of the Chebyshev polynomials $T_n^*(X)$ defined by (8.12.4f).

Theorem 8.12.12. (the second Fourier representation theorem). If X is an r.v. over the interval $(-1,1)$ with distribution function $F(x)$, then

$$F(x)=\begin{cases} 0, & x<-1 \quad (8.12.7a)\\ 1-\dfrac{\theta}{\pi}-\displaystyle\sum_{n=1}^{\infty} b_n\sin(n\theta), & -1\leqslant x\leqslant 1 \quad (8.12.7b)\\ 1, & x>1, \quad (8.12.7c)\end{cases}$$

where $\theta=\mathrm{Cos}^{-1}x$, $b_n=(2/n\pi)E(T_n(X))$, and $E(T_n(X))$ is the expected value of the Chebyshev polynomials T_n defined over the interval $(-1,1)$ by (8.12.5f).

8.12.3 Error Analysis Based on the Fourier Theorems

In using the aforementioned Fourier theorems to evaluate the error incurred when replacing a true distribution function with an approximating distribution function, one must, of course, know the accuracy of the Fourier series involved. As Posten and Woods point out, the accuracy of the relevant Fourier series is governed by errors from three sources:

1. The truncation of the series.
2. The rounding of the Fourier coefficients.
3. The method of evaluating the truncated series.

Since the absolute value of the sine function is less than unity, the effect of the rounding error can generally be minimized by using a guarding digit. That is, rounding the coefficients to m decimal places in the calculation of $\Sigma_n b_n\sin(n\theta)$ should result, as a rule, in a rounding error of approximately one unit in the $(m-1)$st decimal digit. Similarly, Clenshaw [56] has shown that the error associated with the evaluation of a series with the aid of a three-term recurrence relation is negligible if a guarding digit is retained. The question, then, is this: what size error is made if the series is terminated after n terms, where n is not too small but not arbitrarily large? Posten and Woods discuss this problem and present three methods, two for determining bounds on the truncation error and another for approximating the truncation error. As a result, they recommend the following procedure to control the accuracy of the Fourier series in evaluating the error incurred by approximating the true distribution function $F(x)$ by another distribution function $G(x)$:

1. Evaluate the Fourier coefficients until they are no longer significant relative to the desired accuracy. This will eliminate the effects due to truncating the series.
2. Retain one of two guarding digits in all the intermediate computations. This will eliminate the effects due to rounding and the method of evaluating the series. Only the final answer is rounded to the required number of digits.

To apply the Fourier or generalized Fourier theorems, one must evaluate $d_k^*, d_k, \Sigma_{k=1}^n d_k^* \sin(k\theta), \Sigma_{k=1}^n d_k \sin(k\theta)$, where

$$d_k^* = b_{kf}^* - c_{kg}^*, \tag{8.12.8a}$$

$$d_k = b_{kf} - c_{kg}, \tag{8.12.8b}$$

and b_k^*, c_k^*, and b_k, c_k, are Fourier coefficients involving, respectively, the expected values of the Chebyshev polynomials that are valid for the ranges $0 \leqslant x \leqslant 1$ and $-1 \leqslant x \leqslant 1$. Specifically,

$$c_{kg}^* = \frac{2}{k\pi} E_g(T_k^*), \qquad 0 \leqslant x \leqslant 1, \tag{8.12.9a}$$

$$E_g(T_k^*) = \int_0^1 T_k^* g(x)\, dx, \tag{8.12.9b}$$

$$= \sum_{j=0}^{k} \alpha_{kj}^* x^j g(x) \tag{8.12.9c}$$

$$b_{kf}^* = \frac{2}{k\pi} E_f(T_k^*), \qquad 0 \leqslant x \leqslant 1,$$

$$E_f(T_k^*) = \int_0^1 T_k^*(x) f(x)\, dx$$

$$= \sum_{j=0}^{k} \alpha^*{}_{kj} x^j f(x) \tag{8.12.9d}$$

where $T_k^*(x)$ and $T_k(x)$ denote, respectively, the Chebyshev polynomials for the ranges $0 \leqslant x \leqslant 1$ and $-1 \leqslant x \leqslant 1$. Also,

$$c_{kg} = \frac{2}{k\pi} E_g(T_k(x)), \qquad -1 \leqslant x \leqslant 1 \tag{8.12.10a}$$

$$E_g(T_k) = \int_{-1}^{1} T_k(x) g(x)\, dx,$$

$$= \sum_{j=0}^{k} \alpha_{kj} x^j g(x), \qquad -1 \leqslant x \leqslant 1, \tag{8.12.10b}$$

$$b_{kf} = \frac{2}{k\pi} E_f(T_k(x)), \qquad -1 \leqslant x \leqslant 1, \tag{8.12.10c}$$

$$E_f(T_k) = \int_{-1}^{1} T_k f(x)\, dx,$$

$$= \sum_{j=0}^{k} \alpha_{kj} x^j f(x), \qquad -1 \leqslant x \leqslant 1. \tag{8.12.10d}$$

Once these Fourier coefficients are known, finite series $\Sigma_{k=1}^{n} d_k^* \sin(k\theta)$ and $\Sigma_{k=1}^{n} d_k \sin(k\theta)$ corresponding, respectively, to p.d.f.'s defined on the ranges $(0,1)$ and $(-1,1)$, can best be calculated by Goertzel's algorithm [293], which is stated below.

Goertzel's Algorithm

If $S = \Sigma_{k=1}^{n} d_k^* \sin(k\theta)$ for given d_k^* and θ, then $S = U_1 \sin\theta$, where U_1 is found from the recurrence relation

$$U_k = d_k^* + 2(\cos\theta) U_{k+1} - U_{k+2}, \qquad k = n, n-1, \ldots, 1$$

and $U_{n+2} = U_{n+1} = 0$. (The algorithm is, of course, equally valid when d_k^* is replaced by d_k; i.e., when the range of x is $(-1,1)$.)

In the particular case for which $\theta = \mathrm{Cos}^{-1}(2x-1)$, the recurrence relation becomes

$$U_k = d_k^* + (4x-2) U_{k+1} - U_{k+2} \tag{8.12.11a}$$

and

$$S = 2U_1 \sqrt{x - x^2}\,. \tag{8.12.11b}$$

The algorithm is useful because it avoids the necessity of evaluating $\theta = \mathrm{Cos}^{-1}(2x-1)$ or any other trigonometric function.

Returning now to the calculation of b_k^* and c_k^*, one notes that (8.12.9b, d) and (8.12.10b, d) utilize the fact that both $T_k(x)$ and $T_k^*(x)$ are kth degree polynomials in x, hence are expressible in the forms

$$T_k(x)=\sum_{j=0}^{k}\alpha_{kj}x^j, \tag{8.12.12a}$$

$$T_k^*(x)=\sum_{j=0}^{k}\alpha_{kj}^*x^j, \tag{8.12.12b}$$

where the Chebyshev coefficients α_{kj} and α_{kj}^* have been tabulated rather extensively. If tables of Chebyshev polynomials are available, the expected values $E_G(T_k^*)$, $E_F(T_k^*)$ and $E_G(T_k)$, $E_F(T_k)$ may be readily obtained from (8.12.9b, d) and (8.12.10b, d). Then d_k^* and d_k may be evaluated from (8.12.8a, b), (8.12.9a–d), and (8.12.10a–d). If such tables are not available, the recurrence relations

$$T_{k+2}^*(x)=(4x-2)T_{k+1}^*(x)-T_k^*(x), \tag{8.12.13a}$$

where

$$\begin{aligned}T_0^*(x)&=1,\\ T_1^*(x)&=2x-1,\end{aligned} \tag{8.12.13b}$$

and

$$T_{k+1}(x)=2xT_k(x)-T_{k-1}(x), \tag{8.12.14a}$$

where

$$\left.\begin{aligned}T_1(x)&=1\\ T_2(x)&=x^2-1\end{aligned}\right\} \tag{8.12.14b}$$

can be used. Either way, the coefficients b_k^* and c_k^* may be determined by using (8.12.9a–d) in conjunction with (8.12.13a, b), and are expressed in terms of the moments of the approximate and exact distributions about the origin. The coefficient d_k^* may then be obtained from (8.12.8a). Similarly, the coefficients b_k and c_k may be determined either with or without tables of Chebyshev polynomials, after which d_k is obtained from (8.12.8b). In either case, the result is expressed in terms of the moments of the approximate and exact distributions.

It bears stating that if (8.12.13a, b) are utilized in lieu of tables of Chebyshev polynomials, b_k^* and c_k^* are expressible in the forms

$$b_k^* = (-1)^k \frac{2}{\pi} \sum_{j=0}^{k} (-4)^j \frac{(j+k-1)!}{(k-j)!(2j)!} E_f(x^k) \tag{8.12.15a}$$

and

$$c_k^* = (-1)^k \left(\frac{2}{\pi}\right) \sum_{j=0}^{k} (-4)^j \frac{(j+k-1)!}{(k-j)!(2j)!} E_g(x^k), \tag{8.12.15b}$$

from which it follows (8.12.8a) that

$$d_k^* = (-1)^k \left(\frac{2}{\pi}\right) \sum_{j=0}^{k} \frac{(-4)^j (j+k-1)!}{(k-j)!(2j)!} (\mu'_{kf} - \mu'_{kg}), \tag{8.12.16}$$

where μ'_{kg} and μ'_{kf} are, respectively, the kth moments of the approximating and exact p.d.f.'s $g(x)$ and $f(x)$ about the origin.

Now that d_k^* and d_k have been evaluated, it remains to evaluate the truncated series $\Sigma_{k=1}^{n} d_k^* \sin(k\theta)$ and $\Sigma_{k=1}^{n} d_k \sin(k\theta)$. As Posten and Woods point out, the evaluation of the truncated series is not a trivial task. First of all, the value of θ must be calculated, and this calculation must be quite accurate, to eliminate serious error in the calculation of $k\theta$ as k gets larger. Also, it requires the accurate evaluation of the sine function for n different values, which even for a moderate size computer is quite time-consuming.

Goertzel's algorithm, previously stated, seems to be the best method for evaluating the aforementioned truncated series. This is accomplished by solving the recursion formula of Goertzel's algorithm progressively for U_1 and then evaluating $U_1 \sin(\theta)$, since $U_1 \sin(\theta) = S = \Sigma_{k=1}^{n} d_k^* \sin(k\theta)$, the error approximation. When applying Goertzel's algorithm to determine the truncated series in the first general Fourier representation theorem, in which $\theta = \mathrm{Cos}^{-1}(2x-1)$, the recurrence relation becomes

$$U_k = d_k^* + (4x-2)U_{k+1} - U_{k+2}, \tag{8.12.17}$$

and the truncated series

$$S = 2U_1\sqrt{x-x^2} \tag{8.12.17a}$$

then gives the desired error approximation.

To determine the error incurred by using the distribution function $G(x)$ to approximate the distribution function $F(x)$ corresponding to a particular percentage point p, one must first ascertain the value of x. This can be accomplished by various iteration methods. The one recommended by Posten and Woods is based on the work of Traub [383], namely, the

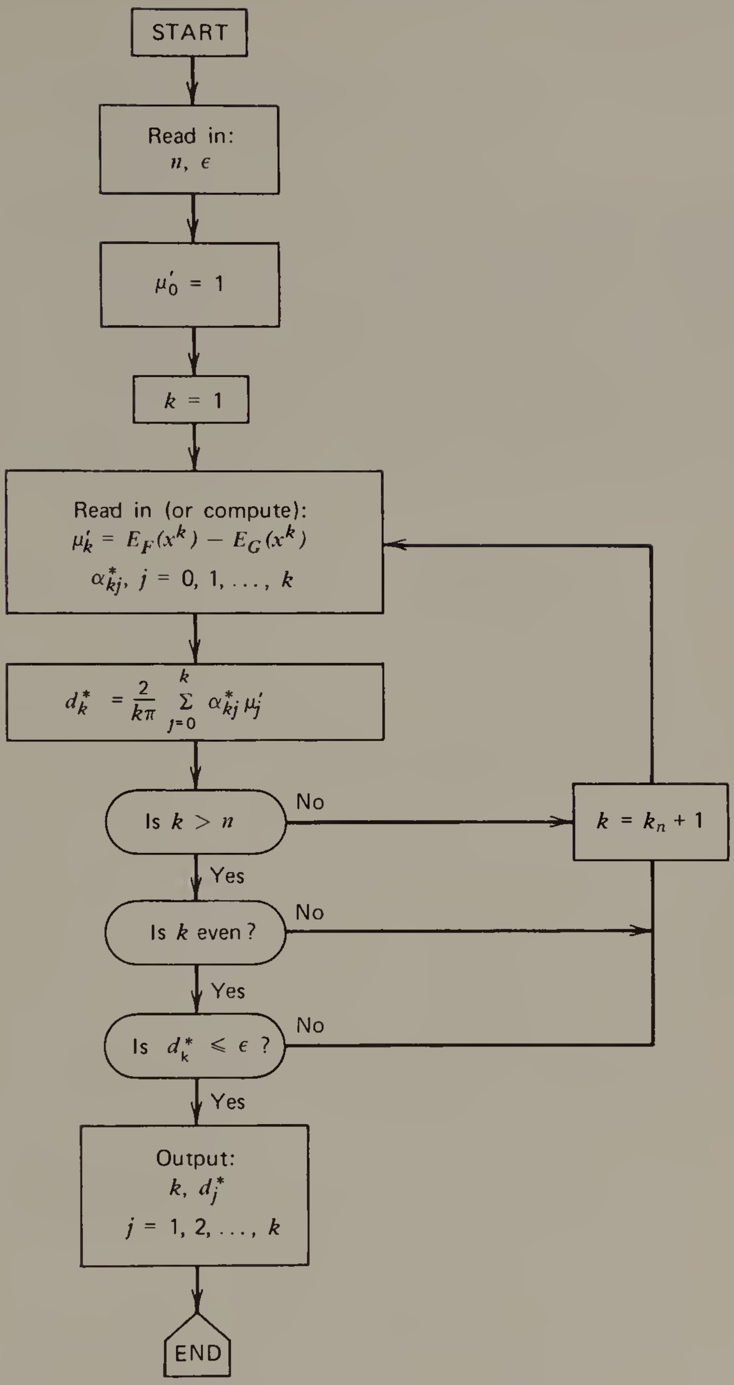


Fig. 8.12.1 Phase I: the calculation of the coefficients d_K^*.

iteration formula (where $y=G(x)$)

$$x_{k+1}=x_k-\frac{y_k-p}{\dfrac{y_k-y_{k-1}}{x_k-x_{k-1}}+\dfrac{y_k-y_{k-2}}{x_k-x_{k-2}}-\dfrac{y_{k-1}-y_{k-2}}{x_{k-1}-x_{k-2}}}, \tag{8.12.18}$$

whose computational efficiency is 1.84, as compared with 1.65 for the Newton-Raphson method and 1.62 for the secant method [293].

Having specified a value of the r.v. x, one can proceed to use the first generalized Fourier representation theorem to determine the error $S=\sum_{k=1}^{n} d_k^* \sin(k\theta)$ in evaluating the distribution function $F(x)$ by $G(x)$ at this value of x. If a computer is used, the evaluation of the truncated series S is logically carried out in three phases, as indicated by the flow charts of Figs. 8.12.1, 8.12.2, and 8.12.3 [293, pp. 51,54,55]. The flow charts are

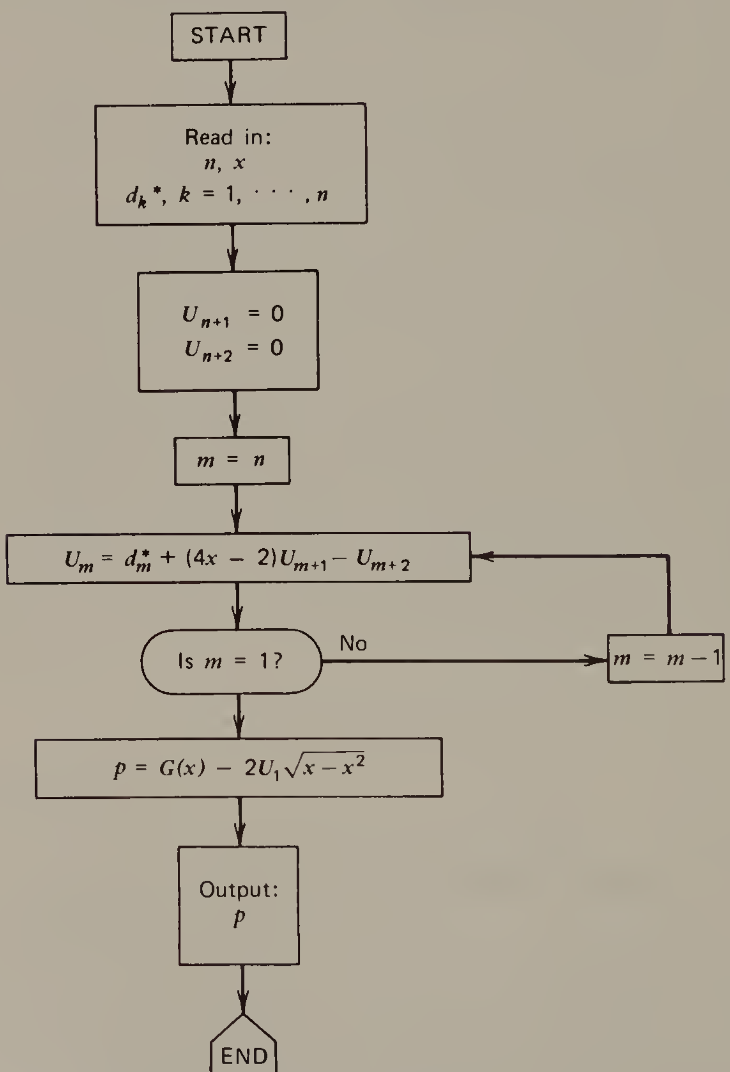


Fig. 8.12.2 Phase II: the evaluation of $p=F(x)$.

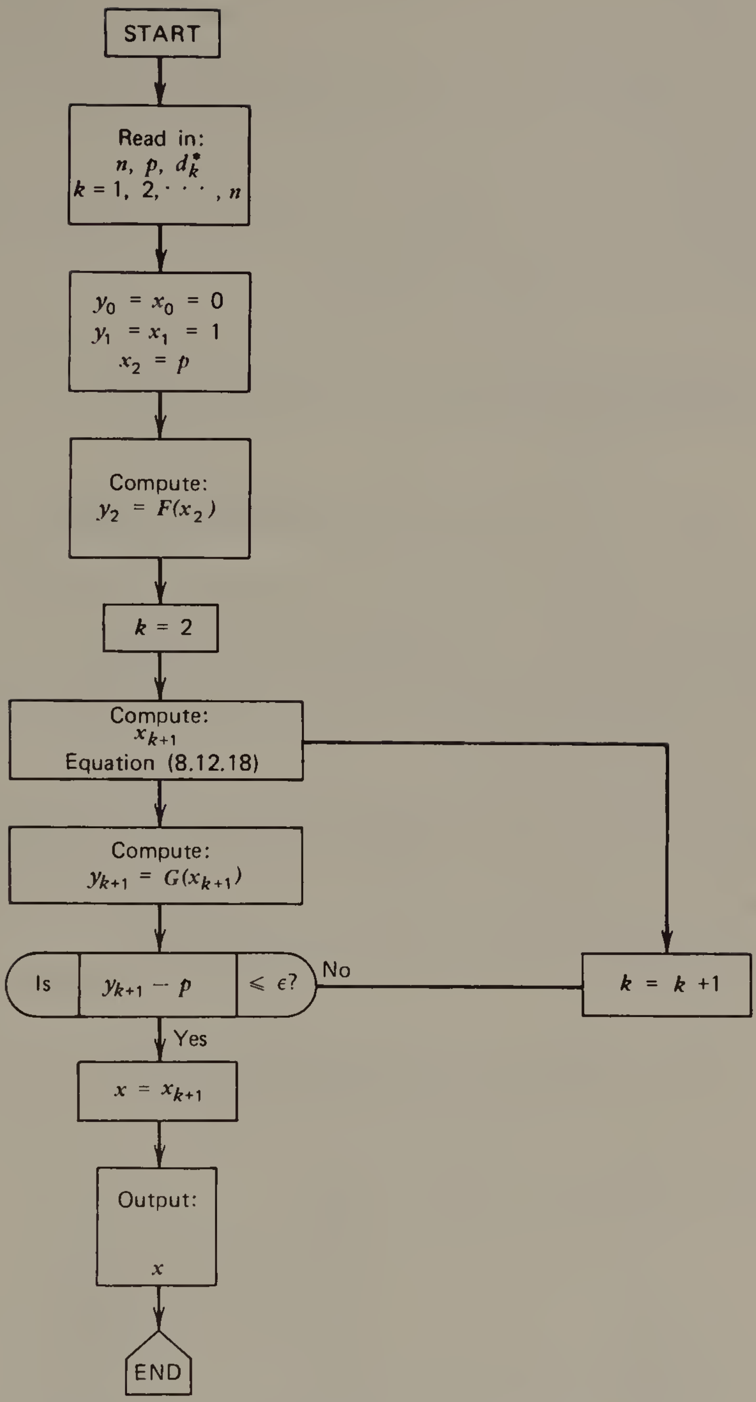


Fig. 8.12.3 Phase III: the computation of a percentage point x.

self-explanatory, except possibly for the two constants m, ε. When working with an infinite series, one must truncate the series after a finite number of terms. The constants m, ε are used to regulate the number of terms in the truncated series, m specifying the minimum number of terms and ε the maximum number.

The flow charts in Figs. 8.12.1 to 8.12.3 pertain to the cases in which the r.v. x ranges over the interval $(0, 1)$. If x ranges over the interval $(-1, 1)$, the following modification must be made:

Fig. 8.12.1: α_{kj} is the coefficient of x^j in $T_k(x)$ instead of $T_k^*(x)$.
Fig. 8.12.3: The equation $p = G(x) - U_1\sqrt{1-x^2}$ is used.

8.13 EXAMPLES OF ERROR EVALUATION

The following examples illustrate some of the aforementioned procedures for evaluating the accuracy of approximation distributions. In particular, two examples are given to illustrate the use of a Fourier series in evaluating the accuracy of approximation distributions when the variable ranges over the intervals $(0, 1)$ and $(0, \infty)$.

Example 8.13.1. Find the beta approximation to the cumulative distribution $H(Y)$ of the product $Y = \Pi_{j=1}^3 X_j$ of three beta r.v.'s having p.d.f.'s (8.8.1) with $a_1 = 5$, $b_1 = 2$, $a_2 = 6$, $b_2 = 2$, $a_3 = 6$, $b_3 = 3$, and evaluate the accuracy of the approximation to four decimal places.

The sth moments of $f_j(x_j)$, $j = 1, 2, 3$ are given by the Mellin transforms

$$M_s(f_j(x_j)) = \frac{\Gamma(a_j + b_j)\Gamma(a_j + s - 1)}{\Gamma(a_j + b_j + s - 1)\Gamma(a_j)}, \tag{8.13.1}$$

so that the sth moment of the p.d.f. $h(y)$ is

$$\mu'_{s-1} = M_s(h(y)) = \prod_{j=1}^{3} \frac{\Gamma(a_j + b_j)\Gamma(a_j + s - 1)}{\Gamma(a_j + b_j + s - 1)\Gamma(a_j)}. \tag{8.13.2}$$

The desired moments μ'_1 and μ'_2 are then

$$M_2(h(y)) = \frac{5}{14}$$

and

$$M_3(h(y)) = \frac{7}{48},$$

respectively, and

$$\begin{aligned}\mu_2 &= \mu_2' - \mu_1'^2 \\ &= \frac{7}{48} - \left(\frac{5}{14}\right)^2 \\ &= \frac{43}{2352}.\end{aligned}$$

Substituting these values for μ_1' and μ_2 in (8.8.2) one obtains the values

$$b = 7.430348 \quad \text{and} \quad a = 4.127971, \tag{8.13.3}$$

which gives the approximating beta p.d.f.

$$h(y) = \frac{1}{B(4.127971, 7.430348)} y^{3.127971}(1-y)^{6.430348}. \tag{8.13.4}$$

Integration of $h(y)$ yields the desired approximating beta c.d.f. (Table 8.13.1B).

To find the accuracy of the approximation, one uses Theorem 8.12.2 and calculates

$$\sum_{k=1}^{11} d_k^* \sin(k\theta), \tag{8.13.5}$$

Table 8.13.1A Data Required for Evaluating Accuracy of the Approximating Beta Distribution

$E_H(y)$	$E_G(y)$	K	d_k^*
1.000000	1.000000	0	0
0.357143	0.357143	1	0
0.145833	0.145833	2	0
0.065993	0.065913	3	0.000547
0.032397	0.032272	4	−0.000742
0.016991	0.016860	5	−0.000252
0.009414	0.009294	6	0.000966
0.005462	0.005361	7	−0.000380
0.003297	0.003215	8	−0.000346
0.002059	0.001993	9	0.000338
0.001325	0.001273	10	−0.000076
0.000875	0.000834	11	0.000005

Table 8.13.1B Accuracy of the Beta Approximation to Four Decimal Places

Y	Beta Distribution Function $H(y)$	Error	$F(y)$
0.	0.	0.	0.
0.05	0.0009	0.0002	0.0007
0.1	0.0125	0.0015	0.0110
0.2	0.1243	0.0012	0.1231
0.3	0.3636	−0.0021	0.3657
0.4	0.6376	−0.0023	0.6399
0.5	0.8451	−0.0001	0.8452
0.6	0.9541	0.0011	0.9530
0.7	0.9920	0.0006	0.9914
0.8	0.9994	0.0001	0.9993
0.9	1.0000	0.0000	1.0000
0.95	1.000000	−0.0000	1.0000
0.99	1.000000	0.0000	1.0000

with F replaced by H, and then applies Goertzel's algorithm. Using (8.13.1), (8.13.2), (8.12.15a, b), and (8.12.16), one first evaluates the Fourier coefficients d_k^* until reaching one that is zero when rounded to four decimal places. The values of b_k^*, c_k^*, and d_k^* are calculated, from which it is seen that $d_{10}^* = -0.000076$ and $d_{11}^* = 0.000005$ (Table 8.13.1A). Thus, d_{11}^* is the first Fourier coefficient that satisfies the accuracy requirement. That is, the first 11 moments are required to evaluate the accuracy of the approximate distribution $G(x)$ to four decimal places. Six decimal places (i.e., two guarding digits) are used throughout the calculations; the final answer is then correct to four decimal places. The results are summarized in Table 8.13.1B for selected values of the r.v. Y.

Example 8.13.2. Using the first five moments of the exact and the approximate distributions, evaluate the accuracy of the Laguerre approximation $G(w)$ to the cumulative distribution $F(w)$ of the sum $W = X_1 + X_2$ of the half-normal and exponential i.r.v.'s with p.d.f.'s

$$f_1(x_1) = \frac{2}{\sqrt{2\pi}} e^{-x_1^2/2}, \qquad 0 \leqslant x_1 < \infty$$

$$f_2(x_2) = e^{-x_2}, \qquad 0 \leqslant x_2 < \infty,$$

where the Laguerre p.d.f. is based on the first three moments of $g(w)$.

Since the range of the r.v. w is from 0 to ∞, one cannot directly apply the generalized Fourier representation theorem (Theorem 8.12.2) to evaluate the accuracy of the Laguerre approximation. It is first necessary to find the Laguerre p.d.f. $g(w)$ whose first three moments are identical with those of $f(w)$. Then the moments of the corresponding transformed density functions $p(y)$ and $f(y)$ must be determined, where $y = e^{-w}$. From a knowledge of these moments, the accuracy of the approximate distribution function may be evaluated.

First, note that the m.g.f.'s of $f_1(x_1)$ and $f_2(x_2)$ are, respectively,

$$m_{x_1}(t) = E[e^{tx_1}] = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{tx_1} e^{-x_1^2/2} dx_1$$

$$= \frac{2}{\sqrt{2\pi}} e^{t^2/2} \int_{-t}^\infty e^{-u^2/2} du = 2\Phi(t) e^{t^2/2},$$

where $u = x_1 - t$,

$$\Phi(t) = \int_{-t}^\infty e^{u^2/2} du, \tag{8.13.6}$$

and

$$M_{x_2}(t) = E[e^{tx_2}] = \int_0^\infty e^{tx_2} e^{-x_2} dx_2 = \frac{1}{1-t}.$$

Hence the m.g.f. for $f(w)$ is

$$M_w(t) = M_{x_1}(t) M_{x_2}(t) = \frac{2}{1-t} \Phi(t) e^{t^2/2},$$

so that

$$\mu'_{k:f} = \left[\frac{d^k}{dt^k} M_w(t) \right] \Bigg|_{t=0}. \tag{8.13.7}$$

It then follows immediately from (8.13.7) that the first three moments of the exact p.d.f. $f(w)$ are

$$\mu'_{2:f} = (4/\sqrt{2\pi}) + \Phi(6) + 4.595769122$$

$$\mu'_{1:f} = \frac{2}{\sqrt{2\pi}} + 1,$$

$$= 1.797884561$$

$$\mu'_{3:f} = \frac{16}{\sqrt{2\pi}} - 2\Phi(0)(-9), \qquad (\Phi(0) = 0.5)$$

$$= \frac{16}{\sqrt{2\pi}} + 9$$

$$= 15.38307649.$$

The Laguerre polynomial (p.d.f.) based on these three moments is

$$g(w) = \sum_{j=0}^{3} d_j L_j^{(r)}(w)\psi(w), \qquad 0 \leqslant w < \infty,\ r = 0, \tag{8.13.8}$$

where

$$d_n = \frac{(-1)^n}{n!(1+r)_n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{(1+r)_n}{(1+r)_k} \mu'_{k:w}, \qquad n = 0, 1, 2, 3, \tag{8.13.9}$$

$$\psi(w) = \frac{w^r e^{-w}}{\Gamma(r+1)}, \tag{8.13.10}$$

and

$$(n)_m = n(n+1)(n+2)\cdots(n+m-1),$$

$$L_0^{(0)}(w) = 1,$$

$$L_1^{(0)}(w) = w - 1,$$

$$L_2^{(0)}(w) = w^2 - 4w + 2,$$

$$L_3^{(0)}(w) = w^3 - 9w^2 + 18w - 6.$$

On evaluating (8.13.9) for $n=0,1,2,3$, one obtains

$$d_0=\mu'_{0:w}$$
$$=1$$
$$d_1=0.797884561$$
$$d_2=-0.1489422805$$
$$d_3=0.0106410136.$$

Also, from (8.13.10) with $r=0$, one has

$$\psi(w)=e^{-w}.$$

Using these results in (8.13.8), one obtains the desired Laguerre p.d.f.

$$g(w)=[0.0106410136w^3-0.2447114029w^2+1.585191928w$$
$$-0.1596152036]e^{-w}. \quad (8.13.11)$$

It remains to determine the first five moments of $p(y)$ and $h(y)$, denoted, respectively, by $\mu_{k:p}$ and $\mu_{k:h}$, $k=1,2,3,4,5$, where $y=e^{-w}$. Specifically, since $w=-\ln y$, the transformed Laguerre density function is

$$p(y)=\left[0.0106410136\left(\ln\frac{1}{y}\right)^3-0.2447114029\left(\ln\frac{1}{y}\right)^2\right.$$
$$\left.+1.585191928\left(\ln\frac{1}{y}\right)-0.1596152036\right],$$

and the kth moment is

$$\mu'_{k:p}=E[y^k]$$
$$=\int_0^1 y^k p(y)\,dy.$$

In particular,

$$\mu'_{1:p}=0.2593029096,$$
$$\mu'_{2:p}=0.1055888212,$$
$$\mu'_{3:p}=0.0517728620,$$
$$\mu'_{4:p}=0.0276714076,$$
$$\mu'_{5:p}=0.0152139928.$$

Similarly, $f(y)$ is transformed into the density function $h(y)$ whose kth moment is

$$\begin{aligned}\mu'_{k:f} &= E[y^k]\\ &= E[e^{-k(x_1+x_2)}]\\ &= \int_0^{\infty}\int_0^{\infty} e_2^{-k(x_1+x_2)} f_1(x_1) f_2(x_2)\,dx_1\,dx_2\\ &= \frac{2}{k+1} e^{k^2/2}[1-\Phi(k)],\end{aligned}$$

where $\Phi(\cdot)$ is defined by (8.13.6). In particular, for $k=1,2,3,4,5$, one has

$$\mu'_{1:h} = 0.2615782918,$$

$$\mu'_{2:h} = 0.1120680006,$$

$$\mu'_{3:h} = 0.0607569727,$$

$$\mu'_{4:h} = 0.0377642066,$$

$$\mu'_{5:h} = 0.0256441.$$

Finally, to evaluate the error of the approximation, one utilizes Goertzel's algorithm

$$U_k = d^*{}_k + (4y-2)U_{k+1} - U_{k+2}$$

to determine, recursively, the quantities U_k, $k=5,4,3,2,1$, from which one then determines the error $S=2U_1\sqrt{y-y^2}$ incurred by using the distribution function $P(y)$ to estimate $H(y)$ for a specified value of y. The d_k^* are, of course, calculated by way of (8.12.16), where μ'_{kh} and μ'_{kp} are the kth moments of $h(y)$ and $p(y)$, respectively.

The values of d_k^* are given in Table 8.13.2, which indicates that the first value of d_k^* that is zero when rounded to two decimal places is d_5^*. Hence the calculations will be carried out to four decimal places (i.e., two guarding digits will be used) and the final calculation S will be rounded to two decimal places. That is, the error values listed in Table 8.13.2 are correct to two decimal places. The error in the distribution function $P(y)$, denoted by $\varepsilon(P(y))$, is evaluated for $y=0.1,0.2,0.3,0.4,0.6,0.7,0.8$ and for those values of y corresponding to $w=E[w]+k\sigma_w$, $k=-1.5$, $-1.0,0,1,2,3$. Since $y=e^{-w}$ varies inversely with w, $P(y)$ corresponds to the complementary distribution function $1-H(w)$, where $w=-\ln y$, $0<y$

Table 8.13.2 Evaluation of the Accuracy of the Laguerre Approximation in Example 8.13.2 to Two Decimal Places

k	d_k^*	W	Y	$G(w)$	Error $\varepsilon(G(w))$	$F(y)=1-[(G(w)-\varepsilon(G(w))]$
1	0.0029	0.0464	0.9546	1.0060	0.01	0.00
2	0.0107	0.2231	0.8000	0.9986	0.02	0.02
3	0.0037	0.3567	0.7000	0.9707	0.02	0.05
4	−0.0070	0.5108	0.6000	0.9249	0.01	0.09
5	0.0007	0.6302	0.5325	0.8771	0.00	0.11
		0.9163	0.4000	0.7546	−0.01	0.24
		1.2040	0.3000	0.6282	−0.01	0.26
		1.6094	0.2000	0.4660	−0.01	0.52
		1.7979	0.1656	0.4005	−0.01	0.59
		2.3026	0.1000	0.2585	−0.00	0.74
		2.9655	0.0515	0.2065	0.00	0.79
		4.1332	0.0160	0.0386	0.00	0.96
		5.3008	0.0050	0.0085	0.00	0.99

$\leqslant 1$. That is,

$$\begin{aligned}\varepsilon(P(y)) &= \varepsilon(1-G(w))\\ &= \varepsilon(1)-\varepsilon(G(w))\\ &= -\varepsilon(G(w)).\end{aligned}$$

Thus, the error in $P(y)$ is numerically equal to that in $G(w)$ but opposite in sign. Finally, since $g(w)$ maps into $p(y)$ and $h(w)$ maps into $f(y)$, it follows that $1-(G(w)-\varepsilon G(w))$ is a valid approximation to the exact distribution function $F(y)$.

To evaluate the accuracy of an approximating distribution of doubly infinite range $(-\infty, \infty)$, one can employ the transformation

$$Y=\begin{cases} -e^{X}, & -\infty < X \leqslant 0\\ e^{-X}, & 0 \leqslant X < \infty\end{cases}$$

and proceed in the same manner. That is, the approximating distribution with p.d.f. $g(w)$ is obtained on the basis of a specified number of moments of the exact distribution with p.d.f. $f(w)$. These distributions are then transformed into new distributions with p.d.f.'s $p(y)$ and $f(y)$, and the required moments of these distributions are determined. From this point the procedure for the evaluation of the accuracy of the approximation $G(w)$ is the same as that for the approximating distribution of singly infinite range $(0, \infty)$.

CHAPTER 9

Distribution Problems in Statistics

The entire field of statistics is intimately connected with statistical inference. Furthermore, the basis for the solution of all problems in statistical inference, whether applied or theoretical, is a knowledge of the distributions of relevant statistics. Thus to reach a decision concerning the value of a population statistic, one must know the sampling distribution of that statistic. For example, in comparing two means from normal populations with unknown but identical variances, one needs to utilize the sampling distribution of the central t variable. If the variances of the normal populations are not identical, a different distribution (such as the Behrens-Fisher distribution) must be used. Although these sampling distributions often may be obtained by several methods (e.g., the m.g.f., the geometric method [215, 129, 104], and the use of joint distributions [201]), the natural and most direct method is that of integral transforms and convolutions. This method provides a unifying approach to the analysis of distribution problems in statistics for the following reasons.

1. It is a well-defined, exact method for the derivation of the distribution of sums, differences, products, quotients, and more generally, algebraic functions of r.v.'s. Moreover, a simple condition (the analyticity of the transforms in a specified strip) ensures the uniqueness of the resultant distributions.
2. It is an analytical method amenable to computer evaluation and analysis, yielding the exact sampling distribution of the relevant statistic. It also provides an important by-product to such a sampling distribution, namely, the moments of the distribution, often when the distribution is itself unknown (as, e.g., in the case of product distributions).
3. It provides m.g.f.'s not subject to the deficiencies of the classical m.g.f. (see Section 3.6). In particular, the characteristic function (Fourier

transform)

$$F_t(f(x)) = E[e^{itx}]$$

generates the moments (about the origin) of the p.d.f. $f(x)$, the mth moment μ'_m being

$$\mu'_m = \frac{1}{i^m m!} \frac{d^m}{dt^m} F_t(f(x))\Big|_{t=0}.$$

Similarly, the Laplace transform

$$L_r(f(x)) = E[e^{-rx}]$$

is also a bona fide m.g.f., since

$$\mu'_m = \frac{(-1)^m}{m!}\left[\frac{d^m}{dr^m} L_r(f(x))\right]\Bigg|_{r=0}.$$

Likewise, the Mellin transform

$$M_s(f(x)) = E[x^{s-1}]$$

is an m.g.f., since

$$\mu'_m = M_s(f(x))\big|_{s=m+1}.$$

4. It affords a direct and simple means of making the transition from a lower to a higher dimension in the analysis of distribution problems involving i.r.v.'s. This is because the basic structural form of the transform method is the same regardless of the dimension of the problem. Thus the derivation of the p.d.f. or distribution function of either the sum, difference, product, or quotient of n i.r.v.'s is achieved by inverting an integral transform consisting of the product of n integral transforms, regardless of the value of n. When the method of residues is applicable (as it usually is), this is a matter of evaluating and summing the residues at the various poles by means of evaluating mth order derivatives at poles of order $m \leqslant n$.

5. It provides a natural and simple means for decomposing the p.d.f. of a sum of i.r.v.'s, some or all of which are of finite range, into component p.d.f.'s, and automatically determines the subrange over which each component p.d.f. is valid (see Sections 3.2.1 and 3.3.1).

6. It provides a direct and efficient method for finding the p.d.f. of algebraic functions of i.r.v.'s. As Chapter 5 pointed out, the relevant transform for an algebraic function of i.r.v.'s can be obtained without the necessity of determining the p.d.f. of the auxiliary variables accumulated along the way. (For example, if $h(u)$ is the unknown p.d.f. of the r.v.

$$U = X_1 X_2 + \frac{1}{X_3},$$

one need not determine the p.d.f.'s of the auxiliary variables $X_1 X_2$ and $1/X_3$ to obtain the Fourier transform of the p.d.f. $h(u)$, from which both the p.d.f. and its moments may be determined.) This is accomplished, of course, by the use of Prasad's theorems (Chapter 5). If the inversion integral yielding the desired p.d.f. is difficult to obtain in exact form, satisfactory approximations to the p.d.f. usually may be obtained from the exact values of the moments by one of the approximating methods of Chapter 8.

7. It expedites the evaluation of the error incurred by approximating a desired p.d.f. with another density function by comparing the exact moments of the former (generated by the relevant integral transform) with the moments of the latter (also generated by a relevant transform). The error of the approximation can be evaluated to any specified accuracy by utilizing a (determinate) number of moments of the approximating and exact (but unknown) density function, as explained in Chapter 8.

8. It provides a simpler and more direct insight into some of the characteristics of a desired but unknown distribution.

(*a*) Examination of the Mellin transform of the density function of the quotient of two noncentral normal i.r.v.'s (Chapter 4) quickly reveals the nonexistence of the mth order moments of the distribution, $m > 0$.

(*b*) In determining whether the distribution of linear functions of dependent normal r.v.'s is normal (Section 9.5), one finds the characteristic function an extremely helpful (and perhaps necessary) tool. One reason for this is that an analytic characteristic function determines the corresponding p.d.f. uniquely. Moment-generating functions lose their usefulness in this problem because of the dependence among the normal variables involved.

9. It provides a convenient and powerful basis for classifying distributions, the potential of which is illustrated by the following three examples.

(*a*) H-function r.v.'s, their products, quotients, and powers, constitute a class or set of distributions that is closed under the operations of multiplication, division, and exponentiation—but not under addition or subtraction. That is, products, quotients, and powers (but in general not sums or differences[41]) of H-function r.v.'s are themselves H-function r.v.'s. The proof of this is readily accomplished in Chapter 6 by proving that the Mellin transforms of such distributions are the Mellin transforms of H-function variates.

(*b*) The set of bivariate r.v.'s X, Y for which the quotient of their coordinates follows some known distribution is characterized by the fact that the Mellin transform of the joint density function satisfies a specific set of conditions. Specific sets of conditions have been derived by various authors (see Section 9.14) for which the distribution of the quotient is (1) Cauchy and (2) Snedecor's F.

(*c*) If the product

$$U = X_1 X_2 \cdots X_n$$

of n i.r.v.'s has a beta distribution, the component r.v.'s $X_j, j = 1, 2, \ldots, n$ are not necessarily beta variables. Kotlarski [183] has determined necessary and sufficient conditions under which the product $Y = X_1 X_2 \cdots X_n$ of n positive i.r.v.'s X_j will have a beta distribution. The conditions pertain to the Mellin transforms of the density functions of the component r.v.'s (Section 9.13).

This chapter discusses the derivation of density and distribution functions of various r.v.'s and of functions of r.v.'s that are important in the field of statistics, from both the theoretical and applied standpoints. The important and useful role of integral transforms and the unifying thread they provide, are clearly evident.

9.1 THE SAMPLING DISTRIBUTION OF MEANS

One of the oldest problems in statistics is that of finding the distribution of the sample arithmetic mean. Two other types of mean, geometric and harmonic, are in use in elementary statistics. Although they are not as important as the arithmetic mean, they do have considerable application to the use of index numbers in the field of economics. This section discusses the use of integral transforms in deriving the sampling distribution of these three means.

[41]Sums of H-function r.v.'s (e.g., gamma r.v.'s) that possess the reproductive property with respect to addition will, of course, be H-function r.v.'s, but sums of H-function r.v.'s in general will not be H-function r.v.'s.

9.1.1 Arithmetic Mean

The derivation of the sampling distribution of the arithmetic mean is best accomplished by utilizing *the characteristic function, the complex Fourier (bilaterial Laplace), or the ordinary Laplace transform*, whichever is most appropriate for the problem at hand. In this connection, the following theorem is useful.

Theorem 9.1.1. The characteristic function (Fourier transform) of the p.d.f. $g(\bar{x})$ of the mean $\bar{x}=(1/n)\Sigma_{j=1}^{n}x_j$ of continuous i.r.v.'s x_j with p.d.f. $f_j(x_j)$, is

$$F_t(g(\bar{x}))=\prod_{j=1}^{n} F_{t/n}(f_j(x_j)). \tag{9.1.1}$$

PROOF. By definition,

$$\begin{aligned}
F_t(g(\bar{x}))&=E[e^{it\bar{x}}]\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left[\frac{it(x_1+x_2+\cdots+x)}{n}\right]f(x_1)f(x_2)f(x_n)\prod_{1}^{n}dx_j\\
&=\prod_{j=1}^{n}\int_{-\infty}^{\infty}e^{itx_j/n}f(x_j)\,dx_j\\
&=\prod_{j=1}^{n}F_{t/n}(f(x_j)).
\end{aligned}\tag{9.1.2}$$

If the r.v.'s x_j are identically distributed with p.d.f. $f(x)$, (9.1.2) becomes

$$F_t(g(\bar{x}))=[F_{t/n}(f(x_j))]^n. \tag{9.1.3}$$

In either case, the desired p.d.f. $g(\bar{x})$ is obtained by evaluating the Fourier inversion integral

$$g(\bar{x})=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-it\bar{x}}F_t(g(\bar{x}))\,dt. \tag{9.1.4}$$

In the same manner, if the i.r.v.'s are of finite or singly infinite range, it is readily established that

$$g(\bar{x})=\frac{1}{2\pi i}\int_{c-i_{\infty}}^{c+i_{\infty}}e^{r\bar{x}}\prod_{j=1}^{n}L_{r/n}(f_j(x_j))\,dr, \tag{9.1.5}$$

where

$$L_r(f_j(x_j)) = \int_0^\infty e^{-rx_j} f_j(x_j)\,dx_j. \tag{9.1.6}$$

The partitioning procedures encountered in deriving the p.d.f. $h(w)$ of sums $W=\Sigma_1^n X_j$ of i.r.v.'s are also inherent in the derivation of the p.d.f. $g(\bar{x})$ of the mean, particularly when the mean involves i.r.v.'s of finite range or a mixture of i.r.v.'s of both finite and infinite ranges. Of course the p.d.f. of $\bar{x}$ is directly obtainable from that of W by utilizing the transformation $\bar{x}=w/n$. Specifically,

$$g(\bar{x}) = nf(n\bar{x}). \tag{9.1.7}$$

Again, as was noted in the derivation of the distribution of sums of i.r.v.'s, the problem is considerably simpler when the i.r.v.'s are identically distributed than when they are not. The following examples are illustrative.

Example 9.1.1. Find the p.d.f. $g(\bar{x})$ of the arithmetic mean $\bar{X}=(1/n)\Sigma_1^n X_j$ of n identically distributed normal i.r.v.'s with p.d.f.

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/\sigma^2}, \qquad -\infty < x < \infty.$$

From (9.1.3),

$$F_t(g(\bar{x})) = \left[F_{t/n}(f(x))\right]^n,$$

where

$$F_t(f(x)) = \frac{1}{\sigma\sqrt{2\pi}} \int_\infty^\infty e^{itx} e^{-x^2/(2\sigma^2)}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2\sigma^4/(2\sigma^2)} \int_{-\infty}^\infty \exp\left[-\frac{(x-it\sigma^2)^2}{2\sigma^2}\right] dx$$

$$= e^{-t^2\sigma^2/2}.$$

Then

$$F_t(g(\bar{x})) = \exp\left[\left(-\frac{t^2\sigma^2}{(2n^2)}\right)^n\right]$$

$$= e^{-t^2\sigma^2/(2n)}$$

and

$$\begin{aligned}
g(\bar{x}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\bar{x}} F_t(g(\bar{x}))\,dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\bar{x}} e^{-t^2\sigma^2/(2n)}\,dt \\
&= \frac{1}{2\pi} e^{-\bar{x}^2/(2\sigma^2/n)} \int_{-\infty}^{\infty} \exp\left[-\frac{\sigma^2}{2n}\left(t+\frac{in\bar{x}}{\sigma^2}\right)^2\right] dt \\
&= \frac{1}{\sqrt{\frac{2\pi}{n}}\;\sigma} e^{-\bar{x}^2/(2\sigma^2/n)}. \qquad (9.1.8)
\end{aligned}$$

Thus, as is well known, $g(\bar{x})$ is a normal distribution with mean 0 and variance σ^2/n. The term $\sigma/\sqrt{n}$ is known as the standard error of the mean.

Example 9.1.2. Find the p.d.f. $g(\bar{x})$of the arithmetic mean $\bar{X}=\frac{1}{3}\sum_{i=1}^{3}X_i$ of three exponential i.r.v.'s with p.d.f.

$$\begin{aligned}
f_1(x_1) &= a_1 e^{-a_1x_1}, & 0 \leqslant x_1 < \infty, \\
f_2(x_2) &= a_2 e^{-a_2x_2}, & 0 \leqslant x_2 < \infty, \\
f_3(x_3) &= a_3 e^{-a_3x_3}, & 0 \leqslant x_3 < \infty.
\end{aligned}$$

To find $g(\bar{x})$, note first that

$$\begin{aligned}
F_r(f_i(x_i)) &= a_i\int_0^{\infty} e^{-rx_i} e^{-a_ix_i}\,dx_i \\
&= \frac{a_i}{a_i + r/3} \qquad i = 1,2,3,
\end{aligned}$$

so that

$$F_r(g(\bar{x})) = \frac{3^3 a_1a_2a_3}{(r+3a_1)(r+3a_2)(r+3a_3)}.$$

Then

$$g(\bar{x})=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{27a_1a_2a_3e^{r\bar{x}}}{(r+3a_1)(r+3a_2)(r+3a_3)}\,dr$$

$$=3a_1a_2a_3\left[\frac{e^{-3a_1\bar{x}}}{(a_2-a_1)(a_3-a_1)}+\frac{e^{-3a_2\bar{x}}}{(a_1-a_2)(a_3-a_2)}\right.$$

$$\left.+\frac{e^{-3a_3\bar{x}}}{(a_1-a_3)(a_2-a_3)}\right],\qquad a_i\neq a_j. \tag{9.1.9}$$

9.1.2 Geometric Mean

Camp [45] has derived the distribution of the geometric mean of a sample of n identically distributed nonnegative i.r.v.'s by applying the geometric method to the logarithm of the geometric mean. His examples were limited to distributions that are "reproductive" with respect to multiplication; that is, the product variable has the same type of distribution as does the original variable. In contrast, the Mellin transform method derives the distribution of the geometric mean directly, without recourse to the logarithm of the geometric mean. It is equally applicable, regardless of whether the (nonnegative) i.r.v.'s are identically distributed or reproductive with respect to multiplication.

The distribution of the geometric mean of n nonnegative i.r.v.'s follows directly from the distribution of products. Thus if $h(y)$ is the product $Y=\prod_{j=1}^{n}X_j$ of n i.r.v.'s X_j with p.d.f.'s $f_j(x_j), j=1,2,\ldots,n$, then by definition the corresponding geometric mean is $U=Y^{1/n}$ and, therefore, has p.d.f.

$$g(u)=nu^{n-1}h(u^n). \tag{9.1.10}$$

On the other hand, $g(u)$ can be obtained directly through the use of the Mellin transform. In particular,

$$M_s(g(u))=M_{(s+n-1)/n}(h(y)), \tag{9.1.11}$$

which follows directly from utilizing properties 2 and 10 (Section 2.8.2) relative to $M_s(h(y))$. Then $g(u)$ is given by the inversion integral

$$g(u)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}u^{-s}M_{(s+n-1)/n}(h(y))\,ds, \tag{9.1.12}$$

where the Bromwich path $(c-i^{\infty}, c+i^{\infty})$ is a straight line in the strip or half-plane in which the Mellin transform $M_s(g(u))$ is analytic. Evaluation of the inversion integral can usually be accomplished by the residue theorem.

An interesting application of the geometric mean occurs in connection with the distribution of the statistic

$$\lambda_{H_1} = L_1^{N/2}$$

$$= \prod_{j=1}^{k} \left(\frac{s_j^2}{s^2} \right)^{(n_j)/2},$$

where n_j is the number of elements in the jth sample, s_j^2 is the variance of the jth sample, and

$$s^2 = \sum_j \frac{n_j}{N} s_j^2,$$

$$N = \sum_{j=1}^{k} n_j.$$

The statistic λ_{H_1} is an appropriate criterion for testing the statistical hypothesis that k random samples of sizes n_j, $j=1,2,\ldots,k$ came from k populations with standard deviations σ_j, $j=1,2,\ldots,k$. When written in the form of $L_1 = \lambda_{H_1}^{2/N}$, where N isthe number of observations in the pooled samples, the criterion becomes the ratio of the weighted geometric to the weighted arithmetic mean of the k sample variances [408, editorial note, p. 124]. In 1937 Wilks derived the first two moments of L_1^{-1}, but stated [408] that "the higher moments of L_1^{-1} become more and more complicated so that there is little hope of finding a workable form of the exact distribution of L_1^{-1}." Actually, the exact distribution of L_1 is an H-function, as we now show, hence can be obtained by using Mellin transforms. The distribution is in the form of an infinite series and can be evaluated to any required degree of accuracy with the use of an electronic computer.

Note that one can, without loss of generality, utilize the standardized form of L_1, namely,

$$\mathfrak{L}_1 = \frac{Y^{1/N}}{V},$$

where

$$Y = \frac{\Pi_{j=1}^{k} n_j s_j^2}{\sigma_j^2}$$

$$\sum_{j=1}^{k} n_j = N$$

is the product of the variances s_j^2 of samples of size n_j drawn at random from k normal populations $N(\mu_j, \sigma_j^2)$, and

$$V = \frac{1}{N} \frac{\Sigma_{j=1}^{k} n_j s_j^2}{\sigma_j^2}.$$

Denote the p.d.f.'s of $\chi_j^2 = n_j s_j^2 / \sigma_j^2$, V, and $\mathfrak{L}_1$, respectively, by $f_j(\chi_j^2)$, $g(v)$, and $h(l_1)$, and note that

$$f_j(\chi_j^2) = ce^{-\chi_j^2/2}(\chi_j^2)^{(n_j-3)/2}, \qquad \chi_j^2 > 0,$$

where

$$c = \frac{1}{2^{(n_j-1)/2}\Gamma((n_j-1)/2)}.$$

That is, $f_j(\chi_j^2)$ is a chi-square distribution with $n_j - 1$degrees of freedom; but more important, it is also an H-function variate. Similarly, the sum of k chi-square variables each with $n_j - 1$ degrees of freedom is also a chi-square variable with $\Sigma_{j=1}^{k}(n_j - 1)$ degrees of freedom. Thus V is an H-function r.v., as discussed in connection with the half-Student distribution in Section 6.3.4. Also, Y is an H-function r.v., and since a rational power of an H-function r.v. is an H-function r.v., $Y^{1/N}$ is an H-function r.v. Finally, since the quotient of two H-function r.v.'s is an H-function r.v., it follows that

$$\mathfrak{L}_1 = \frac{Y^{1/N}}{V}$$

is an H-function r.v. By a straightforward application of Theorems 6.4.1, 6.4.2, and 6.4.3, one can show (Exercise 9.30) that $\mathfrak{L}_1$ is an H-function r.v.

with p.d.f.

$$h(l_1)=\frac{2^{\frac{N-K}{N}}}{N\prod_{j=1}^{K}\Gamma\left(\frac{n_j-1}{2}\right)\Gamma\left(\frac{N-K}{2}\right)}$$

$$\times\mathbf{H}_{1,K}^{K,1}\left[\frac{2^{\frac{N-K}{N}}}{N},\frac{l_1}{N}\middle|\begin{matrix}\left(-\left[\frac{N-K}{2}\right],1\right)\\ \left(\frac{n_1-1}{2}-\frac{1}{N},\frac{1}{N}\right),\ldots,\left(\frac{n_K-1}{2}-\frac{1}{N},\frac{1}{N}\right)\end{matrix}\right].$$

(9.1.13)

The analytical (series) form of the p.d.f. $h(l_1)$ can be obtained by using the model developed in Chapter 7, which enables one to evaluate any H-function inversion integral whose parameters are known. Likewise, the corresponding c.d.f. $H(l_1)$ can be obtained in analytical (series) form by a companion model developed in Chapter 7, which again requires only a knowledge of the parameters in the H-function $H(l_1)$ as given by (9.1.13).

It bears stating that if one sets $\sigma_j^2=\sigma^2$, $j=1,2,\ldots,k$, the distribution function of $\mathcal{L}_1$ can be used to test the null hypothesis that k samples came from k normal populations with identical variances. In fact, it was Neyman and Pearson who, in 1931, discussed in some detail a method for testing this null hypothesis based on the statistic L_1 [271]. Later, Mood [259] examined the behavior of L_1 for large values of k. In 1932 Wilks [405] was concerned with devising a statistic for use in testing this null hypothesis in connection with a multivariate analysis of variance test. As a result, he proposed what is now variously called Wilks's statistic, Wilks's Λ criterion, and Wilks's likelihood ratio criterion. (Wilks himself designated the criterion by W in his 1932 *Biometrika* paper.) This criterion provided a multivariate generalization of what is today called the analysis of variance F test. Specifically, if the "among" sum of squares in analysis of variance is generalized to a $p\times p$ matrix A of sums of squares and products and the "within" sum of squares is likewise generalized to a $p\times p$ matrix B, the criterion is defined as the ratio of determinants. In particular,

$$\Lambda=\frac{\det(B)}{\det(A+B)}.$$

Wilks derived in integral form the distribution of Λ that holds when the null hypothesis is true.

More recently, a number of authors have derived the null density and distribution functions of Λ for various special cases, in particular, the distributions for the special cases p or $q \leqslant 2$, and p and $q \leqslant 4$ (see Anderson [14, Chapter 8]). Also, Wilks's criterion Λ, as defined above, is a special case of the statistic analyzed by Brookner and Wald [41] when the number of groups is two. Later Rao [302] obtained the first three terms of a series more rapidly convergent than the infinite series expansion derived by Brookner and Wald for the exact cumulative distribution. Then in 1966 Schatzoff [320] proved that under the null hypothesis, both the density and distribution functions of Λ have exact closed form representations when p or q is even. This result followed from their observation that Λ was distributed as a product of p beta i.r.v.'s and that consequently, $-\log\Lambda$ was distributed as a sum of p i.r.v.'s. Schatzoff used the process of successive convolution to obtain a recursive algorithm for determining the density and distribution functions on a digital computer. He also constructed tables of correction factors for converting chi-square percentiles to exact percentiles of a logarithmic function of Λ.

Thus although considerable work has been done relative to the derivation of the null density and distribution functions of Λ, these distributions have not been derived for the general case. Furthermore, relatively little progress has been made thus far in the way of deriving the exact density and distribution functions of Λ when the null hypothesis is not true.

Example 9.1.2. Find the sampling distribution of the geometric mean of n identically distributed uniform i.r.v.'s

$$f(x_i)=1, \qquad 0 \leqslant x_i \leqslant 1, \qquad i=1,2,\ldots,n.$$

In Chapter 4, the Mellin transform of $Y=\Pi_1^n X_j$ was found to be

$$M_s(h(y))=\frac{1}{s^n}.$$

Hence from (9.1.11), it follows that

$$M_s(g(u))=\frac{1}{\left[(s+n-1)/n\right]^n}. \tag{9.1.14}$$

Inversion of the Mellin transform (9.1.14) then yields

$$g(v)=\frac{n}{(n-1)!}v^{n-1}\left(\ln\frac{1}{v^n}\right)^{n-1}, \qquad 0 \leqslant v \leqslant 1. \tag{9.1.15}$$

Standard Error of the Geometric Mean

Norris [273] has shown that the standard error of the geometric mean of a random sample of n positive i.r.v.'s $X_1, X_2, \ldots, X_n$ is

$$\sigma_G = \frac{\theta_1 \sigma_{\ln x}}{\sqrt{n}},$$

where θ_1 is the population geometric mean of the variates and $\sigma_{\ln x}$ is the standard deviation of the logarithms in the population as given by

$$\sigma_{\ln x} = \left[E\left\{ \left[\ln x - E(\ln x) \right]^2 \right\} \right]^{1/2}.$$

The estimate of the standard deviation of the geometric mean is

$$s_G = G \frac{s_{\ln x}}{\sqrt{n-1}},$$

where G is the sample geometric mean; that is, G is also the estimate of θ_1, and $s_{\ln x}$ the estimate of $\sigma_{\ln x}$, with $n-1$ degrees of freedom. The derivation of the formula for s_G can be accomplished by applying the central limit theorem, which is not done here. The interested reader is referred to the paper by Norris.

It perhaps bears stating at this point that for a positive r.v. X,

$$E\sqrt{X} < \sqrt{E(X)},$$

as Murthy and Pillai have shown [266]; and more generally, as proved by Sclove et al. [324], that

$$E[X^s] \geqslant (E[X])^s$$

for all real s not in $(0, 1)$.

9.1.3 Harmonic Mean

The harmonic mean of n i.r.v.'s X_j with p.d.f.'s $f_j(x_j)$ is defined to be the reciprocal of the arithmetic mean of the reciprocals $1/X_j$:

$$U = \frac{1}{\frac{1}{n} \sum_{j=1}^{n} \frac{1}{X_j}}. \tag{9.1.16}$$

Thus the p.d.f. $k(u)$ of the r.v. U can be found in two steps. The first step is to utilize the Fourier (or Laplace) transforms (when they exist) of the p.d.f.'s $h_j(w_j)$ of the reciprocal r.v.'s $W_j = 1/X_j$ to find the p.d.f. $h(w)$ of $W = (1/n)\Sigma_{j=1}^{n}(1/X_j)$. One can then find the p.d.f. $k(u)$ from $h(w)$ by means of the transformation $U = 1/W$, which is the second step.

Thus if X_j is defined on the interval $(0,\infty)$ and has p.d.f. $f_j(x_j)$, and the reciprocal variable $W_j = 1/X_j$ has p.d.f. $g(w_j)$, then

$$h(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{wr} \prod_{j=1}^{n} L_{r/n}(h(w))\,dr, \tag{9.1.17}$$

where

$$L_r(h(w)) = \prod_{j=1}^{n} L_r(g_j(w_j)), \qquad 0 < w < \infty, \tag{9.1.18}$$

$$L_r(g(w_j)) = \int_0^\infty e^{-w_j r} g_j(w_j)\,dw_j \tag{9.1.19}$$

and the Bromwich path $(c - i\infty, c + i\infty)$ is any vertical line parallel to the imaginary axis and lying within the strip of analyticity of $L_r(h(w))$. Finally, letting $U = 1/W$, one obtains

$$k(u) = h\left(\frac{1}{u}\right)\left|\frac{dw}{du}\right|, \tag{9.1.20}$$

which is the desired p.d.f. of the harmonic mean of X_j, $j = 1,2,\ldots,n$.

If the r.v.'s X_j are restricted to the range (a_j, b_j), the integral (9.1.19) is evaluated between finite limits instead of zero and infinity. The limits of integration of the integral (9.1.19) are changed accordingly, and the relevant Bromwich path is chosen for the inversion integral (9.1.17).

It is well known and rather easily proved [110, 266] that if X is an r.v. that is positive and not identically equal to a constant,

$$E\left[\frac{1}{X}\right] > \frac{1}{E[X]}.$$

More generally, for such an r.v.

$$E[X^{-1}] \geqslant \frac{E[X^{a-1}]}{E[X^a]}, \qquad a > 0$$

as Gurland has shown [139].

Standard Error of the Harmonic Mean

The standard error of the harmonic mean of a random sample of n positive i.r.v.'s $X_1, X_2, \ldots, X_n$ is [273]

$$\sigma_H = \frac{\theta_2^2 \sigma_{1/X}}{\sqrt{n}},$$

where the population harmonic mean of the variates is

$$\theta_2 = \frac{1}{\alpha} = \left[E\left(\frac{1}{X}\right)\right]^{-1},$$

so that the standard deviation of $1/X$ in the population is

$$\sigma_{1/X} = \left[E\left\{\left[1 - E\left(\frac{1}{X}\right)\right]^2\right\}\right]^{1/2}.$$

The estimate of the standard error of the harmonic mean, with $n-1$ degrees of freedom, is

$$s_H = \frac{1}{a^2}\frac{s_{1/x}}{\sqrt{n-1}},$$

$a = 1/H = (1/n)\Sigma_1^n(1/X_i)$ being the estimate of α, and s_{1/X_i} the standard deviation of the reciprocals of the sample items.

As in the case of the geometric mean, the formula above for s_H is stated without proof. The interested reader is again referred to the Norris paper for the proof.

9.2 THE t, F, AND CHI–SQUARE DISTRIBUTIONS

Although the central and noncentral t, F, and chi-square distributions may be obtained in various ways, the most direct and satisfactory method is that of integral transforms, as is evident in the following sections.

9.2.1 The Student-Fisher t Distribution

Consider the ratio

$$T = \frac{U}{\sqrt{(V/m)}}$$

of the standard normal r.v. U with p.d.f. $f(u)$ and a chi-square r.v. V with m degrees of freedom and p.d.f. $g(v)$, where U and V are independent. Let $h(t)$ denote the p.d.f. of t. It is now shown that

$$M_s(h^+(t)) = M_s(f^+(u))M_{-s+2}(g(v))$$

$$= \frac{m^{\frac{s}{2}}2^{\frac{s-1}{2}}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{-s+m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}, \qquad 0<\mathrm{Re}(s)<m+1,$$

which, on inversion, yields

$$h(t) = \frac{\Gamma\left(\frac{m+1}{2}\right)\left(1+\frac{t^2}{m}\right)^{-\frac{(m+1)}{2}}}{\sqrt{(\pi m)}\,\Gamma\left(\frac{m}{2}\right)}.$$

To find the Mellin transform of $h(t)$, one needs first to find the Mellin transform of the product of $f(u)$ and $g(1/(\chi/\sqrt{m}))$, where $\chi = \sqrt{V}$ is the chi r.v. with m degrees of freedom and p.d.f. $k(\chi)$. Now,

$$M_s(f^+(u)) = \frac{2^{(s-3)/2}}{\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right), \qquad \mathrm{Re}(s)>0$$

and (Table D.2, formula 6).

$$M_s(k(\chi)) = \frac{2^{(s-1)/2}}{\Gamma(m/2)}\Gamma\left(\frac{s+m-1}{2}\right), \qquad \mathrm{Re}(s)>-(m-1).$$

Then

$$M_s\left(g\left(\frac{1}{\chi}\right)\right) = \frac{2^{(-s+1)/2}}{\Gamma(m/2)}\Gamma\left(\frac{-s+m+1}{2}\right), \qquad \mathrm{Re}(s)>-(m+1),$$

and from the scaling property (Section 2.8.2),

$$M_s\left(g\left(\frac{1}{\chi/\sqrt{m}}\right)\right) = \frac{m^{s/2}2^{(-s+1)/2}}{\Gamma(m/2)}\Gamma\left(\frac{-s+m+1}{2}\right), \qquad \mathrm{Re}(s)>-(m+1).$$

Hence

$$M_s\left(h^+\left(\frac{t}{\sqrt{m}}\right)\right)=M_s(f^+(u))M_s\left(g\frac{1}{(\chi/\sqrt{m}\,)}\right)$$

$$=\frac{m^{s/2}\Gamma\left(\frac{(-s+m+1)}{2}\right)\Gamma\left(\frac{s}{2}\right)}{2\sqrt{\pi}\,\Gamma\left(\frac{m}{2}\right)}. \qquad (9.2.1)$$

Now it is well known [95 (19), p. 310] that

$$M_s((1+\alpha t)^{-r})=\alpha^{-s}B(s,r-s), \qquad 0<\mathrm{Re}(s)<\mathrm{Re}(r),\ \alpha>0,\ 0\leqslant t<\infty$$

$$=\frac{\alpha^{-s}\Gamma(s)\Gamma(r-s)}{\Gamma(r)}.$$

From the exponentiation property (Section 2.8.2), it follows that

$$M_s((1+\alpha t^2)^{-r})=\frac{\alpha^{-s/2}\Gamma(s/2)\Gamma(r-s/2)}{\Gamma(r)}, \qquad (9.2.1a)$$

with the same constraints as above on s, r, α, and t. If $\alpha=1/m$ and $r=(m+1)/2$, (9.2.1a) becomes

$$M_s\left(\left(1+\frac{t^2}{m}\right)^{-(m+1)/2}\right)=\frac{m^{s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{-s+m+1}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)}. \qquad (9.2.1b)$$

Thus (9.2.1) is expressible in the form

$$M_s\left(h^+\left(\frac{t}{\sqrt{m}}\right)\right)=\frac{\Gamma\left(\frac{m+1}{2}\right)}{2\sqrt{\pi}\,\Gamma\left(\frac{m}{2}\right)}M_s\left(1+\frac{t^2}{m}\right)^{-(m+1)/2}.$$

Or equivalently,

$$M_s(h^+(t))=\frac{\Gamma\left(\frac{m+1}{2}\right)}{2\sqrt{\pi m}\,\Gamma\left(\frac{m}{2}\right)}M_s\left(1+\frac{t^2}{m}\right)^{-(m+1)/2}, \qquad 0\leqslant t<\infty.$$

Hence $h^+(t)$ is obtained by evaluating the Mellin inversion integral

$$h^+(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-s/2}M_s(h^+(t))d\left(\frac{s}{2}\right), \qquad 0<c<r, \qquad 0\leqslant t<\infty,$$

which, in light of (9.2.1a) becomes

$$h^+(t)=\frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{\pi m}\,\Gamma\left(\frac{m}{2}\right)\left(1+\frac{t^2}{m}\right)\frac{m+1}{2}}, \qquad 0\leqslant t<\infty.$$

Furthermore, since the standardized normal distribution is symmetrical about the origin, it follows immediately that

$$h^-(t)=h^+(t)=\frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{\pi m}\,\Gamma\left(\frac{m}{2}\right)\left(1+\frac{t^2}{m}\right)^{(m+1)/2}}, \qquad -\infty<t\leqslant 0.$$

Consequently,

$$h(t)=\frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{\pi m}\,\Gamma\left(\frac{m}{2}\right)\left(1+\frac{t^2}{m}\right)^{(m+1)/2}}, \qquad -\infty<t<\infty,$$

which is the well-known Student-Fisher t distribution.

9.2.2* The Noncentral t Distribution

Let $f_1(U)$ be the p.d.f. of the normal r.v. with mean δ and variance 1, and $f_2(V)$ the p.d.f. of a chi-square r.v. V with m degrees of freedom, where U and V are stochastically independent. Then the quotient

$$T=\frac{U}{\sqrt{V/m}} \tag{9.2.2}$$

is said to have a noncentral t distribution, denoted by $h(t)$, with m degrees of freedom and noncentrality parameter δ. This noncentral t distribution is of considerable importance in hypothesis testing, since it yields the power of the widely used t tests [179, p. 255]. The classical t distribution is, of course, the special case for which $\delta=0$.

To obtain the p.d.f. $h(t)$ of the noncentral t distribution by means of Mellin transforms—which would seem to be the natural approach to take—one can evaluate the Mellin inversion integral

$$h(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(t)^{-s}M_s(h(t))\,ds$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(t)^{-s}\frac{e^{-\delta^2/4}\Gamma(s)D_{-s}(-\delta)\Gamma\left(\frac{m+1-s}{2}\right)}{\sqrt{2\pi}\sqrt{m}\,\Gamma\left(\frac{m}{2}\right)}\,ds,\qquad \operatorname{Re}(s)>0,$$

$$(9.2.3)$$

where $D_{-s}(-\delta)$ denotes the parabolic cylinder function (defined in Appendix D.1). Likewise, to obtain the distribution function $H(t)$—which is of primary interest—one would evaluate the corresponding inversion integral

$$H(t)=1-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(t)^{-s}\frac{M_{s+1}(f(t))}{s}\,ds. \tag{9.2.4}$$

Clearly, the evaluation of the inversion integrals (9.2.2) and (9.2.3) is a rather arduous task. The difficulty stems from the complexity of the parabolic cylinder function, which is not easily generated on a computer, even when it is expressed in terms of the more manageable gamma and hypergeometric functions. Expressions for the p.d.f. and c.d.f. that are equivalent to (9.2.3) and (9.2.4), respectively, can be obtained in a simpler manner by expressing the Mellin transform of $f(U)$ in the form

$$\begin{aligned} M_s(f_1(U)) &= M_s(f_1^-(-U)), \qquad -\infty\leqslant U<0\\ &= M_s(f_1^+(U)), \qquad 0<U\leqslant\infty, \end{aligned}$$

where

$$M_s(f_1^+(U))=\frac{1}{\sqrt{2\pi}}2^{(s/2)-1}e^{-\delta^2/2}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\delta}{\sqrt{2}}\right)^j\Gamma\left(\frac{s+j}{2}\right),$$

$$M_s(f_1^-(U))=\frac{1}{\sqrt{2\pi}}2^{(s/2)-1}e^{-\delta^2/2}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{-\delta}{\sqrt{2}}\right)^j\Gamma\left(\frac{s+j}{2}\right).$$

(See Section 4.6.2, (4.6.39) and (4.6.42).) Note that $\sqrt{V/m} = \chi/\sqrt{m}$, where χ is a chi-variable with m degrees of freedom. Let $g(w)$ be the p.d.f. of $W=(\chi/\sqrt{m})^{-1}$; then

$$M_s(g(w))2^{(-s+1)/2}m^{\left(-s+\frac{1}{2}\right)}\frac{\Gamma\left(\frac{-s+m+1}{2}\right)}{\Phi\left(\frac{m}{2}\right)}$$

Also,

$$M_s(h^-(-t))=M_s(f^-(U))M_s(g(W)), \qquad -\infty<t\leqslant 0,$$

$$M_s(h^+(t))=M_s(f^+(U))M_s(g(W)), \qquad 0\leqslant t<\infty,$$

so that

$$h^-(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-s}M_s(h^-(t))\,ds, \qquad 0<c<m+1$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[(t)^{-s}e^{-\delta^2/2}m^{\left(-s+\frac{1}{2}\right)}\right.$$

$$\left.\times\sum_{j=0}^{\infty}\frac{1}{j!}(-\delta\sqrt{2})^j\Gamma\left(\frac{s+j}{2}\right)\frac{\Gamma\left(\frac{-s+m+1}{2}\right)}{\sqrt{2\pi}\,\Gamma\left(\frac{m}{2}\right)}\right]ds \tag{9.2.5}$$

and

$$h^+(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-s}M_s(h^+(t))\,ds, \qquad 0<c<m+1$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\frac{t^{-s}e^{-\delta^2/2}}{\sqrt{2\pi}}m^{\left(-s+\frac{1}{2}\right)}\right.$$

$$\left.\times\sum_{j=0}^{\infty}\left(\frac{1}{j!}\right)(\delta\sqrt{2})^j\Gamma\left(\frac{s+j}{2}\right)\frac{\Gamma\left(\frac{-s+m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}\right]ds. \tag{9.2.6}$$

Consider the integrals corresponding to (9.2.5) and (9.2.6) but evaluated over the contours C_L and C_R (Fig. 2.9.1). It is clear that the integrand in each of these integrals is an infinite series, and the question arises as to whether the series can be integrated termwise over the indicated contours (i.e., whether the integral and summation signs may be interchanged). That this interchange can be made without changing the value of the contour integrals can be shown by establishing that each infinite series is uniformly convergent with respect to s and noting that each term in either series is continuous on C_L and C_R. Application of a well-known theorem [415, Theorem 5, p. 351] in complex variable theory, which states that the integral of a uniformly convergent series of continuous functions along any curve C can be found by termwise integration of the series, then validates the interchange of the summation and integral signs. Specifically,

$$h^-(t)=\sum_{j=0}^{\infty}\left(\frac{1}{2\pi i}\right)\int_{c-i\infty}^{c+i\infty} t^{-s}e^{-\delta^2/2}m^{\left(-s+\frac{1}{2}\right)}$$

$$\times\frac{1}{j!}(-\delta\sqrt{2}\,)^j\Gamma\left(\frac{s+j}{2}\right)\frac{\Gamma\left(\dfrac{-s+m+1}{2}\right)}{\sqrt{2\pi}\,\Gamma\left(\dfrac{m}{2}\right)}\,ds \qquad (9.2.7)$$

$$h^+(t)=\sum_{j=0}^{\infty}\left(\frac{1}{2\pi i}\right)\int_{c-i\infty}^{c+i\infty} t^{-s}e^{-\delta^2/2}m^{\left(-s+\frac{1}{2}\right)}$$

$$\times\left(\frac{1}{j!}\right)(\delta\sqrt{2}\,)^j\Gamma\left(\frac{s+j}{2}\right)\frac{\Gamma\left(\dfrac{-s+m+1}{2}\right)}{\sqrt{2\pi}\,\Gamma\left(\dfrac{m}{2}\right)}\,ds. \qquad (9.2.8)$$

When $|t|<1$ each of the integrals in (9.2.7) and (9.2.8) satisfies the conditions of Jordan's lemma when evaluated over the Bromwich contour C_L. Similarly, when $|t|>1$, the conditions of Jordan's lemma are satisfied for evaluation of the integral over the contour C_R. Thus in either case, the value of the integral over the circular arc approaches zero as the radius of the circle approaches infinity, except possibly for the value $t=1$. However when $t=1$, it can be shown that the value of the inversion integral also approaches zero on the circular arc as the radius approaches infinity.

Consequently, both inversion integrals can be evaluated by the method of residues in the manner shown below.

Consider first the integral (9.2.7) when $-1 \leqslant t \leqslant 0$, and utilize the identity

$$\left(\frac{s+j}{2}+k\right)\Gamma\left(\frac{s+j}{2}\right) \equiv \frac{\Gamma\left(\frac{s+j}{2}+k+1\right)}{\left(\frac{s+j}{2}\right)\left(\frac{s+j}{2}+1\right)\cdots\left(\frac{s+j}{2}+k-1\right)} \tag{9.2.9}$$

when integrating over the contour C_L. (The denominator in (9.2.9) will be defined to be 1 when $j=0$, so that the identity holds for all values of j.) Then, from the residue theorem,

$$h_1^-(t) = \sum_{j=0}^{\infty} R_{j1}, \qquad -1 \leqslant t \leqslant 0,$$

where R_{j1} denotes the sum of the residues at the poles $s=-(j+2k)$, $k=0,1,2,\ldots$. That is,

$$R_{j1} = \sum_{k=0}^{\infty} \left(\frac{s+j}{2}+k\right) I^-(t)\Bigg|_{s=-(j+2k)}, \tag{9.2.10}$$

where $I^-(t)$ is the integrand in (9.2.7). Utilizing the identity (9.2.9) in (9.2.10) one has

$$R_{j1} = \sum_{k=0}^{\infty} t^{j+2k} e^{-\delta^2/2} m^{\left(j+2k+\frac{1}{2}\right)}$$

$$\times \frac{(-1)^k}{k!\,j!}(-\delta\sqrt{2}\,)^j \frac{\Gamma\left(\frac{m+1+j+2k}{2}\right)}{\sqrt{2\pi}\,\Gamma\left(\frac{m}{2}\right)}. \tag{9.2.11}$$

Similarly, consider the integral (9.2.8) when $0 \leqslant t \leqslant 1$, and again utilize the identity (9.2.9) when integrating over the contour C_L. Again, from the residue theorem,

$$h_1^+(t) = \sum_{j=0}^{\infty} R'_{j1}, \qquad 0 \leqslant t \leqslant 1,$$

where R'_{j1} denotes the sum of the residues at the poles $s=-(j+2k)$, $k=0,1,2,\ldots$. Specifically,

$$R'_{j1}=\sum_{k=0}^{\infty}\left(\frac{s+j}{2}+k\right)I^{+}(t)\bigg|_{s=-(j+2k)}, \tag{9.2.12}$$

where $I^{+}(t)$ denotes the integrand in (9.2.8). Application of the identity (9.2.9) to (9.2.12), yields the residue

$$R'_{j1}=\sum_{k=0}^{\infty}\frac{t^{j+2k}}{\sqrt{2\pi}\,\Gamma\left(\frac{m}{2}\right)}e^{-\delta^2/2}m^{\left(j+2k+\frac{1}{2}\right)}\frac{(-1)^k}{k!j!}(\delta\sqrt{2}\,)^j\Gamma\left(\frac{m+1+j+2k}{2}\right). \tag{9.2.13}$$

It remains to evaluate the inversion integrals (9.2.7) and (9.2.8) when $|t|>1$. This can be accomplished in a similar manner by utilizing the identity

$$\left(\frac{-s+m+1}{2}+k\right)\Gamma\left(\frac{-s+m+1}{2}\right)$$
$$=\frac{\Gamma\left(\frac{-s+m+1}{2}+k+1\right)}{\left(\frac{-s+m+1}{2}\right)\left(\frac{-s+m+1}{2}+1\right)\cdots\left(\frac{-s+m+1}{2}+k-1\right)} \tag{9.2.14}$$

and evaluating the integrals (9.2.7) and (9.2.9) by means of the residue theorem over the contour C_R (Fig. 2.9.1b). (Again the denominator on the right-hand side of (9.2.14) will be defined to be 1 when $k=0$, so that (9.2.14) is true for all values of k.) Thus

$$h_2^{-}(t)=\sum_{j=0}^{\infty}R_{j2},\qquad -\infty<t\leqslant -1,$$

where R_{j2} denotes the sum of the residues at the poles $s=m+1+2k$, $k=0,1,2,\ldots$. That is,

$$R_{j2}=\left(\frac{-s+m+1}{2}+k\right)I^{-}(t)\bigg|_{s=m+1+2k},$$

where $I^{-}(t)$ as above denotes the integrand in (9.2.7) with $\Gamma(-s+m+1)/2$ replaced by its equivalent as obtained from (9.2.14). Application of the residue theorem then yields

$$R_{j2}=\sum_{k=0}^{\infty} t^{-(m+1+2k)}\frac{e^{-\delta^2/2}}{\sqrt{2\pi}}m^{\left(-m-2k+\frac{1}{2}\right)}$$

$$\times(-\delta\sqrt{2}\,)^j\frac{(-1)^k}{j!k!}\frac{\Gamma\left(\dfrac{m+1+j+2k}{2}\right)}{\Gamma\left(\dfrac{m}{2}\right)}. \tag{9.2.15}$$

Similarly,

$$h_2^+(t)=\sum_{j=0}^{\infty} R'_{j2}, \qquad 1\leqslant t<\infty,$$

where R_{j2} is now the residue

$$R'_{j2}=\sum_{k=0}^{\infty}\left(\frac{-s+m+1}{2}+k\right)I^+(t)\Big|_{s=m+1+2k}$$

and $I^+(t)$ as above now denotes the integrand in (9.2.8) with $\Gamma[(-s+m+1)/2+k]$ replaced by its equivalent as obtained from (9.2.14). The result is

$$R'_{j2}=\sum_{k=0}^{\infty}\frac{t^{-(m+1+2k)}}{\sqrt{2\pi}}e^{-\delta^2/2}m^{(-m-2k+1)/2}(\delta\sqrt{2}\,)^j\frac{(-1)^k}{j!k!}$$

$$\times\frac{\Gamma\left(\dfrac{m+1+j+2k}{2}\right)}{\sqrt{2\pi}\,\Gamma\left(\dfrac{m}{2}\right)}. \tag{9.2.16}$$

Combining all these results, one has

$$\begin{aligned} h(t)&=h_2^-(t), && -\infty<t<-1\\ &=h_1^-(t), && -1\leqslant t<0\\ &=h_1^+(t), && 0\leqslant t<1\\ &=h_2^+(t), && 1\leqslant t<\infty, \end{aligned} \tag{9.2.17}$$

where

$$h_2^-(t)=\sum_{j=0}^{\infty} R_{j2}, \qquad -\infty<t<-1,$$

$$h_1^-(t)=\sum_{j=0}^{\infty} R_{j1}, \qquad -1\leqslant t<0,$$

$$h_1^+(t)=\sum_{j=0}^{\infty} R'_{j1}, \qquad 0\leqslant t<1,$$

$$h_2^+(t) = \sum_{j=0}^{\infty} R'_{j2}, \qquad 1 \leqslant t < \infty.$$

Finally, because of the relationship (4.3.10b), namely,

$$M_s(G(t)) = \frac{M_{s+1}(h(t))}{s} \tag{9.2.18}$$

between the Mellin transform of the complementary distribution function $G(t) = 1 - H(t)$ and that of the p.d.f. $h(t)$, it is readily shown that

$$\begin{aligned} H(t) &= 1 - G_2^+(t), \qquad 1 \leqslant t < \infty \\ &= 1 - G_1^+(t) - G_2^+(1), \qquad 0 \leqslant t < 1 \\ &= 1 - G_1^-(t) - G_1^+(0) - G_2^+(1), \qquad -1 \leqslant t < 0 \\ &= 1 - G_2^-(t) - G_1^-(t) - G_2^+(1) - G_1^+(0) \qquad t < -1. \end{aligned} \tag{9.2.19}$$

where

$$G_2^-(t) = \frac{e^{-\delta^2/2}}{\sqrt{2}} + \sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\left[\frac{(-1)^k t^{-(m+2k)} e^{-\delta^2/2}}{j!k!\Gamma(m/2)\sqrt{2\pi}} m\left(-m-2k+\tfrac{1}{2}\right)\right.$$

$$\left.\times(-\delta\sqrt{2}\,)^j \Gamma\left(\frac{m+2k+j+1}{2}\right)\right], \qquad -\infty < t < 1,$$

$$G_1^-(t) = \frac{e^{-\delta^2/2}}{\sqrt{2}} + \sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\left[\frac{(-1)^k t^{j+1+2k} e^{-\delta^2/2}}{j!k!\Gamma(m/2)\sqrt{2\pi}} m\left(j+2k+\tfrac{1}{2}\right)\right.$$

$$\left.\times(-\delta\sqrt{2}\,)^j \Gamma\left(\frac{m+2k+j+2}{2}\right)\right], \qquad -1 \leqslant t < 0,$$

$$G_1^+(t) = \frac{e^{-\delta^2/2}}{\sqrt{2}} + \sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\left[\frac{(-1)^k t^{j+1+2k} e^{-\delta^2/2}}{j!k!\Gamma(m/2)\sqrt{2\pi}} m\left(j+2k+\tfrac{1}{2}\right)\right.$$

$$\left.\times(\delta\sqrt{2}\,)^j \Gamma\left(\frac{j+2k+m+2}{2}\right)\right], \qquad 0 \leqslant t < 1,$$

$$G_2^+(t) = \frac{e^{-\delta^2/2}}{\sqrt{2}} + \sum_{j=1}^{\infty}\sum_{i=0}^{\infty}\left[\frac{(-1)^k t^{-(m+2k)}}{j!k!\Gamma(m/2)\sqrt{2\pi}} e^{-\delta^2/2} m\left(-m-2k+\tfrac{1}{2}\right)\right.$$

$$\left.\times(\delta\sqrt{2}\,)^j \Gamma\left(\frac{m+2k+j+1}{2}\right)\right], \qquad 1 \leqslant t < \infty.$$

As can be seen, the complementary distribution function $G(t)$, as well as $H(t)$, is expressed in terms of gamma functions. It is interesting to note that both Hawkins [146] and Craig [70] have shown that this complementary distribution function is expressible in terms of incomplete gamma functions, namely,

$$P[T>t|m,\delta]=G(t)$$

$$=\frac{1}{2}e^{-\delta^2/2}\sum_{j=0}^{\infty}\frac{(\delta\sqrt{2}\,)^j}{\Gamma(j/2+1)}I_\alpha\left\{\frac{m}{2},\frac{j+1}{2}\right\},$$

where $\alpha=n/(n+t^2)$, and $I_x(a,b)$ is the incomplete beta function

$$I_x(a,b)=\int_0^x u^{a-1}(1-u)^{b-1}\,du. \tag{9.2.20}$$

The reader is referred to the papers by Hawkins and Craig for details of the derivation of this result.

It is also noteworthy (as Rahman and Saleh point out [299]) that the noncentral t distribution function does not seem to be expressible as a single special function in the general case. This is in sharp contrast with the noncentral chi-square and noncentral F distribution functions, each of which is a special case of a Bessel type I distribution (see Section 9.9.7).

It bears stating that the distribution function (9.2.19) can be used to test the hypothesis that two random samples came from two normal populations whose variances are σ_1^2 and σ_2^2 and whose means differ by an amount δ. This involves utilizing the statistic T as defined by (9.2.2), where

$$U=\frac{\bar{X}_1-\bar{X}_2}{\sqrt{\sigma_1^2/n_1+\sigma_2^2/n_2}},$$

$$V=\left[\frac{\left(\dfrac{n_1s_1^2}{\sigma_1^2}+\dfrac{n_2s_2^2}{\sigma_2^2}\right)}{(n_1+n_2-2)}\right]^{1/2},$$

and $\bar{X}_1$, $\bar{X}_2$, s_1^2, s_2^2 are the means and variances of two samples of sizes n_1, n_2 drawn at random from two normal populations with means and variances

$\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$, where $\mu_1 - \mu_2 = \delta$. Since U is now a normal r.v. with mean

$$\delta' = \frac{\delta}{\left(\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}\right)^{1/2}}$$

and variance 1, while V has a chi-square distribution with $m = n_1 + n_2 - 2$ degrees of freedom, and U and V are independent, the density and distribution functions (9.2.17) and (9.2.19), with δ replaced with δ', are applicable for testing whether the means of the aforementioned normal populations differ by an amount δ (see, e.g., ref. 166).

9.2.3 Analogues of Student's *t*

Various authors—Walsh [390], Siddiqui [337], Birnbaum and Vincze [29], Birnbaum and Friedman [114], and others—have considered a different statistic similar to Student's t, usually based on order statistics, which can be applied under very general conditions. More or less typical of these analogues of Student's t is the one originally studied by Siddiqui and recently tabulated by Birnbaum and Friedman [114]. It is defined, for a given integer r, $1 \leqslant r \leqslant m$, as

$$S'_{m,r} = \frac{X_{(m+1)} - \mu}{X_{(m+1+r)} - X_{(m+1-r)}},$$

where X is an r.v. with continuous distribution function $F(x) = \Pr[X \leqslant x]$, $X_{(1)} \leqslant X_{(2)} \leqslant \cdots \leqslant X_{(2m+1)}$ is an ordered sample of size $2m+1$, μ is the population median of X, and $V = X_{(m+1)}$ is the sample median. It is interesting to note the similarity between the structure of this analogue and that of Student's t. As pointed out by Birnbaum and Friedman, in each case the numerator is the difference between a population location parameter and a sample estimate of that parameter, whereas the denominator is a sample estimate of a scale parameter, the latter being the interquartile range in the case of the analogue. Unlike the t-statistic, $S_{m,r}$ has the practical advantage that it can be computed when only very few order statistics are available, even when all the other values are "censored" out. Birnbaum [28, 29] has shown that if the r.v. X has the distribution function $F(x)$ and p.d.f. $f(x) = F'(x)$, the distribution function of $S_{m,r}$ is given by

$$P[S_{m,r} \leqslant \lambda] = 1 - P(\lambda),$$

where

$$P(\lambda)=\frac{(2m+1)!}{[(m-r)!(r-1)!]^2}\int_{v=0}^{\infty}\int_{u=(\lambda-1)/\lambda}^{v}\int_{w=v}^{u+(v/\lambda)}f(u)f(v)f(w)$$

$$\cdot F^{m-r}(u)[F(v)-F(u)]^{r-1}[F(w)-F(v)]^{r-1}[1-F(w)]^{m-r}dw\,du\,dv.$$

Birnbaum and Friedman [114] have constructed tables for this probability distribution function giving exact critical values of $S_{m,r}$ for small sample sizes ($n\leqslant 10$), under the assumption that X has a normal probability distribution. A table of asymptotic critical values for $S_{m,r}$ is also given, which can be used for large sample sizes even when X is not normally distributed. As Birnbaum and Friedman point out, these tables permit the practical use of $S_{m,r}$ in some situations in which Student's t cannot be used.

9.2.4 The Snedecor F Distribution

Let U and V be independent chi-square r.v.'s with m and n degrees of freedom, and p.d.f.'s $f(u)$ and $g(v)$, respectively. Then

$$f(u)=\frac{1}{2^{m/2}\Gamma(m/2)}u^{(m/2)-1}e^{-u/2},\qquad 0\leqslant u<\infty,$$

$$g(v)=\frac{1}{2^{n/2}\Gamma(n/2)}v^{(n/2)-1}e^{-v/2},\qquad 0\leqslant v<\infty,$$

$$M_s(f(u))=\frac{2^{s-1}\Gamma(s+m/2-1)}{\Gamma(m/2)},\qquad \operatorname{Re}(s)>1-\frac{m}{2},$$

$$M_s(g(v))=\frac{2^{s-1}\Gamma(s+n/2-1)}{\Gamma(n/2)},\qquad \operatorname{Re}(s)>1-\frac{n}{2}.$$

If one denotes the p.d.f. of $F=(U/m)/(V/n)$ by $h(F)$, it follows that

$$M_s(h(F))=M_s\left(f\left(\frac{u}{m}\right)\right)M_{-s+2}\left(g\left(\frac{v}{n}\right)\right)$$

$$=\frac{\Gamma\left(s+\frac{m}{2}-1\right)\Gamma\left(-s+\frac{n}{2}+1\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)},$$

which is the Mellin transform [95, (15), p. 349] of the well-known F distribution

$$h(F)=\frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}\left(\frac{m}{n}\right)^{m/2}\frac{F^{(m-2)/2}}{\left(1+\frac{m}{n}F\right)^{\frac{m+n}{2}}},$$

$$0\leqslant F<\infty,\qquad m>0,\qquad n>0.$$

9.2.5* The Noncentral F Distribution

If the r.v.'s U and V possess independent noncentral chi-square distributions, then the p.d.f. $h(w)$ of the r.v.

$$W=\frac{U}{V}$$

is called the noncentral F density function [167]. It may be derived in various ways, but the natural and preferable approach is to use the Mellin transform. In Section 9.9.7 the distribution of the ratio of two noncentral chi-square i.r.v.'s is derived, the result being the noncentral F distribution whose parameters are determined from those of the original noncentral chi-square distributions. The reader is referred to Section 9.9.7 for the explicit form of the resulting noncentral F distribution.

9.2.6 The Chi-Square Distribution

Let x_j be a standardized normal r.v. with p.d.f.

$$f(x_j)=\frac{1}{\sqrt{2\pi}}e^{-x_j^2/2},\qquad -\infty<x_j<\infty.$$

Consider the p.d.f. $g(w)$, where $w=\sum_1^n x_j^2$. Since the characteristic function of $f(x_j)$ is

$$F_t(f(x_j))=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{itx_j^2}e^{-x_j^2/2}dx_j$$

$$=(1-2it)^{-1/2},$$

the characteristic function of $g(w)$ is

$$F_t(g(w)) = (1-2it)^{-n/2},$$

which is readily recognized (Table D.2) as the characteristic function of a chi-square distribution with n degrees of freedom. (The reader is asked to verify this by inverting the characteristic function $F_t(g(w))$; see Exercise 9.29.)

That is, the sum of the squares of n normally and independently distributed r.v.'s with mean zero and standard deviation one has, as is well known, a chi-square distribution with n degrees of freedom.

9.2.7* The Noncentral Chi-Square Distribution

As was seen in the preceding section, the sum

$$W = \sum_{1}^{n} X_j^2$$

of n standardized normal i.r.v.'s $N(0,1)$ has a chi-square distribution with n degrees of freedom. It is natural, then, to label the p.d.f. $h(W)$ of the sum $W = \sum_1^n X_j^2$ of the squares of n noncentral normal i.r.v.'s $N(\mu_j, \sigma)$ as a noncentral chi-square distribution. This distribution is, of course, well known and can be derived in various ways. However, the natural approach is to use the characteristic function (Fourier transform), which, as McNolty [249] points out, has considerable utility for this type of problem.

McNolty has derived the noncentral chi-square distribution by means of the characteristic function. His procedure is presented here,[42] including some further details of the derivation.[43]

Consider the r.v.

$$W = \sum_{j=1}^{n} X_j^2$$

where the i.r.v.'s X_j have the p.d.f. $f(x_j) = (1/\sqrt{2\pi})\exp[-(x_j - \mu_j)^2]/2\sigma^2$, $\mu_j \neq 0$, and denote the p.d.f. of W by $g(w)$. Then the characteristic function

[42]With the permission of the Executive Secretary of the Institute of Mathematical Statistics.

[43]The procedure in this section differs somewhat from that of McNolty because of an error in one part of the cited reference. This error was corrected by McNolty and the corrected version conveyed to the author by letter.

of $f_j(x_j)$ is

$$F_t(f_j(x_j)) = \int_{-\infty}^{\infty} e^{itx_j^2} f_j(x_j)\,dx_j$$
$$= \frac{1}{(1-2it\sigma^2)} \exp\left(\frac{it\mu_j^2}{1-2it\sigma^2}\right),$$

so that, since the x_j are independent,

$$F_t(g(w)) = \frac{1}{(1-2it\sigma^2)^{n/2}} \exp\left(\frac{itr^2}{1-2it\sigma^2}\right),$$

where $r^2 = \sum_1^n \mu_j^2$. The p.d.f. $g(w)$ is then obtained by evaluating the inversion integral

$$g(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itw} \frac{1}{(1-2it\sigma^2)^{n/2}} \exp\left(\frac{itr^2}{1-2it\sigma^2}\right) dt. \qquad (9.2.21)$$

Letting $z = (\sqrt{w}\,/r)(1-2it\sigma^2)$, (9.2.21) becomes

$$g(w) = \frac{1}{4\pi i\sigma^2}\left(\frac{\sqrt{w}}{r}\right)^{(n-2)/2} \exp\left[\frac{-(w+r^2)}{2\sigma^2}\right]$$
$$\cdot \int_{(\sqrt{w}\,/r)-i\infty}^{(\sqrt{w}\,/r)+i\infty} \exp\left[\frac{\sqrt{w}\,r}{2\sigma^2}\left(\frac{z^2+1}{z}\right)\right] \frac{dz}{z^{n/2}}, \qquad w>0. \quad (9.2.22)$$

The problem is to evaluate this integral.

Since (as we show shortly) there is a branch point [229, p. 14] at the origin when n is odd and no branch point anywhere when n is even, the two cases must be treated separately. Consider first the evaluation of the integral in (9.2.22) over the indicated Bromwich path when n is even. Note that this Bromwich path is part of the (closed) Bromwich contour

$$C_L = \sum_{j=1}^{4} \Gamma_j \qquad \text{(Fig. 9.2.1}a\text{)},$$

where Γ_2 and Γ_4 contain the end points of the vertical diameter but Γ_3 does

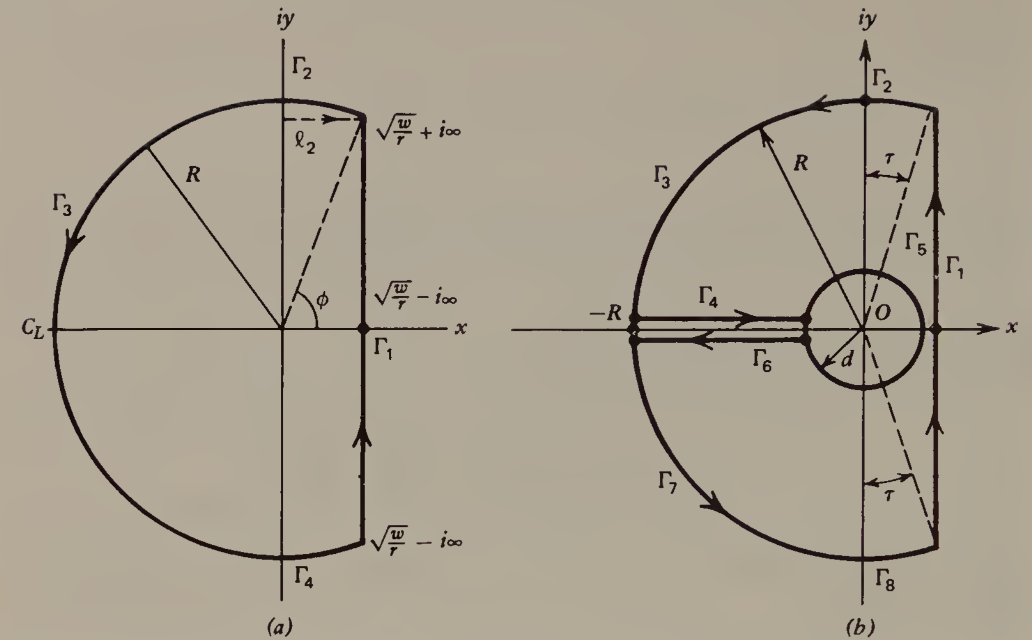


Fig. 9.2.1 Bromwich contours utilized in deriving the noncentral chi-square distribution.

not. Note also that

$$\int_{C_L} \exp\left[\frac{\sqrt{w}\, r}{2\sigma^2}\left(\frac{z^2+1}{z}\right)\right]\frac{dz}{z^{n/2}} = 2\pi i R_1, \tag{9.2.23}$$

where

$$R_1 = \frac{1}{(n/2-1)!}\left[\frac{d^{(n/2)-1}}{dz^{(n/2)-1}}\exp\left\{\frac{\sqrt{w}\, r}{2\sigma^2}\left(\frac{z^2+1}{z}\right)\right\}\right]\Bigg|_{z=0}$$

$$= I_{(n/2)-1}\left(\frac{\sqrt{w}\, r}{\sigma^2}\right) \tag{9.2.24}$$

(a modified Bessel function of the first kind of order $(n/2)-1$) is the residue at the pole $z=0$, the only pole occurring in the integrand. Let I_j denote the integral

$$I_j = \int_{\Gamma_j} \exp\left[\frac{\sqrt{w}\, r}{2\sigma^2}\left(\frac{z^2+1}{2}\right)\right]\frac{dz}{z^{n/2}}, \qquad j=2,3,4 \qquad \text{(Fig. 9.2.1}a\text{)}. \tag{9.2.25}$$

Then, if

$$\lim_{R\to\infty} |I_j| = 0, \qquad j=2,3,4$$

the integral in (9.2.22) becomes equal, as R tends to infinity, to the integral (9.2.23), and (9.2.22) becomes, for n even,

$$g(w)=\frac{1}{2\sigma^2}\left(\frac{\sqrt{w}}{r}\right)^{(n/2)-1}\exp\left[-\frac{1}{2\sigma^2}(w+r^2)\right]I_{(n/2)-1}\left(\frac{\sqrt{w}\,r}{\sigma^2}\right) \tag{9.2.26}$$

$$=\frac{1}{2\sigma^2}\left(\frac{\sqrt{w}}{r}\right)^{(n/2)-1}\exp\left[-\frac{1}{2\sigma^2}(w+r^2)\right]$$

$$\times\sum_{j=0}^{\infty}\frac{(\sqrt{w}\,r/2\sigma^2)^{2j+(n/2)-1}}{\Gamma(j+1)\Gamma(j+(n/2))},\qquad 0\leqslant w<\infty. \tag{9.2.26a}$$

It remains, therefore, to establish that

$$\lim_{R\to\infty} I_j=0,\qquad j=2,3,4.$$

Note first that the value of the integral (9.2.25) over each arc Γ_j is greatest when $n=2$, which is the smallest even integer being considered. Thus it is sufficient to show that when $n=2$, the integral (9.2.25), $j=2,3,4$ approaches zero as R goes to infinity. Now on the arc Γ_3, one has $z=\mathrm{R}e^{i\theta}$ and $dz=i\mathrm{R}e^{i\theta}\,d\theta$. Then

$$I_3=\int_{\Gamma_3} e^{az+a/z}\frac{dz}{z}$$

$$=\lim_{\delta\to 0}\int_{(\pi/2)+\delta}^{(3\pi/2)-\delta}\exp\left(aR\cos\theta+iR\sin\theta+\frac{a}{R}\cos\theta-i\frac{a}{R}\sin\theta\right)\cdot\left(\frac{iRe^{i\theta}}{iRe^{i\theta}}\right)d\theta$$

and

$$|I_3|\leqslant\lim_{\delta\to 0}\int_{(\pi/2)+\delta}^{(3\pi/2)-\delta}\exp\left(aR\cos\theta+\left(\frac{a}{R}\right)\cos\theta\right)d\theta.$$

Note also that since θ is on the open interval $\pi/2<\theta<3\pi/2$, $\cos\theta<0$, and

$$\lim_{\substack{R\to\infty\\ \delta\to 0}}\int_{(\pi/2)+\delta}^{(3\pi/2)-\delta}\exp\left(aR\cos\theta+\left(\frac{a}{R}\right)\cos\theta\right)d\theta=0,$$

hence

$$\lim_{R\to\infty}|I_3|=0.$$

Next, consider the integral

$$I_2=\int_{\phi}^{\pi/2}\exp\left[\frac{\sqrt{w}\,r}{2\sigma^2}\left(\frac{z^2+1}{z}\right)\right]\frac{dz}{z^{(n/2)/2}}.$$

Again, for simplicity, let $a=\sqrt{w}\,r/2\sigma^2$; then, for large values of R,

$$I_2=\int_{\phi}^{\pi/2}e^{az+a/z}\frac{dz}{z}\sim I_2',$$

where

$$\begin{aligned}I_2'&=\int_{l_2}e^{az+a/z}\frac{dz}{z}\\&=-\int_0^{\sqrt{w}/r}e^{az+a/z}\frac{dz}{z}\end{aligned}$$

and l_2 is a line segment perpendicular to Γ_{I} (Fig. 9.2.1a). Note that

$$\begin{aligned}e^{a/z}&=e^{a/(x+iy)}\\&=\exp\left(\frac{ax}{x^2+y^2}-\frac{iay}{x^2+y^2}\right).\end{aligned}$$

Hence

$$I_2'=\int_0^{\sqrt{w}/r}e^{a(x+iy)}\exp\left(\frac{e^{ax}}{(x^2+y^2)}-\frac{iay}{x^2+y^2}\right)\frac{dx}{\sqrt{x^2+y^2}},$$

so that

$$|I_2'|<\int_0^{\sqrt{w}/r}e^{ax}e^{ax/(x^2+y^2)}\frac{dx}{\sqrt{x^2+y^2}}.$$

Note that for the specific Bromwich path indicated, x is a positive finite constant. Also, $y\to\infty$ necessarily as $R\to\infty$, and vice versa. Thus, by choosing y sufficiently large,

$$e^{ax/(x^2+y^2)}/\sqrt{x^2+y^2}<\epsilon,$$

where ϵ is any positive number, however small, in which case

$$\lim_{R\to\infty} |I_2'| = \lim_{y\to\infty} |I_2'| < \epsilon \int_0^{\sqrt{w}/r} e^{ax}\,dx,$$

and

$$\lim_{R\to\infty} |I_2'| = 0.$$

And since $l_2 \to \Gamma_2$ as $R \to \infty$, it follows that

$$\lim_{R\to\infty} I_2 = \lim_{R\to\infty} I_2'$$
$$=0.$$

Similarly, it can be shown that the value of the integral in (9.2.25) over the arc Γ_4 approaches zero as R goes to infinity, from which the validity of the p.d.f. $g(w)$, as given by (9.2.26a), follows when n is even.

Consider next the evaluation of the integral in (9.2.22) when n is odd. To expedite the evaluation in this case, write (9.2.22) in the form

$$g(w) = \frac{1}{4\pi i\sigma^2}\left(\frac{\sqrt{w}}{r}\right)^{(n-2)/2} \exp\left[-\frac{(w+r^2)}{2\sigma^2}\right]\cdot \sum_{m=0}^{\infty}\left(\frac{\sqrt{w}\,r}{2\sigma^2}\right)^m$$
$$\cdot \frac{1}{m!}\int_{(\sqrt{w}/r)-i\infty}^{(\sqrt{w}/r)+i\infty} \exp\left(\frac{\sqrt{w}\,r}{2\sigma^2}z\right)\frac{dz}{z^{m+(n/2)}}, \qquad n \text{ odd} \qquad (9.2.22a)$$

and let $m+n/2=\lambda+1$. Note that when $0<\lambda+1<1$ (or equivalently, when $-1<\lambda<0$), a branch point [229, p. 14] occurs at the origin. That is, when $0<\lambda+1<1$, the exponent in $z^{\lambda+1}$ has a value between 0 and 1, so that (as we show shortly) $z^{\lambda+1}$—hence the integral in (9.2.23)—is not single valued at the origin for the usual contour of integration C_L (Fig. 2.9.1a). The importance of this insofar as the problem at hand is concerned stems from the inapplicability of the residue theorem (as well as many other theorems) when the integrand of the inversion integral is not single valued at all points on the contour of integration. When the integrand is not single valued, it is necessary to replace the closed contour (in this case, C_L) by another contour $\Gamma = \Sigma_1^8 \Gamma_j$ (Fig. 9.2.1b) having two branches on each of which $z^{\lambda+1}$ (hence, the integrand in (9.2.22)) is single valued.

To see the necessity for utilizing such a contour, consider the situation when $\lambda+1=\frac{1}{2}$. Write

$$w = z^{1/2}$$
$$= R^{1/2}e^{i\theta/2}$$

and begin with the point $A:(R,O)$, that is, the point on a circle of radius R and centered at the origin, for which $\theta=0$ and in which case $w=R^{1/2}$. Now let θ increase counterclockwise to $\theta=2\pi$. For $\theta=2\pi$, one notes that $w=-R^{1/2}$ (since $e^{i\pi}=-1$). Thus even though θ has made a complete circuit from zero to 2π inclusive, the value of w is not the same as its initial value (i.e., one has not yet returned to the initial point A). However when θ increases to 4π, one arrives at the beginning point $A:(R,2\pi)=(R,O)$. Clearly, w (hence the integrand in (9.2.24)) is not single valued at the origin, in which case one says that there is a branch point at the origin. That is, the values $0\leqslant\theta<2\pi$ generate one branch of the function $w=z^{1/2}$, whereas the values $2\pi\leqslant\theta<4\pi$ generate the other branch of the function. Note that w is single valued on each branch, since the value $\theta=4\pi$ is excluded. Thus to exclude the value $\theta=4\pi$, one can use the contour $\Gamma=\Sigma_{j=1}^{8}\Gamma_j$ (Fig. 9.2.1b), which does not cross the real axis and on which the integrand of (9.2.23) is single valued.

Since the integrand in (9.2.22) is single valued at all points on the contour Γ and also has no poles on or inside this contour, it follows from Cauchy's theorem that the integral over Γ is zero. That is,

$$\lim_{\substack{R\to\infty\\ d\to 0}}\int_\Gamma \exp\left(\frac{\sqrt{w}\,r}{2\sigma^2}z\right)\frac{dz}{z^{\lambda+1}}=\lim_{R\to\infty}\left\{\int_{\Gamma_1}+\int_{\Gamma_2}+\int_{\Gamma_3}+\int_{\Gamma_7}+\int_{\Gamma_8}\right\}$$

$$+\lim_{d\to 0}\int_{\Gamma_5}+\lim_{\substack{R\to\infty\\ d\to 0}}\left\{\int_{\Gamma_4}+\int_{\Gamma_6}\right\}=0. \qquad (9.2.27)$$

Now, in the same manner as before, it can be shown that

$$\lim_{R\to\infty}\int_{\Gamma_2}=\lim_{R\to\infty}\int_{\Gamma_8}=0 \qquad \text{for } \lambda>-1$$

and

$$\lim_{R\to\infty}\int_{\Gamma_3}=\lim_{R\to\infty}\int_{\Gamma_7}=0 \qquad \text{for } \lambda>-1,$$

and clearly

$$\lim_{d\to 0}\int_{\Gamma_5}=0 \qquad \text{for } \lambda<0,$$

so that

$$\int_{(\sqrt{w}/r)-i\infty}^{(\sqrt{w}/r)+i\infty}\exp\left(\frac{\sqrt{w}\,r}{2\sigma^2}z\right)\frac{dz}{z^{\lambda+1}}=-\lim_{\substack{R\to\infty\\ d\to 0}}\left[\int_{\Gamma_4}+\int_{\Gamma_6}\right].$$

Hence

$$\int_{(\sqrt{w}/r)-i\infty}^{(\sqrt{w}/r)+i\infty} \exp\left(\frac{\sqrt{w}\,r}{2\sigma^2}z\right)\frac{dz}{z^{\lambda+1}} = [-(I_4+I_6)], \qquad (9.2.28)$$

where

$$I_4 = \lim_{\substack{R\to\infty\\ d\to 0}} \int_{\Gamma_4} \exp\left(\frac{\sqrt{w}\,r}{2\sigma^2}z\right)\frac{dz}{z^{\lambda+1}},$$

$$I_6 = \lim_{\substack{R\to\infty\\ d\to 0}} \int_{\Gamma_6} \exp\left(\frac{\sqrt{w}\,r}{2\sigma^2}z\right)\frac{dz}{z^{\lambda+1}}.$$

It remains, then, to evaluate the integrals I_4 and I_6. Note that z takes on only real values over both the contours Γ_4 and Γ_6; that is, $z=x$ and $\theta=\theta_0$, where $\theta_0=\pi$ and $\theta_0=-\pi$ in the limit as $d\to 0$. Thus,

$$\begin{aligned} z &= xe^{i\theta}, \\ dz &= e^{i\theta_0}dx \\ &= [\cos\theta_0 + i\sin\theta_0]\,dx, \\ z^{-(\lambda+1)} &= x^{-(\lambda+1)}[\cos(\lambda+1)\theta_0 - i\sin(\lambda+1)\theta_0], \end{aligned}$$

and we have

$$\begin{aligned} I_4 &= (\cos(\lambda+1)\pi - i\sin(\lambda+1)\pi)\int_{-\infty}^{0} \exp\left(\frac{\sqrt{w}\,r}{2\sigma^2}\right)\frac{dx}{x^{\lambda+1}}, \\ &= (-\cos(\lambda+1)\pi + i\sin(\lambda+1)\pi)\int_{-\infty}^{0} \exp\left(\frac{\sqrt{w}\,r}{2\sigma^2}\right)\frac{dx}{x^{\lambda+1}}, \quad -1<\lambda<0. \end{aligned}$$

Similarly, remembering that the integration is carried out in opposite directions over the contours Γ_4 and Γ_6, we have

$$\begin{aligned} I_6 &= (\cos(\lambda+1)(-\pi)) - \sin(\lambda+1)(-\pi))\int_{0}^{\infty} \exp\left(\frac{\sqrt{w}\,r}{2\sigma^2}\right)\frac{dx}{x^{\lambda+1}} \\ &= (\cos(\lambda+1)\pi + i\sin(\lambda+1)\pi)\int_{0}^{\infty} \exp\left(\frac{\sqrt{w}\,r}{2\sigma^2}\right)\frac{dx}{x^{\lambda+1}}, \quad -1<\lambda<0. \end{aligned}$$

Thus, for a circle of fixed radius $x=R$ (Fig. 9.2.1*b*), it follows that

$$-(I_4+I_6)=-2i(\sin\lambda\pi)\int_0^{\infty}x^{-(\lambda+1)}\exp\left(\frac{\sqrt{w}\,r}{2\sigma^2}x\right)dx,\quad -1<\lambda<0.$$

Hence (9.2.27) becomes

$$\int_{(\sqrt{w}/r)-i\infty}^{(\sqrt{w}/r)+i\infty}\exp\left(\frac{\sqrt{w}\,r}{2\sigma^2}z\right)\frac{dz}{z^{\lambda+1}}=-2i\sin\lambda\pi$$

$$\times\int_0^{\infty}x^{-(\lambda+1)}\exp\left(-\frac{\sqrt{w}\,r}{2\sigma^2}x\right)dx,\qquad -1<\lambda<0$$

$$=-2i\sin\lambda\pi\cdot\left(\frac{\sqrt{w}\,r}{2\sigma^2}\right)^{\lambda}\int_0^{\infty}u^{-(\lambda+1)}e^{-u}\,du$$

$$=-2i\sin(\lambda\pi)\Gamma(-\lambda)\left(\frac{\sqrt{w}\,r}{2\sigma^2}\right)^{\lambda}=\frac{2\pi i}{\Gamma(\lambda+1)}\left(\frac{\sqrt{w}\,r}{2\sigma^2}\right)^{\lambda},\qquad (9.2.29)$$

since for any noninteger argument y [18]

$$\sin\pi y=\frac{-\pi}{y\Gamma(y)\Gamma(-y)}$$

$$=\frac{-\pi}{\Gamma(y+1)\Gamma(-y)}.$$

Equation 9.2.28 is valid for $-1<\lambda<0$, and since both sides of (9.2.28) are analytic functions of λ throughout the region $-\infty<\lambda<\infty$, it follows by analytic continuation [229, p. 122] that (9.2.29) holds for $-\infty<\lambda<\infty$ and consequently, for the $m+n/2$ values of λ involved in (9.2.22a).

Thus (9.2.22a) is now expressible in the form

$$g(w)=\left(\frac{1}{2\sigma^2}\right)^{(n/2)}w^{(n-2)/2}\exp\left[-\frac{1}{2\sigma^2}(w+r^2)\right]$$

$$\cdot\sum_{m=0}^{\infty}\frac{\left(\dfrac{wr^2}{2^2\sigma^4}\right)^m}{\Gamma(m+1)\Gamma\left(m+\dfrac{n}{2}\right)},\qquad n\quad\text{odd.}\qquad (9.2.30)$$

But (9.2.26a) with n even can be written in precisely the same form as

(9.2.30), so that (9.2.30) is the density function for w when n is either even or odd.

Finally, letting $a=r^2/2\sigma^2$ and $\chi'^2=w/\sigma^2$, (9.2.30) takes on the more familiar form of the noncentral chi-square distribution, namely,

$$f(\chi'^2)=2\left(\frac{\chi'^2}{2}\right)^{(n-2)/2}\exp(-a)\exp\left(\frac{-\chi'^2}{2}\right)$$

$$\cdot\sum_{m=0}^{\infty}\frac{(a\chi'^2)^m}{m!\Gamma\left(m+\frac{n}{2}\right)},\qquad 0\leqslant\chi'^2<\infty. \tag{9.2.31}$$

9.3 THE DISTRIBUTION OF THE VARIANCE OF A SAMPLE FROM A NORMAL POPULATION $N(\mu,\sigma^2)$[44]

The following lemma, due to R. A. Fisher [104], is particularly useful in deriving the distribution of the variance and covariance.

Lemma 9.3

Consider the sample as a point (or points, in the case of multivariate distributions) in an n-dimensional Euclidean space. Then, if the probability density at any point is a function of the distance from the origin, the mean value of a function of the distance from the origin and of other geometric invariants of the system for $x_j, y_j, \ldots, j=1,2,\ldots,n$ satisfying the conditions $\Sigma_1^n x_j=0, \Sigma_1^n y_j=0,\ldots,$ will be the same as for the same function for independent variables in $(n-1)$-dimensional space.

Consider now the distribution of the variance of a random sample of n items $X_1, X_2,\ldots,X_n$ from the normal population

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\qquad -\infty<x<\infty$$

and let $W=X_1^2+\cdots+X_n^2=nS^2$, subject to $\Sigma_1^n X_j=0$. Then from Lemma

[44]The derivations in Sections 9.3 and 9.4 are essentially those of Kullback [193] and are reprinted with the permission of the Executive Secretary of the Institute of Mathematical Statistics.

9.3, the characteristic function of $g(w)$ is

$$F_t(g(w)) = \left[\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2} + itx^2\right) dx\right]^{n-1}$$

$$= (1-2it)^{-(1/2)(n-1)}, \tag{9.3.1}$$

which is readily recognized (Table D.2) as the characteristic function of a chi-square distribution with $n-1$ degrees of freedom. That is,

$$g(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-itw}}{(1-2it)^{(n-1)/2}}\, dt \tag{9.3.2}$$

$$= \frac{w^{(n-3)/2} e^{-(w/2\sigma^2)}}{(2\sigma^2)^{(n-1)/2}\Gamma[(n-1)/2]}. \tag{9.3.2a}$$

If $h(s^2)$ denotes the p.d.f. of the r.v. S^2, it follows immediately from (9.3.2a) and the relationships $W = nS^2$, $dw = ndS^2$ (where dw and dS^2 are differentials) that

$$h(s^2) = \left(\frac{n}{2\sigma^2}\right)^{(n-1)/2} \frac{(s^2)^{(n-3)/2} e^{-(ns^2/2\sigma^2)}}{\Gamma[(n-1)/2]}. \tag{9.3.3}$$

9.4* THE DISTRIBUTION OF THE COVARIANCE OF A SAMPLE OF N FROM A BIVARIATE NORMAL POPULATION

Let

$$U = \frac{\rho}{(1-\rho^2)\sigma_x\sigma_y} \sum_{j=1}^{n} X_j Y_j, \tag{9.4.1}$$

where X_j and Y_j are r.v.'s having the joint p.d.f.

$$f(x_j, y_j) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left\{\frac{x_j^2}{\sigma_x^2} - \frac{2\rho x_j y_j}{\sigma_x\sigma_y} + \frac{y_j^2}{\sigma_y^2}\right\}\right],$$

$$-\infty < x_j < \infty, \qquad -\infty < y_j < \infty$$

and U has p.d.f. $g(u)$. Then the joint characteristic function of $f(x_j,y_j)$ is

$$F_t(f(x_j,y_j))=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp\left[-\frac{1}{2(1-\rho^2)}\left[\frac{x_j^2}{\sigma_x^2}-\frac{2(1+it)\rho x_jy_j}{\sigma_x\sigma_y}+\frac{y_j^2}{\sigma_y^2}\right]\right]dx_jdy_j}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}.$$

If one now imposes the conditions

$$\sum_j x_j=0 \qquad \text{and} \qquad \sum_j y_j=0$$

and lets $S_{xy}=\Sigma_j x_jy_j$, then

$$U=\frac{n\rho S_{xy}}{(1-\rho^2)\sigma_x\sigma_y},$$

and the characteristic function of $g(u)$ is

$$F_t(g(u))=\frac{(1-\rho^2)^{(n-1)/2}}{\left[1-\rho^2(1+it)^2\right]^{(n-1)/2}}. \tag{9.4.2}$$

Then,

$$g(u)=\frac{(1-\rho^2)^{(n-1)/2}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itu}}{\left[1-\rho^2(1+it)^2\right]^{(n-1)/2}}dt. \tag{9.4.3}$$

Now consider the integral

$$I=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itu}dt}{\{[1-\rho(1+it)][1+\rho(1+it)]\}^{(n-1)/2}} \tag{9.4.4}$$

and let $1-\rho(1+it)=-\rho z/u$; then

$$I=\frac{u^{(n-3)/2}e^{u(\rho-1)/\rho}}{(2\rho)^{(n-1)/2}2\pi i}\int_{-[(1-\rho)/\rho]u-i\infty}^{-[(1-\rho)/\rho]u+i\infty}\frac{e^{-z}dz}{(-z)^{(n-1)/2}(1+z\rho/2u)^{(n-1)/2}}. \tag{9.4.5}$$

Since it can be shown that

$$\lim_{z\to\infty}\left|z^m\frac{e^{-z}}{(-z)^{(n-1)/2}(1+z\rho/2u)^{(n-1)/2}}\right|\to 0,$$

the integral is convergent and is expressible as

$$I=\frac{-u^{(n-3)/2}e^{u(\rho-1)/\rho}}{(2\rho)^{(n-1)/2}2\pi i}\int_{\infty}^{(0+)}\frac{e^{-z}dz}{(-z)^{(n-1)/2}(1+z\rho/2u)^{(n-1)/2}},\qquad (9.4.6)$$

where $\int_{\infty}^{(0+)}$ indicates that the path of integration starts at infinity on the real axis, encircles the origin in the positive direction, and returns to the starting point (see, e.g., ref. 402, pp. 239, 333). Note that $(1-\rho)/\rho<2/\rho$, so that the point $z=-2\rho/u\rho$ is outside the contour, in which case

$$-\frac{1}{2\pi i}\int_{\infty}^{(0+)}\frac{e^{-z}dz}{(-z)^{(n-1)/2}(1+z\rho/2u)^{(n-1)/2}}dz$$

$$=\frac{e^{u/\rho}}{\Gamma[(n-1)/2]}\,{}_1F_1\left(0,\frac{-(n-2)}{2},\frac{2u}{\rho}\right),$$

$$(9.4.7)$$

where ${}_1F_1\ (a,c,z)$ is the confluent hypergeometric function defined in Appendix D.2. Furthermore, since ${}_1F_1\ (0,m,z)={}_1F_1(0,-m,z)$, (9.4.3) becomes

$$g(u)=(1-\rho^2)^{(n-1)/2}\frac{u^{(n-3)/2}e^{u}\,{}_1F_1\left(0,\dfrac{n-2}{2},\dfrac{2u}{\rho}\right)}{\Gamma\left(\dfrac{n-1}{2}\right)(2\rho)^{(n-1)/2}}.\qquad (9.4.8)$$

If one now begins with the definition

$$K_m(x)=\frac{\sqrt{\pi}\,x^m}{2^m\sqrt{m+\frac{1}{2}}}\int_1^{\infty}e^{-xt}(t^2-1)^{m-(1/2)}dt$$

for the Bessel function of the second kind and imaginary argument, it can be shown that

$$K_m(x)=\sqrt{\pi}\,x^{-1/2}2^{-1/2}\,{}_1F_1(0,m,2x),$$

so that

$$g(u)=\frac{(1-\rho^2)^{(n-1)/2}u^{(n-2)/2}e^{u}K_{(n-2)/2}\left(\frac{u}{\rho}\right)}{\sqrt{\pi}\,\Gamma\!\left(\frac{(n-1)}{2}\right)2^{(n-2)/2}\rho^{n/2}}. \tag{9.4.9}$$

Finally, setting

$$V=\frac{U}{\rho}=\frac{nS_{xy}}{(1-\rho^2)\sigma_x\sigma_y},$$

one finds the p.d.f. $g(v)$ of the covariance V to be

$$g(v)=\frac{(1-\rho^2)^{(n-1)/2}e^{\rho v}v^{(n-2)/2}K_{(n-2)/2}(v)}{\sqrt{\pi}\,2^{(n-2)/2}\Gamma[(n-1)/2]}, \tag{9.4.10}$$

which agrees with the results of Pearson et al. [281].

9.5 THE DISTRIBUTION OF LINEAR FUNCTIONS OF NORMAL VARIABLES

Authors of textbooks in elementary mathematical statistics or probability theory usually prove that linear combinations of independent normally distributed r.v.'s are normally distributed. Because of this, it is often loosely stated that "linear combinations of normally distributed r.v.'s are normally distributed." It is true, of course, that if the summands have a multivariate normal distribution, linear combinations of these r.v.'s are also normally distributed [313]. However it is not difficult to cite examples, as several authors have done [313, 175, 4, 141], of linear combinations of normally distributed r.v.'s that are not normally distributed.

More specifically, if U and V are normal i.r.v.'s, it is well known that $U+V$ is also normally distributed. Conversely, a well-known theorem of Cramer [73] states that if U, V are independent and $U+V$ is normal, each U and V must be normal. If, however, the assumption of independence of U and V is dropped, one could, as Kale has shown, have the following possibilities [175]:

1. U and V are marginally normal and $U+V$ is also normal.
2. U and V are marginally normal but $U+V$ is not normal.
3. U and V are marginally not normal but $U+V$ is normal.

For a proof of these results, the reader is referred to the paper by Kale, who points out that situation 1 is readily verified, but situations 2 and 3 are not easy to come by. Rosenberg [313] has given an example of 2 but not of 3. Ferguson [101] notes that most textbook examples of (U, V) marginally normal but not jointly normal are examples of case 2. Because of this, Kale [175] considers a class of p.d.f.'s that are useful in demonstrating both situations 2 and 3. His approach, utilizing the characteristic function, is to construct an example of r.v.'s X, Y that are both normal but are such that no linear combination $aX+bY$, $(ab\neq 0)$ is normal. Further examples are given by Behboodian [24], Albert and Tittle [4], and Ruymgaart [317].

9.6* THE BEHRENS–FISHER DISTRIBUTION

It is well known that when one uses the Student-Fisher t test to determine whether two sample means came from normal populations whose means differ by a stated amount, he must assume that the variances of these normal populations are identical. If this assumption is not satisfied, the t test is not valid.

Various alternative tests—both exact and approximate—have been suggested for use when the population variances are not identical. The best known of these tests is probably the Behrens-Fisher t test [161]. It utilizes the d statistic defined by

$$d = t_1 \sin\theta - t_2 \cos\theta, \tag{9.6.1}$$

which is the weighted difference of two independent t variables with different degrees of freedom ν_1 and ν_2, and was first obtained by Behrens [25] in integral form. In 1935 Fisher [106] verified this result and extended Behrens' theory. A table was constructed by Sukhatme [372] in 1938 covering values of ν_1 and ν_2 at 6, 8, 12, 24, and ∞ and values of $\tan\theta$ for $\Theta = 0°(15°)90°$. A few years later Fisher [107] derived asymptotic expansions for calculating the probabilities in any specific case and provided a further set of tables for the cases when either ν_1 or ν_2 is small and odd. Ruben [316] attempted, with partial success, to obtain the exact distribution of the d-statistic. He succeeded only in expressing it in integral form while interpreting the d-statistic as the ratio of two i.r.v.'s, the numerator of which is a t-variable and the denominator a function of a beta variable. He then gave explicit forms of the distribution of the d-statistic as a finite or infinite series involving incomplete beta function ratios or incomplete gamma function ratios for the particular cases in which (1) $\nu_1 = \nu_2$ with $\Theta = 45°$ and (2) one of the components is normal.

Several years after Ruben published his results, Rahman and Saleh [299] used characteristic functions to establish that the p.d.f. of the d-statistic can be written in several equivalent integral forms, and at least one of them is recognizable as an Appell function. Specifically, they noted that the characteristic functions of the two i.r.v.'s t_k, $k=1,2$, in (9.6.1) are

$$\phi_k(x)=\int_{-\infty}^{\infty}\exp(it_k x)f(t_k)\,dt_k,\qquad k=1,2$$

where

$$f(t_k)=\frac{1}{\sqrt{\nu_k B\left(\frac{1}{2},\frac{\nu_k}{2}\right)}\left[1+\frac{t_k^2}{\nu_k}\right]^{(\nu_k+1)/2}},\qquad k=1,2,$$

so that the characteristic function of the d-statistic (9.6.1) is expressible in the form

$$\phi(x)=\phi_1(x\sin\theta)\phi_2(-x\cos\theta)$$
$$=\frac{2^{2-(\nu_1+\nu_2)/2}a_1^{\nu_1/2}a_2^{\nu_2/2}x^{(\nu_1+\nu_2)/2}K_{\nu_{1/2}}(a_1x)K_{\nu_{2/2}}(a_2x)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}$$

where $K_{\nu_{1/2}}(a_1x)$ and $K_{\nu_{2/2}}(a_2x)$ are modified Hankel functions, while $a_1=\sqrt{\nu_1}\;x\sin\theta$, $a_2=\sqrt{\nu_2}\;x\cos\theta$ and $B_2(\frac{1}{2},k/2)$ is the familiar Beta function. [See Equations (4.4.42a,b).] They then inverted this characteristic function and expressed the resultant p.d.f. of the d-statistic in terms of an Appel function which lends itself to easy computation of the tail probability for any pair (ν_1,ν_2) degrees of freedom. In particular, after considerable detailed analysis, they established the fact that the p.d.f. of the d-statistic (9.6.1) is expressible in the convergent form (where t is written for d)

$$f(t)=\frac{1}{\sqrt{\pi}\,a_1}\left(\frac{a_2}{a_1}\right)^{-(\nu_2+1)/2}\tau_1^{-(\nu_1+1)/2}\frac{B\left(\frac{\nu_1+1}{2},\frac{\nu_2+1}{2}\right)\Gamma\left(\frac{\nu_1+\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}$$
$$\times F_1\left\{\frac{\nu_1+1}{2},\frac{-(\nu_1+\nu_2)}{2},\frac{\nu_1+\nu_2+1}{2},\frac{\nu_1+\nu_2+2}{2};1-\frac{a_2}{a_1\tau_1},1-\frac{1}{\tau_1^2}\right\},\tag{9.6.2}$$

where τ_i are functions of t [299] and $F_1\{\cdot\}$ denotes the Appell function

$$F_1(\alpha,\beta,\beta',\gamma;x,y)=\sum_{m,n}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n x^m y^n}{(\nu)_{m+n}m!n!}$$

$$=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}du. \quad (9.6.3)$$

As Rahman and Saleh point out, in the special case $\nu_1=\nu_2=\nu$, the p.d.f. of the d-statistic becomes

$$f(t)=\frac{B\left(\frac{\nu+1}{2},\frac{\nu+1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\nu\pi}\left(\Gamma\left(\frac{\nu}{2}\right)\right)^2}\csc\theta\cot^\nu\theta$$

$$\times{}_2F_1\left(\left(\nu+\frac{1}{2}\right),\frac{\nu+1}{2};\frac{\nu}{2}+1;\left(\frac{-t^2}{\nu\sin^2\theta}\right)\right), \quad (9.6.4)$$

where ${}_2F_1(a,b;\ c,x)$ is the ordinary hypergeometric function defined in Appendix D.1. This leads directly to Ruben's result [316] for the special case $\nu_1=\nu_2=\nu$ and $\theta=45°$.

Unfortunately, as Rahman and Saleh note, the distribution function of the d-statistic does not seem to be expressible as a single special function in the general case. They go on to state that the distribution function can be written as an infinite series of Appell functions, but without presenting the series.

Chapman [49] derived a series representation for the characteristic function of $W_{n,n}$ when n is an odd integer and indicated briefly how one might obtain the corresponding p.d.f. $f_{n,n}(w)$ from this, where

$$W_{m,n}=U_m-V_n$$

and U_m, V_n are classical t variables with $m>0$ and $n>0$ degrees of freedom. Ghosh [120] gives formulas and values of the percentage points and the incomplete probability integral for the distribution of the sum U_m+V_n or difference U_m-V_n of two independent t variables.

Confidence limits and approximate solutions to the Behrens-Fisher problem have been developed by various workers (e.g., Scheffe [321, 322] and Murphy [265]). In particular, Lee and Gurland [202] have developed a technique for obtaining the size and power of a wide class of tests, including virtually all those appearing in the literature on the Behrens-

Fisher problem. Similarly, Scheffe [322] discusses some six solutions (both exact and approximate) of the Behrens-Fisher problem in terms of interval estimation and testing. Included in his discussion are approximations for the power of three of these tests that he prefers and judges to be practical.

9.7 DISTRIBUTION OF THE PRODUCT AND QUOTIENT OF ORDER STATISTICS

If the r.v.'s $X_1, X_2, \ldots, X_n$ are rearranged in ascending order of magnitude

$$X_{(1)} \leqslant X_{(2)}, \ldots, \leqslant X_{(n)},$$

then $X_{(i)}$ is called the ith order statistic, $i = 1, 2, \ldots, n$. The unordered r.v.'s X_i are usually—but not necessarily—statistically independent and identically distributed. In contrast, the ordered r.v.'s $X_{(i)}, i = 1, 2, \ldots, n$ are necessarily dependent. Although the distributions of products and quotients found in the literature deal primarily with i.r.v.'s, the distribution of the product and quotient of the extreme order statistics $X_{(1)}$, $X_{(n)}$ and that of consecutive order statistics, $X_{(i)}$, $X_{(i+1)}$ are often found useful in ranking [129] and selection [130] problems.

Subrahmanian [371] has derived the distribution of the product and quotient of order statistics from a uniform distribution, and some of the results are summarized in this section. He has also derived the distribution of the product and quotient of order statistics from a negative exponential distribution (Exercises 9.9 and 9.10). In a paper recently submitted for publication, Trudel and Malik [236] have derived the distribution of the product and quotient of order statistics from Pareto, power and Weibull distributions. The results for the Pareto and Weibull distributions are stated (with the permission of the authors) in Sections 9.7.2 and 9.7.3. For proofs of these results, the reader is referred to the paper by Trudel and Malik.

For convenience in future reference, the reader is reminded that the joint p.d.f. of the ith and jth order statistics $X_{(i)}, X_{(j)} (1 \leqslant i < j \leqslant n)$ is [76, p. 9]

$$\begin{aligned} g(x_{(i)}, x_{(j)}) = {} & \frac{n!}{(i-1)!(j-i-1)!(n-j)!} [F(x_i)]^{i-1} \\ & \times \left[F(x_j) - F(x_i)^{j-i-1} \cdot [1 - F(x_j)]^{n-j} f(x_i) f(x_j) \right], \end{aligned} \tag{9.7.1}$$

where $f(x)$ is the p.d.f. of the r.v. X.

To obtain results for products and quotients of extreme order statistics, one merely sets $i=1$ and $j=n$ in the general formulas for ith and jth order statistics.

9.7.1* Distribution of the Product and Quotient of Order Statistics from the Uniform Distribution

Let $X_{(i)}X_{(j)}, (i<j)$ be the ith and jth order statistics based on a random sample of size n from the uniform distribution over the interval (0, 1). The joint p.d.f. of $X_{(i)}, X_{(j)}$ is then [371]

$$g(x_{(i)}, x_{(j)}) = \frac{n!}{(i-1)!(j-r-1)!(n-j)!} x_{(i)}^{i-1}$$

$$\times (1-x_{(j)})^{n-j}(x_{(j)}-x_{(i)})^{j-i-1}, \qquad 0<x_{(i)}<x_{(j)}<1 \quad (9.7.2)$$

and zero elsewhere. The Mellin transform of (9.7.2) is

$$M(s_1, s_2) = \frac{n!}{(i-1)!} \frac{\Gamma(s_1+i-1)\Gamma(s_1+s_2+j-2)}{\Gamma(s_1+j-1)\Gamma(s_1+s_2+n-1)}. \qquad (9.7.3)$$

Putting $s_1 = s_2 = t$, one obtains the Mellin transform of the p.d.f. $h(y)$ of the product $Y = X_{(i)}X_{(j)}$ of the ith and jth order statistics, namely,

$$M_t(h(y)) = \frac{n!}{(i-1)!} \frac{\Gamma(t+i-1)\Gamma(2t+j-2)}{\Gamma(t+j-1)\Gamma(2t+n-1)}. \qquad (9.7.4)$$

The inverse of the transform (9.7.4) is (see Consul [61])

$$h(y) = \frac{1}{2} \frac{n!y^{(j/2)-1}}{(i-1)!(n-j)!(j-i)!} (1-y)^{j-i} \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} y^{k/2}$$

$$\cdot {}_2F_1\left[j-i, \frac{j+k}{2} - i+1; j-i+1; 1-y\right], \qquad 0<y<1$$

$$= 0 \qquad \text{elsewhere.} \qquad (9.7.5)$$

In (9.7.5), ${}_2F_1(a,b,c,w)$ is the well-known hypergeometric function (Appendix D.1).

If $i=1$, $j=n$, (9.7.5) becomes the p.d.f. of the product of extreme order statistics, namely,

$$h(y)=\frac{1}{2}ny^{(n/2)-1}(1-y)^{n-1}{}_2F_1\left[n-1,\frac{n}{2};n;1-y\right], \qquad 0<y<1. \quad (9.7.6)$$

To obtain the Mellin transform of the p.d.f. $g(v)$ of the quotient $V=X_{(i)}/X_{(j)}$, one simply sets $s_1=t$, $s_2=-t+2$ in (9.7.3) so that

$$M_t g(v)=\frac{(j-1)!\Gamma(t+i-1)}{(i-1)!\Gamma(t+j-1)}. \tag{9.7.7}$$

The inverse of (9.7.7) is then seen to be [95 (20), p. 349]

$$g(v)=\frac{\Gamma(j)}{\Gamma(i)\Gamma(j-i)}v^{i-1}(1-v)^{j-i-1}, \qquad 0<v<1. \tag{9.7.8}$$

9.7.2* Distribution of Product and Quotient of Order Statistics from a Pareto Distribution

The p.d.f. and c.d.f. of the Pareto distribution are, respectively,

$$f(x)=va^v x^{-(v+1)}$$

and

$$F(x)=1-\left(\frac{x}{a}\right)^{-v},$$

where $x \geqslant a$ and $a,v>0$. The joint p.d.f. (9.7.1) then becomes

$$g(x_{(i)},x_{(j)})=kv^2a^{2v}x_{(i)}^{-(v+1)}x_{(j)}^{-(v+1)}\left[1-\left(\frac{x_i}{a}\right)^{-v}\right]^{i-1}$$
$$\cdot\left[\left(\frac{x_{(i)}}{a}\right)^{-v}-\left(\frac{x_{(j)}}{a}\right)^{-v}\right]^{j-i-1}\left[\left(\frac{x_{(j)}}{a}\right)^{-v}\right]^{n-j},$$

where $x_{(i)} \leqslant x_{(j)}$, $x_{(i)} \geqslant a$, $0<i\leqslant j\leqslant n$, $a>0$, $v>0$, and

$$k=\frac{n!}{(i-1)!(j-i-1)!(n-j)!}. \tag{9.7.9}$$

The p.d.f. $h(y)$ of the product

$$Y=X_{(i)}X_{(j)}$$

is then [236]

$$h(y)=\sum_{r=0}^{j-i-1}\sum_{s=0}^{i-1}(-1)^{r+s+1}\binom{j-i-1}{r}\binom{i-1}{s}$$

$$\cdot\frac{n!va^{2v(r+n-j+1)}}{(i-1)!(j-i-1)!(n-j)!(2r+n+i-2j-s+1)}$$

$$\cdot y^{v(j-r-n-1)-1}\left[1-\left(\frac{a}{\sqrt{y}}\right)^{-v(2r+n+i-2j-s+1)}\right],$$

where $0<i<j\leqslant n$, $0<u<\infty$, and $a,v>0$. Similarly, the p.d.f. $h(u)$ of the quotient

$$U=\frac{X_{(i)}}{X_{(j)}}$$

is [236]

$$h(u)=\frac{v}{a^vB(n-j+1,j-1)}u^{v(n-j+1)-1}(1-u^v)^{j-i-1},$$

with $0<i<j\leqslant n, 0<u\leqslant 1, a,v>0$.

9.7.3* Distribution of the Product and Quotient of Order Statistics from a Weibull Distribution

It is well known that the p.d.f. and c.d.f. of the Weibull distribution are expressible, respectively, in the forms

$$f(x)=\frac{\alpha}{\theta}x^{\alpha-1}\exp\left(\frac{-X^\alpha}{\theta}\right),\quad x,\alpha,\theta>0,$$

$$F(x)=1-\exp\left(\frac{-X^\alpha}{\theta}\right),\qquad x,\alpha,\theta>0.$$

Let $X_{(i)}$ and $X_{(j)}$ be the ith and jth order statistics ($i<j$) for a random sample of size n drawn from this Weibull population. Then the joint distribution $g(x_{(i)},x_{(j)})$ of these order statistics is given by

$$g(x_{(i)},x_{(j)})=K\alpha^2(x_{(i)},x_{(j)})^{\alpha-1}\exp\left[-\left\{\left(\frac{x_{(j)}}{\theta}\right)(n-j+1)+\frac{x_{(i)}^\alpha}{\theta}\right\}\right]$$

$$\cdot\left[1-\exp\left(\frac{-x_{(i)}^\alpha}{\theta}\right)\right]^{i-1}\left[\exp\left\{\left(\frac{-x_{(i)}^\alpha}{\theta}\right)\right\}-\exp\left\{\frac{-x_{(j)}^\alpha}{\theta}\right\}\right]^{j-i-1},$$

where $0<x_{(i)}<x_{(j)}<\infty$, $0<i<j<n$, $\alpha,\theta>0$, and k is given by (9.7.9). Again, letting

$$Y=X_{(i)}X_{(j)}, \qquad i<j$$

and

$$U=\frac{X_{(i)}}{X_{(j)}}, \qquad i<j,$$

Trudel and Malik [236] establish that the corresponding p.d.f.'s are, respectively,

$$\begin{aligned} h(y)=\frac{K\alpha}{\theta^2}y^{\alpha-1}\int_0^1\frac{1}{w}\exp\left\{-\left(\frac{y^{\alpha/2}}{\theta}\right)\left[\frac{n-j+1}{w}+w\right]\right\} \\ \cdot\left[1-\exp\left\{\frac{-y^{\alpha/2}w}{\theta}\right\}\right]^{i-1}\cdot\left[\exp\left\{\frac{-(y^{\alpha/2}w)}{\theta}\right\}\right. \\ \left.-\exp\left\{\frac{(-y^{\alpha/2})}{w\theta}\right\}\right]^{j-i-1} dw, 0<i<j\leqslant n, 0<u<\infty, \alpha,\theta>0 \end{aligned}$$

and

$$\begin{aligned} h(u)=k\sum_{r=0}^{j-i-1}\sum_{s=0}^{i-1}(-1)^{r+s}\binom{j-i-1}{r}\binom{i-1}{s} \\ \cdot\alpha u^{\alpha-1}\left[(n-j+r+1)+(j-i-r+s)u^{\alpha}\right]^{-2}, \end{aligned}$$

where k is given by (9.7.9). Note that for $\alpha=1$ one obtains the p.d.f.'s $h(y)$ and $h(u)$ of the product and quotient of order statistics for an exponential

9.7.4 Approximations to the Distributions of Order Statistics

Tiku and Malik [378] have obtained approximations to the distributions of order statistics based on the chi-square and t distributions, which are easy to compute and provide reasonably accurate values for the percentage points and probability integrals of the distributions. For a discussion of the derivation of these approximations, the reader is referred to their paper.

9.8* BESSEL FUNCTION DISTRIBUTIONS

It is interesting—and somewhat surprising—that Bessel function r.v.'s play a rather important role in statistics. In the first place, a number of important and well-known distributions [190] are Bessel function distributions, mostly of types I and II. Specifically, as we show later, the randomized gamma, chi-square, and noncentral chi-square distributions are special cases of the type I Bessel function with p.d.f.

$$f(x,\beta,\theta,\lambda)=Cx^{(\lambda-1)/2}e^{-\theta x}I_{\lambda-1}(\beta\sqrt{x}),\qquad x\geqslant 0,\tag{9.8.1}$$

where

$$C=\left(\frac{2}{\beta}\right)^{\lambda-1}\theta^{\lambda}e^{-\beta^2/4\theta},\qquad \theta>0,\lambda>0,\qquad \beta\geqslant 0\tag{9.8.2}$$

and I_ν is the modified Bessel function of the first kind given by

$$I_\nu(w)=\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+\nu+1)}\left(\frac{w}{2}\right)^{2m+\nu}.\tag{9.8.3}$$

Similarly, the chi, noncentral chi with two degrees of freedom, Rayleigh, and folded normal [166, p. 93] are all special cases of the type II Bessel function with p.d.f.

$$g(x,\beta,\theta,\lambda)=Dx^{\lambda}e^{-(\theta x^2)/2}I_{\lambda-1}(\beta x),\qquad x\geqslant 0,\tag{9.8.4}$$

where

$$D=\left(\frac{1}{\beta}\right)^{\lambda-1}\theta^{\lambda}e^{-\beta^2/2\theta},\qquad \theta>0,\qquad \lambda>0,\qquad \beta\geqslant 0\tag{9.8.5}$$

and $I_\nu(w)$ is defined as in (9.8.3). Moreover, various other distributions fall into the Bessel function categories,, as, for example, radial distributions in engineering [251], distributions associated with military operations research problems involving radar discrimination [255], and distributions used in the analysis of urban population problems [399]. Several distributions encountered in "randomization," such as randomized gamma distribution and distributions arising in connection with first passage problems in stochastic processes [100], are special cases of the Bessel function distributions.

The section that follows gives the distributions of sums, differences, products, quotients, and linear functions of type I Bessel i.r.v.'s, as well as the distribution of products and quotients of type II Bessel i.r.v.'s, citing important special cases relevant to distributions in statistics. The results are essentially a summary of those given in a paper by Kotz and Srinivasan [190] and are presented here[45] because this paper is not readily available.

9.9 DISTRIBUTION OF SUMS, DIFFERENCES, PRODUCTS, QUOTIENTS, AND LINEAR FUNCTIONS OF BESSEL FUNCTION INDEPENDENT RANDOM VARIABLES

Bessel functions are closely related to a number of important distributions in statistics. For example, several common distributions in statistics will be seen to be special cases of type I and type II Bessel function i.r.v.'s. Also, the reproductive property of type I Bessel i.r.v.'s will be noted. This property, together with a knowledge of the distribution of the product and quotient of type I and type II Bessel i.r.v.'s, provides a direct means of readily obtaining the distribution of sums, differences, products, and quotients of certain important i.r.v.'s (such as chi-square, noncentral chi-square, and Rayleigh r.v.'s) which are special cases of type I or type II Bessel function r.v.'s. The results in Sections 9.9.1–9.9.9 are due primarily to Laha[46] [195] and Kotz and Srinivasan [190].

9.9.1.* The Distribution of the Sum and Mean of n Bessel Type I Independent Random Variables

The p.d.f. $f(x,\beta,\theta,\lambda)$ of a Bessel type I r.v. as defined by Watson [392] has been given in (9.8.1). (Throughout this section, for simplicity, this p.d.f. is denoted by $f(x)$.) Its characteristic function is, by definition,

$$\begin{aligned} F_t(f(x)) &= \int_0^\infty e^{itx} f(x)\,dx \\ &= \text{const}\left(\frac{\beta}{2}\right)^{\lambda-1} \frac{1}{(\theta-it)^\lambda} \exp\left[\frac{\beta^2}{4(\theta-it)}\right]. \end{aligned} \tag{9.9.1}$$

Putting $t=0$ and noting that

$$F_0(f(x)) = \int_0^\infty f(x)\,dx = 1,$$

[45]With the permission of the Editor of the Institute of Statistical Mathematics, Tokyo.

[46]The results obtained by Laha in Sections 9.9.1 to 9.9.9 were derived by him in a paper entitled "On Some Properties of Bessel Function Distributions," *Bulletin of the Calcutta Mathematical Society*, Vol. 46 (1954), pp. 59–71, and are given here with the permission of the Editor of that journal.

one has

$$\left(\frac{2}{\beta}\right)^{\lambda-1}\theta^{\lambda}e^{-\beta^2/4\theta}=\text{const.} \tag{9.9.2}$$

Thus the distribution of the Bessel r.v. X will be given by

$$dF(x)=\left(\frac{2}{\beta}\right)^{\lambda-1}x^{(\lambda-1)/2}\theta^{\lambda}e^{-\beta^2/4\theta}e^{-\theta x}I_{\lambda-1}(\beta x^{1/2})\,dx, \tag{9.9.3}$$

so that the corresponding characteristic function becomes

$$F_t(f(x))=\left(1-\frac{it}{\theta}\right)^{-\lambda}\exp\frac{it\beta^2}{4\theta^2(1-it/\theta)}. \tag{9.9.4}$$

Also, since the r.v.'s are independent, the characteristic function of the p.d.f. $g(w)$ of the sum $W=X_1+X_2+\cdots+X_n$ is

$$\begin{aligned} F_t(g(w)) &= \prod_{j=1}^{n} F_t(f(x_j)) \\ &=(1-it)^{\lambda_0}\exp\left[\frac{it\beta_0^2}{4(1-it)}\right], \end{aligned} \tag{9.9.5}$$

where

$$\lambda_0=\sum_{j=1}^{n}\lambda_j,$$

$$\beta_0^2=\sum_{j=1}^{n}\beta_j^2.$$

Laha [195] inverts this characteristic function to obtain

$$g(w)=\left(\frac{4w}{\beta_0^2}\right)^{(\lambda_0-1)/2}\exp\left\{-w-\frac{\beta_0^2}{4}\right\}I_{\lambda_0-1}(\beta_0 w^{1/2}), \tag{9.9.6}$$

which is seen to be a type I Bessel function. Thus he has established the reproductive property of type I Bessel i.r.v.'s, which for convenience in future referencing is stated below in theorem form.

Theorem 9.9.1. The distribution of the sum of n type I Bessel i.r.v.'s is a type I Bessel r.v.

Laha also used the characteristic function to establish the following corollaries to Theorem 9.9.1 [195].

Corollary 9.9.1a. The distribution of the mean of samples of n i.r.v.'s drawn at random from a type I Bessel population is a type I Bessel r.v.

Corollary 9.9.1b. The sampling distribution of the mean of n i.r.v.'s drawn at random from a gamma population is a gamma distribution.

Corollaries 9.9.1a and 9.9.1b were previously derived by Bose [33] and Irwin [158], respectively.

9.9.2* The Distribution of the Sum of p Noncentral Chi-Square Independent Random Variables

Theorem 9.9.2*. The sum of p noncentral chi-square i.r.v.'s with parameters δ_j^2 and with n_j degrees of freedom $(j=1,2,\ldots,p)$ has a noncentral chi-square distribution with parameters $\delta^2=\Sigma_1^p\delta_j^2$ and with $n=\Sigma_1^p n_j$ degrees of freedom.

As Laha [195] points out, once it is established that the distribution of the noncentral chi-square r.v. is an I-type Bessel function (as Section 9.9.7 points out), Theorem 9.9.2 follows from the reproductive property of type I Bessel r.v. Tang earlier (1938) established this reproductive property of the noncentral chi-square r.v. by another method [374].

9.9.3* The Distribution of the Difference of Type I Bessel and Gamma Independent Random Variables

Theorem 9.9.3. The distribution of the difference of two type I Bessel i.r.v.'s is expressible in terms of Whittaker functions.

Laha establishes this theorem by finding the distribution of the difference $U=X_1-X_2$ of two type I Bessel i.r.v.'s X_j with p.d.f.'s

$$f(x_j)=\text{const}\, x_j^{(\lambda_j-1)/2}e^{-x_j}I_{\lambda_j-1}\left(\beta_j x_j^{1/2}\right), \qquad j=1,2. \tag{9.9.7}$$

To avoid unnecessary complication, he considers the case for which $\beta_1=\beta_2=\beta$; the Fourier transform of the p.d.f. $g(u)$ of the difference $u=x_1-x_2$ then becomes

$$F_t(g(u))=e^{-\beta^2/2}\exp\left[\frac{\beta^2}{2(1+it)(1-it)}\right](1-it)^{-\lambda_1}(1+it)^{-\lambda_2}. \tag{9.9.8}$$

Inversion of the Fourier transform (9.9.8) is expedited by using the substitution $1-it=-z/u$ and expanding the exponent in the resultant integrand as a power series in u.

Utilizing the facts [402, p. 340] that

$$\frac{1}{2\pi i}\int_{-u-i\infty}^{-u+i\infty}\frac{e^{-z}}{(-z)^{\lambda_1+r}(1+z/2u)^{\lambda_2+r}}\,dz=\frac{e^{u}(2u)^{(1/2)(\lambda_2-\lambda_1)}}{\Gamma(\lambda_1+r)}$$

$$\times W_{1/2(\lambda_1-\lambda_2),\,1/2(1-\lambda_1-\lambda_2-2r)}(2u) \tag{9.9.9}$$

and that $W_{K,m}(u)=W_{K,-m}(u)$, where $W_{K,m}(u)$ is Whittaker's function, one finds that the inversion of the characteristic function (9.9.8) yields

$$g(u)=\frac{e^{-\beta^2/2}u^{\frac{\lambda_1+\lambda_2}{2}-1}}{2^{(\lambda_1+\lambda_2)/2}}\sum_{r=0}^{\infty}\frac{\beta^{2r}u^{r}}{r!\Gamma(\lambda_1+r)2^{2r}}$$

$$\times W_{1/2(\lambda_1-\lambda_2),\,1/2(d_1+\lambda_2+2r-1)}(2u), \tag{9.9.10}$$

which establishes Theorem 9.9.3.

For the particular case $\lambda_1=\lambda_2=\lambda$, the relationship

$$W_{0,m}(2u)=\left(\frac{2u}{\pi}\right)^{1/2}K_m(u)$$

holds, where $K_m(u)$ is the modified Bessel function of the second kind (Appendix D.1), in which case (9.9.10) reduces to the form

$$g(u)=e^{-\beta^2/2}\frac{u^{(1/2)(2\lambda-1)}}{2^{(1/2)(2\lambda-1)}}\cdot\frac{1}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{2^{2r}u^{r}K_{\lambda+r-1/2}(u)}{r!\Gamma(\lambda_1+r)2^{2r}} \tag{9.9.11}$$

Corollary 9.9.3a. The p.d.f. $g(u)$ of the difference $U=X_1-X_2$ of two gamma i.r.v.'s X_1 and X_2 is expressible in terms of Whittaker functions.

As Laha points out, this result follows immediately by letting $\beta_j\to 0$ in (9.9.7), in which case (9.9.10) reduces to

$$g(u)=\frac{u^{(1/2)(2\lambda-1)}}{2^{(1/2)(2\lambda-1)}}\cdot\frac{1}{\pi^{1/2}}\frac{1}{\Gamma(\lambda)}K_{\lambda-(1/2)}(u) \tag{9.9.12}$$

and becomes the p.d.f. of the difference of two gamma i.r.v.'s.

Corollary 9.9.3b. The distribution of the difference of a type I Bessel variate and a gamma variate is expressible in terms of Whittaker functions.

Laha establishes this corollary by putting $\beta_1=\beta$ and $\beta_2=0$, thereby obtaining the characteristic function of the p.d.f. $g(u)$ of the difference $U=X_1-X_2$ of type I Bessel and gamma i.r.v.'s:

$$F_t(g(u))=e^{-\beta^2/4}\exp\frac{\beta^2}{4(1-it)}(1-it)^{-\lambda_1}(1+it)^{-\lambda_2}. \qquad (9.9.13)$$

As before, he utilizes the transformation $1-it=-z/u$, expands the exponent in the inversion integral, and integrates termwise (which is permissible, since the series is uniformly convergent), the result being

$$g(u)=e^{-\beta^2/4}\frac{u^{(1/2)(\lambda_1+\lambda_2)-1}}{2^{(1/2)(\lambda_1+\lambda_2)}}\sum_{r=0}^{\infty}\left(\frac{\beta}{2}\right)^{2r}\left(\frac{u}{2}\right)^{r/2}$$

$$\times\frac{1}{r!\Gamma(\lambda_1+r)}W_{(\lambda_1-\lambda_2+r)/2,(\lambda_1+\lambda_2+r-1)/2}(2u), \qquad (9.9.14)$$

which establishes the corollary.

Laha notes that in the particular case when $\lambda_1=\lambda_2=\lambda$, the density function (9.9.14) reduces to

$$g(u)=e^{-\beta^2/4}\frac{u^{\lambda-1}}{2^{\lambda}}\sum_{r=0}^{\infty}\left(\frac{\beta}{2}\right)^{2r}\left(\frac{u}{2}\right)^{r/2}\frac{1}{r!\Gamma(\lambda+r)}W_{(1/2)r,(1/2)(2\lambda-1+r)}(2u), \qquad (9.9.15)$$

and for $\lambda=\frac{1}{2}$, the p.d.f. (9.9.14) simplifies further to become

$$g(u)=e^{-\beta^2/4}\frac{1}{\sqrt{2\pi u}}\sum_{r=0}^{\infty}\frac{\beta^{2r}}{(2r)!}\left(\frac{u}{2}\right)^{r/2}W_{(r/2,r/2)}(2u). \qquad (9.9.16)$$

9.9.4* The Distribution of a Linear Function of Type I Bessel Independent Random Variables

Let X_j be m type I Bessel i.r.v.'s with p.d.f. $f(x_j)$ as given by (9.9.7), $j=1,2,\ldots,m$. One seeks the p.d.f. $g(u)$ of the linear function

$$u=l(l_1x_1+l_2x_2+\cdots+l_mx_m), \qquad (9.9.17)$$

where, without loss of generality, one may assume that $l_j>1$ for every j.

From (9.9.4) it follows that the characteristic function of $f(x)$ is

$$F_t(f(x_j)) = e^{-\beta_j^2/4} \sum_{r=0}^{\infty} \frac{\beta_j^{2r}}{2^{2r} r!} (1-it)^{-(\lambda_j + r)}, \qquad j = 1, 2, \ldots, m, \quad (9.9.18)$$

so that the characteristic function for the p.d.f. of $g(u/l)$, where $l^2 = \sum_{j=1}^{m} l_j^2$, is

$$F_t\left(g\left(\frac{u}{l}\right)\right) = \prod_{j=1}^{m} e^{-\beta_j^2/4} \sum_{r=0}^{\infty} \frac{\beta_j^{2r}}{2^{2r} r!} (1 - il_j t)^{-(\lambda_j + k)}$$

$$= \exp\left[-\sum_{j=1}^{m} \frac{\beta_j^2}{4}\right] \sum_{r_1, r_2, \ldots r_m = 0}^{\infty} \frac{\beta_1^{2r_1} \beta_2^{2r_2} \cdots \beta_m^{2r_m}}{2^{2r_1 + \cdots + 2r_m} r_1! \cdots r_m!}$$

$$\cdot (1 - il_1 t)^{-(\lambda_1 + r_1)} \cdots (1 - il_m t)^{-(\lambda_m + r_m)}. \quad (9.9.19)$$

Inverting this characteristic function, Laha establishes the following theorem.

Theorem 9.9.4. The p.d.f. $g(u/l)$ of the linear function (9.9.17), $l_j > 1$ for every j, is

$$g\left(\frac{u}{l}\right) = \exp\left[-\frac{1}{4} \sum_{j=1}^{m} \beta_j^2\right] \sum_{r_1, r_2, \ldots, r_m = 0}^{\infty} \frac{\beta_1^{2r_1} \cdots \beta_m^{2r_m}}{2^{2r_1 + \cdots + 2r_m} r_1! \cdots r_m!}$$

$$\cdot \sum_{r=0}^{\infty} a_r \frac{e^{-u/l} (u/l)^{\Sigma_j (\lambda_j + r_j) + r - 1}}{\Gamma(\lambda_1 + \lambda_2 + \cdots + \lambda_m + r_1 + \cdots + r_m + r)}. \quad (9.9.20)$$

Corollary 9.9.5. The p.d.f. $g(u/\lambda)$ of a linear function (9.9.17) of gamma variates, where x_j has p.d.f.

$$f(x_j) = \frac{1}{\Gamma(l)} x_j^{l_j - 1} e^{-x_j}, \qquad 0 \leqslant x_j < \infty \qquad \text{and} \qquad l_j > 1$$

is

$$g\left(\frac{u}{l}\right) = \sum_{r=0}^{\infty} a_r \frac{e^{-u/l} (u/l)^{\lambda_1 + \cdots + \lambda_m + r - 1}}{\Gamma(\lambda_1 + \cdots + \lambda_m + r)}, \qquad 0 \leqslant u < \infty,$$

where

$$a_r = \sum_{r_1+r_2+\cdots+r_n} \{a_{lr_1}a_{2r_2}\cdots a_{mr_m}\}$$

and

$$a_{rj} = l_j^{-(\lambda_j+r_j)} \sum_{r=0}^{\infty} \frac{(\lambda_j+r_j)(\lambda_j+r_j+r-1)}{r!}\left(1-\frac{1}{l_j}\right)^r,$$

$$j=1,2,\ldots,m.$$

This corollary follows immediately as the special case of Theorem 9.9.4 for which $\beta_j=0, j=1,2,\ldots,m$.

Exponentially Correlated Gamma Variables

Corollary 9.9.5 pertains to linear functions of independent gamma r.v.'s. Kotz and Adams [189] point out that the distribution of the sum of correlated gamma variables has many applications in engineering, meteorology, and insurance [188, 138, 387, 399] and therefore merits study. Gurland [138] has derived the distribution of a correlated sum of gamma r.v.'s when there is a constant correlation between each pair of variables in the sum. Kotz and Adams extend this result for the case of "an exponential autocorrelation scheme" between the variables, where each one of the variables has the marginal density given by

$$\begin{aligned} f(x) &= \left[\Gamma(r)\theta^r\right]^{-1} e^{-x/\theta} x^{r-1}, \qquad x \geqslant 0 \\ &= 0, \qquad x<0. \end{aligned} \tag{9.9.21}$$

They derive the distributions of the sum of such correlated gamma variables by utilizing the characteristic function. Specifically, they derive the distribution of the sum of identically distributed gamma r.v.'s correlated according to an "exponential" autocorrelation law

$$\rho_{kj} = \rho^{|k-j|} \qquad (k,j=1,2,\ldots,n),$$

where ρ_{kj} is the correlation coefficient between the kth and the jth random variables and $0<\rho<1$ is a given number.

To derive this distribution, they began by considering the characteristic function

$$\phi(t_1,t_2,\ldots,t_n) = |I - i\theta TV|^{-r},$$

where θ and r are positive constants, I is the $n\times n$ identity matrix, T is the $n\times n$ diagonal matrix with the elements $t_{jj}=t_j$, and V is an arbitrary $n\times n$ positive definite matrix. This characteristic function leads to a joint p.d.f. whose marginals are given by (9.9.21) and whose matrix of second moments is some positive definite matrix, denoted by V^*. If the elements of V are given by $v_{ij}=\rho^{|i-j|}$, $i,j=1,2,\ldots,n$, it is readily verified by differentiating the foregoing characteristic function $\phi(t_1,t_2,\ldots,t_n)$ that the corresponding elements of V^* will be given by

$$v_{ij}^*=r\theta^2\rho^{2|i-j|}, \qquad i,j=1,2,\ldots,n.$$

The characteristic function of the distribution of the sum of the random variables whose joint distribution has the characteristic function $\phi(t_1,t_2,\ldots,t_n)$ is

$$\phi(t)=|I-i\theta tV|^{-r},$$

which is expressible in the form

$$\phi(t)=\prod_{j=i}^{n}(1-i\theta\lambda_j t)^{-r}, \tag{9.9.22}$$

where the λ_j are the characteristic roots of the matrix V.

The distribution function of the gamma variable with positive parameters r and θ, denoted here by $F_r(\cdot)$, is given by

$$\begin{aligned} F_r(x)&=\frac{1}{\theta^r\Gamma(r)}\int_0^x e^{-u/\theta}u^{r-1}\,du, \qquad x\geqslant 0\\ &=0, \qquad x<0. \end{aligned}$$

Let Y be the r.v. whose characteristic function is given by (9.9.22). Then Y has the same distribution as the r.v.

$$X=\sum_{j=1}^{n}\lambda_j X_j,$$

where $X_j, j=1,2,\ldots,n$ are identically distributed i.r.v.'s, each following the aforementioned gamma distribution $F_r(x)$ with parameters r and θ. Using a method developed by Pitman and Robbins [310] and a theorem established by Box [36, Theorem 2.3] Kotz and Adams then readily obtain the distribution function of Y, namely,

$$\begin{aligned} P[Y<y]&=\sum_{k=0}^{\infty}c_k F_{nr+k}\left(\frac{y}{\lambda^*}\right)\\ &=\sum_{k=0}^{\infty}\frac{c_k}{\theta\Gamma(nr+k)}\int_0^{y/\lambda^*}\left(\frac{u}{\theta}\right)^{nr+k-1}e^{-u/\theta}\,du, \end{aligned}$$

where $\lambda^* = \min_j \lambda_j$ and the coefficients c_k are determined by the identity

$$\prod_{j=1}^{n}\left\{\frac{\lambda_j}{\lambda^*}\right\}\left\{\left[1-\left(1-\frac{\lambda^*}{\lambda_j}\right)Z\right]\right\}=\sum_{k=0}^{\infty} c_k Z^k.$$

The upper bound on the error of truncation, given by Pitman and Robbins [310], is

$$0 \leqslant \Pr(Y<y)-\sum_{p_1}^{p_2} c_k F_{nr+k}\left(\frac{y}{\lambda^*}\right) \leqslant 1-\sum_{p_1}^{p_2} c_k.$$

Kotz and Adams show that the characteristic roots $\lambda_j (j=1,2,\ldots,n)$ of the matrix $V=\{\rho^{|i-j|}\}$ can be calculated from the formula

$$\lambda_j=\left(1-2\rho\cos\theta_j+\rho^2\right)^{-1}(1-\rho^2), \qquad j=1,2,\ldots,n,$$

where θ_j are the values that satisfy one or the other of the equations

$$\sin\left(\frac{n+1}{2}\right)\theta=\rho\sin\left(\frac{n-1}{2}\right)\theta$$

and

$$\cos\left(\frac{n+1}{2}\right)\theta=\rho\cos\left(\frac{n-1}{2}\right)\theta.$$

9.9.5* Distribution of the Product of Two Type I Bessel Independent Random Variables

As already stated, a random variable X is said to have a type I Bessel function distribution if its p.d.f. $f(x)$ is given by (9.8.1), where $I_\nu(w)$ is the modified Bessel function of the first kind defined in Appendix D.1. The Mellin transform of $f(x)$ is

$$M_s(f(x))=C\int_0^\infty x^{s-1}x^{(\lambda-1)/2}e^{-\theta x}I_{\lambda-1}(\beta\sqrt{x}\,)\,dx.$$

On substituting the series expression (9.8.3) for $I_{\lambda-1}(\beta\sqrt{x}\,)$ and integrating termwise (since the series is clearly uniformly convergent), one obtains

$$M_s(f(x))=C\sum_{m=0}^{\infty}\frac{(\beta/2)^{2m+\lambda-1}}{m!\Gamma(m+\lambda)}\int_0^\infty e^{-\theta x}x^{s+m+\lambda-2}\,dx$$

$$=\frac{e^{-\beta^2/4\theta}}{\theta^{s-1}}\sum_{m=0}^{\infty}\frac{\left[\beta/(2\sqrt{\theta}\,)\right]^{2m}\Gamma(s+m+\lambda-1)}{m!\Gamma(m+\lambda)}. \qquad (9.9.23)$$

Consider now the product $Y = X_1 X_2$ of two type I Bessel function i.r.v.'s X_1 and X_2, and denote the p.d.f. of Y by $h(y)$. Then

$$
\begin{aligned}
M_s(h(y)) &= M_s(f(x_1)) M_s(f(x_2)) \\
&= \frac{\exp\{-[\beta_1^2/(4\theta_1)] + \beta_2^2/(4\theta_2)\}}{(\theta_1\theta_2)^{s-1}} \sum_{k=0}^{\infty} \frac{\left[\beta_1/(2\sqrt{\theta_1}\,)\right]^{2k} \Gamma(s+k+\lambda_1-1)}{k!\Gamma(k+\lambda_1)} \\
&\quad \cdot \sum_{m=0}^{\infty} \frac{\left[\beta_2/(2\sqrt{\theta_2}\,)\right]^{2m} \Gamma(s+m+\lambda_2-1)}{m!\Gamma(m+\lambda_2)} \\
&= C(\theta_1,\theta_2,\beta_1,\beta_2,s) \sum_{j=0}^{\infty} \sum_{i=0}^{j} \frac{\left[\beta_1/(2\sqrt{\theta_1}\,)\right]^{2i} \left[\beta_2/(2\sqrt{\theta_2}\,)\right]^{2(j-i)}}{i!(j-1)!\Gamma(i+\lambda_1)\Gamma(j-i+\lambda_2)} \\
&\quad \cdot \Gamma(s+i+\lambda_1-1)\Gamma(s+j-i+\lambda_2-1),
\end{aligned}
\tag{9.9.24}
$$

where

$$
C(\theta_1,\theta_2,\beta_1,\beta_2,s) = \frac{\exp\{-[\beta_1^2/(4\theta_1) + \beta_2^2/(4\theta_2)]\}}{(\theta_1\theta_2)^{s-1}}.
$$

Kotz and Srinivasan [190] have evaluated the inversion integral

$$
h(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^{-s} M_s(h(y))\, ds \tag{9.9.25}
$$

over the Bromwich path $(c - i\infty, c + i\infty)$, consisting of any line parallel to the imaginary axis and lying within the strip of analyticity of $M_s(h(y))$, to obtain the desired p.d.f. of the product variable y. Their evaluation of the inversion integral above centers primarily about the evaluation of the integral

$$
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^{-s}\Gamma(s+i+\lambda_1-1)\Gamma(s+j-i+\lambda_2-1)}{(\theta_1\theta_2)^{s-1}}\, ds, \qquad c>0. \tag{9.9.26}
$$

Letting $u = y\theta_1\theta_2$, they reduce the integral (9.9.26) to the form

$$
\frac{\theta_1\theta_2}{2\pi i} \int_{c-i\infty}^{c+i\infty} u^{-s}\Gamma(s+i+\lambda_1-1)\Gamma(s+j-i+\lambda_2-1)\, ds, \tag{9.9.27}
$$

which can now be evaluated by using a table of integral transforms [95 (17), p. 349] and becomes

$$
2\theta_1\theta_2 u^{(\lambda_1+\lambda_2+j-2)/2} K_{(\lambda_1-\lambda_2+2i-j)/2}(2\sqrt{u}\,), \tag{9.9.28}
$$

where $K_\nu(x)$ (tabulated in ref. 2) is the modified Bessel function of the third kind (Appendix D.1). Kotz and Srinivasan now complete the evaluation of the inversion integral (9.9.25), the resultant p.d.f. is

$$h(y)=2\left\{\exp\left[-\left(\frac{\beta_1^2}{4\theta_1}+\frac{\beta_2^2}{4\theta_2}\right)\right]\right\}$$

$$\times(\theta_1\theta_2)^{\frac{(\lambda_1+\lambda_2)}{2}}y^{\frac{\lambda_1+\lambda_2-2}{2}}\cdot\sum_{j=0}^{\infty}\left(\frac{\theta_1 y}{\theta_2}\right)^{j/2}\left(\frac{\beta_2}{2}\right)^{2j}$$

$$\cdot\sum_{i=0}^{j}\frac{\left[(\beta_1\sqrt{\theta_2})/(\beta_2\sqrt{\theta_1})\right]^{2i}K_{\lambda_1-\lambda_2+2i-j}(2\sqrt{\theta_1\theta_2 y})}{i!(j-i)!\Gamma(i+\lambda_1)\Gamma(j-i+\lambda_2)},\quad y\geqslant 0. \quad (9.9.29)$$

9.9.6* Distribution of the Quotient of Type I Bessel Independent Random Variables

To derive the p.d.f. $g(v)$ of the quotient $V=X_1/X_2$ of two type I Bessel i.r.v.'s, Kotz and Srinivasan [190] express the Mellin transform of $g(v)$ in terms of the Mellin transforms of the density functions X_1 and X_2:

$$M_s(g(v))=M_s(f(x_1;\beta_1,\theta_1,\lambda_1))M_{-s+2}(f(x_2;\beta_2,\theta_2,\lambda_2)). \quad (9.9.30)$$

Applying (9.9.23) to the right-hand side of (9.9.30) gives

$$M_s(g(v))=\exp\left\{-\left[\frac{\beta_1^2}{4\theta_1}+\frac{\beta_2^2}{4\theta_2}\right]\right\}\sum_{k=0}^{\infty}\frac{\left[\beta_1/(2\sqrt{\theta_1})\right]^{2k}\Gamma(s+k+\lambda_1-1)}{k!\Gamma(k+\lambda_1)\theta_1^{s-1}}$$

$$\cdot\sum_{m=0}^{\infty}\frac{\left[\beta_2/(2\sqrt{\theta})\right]^{2m}\Gamma(-s+m+\lambda_2+1)}{m!\Gamma(m+\lambda_2)\theta_2^{1-s}}$$

$$=\exp\left\{-\left[\frac{\beta_1^2}{4\theta_1}+\frac{\beta_2^2}{4\theta_2}\right]\right\}\sum_{j=0}^{\infty}\left[\frac{\beta_2}{(2\sqrt{\theta_2})}\right]^{2j}$$

$$\cdot\sum_{i=0}^{j}\left[\frac{\beta_1\sqrt{\theta_2}}{\beta_2\sqrt{\theta_1}}\right]^{2i}\frac{1}{i!(j-i)!\Gamma(i+\lambda_1)\Gamma(j-i+\lambda_2)}$$

$$\times\frac{\Gamma(s+i+\lambda_1-1)\Gamma(-s+j-i+\lambda_2+1)}{\theta_1^{s-1}\theta_2^{1-s}}. \quad (9.9.31)$$

Kotz and Srinivasan accomplished the inversion of the Mellin transform (9.9.31) by noting that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}\frac{\Gamma(s+i+\lambda_1-1)\Gamma(-s+j-i+\lambda_2+1)}{\theta_1^{s-1}\theta_2^{1-s}}\,ds$$

$$=\frac{\theta_1}{\theta_2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{y\theta_1}{\theta_2}\right)^{-s}\Gamma(s+i+\lambda_1-1)\Gamma(-s+j-i+\lambda_2+1)\,ds, \tag{9.9.32}$$

which is seen [95 (15), p. 349] to be equivalent to

$$\frac{\theta_1}{\theta_2}\Gamma(\lambda_1+\lambda_2+j)\left(\frac{y\theta_1}{\theta_2}\right)^{i+\lambda_1-1}\left(1+\frac{y\theta_1}{\theta_2}\right)^{-\lambda_1-\lambda_2-j}.$$

Using this result, they then completed the inversion of the Mellin transform (9.9.31), obtaining

$$g(v)=\frac{\exp\{-[\beta_1^2/(4\theta_1)+\beta_2^2/(4\theta_2)]\}y^{\lambda_1-1}\theta_1^{\lambda_1}\theta_2^{\lambda_2}}{(\theta_1 y+\theta_2)^{\lambda_1+\lambda_2}}$$

$$\cdot\sum_{j=0}^{\infty}\left(\frac{\beta_2}{2}\right)^{2j}\frac{\Gamma(\lambda_1+\lambda_2+j)}{(\theta_1 y+\theta_2)^j}\sum_{i=0}^{j}\left(\frac{\beta_1}{\beta_2}\right)^{2i}\frac{y^i}{i!(j-i)!\Gamma(i+\lambda_1)\Gamma(j-i+\lambda_2)},$$

$$y\geqslant 0. \tag{9.9.33}$$

9.9.7 Special Cases of Type I Bessel Functions

Kotz and Srinivasan [190] have identified the following distributions as special cases of type I Bessel functions.

Chi-Square distribution

Setting $\beta=0$, $\lambda=n/2$, and $\theta=\frac{1}{2}$ in (9.8.1), one obtains the well-known chi-square p.d.f. with n degrees of freedom (cf. Table D.2, probability law 5).

Distribution of Product and Quotient of Two Chi-Square Independent Random Variables

Setting $\beta_1=\beta_2=0$, $\lambda_1=n_1/2, \lambda_2=n_2/2, \theta_1=\theta_2=\frac{1}{2}$ in (9.9.29) yields the p.d.f. $h(y)$ of the product $Y=X_1X_2$ of two independent chi-square variables with n_1 and n_2 degrees of freedom, namely,

$$h(y)=\frac{y^{[(n_1/4)+(n_2/4)-1]}K_{(n_1-n_2)/4}(\sqrt{y})}{2^{[(n_1+n_2/2-1)]}\Gamma(n_1/2)\Gamma(n_2/2)}, \qquad y\geqslant 0. \tag{9.9.34}$$

By making the same substitutions in (9.9.33), one arrives at the p.d.f. $g(y)$ of the quotient $Y=X_1/X_2$ of two chi-square i.r.v.'s with n_1 and n_2 degrees of freedom, namely,

$$g(y)=\frac{\Gamma(n_1/2+n_2/2)}{\Gamma(n_1/2)\Gamma(n_2/2)}\frac{y^{(n_1/2)-1}}{(1+y)^{(n_1+n_2)/2}}, \qquad y\geqslant 0, \tag{9.9.35}$$

which is the well-known F distribution with n_1,n_2 degrees of freedom (see Section 9.2.4).

Noncentral Chi-Square Distribution

Putting $\lambda=n/2$ and $\theta=\frac{1}{2}$ in (9.8.1), one obtains the p.d.f. of the noncentral chi-square variable χ'^2 with noncentrality parameter β and n degrees of freedom:

$$g(\chi'^2)=\frac{1}{2\beta^{(n/2)-1}}e^{-(\beta^2+\chi'^2)/2}(\chi'^2)^{((n/2)-1)/2}I_{(n/2)-1}\left(\beta\sqrt{\chi'^2}\right), \qquad \chi'^2\geqslant 0. \tag{9.9.36}$$

Distribution of Product and Quotient of a Noncentral Chi-Square (β_1, n_1) Variable and a Noncentral Chi-Square (β_2, n_2) Variable

Substituting $\lambda_1=n_1/2, \lambda_2=n_2/2, \theta_1=\theta_2=\frac{1}{2}$ in (9.9.29) and (9.9.33), one arrives at the p.d.f.'s $h(z)$ and $g(y)$ for the product and quotient, respectively, of a chi-square (β_1,n_1) and a chi-square (β_2,n_2) variate where β_1, β_2 are noncentrality parameters and n_1,n_2 denote degrees of freedom. Specifically,

$$h(y)=\exp\left[-\left(\frac{\beta_1^2}{2}+\frac{\beta_2^2}{2}\right)\right]\frac{y(n_1/4+n_2/4)-1}{2(n_1/2+n_2/2)-1}\sum_{j=0}^{\infty}\left(\frac{\beta_2}{2}\right)^{2j}y^{j/2}$$
$$\times\sum_{i=0}^{j}\frac{(\beta_1/\beta_2)^{2i}K_{((n_1/2)-(n_2/2)+2i-j)/2}\sqrt{y}}{i!(j-i)!\Gamma(n_1/2+i)\Gamma(n_2/2+j-i)}, \qquad y\geqslant 0 \tag{9.9.37}$$

and

$$g(v)=\frac{\exp-\{(\beta_1^2/2+\beta_2^2/2)\}\,v^{n_1-1/2}}{(1+v)^{(n_1+n_2)}/2}\sum_{j=0}^{\infty}\left(\frac{\beta_2}{\sqrt{2}}\right)^{2j}\Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}+j\right)$$

$$\cdot\sum_{i=0}^{j}\frac{(\beta_1/\beta_2)^{2i}v^i}{i!(j-i)!\Gamma(n_1/2+i)\Gamma(n_2/2+j-i)},\qquad v\geqslant 0,\qquad (9.9.38)$$

where $K_\nu(\cdot)$ is the modified Bessel function of the third kind (Appendix D.1).

Randomized Gamma Distribution

On setting $\beta=2\sqrt{\mu}$, $\theta=1$, and $\lambda=\rho+1$ in (9.8.1), one obtains the p.d.f. $f(x)$ of the randomized gamma distribution [100]:

$$f(x)=e^{-(\mu+x)}\sqrt{\left(\frac{x}{\mu}\right)^{\rho}}\;I_\rho(2\sqrt{\mu x}\,),\qquad x\geqslant 0,\qquad (9.9.39)$$

where $I_\rho(\cdot)$ is the modified Bessel function of the first kind (Appendix D.1).

Distribution of Product and Quotient of Two Independent Randomized Gamma Variates

To obtain the distribution of the product $Y=X_1X_2$ and quotient $V=X_1/X_2$ of two independent randomized gamma variates with respective parameters $\mu_1,\rho_1,\mu_2,\rho_2$, one makes the substitutions $\beta_i=2\sqrt{\mu_i}$ $\theta_i=1$, and $\lambda_i=\rho_i+1$, $i=1,2$, in (9.9.29) and (9.9.32), respectively. The p.d.f. for the product is then found to be

$$h(y)=2e^{-(\mu_1+\mu_2)}y^{(\rho_1+\rho_2)/2}\sum_{j=0}^{\infty}y^{j/2}\mu_2^j$$

$$\cdot\sum_{i=0}^{j}\frac{(\mu_2/\mu_1)^iK_{(\rho_1-\rho_2+2i-j)/2}(2\sqrt{y}\,)}{i!(j-i)!\Gamma(i+\rho_1+1)\Gamma(j-i+\rho_2+1)},\qquad y\geqslant 0,\quad (9.9.40)$$

where $K_\nu(\cdot)$ is the modified Bessel function of the third kind (Appendix D.1), whereas that for the quotient is

$$g(v)=\frac{e^{-(\mu_1+\mu_2)}v^{\rho_1}}{(1+v)^{\rho_1+\rho_2+2}}\sum_{j=0}^{\infty}\frac{\mu_2^j\Gamma(\rho_1+\rho_2+2+j)}{(1+v)^j}$$

$$\cdot\sum_{i=0}^{j}\frac{\left(\dfrac{\mu_1}{\mu_2}\right)^i v^i}{i!(j-i)!\Gamma(i+\rho_1+1)\Gamma(j-i+\rho_2+1)},\qquad v\geqslant 0. \tag{9.9.41}$$

9.9.8* Distribution of Product and Quotient of Type II Bessel Independent Random Variables

An r.v. is said to have a type II Bessel function distribution if its p.d.f. is given by

$$f(x;\beta,\theta,\lambda)=Dx^{\lambda}e^{-\theta x^2/2}I_{\lambda-1}(\beta x),\qquad x\geqslant 0, \tag{9.9.42}$$

where

$$D=\left(\frac{1}{\beta}\right)^{\lambda-1}\theta^{\lambda}e^{-\beta^2/2\theta},\qquad \theta>0,\qquad \lambda>0,\qquad \beta\geqslant 0 \tag{9.9.43}$$

and $I_\nu(w)$ is the modified Bessel function of the first kind (Appendix D.1).

The p.d.f.'s $h(y)$ and $g(v)$ of the product $Y=X_1X_2$ and quotient $V=X_1/X_2$ of two type II Bessel i.r.v.'s are obtained by following the same procedures as were used to determine the p.d.f.'s of the product and quotient of two type I Bessel i.r.v.'s. The results below were established by Kotz and Srinivasan [190] and are stated here without proof.

$$h(y)=\exp\left[-\left(\frac{\beta_1^2}{2\theta_1}+\frac{\beta_2^2}{2\theta_2}\right)\right](\Theta_1\Theta_2)^{(\lambda_1+\lambda_2)/2}\frac{y^{\lambda_1+\lambda_2-1}}{2^{\lambda_1+\lambda_2-2}}$$

$$\cdot\sum_{j=0}^{\infty}\left(\frac{\beta_2}{\sqrt{2}}\right)^{2j}\left(\frac{\theta_1}{\theta_2}\right)^{j/2}y^j$$

$$\cdot\sum_{i=0}^{j}\frac{(\beta_1/\beta_2)^{2i}(\Theta_2/\Theta_1)^iK_{(\lambda_1-\lambda_2+2i-j)/2}\left(y\sqrt{\theta_1\theta_2}\right)}{i!(j-i)!\Gamma(i+\lambda_1)\Gamma(j-i+\lambda_2)},\qquad y\geqslant 0 \tag{9.9.44}$$

and

$$g(v)=2\exp\left[-\left(\frac{\beta_1^2}{2\theta_1}+\frac{\beta_2^2}{2\Theta_2}\right)\right]\frac{\theta_1^{\lambda_1}\theta_2^{\lambda_2}y^{2\lambda_1-1}}{(\theta_1 y^2+\theta_2)^{\lambda_1+\lambda_2}}$$

$$\cdot\sum_{j=0}^{\infty}\frac{(\beta_2/\sqrt{2})^{2j}\Gamma(\lambda_1+\lambda_2+j)}{(\theta_1 y^2+\theta_2)^j}$$

$$\cdot\sum_{i=0}^{j}\frac{(\beta_1/\beta_2)^{2i}v^{2i}}{i!(j-i)!\Gamma(i+\lambda_1)\Gamma(j-i+\lambda_2)},\qquad v\geqslant 0,\quad (9.9.45)$$

where $K_\nu(\cdot)$ is the modified Bessel function of the third kind (Appendix D.1).

9.9.9 Special Cases of Type II Bessel Functions

As Kotz and Srinivasan [190] note, the distributions listed below are specific cases of type II Bessel functions.

Chi-Distribution

Setting $\beta=0$, $\theta=\theta'/\sigma^2$, and $\lambda=\theta'$ (then writing θ for θ') in (9.9.42) gives

$$g(x;\theta,\sigma^2)=\frac{2}{\Gamma(\theta)}\left(\frac{\theta}{2\sigma^2}\right)^{\theta}x^{2\theta-1}e^{-\theta x^2/(2\sigma^2)},\qquad x\geqslant 0,\qquad (9.9.46)$$

which can then be transformed into the chi distribution with n degrees of freedom by means of the substitutions $\theta=n/2$ and $\sigma=\frac{1}{2}$.

Distribution of Product and Quotient of Chi Independent Random Variables

Let X_1 and X_2 be i.r.v.'s having the p.d.f. (9.9.46) with parameters θ_1, σ_1^2 and θ_2, σ_2^2, respectively. Then, by making corresponding substitutions in (9.9.44), one obtains the p.d.f. $h(y)$ of $Y=X_1X_2$

$$h(y)=\frac{2\left[\sqrt{\theta_1\theta_2}\,/(\sigma_1\sigma_2)\right]^{\theta_1+\theta_2}(y/2)^{\theta_1+\theta_2-1}}{\Gamma(\theta_1)\Gamma(\theta_2)}$$

$$\times K_{(\theta_1-\theta_2)/2}\frac{y\sqrt{\theta_1\theta_2}}{\sigma_1\sigma_2},\qquad y\geqslant 0,\qquad (9.9.47)$$

where again $K_\nu(\cdot)$ is the modified Bessel function of the third kind. Similarly, for the quotient $V=X_1/X_2$ one has from (9.9.45) the p.d.f. of V:

$$g(v)=\frac{2\Gamma(\theta_1+\theta_2)(\theta_1\sigma_2^2)^{\theta_1}(\theta_2\sigma_1^2)^{\theta_2}v^{2\theta_1-1}}{\Gamma(\theta_1)\Gamma(\theta_2)(\theta_1\sigma_2^2v^2+\theta_2\sigma_1^2)^{\theta_1+\theta_2}}, \qquad v\geqslant 0. \qquad (9.9.48)$$

Kotz and Srinivasan point out [190] that the Maxwell-Boltzmann and Rayleigh distributions, which are especially useful as radial distributions in engineering and physical problems, are special cases of the distribution (9.9.46).

Distribution of Product and Quotient of the Noncentral Chi Distribution

If one lets $\beta=\beta'/\sigma^2$, $\theta=1/\sigma^2$, and $\lambda=1$ in (9.9.42), the result is the noncentral chi distribution with p.d.f. with two degrees of freedom:

$$f(x;\beta,\sigma^2)=\frac{x}{\sigma^2}\exp\left\{-\frac{1}{2\sigma^2}(\beta^2+x^2)\right\}I_0\left(\frac{\beta x}{\sigma^2}\right), \qquad x\geqslant 0. \ (9.9.49)$$

It follows from (9.9.44) and (9.9.45), respectively, that the p.d.f.'s $h(y)$ and $g(v)$ of the product $Y=X_1X_2$ and quotient $V=X_1/X_2$ of two independent noncentral chi variables are

$$h(y)=\exp\left\{-\left[\frac{\beta_1^2}{2\sigma_1^2}+\frac{\beta_2^2}{2\sigma_2^2}\right]\right\}\frac{y}{\sigma_1^2\sigma_2^2}\sum_{j=0}^{\infty}\left[\frac{\beta_2}{\sqrt{2}\,\sigma_2^2}\right]^{2j}(y^{\sigma_2/\sigma_1})^j$$

$$\cdot\sum_{i=0}^{j}\frac{[\beta_1\sigma_2/(\beta_2\sigma_1)]^{2i}}{(i!)^2((j-i)!)^2}K_{i-(j/2)}\left(\frac{y}{\sigma_1\sigma_2}\right), \qquad y\geqslant 0 \qquad (9.9.50)$$

$$g(v)=2\sigma_1^2\sigma_2^2\frac{\exp\{-[\beta_1^2/(2\sigma_1^2)+\beta_2^2/(2\sigma_2^2)]\}v}{(\sigma_2^2v^2+\sigma_1^2)^2}$$

$$\cdot\sum_{j=0}^{\infty}\frac{[\beta_2\sigma_1/\sqrt{2}\,\sigma_2]^{2j}(j+1)!}{(\sigma_2^2v^2+\sigma_1^2)^j}$$

$$\cdot\sum_{i=0}^{j}\frac{(\beta_1 v/\beta_2)^{2i}(\sigma_2/\sigma_1)^{4i}}{(i!)^2((j-i)!)^2}, \qquad v\geqslant 0. \qquad (9.9.51)$$

Folded or Half-Gaussian Distribution

On making the substitutions $\beta=0$, $\theta=1/\sigma^2$, and $\lambda=\frac{1}{2}$ in (9.9.42), one has

$$f(x;\sigma^2)=\frac{2}{\sigma\sqrt{2}}e^{-x^2/(2\sigma^2)}, \qquad 0\leqslant x<\infty,$$

which is the folded or half-Gaussian distribution. The p.d.f.'s of the product $Y=X_1X_2$ and the quotient $V=X_1/X_2$ of two folded Gaussian i.r.v.'s are, respectively,

$$h(y)=\frac{2}{\pi\sigma_1\sigma_2}K_0\left(\frac{y}{\sigma_1\sigma_2}\right), \qquad y\geqslant 0 \tag{9.9.52}$$

and

$$g(v)=\frac{2\sigma_1\sigma_2}{\pi(\sigma_1^2+\sigma_2^2v^2)}, \qquad v\geqslant 0. \tag{9.9.53}$$

Other Distributions Related to Type II Bessel Variables

Kotz and Srinivasan [190] have shown that the generalized gamma r.v. as defined by Stacy (see [363]) is also a type II Bessel variable. They used the methods of this section to derive the p.d.f.'s of the product and quotient of two generalized gamma i.r.v.'s but did not publish their results. Malik [231] derived the p.d.f. $h(u)$ of the product $U=X_1X_2$ of two generalized gamma i.r.v.'s X_1,X_2 having p.d.f.'s with the same shape parameter $p_1=p_2=p$:

$$f(x_j)=\left(p\Gamma\left(\frac{d_j}{p}\right)a_j^{d_j}\right)^{-1}x_j^{d_j-1}e^{-(x_j/a_j)p},$$

$$x>0, \qquad a_j,d_j,p_j>0, \qquad j=1,2. \tag{9.9.54}$$

In particular, he utilized the Mellin convolution

$$\begin{aligned}h(u)&=\int_0^\infty\left(\frac{1}{x_2}\right)f_1(x_2)f_2\left(\frac{u}{x_2}\right)dx_2\\&=p^2u^{d_2-1}\left(\Gamma\left(\frac{d_1}{p}\right)\Gamma\left(\frac{d_2}{p}\right)a_1^{d_1}a_2^{d_2}\right)^{-1}\\&\quad\cdot\int_0^\infty\exp\left(-\left(\frac{x_2}{a_1}\right)^p-\left(\frac{u}{a_2x_2}\right)^p\right)x_2^{-(d_2-d_1+1)}dx_2.\end{aligned}$$

Making the transformation $(x_2/a_1)^p = t$ and noting that

$$\int_0^\infty e^{-t-z^2/4t}t^{-(\nu+1)}\,dt = 2\left(\frac{z}{2}\right)^{-\nu}K_\nu(z),$$

where $K_\nu(z)$ is the modified Bessel function of the second kind of order ν (Appendix D.1), he showed that the foregoing density function reduces to

$$h(u) = 2pu^{d_2-1}\left[\Gamma\left(\frac{d_1}{p}\right)\Gamma\left(\frac{d_2}{p}\right)(a_1a_2)^{d_2}\right]^{-1}\left\{\left(\frac{u}{a_1a_2}\right)^{p/2}\right\}^{-(d_2/p)-(d_1/p)}$$
$$\times\left[\mathbf{K}_{(d_2/p)-(d_1/p)}\left(2\left(\frac{u}{a_1a_2}\right)\right)^{p/2}\right].$$

Making the transformation $Z = 2(t/a_1a_2)^{p/2}$ and using the fact [389, p. 80] that for some integer $n \geqslant 0$

$$K_{n+(i/2)}(Z) = \left(\frac{\pi}{2Z}\right)^{1/2}e^{-Z}\sum_{r=0}^{n}(n+r)!\left[r!(n-r)!(2Z)^r\right]^{-1},$$

Malik showed that the distribution function above becomes

$$H(u) = 2^{2-(d_1/p)-(d_2/p)}\left[\Gamma\left(\frac{d_1}{p}\right)\Gamma\left(\frac{d_2}{p}\right)\right]^{-1}\left(\frac{\pi}{2}\right)^{1/2}$$
$$\cdot\sum_{r=0}^{n}(n+r)!\left[r!(n-r)!2^r\right]^{-1}\Gamma\left(\frac{d_1}{p}+\frac{d_2}{p}-r-\frac{1}{2}\right)$$
$$\cdot I\left[2\left(\frac{u}{a_1a_2}\right)^{p/2}, \frac{d_1}{p}+\frac{d_2}{p}-r-\frac{3}{2}\right],$$

where

$$I(v,p) = \frac{\int_0^v e^{-v}v^p\,dv}{\int_0^\infty e^{-v}v^p\,dv} \tag{9.9.55}$$

is the incomplete gamma function, which has been tabulated by Pearson [283]. (It should not be confused with the well-known gamma function, previously defined by (4.4.43).)

Similarly, by using a Mellin (quotient) convolution, Malik [230] derived the quotient of two generalized gamma i.r.v.'s. Using characteristic functions, he later derived [235] the exact distribution of a linear function and the ratio of two such linear functions of generalized gamma i.r.v.'s X_j whose p.d.f.'s are given by (9.9.54). Denoting the linear function by

$$Y=\sum_{j=1}^{n}\lambda_j X_j$$

and its p.d.f. by $g(y)$, he found its characteristic function to be

$$F_t(g(y))=\sum_{r_1,r_2,\dots,r_n}^{\infty} P_n(a_j,d_j,p_j,r_j),$$

where

$$P_n(a_j,d_j,p_j,r_j)=\prod_{j=1}^{n}\frac{\Gamma\left(\dfrac{d_j}{p_j}+\dfrac{r_j}{p_j}\right)a_j^{r_j}(i\lambda_j t)^{r_j}}{r_j!\,\Gamma\left(\dfrac{d_j}{p_j}\right)}.$$

Making use of the fact that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ity}(it)^{R_n}\,dt$$
$$=\sum_{r=0}^{R_n}\binom{R_n}{r}(-1)^{R_n}\left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty}e^{-ity}(1-it)^{-(r-R_n)}\,dt$$

(where $R_n=r_1+r_2+\cdots+r_n$) and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}(1-it)^{-r}\,dt=\frac{e^{-x}x^{r-1}}{\Gamma(r)},$$

he then inverted this characteristic function and obtained

$$g(y)=\sum_{r_1,r_2,\dots,r_n}^{\infty}P_n(a_j,d_j,p_j,r_j)\sum_{r=0}^{R_n}\binom{R_n}{r}(-1)^{r+R_n}\frac{e^{-y}y^{r-R_n-1}}{\Gamma(r-R_n)}.$$

His derivation of the p.d.f. $h(u)$ of the ratio

$$U=\frac{\sum_{j=1}^{m}\lambda_j X_j}{\sum_{j=1}^{n}\lambda_j' X_j'},$$

where X_j and X_j' are generalized gamma i.r.v.'s having the p.d.f. (9.9.54), is somewhat longer and is not given here. Incidentally, Malik points out that there are several problems in the physical sciences calling for the distribution of a product, quotient, linear function, and the ratio of two independent linear functions of generalized gamma i.r.v.'s.

9.10* THE DISTRIBUTION OF A LINEAR COMBINATION OF A FINITE NUMBER OF TRUNCATED INDEPENDENT EXPONENTIAL VARIABLES

Using the Laplace transform, Nath [269] has derived the distribution of any linear combination of a finite number of truncated exponential variates from n populations, either distinct or identical. Specifically, he has shown that if X_i, $i=1,2,\ldots,n$ are truncated exponential i.r.v.'s with

$$\begin{aligned} f(x_i) &= \alpha_i(1-e^{-\alpha_i\theta})^{-1}e^{-\alpha_i x_i}, \qquad \alpha_i>0, 0<x_i<\theta_i<\infty \\ &=0, \qquad \text{elsewhere,} \end{aligned}$$

then the p.d.f. of the sum

$$Y=\sum_{i=1}^{n} C_i X_i,$$

where the C_i's are arbitrary positive constants, is

$$\begin{aligned} g(y)=\prod_{i=1}^{n}(K_i\delta_i)\Bigg[\psi(y)-\sum_{j_1=1}^{n} {}^{-\delta_{j_1}\phi d_{j_1}}\psi(y-\phi_{j_i}) \\ +\sum_{j_1<j_2=1}^{n} e^{-(\delta_{j_1}\phi_{j_1}+\delta_{j_2}\phi_{j_2})}\psi(y-\phi_{j_1}-\phi_{j_2}) \\ +\cdots+(-1)^{n-1}\sum_{j_1<j_2<\ldots<j_{n-1}=1}^{n}\exp\left(-\sum_{h=1}^{n-1}\delta_{j_h}\phi_{j_h}\right)\psi\left(y-\sum_{h=1}^{n-1}\phi_{j_h}\right)\Bigg], \end{aligned}$$

where

$$\begin{aligned} K_i &= (1-e^{-\alpha_i\theta_i})^{-1}, \\ \phi_i &= C_i\theta_i, \\ \delta_i &= \frac{\alpha_i}{C_i}, \qquad C_i>0, \\ \psi(*) &= \sum_{i=1}^{n}\left\{e^{-*\delta_i}\prod_{\substack{t=1 \\ t\neq i}}^{n}{}' \frac{1}{\delta_t-\delta_i}\right\}H(*), \end{aligned}$$

and $H(*)$ denotes the function that is one whenever its argument (*) is nonnegative and zero otherwise. This result was accomplished by writing the Laplace transform in the form

$$L_r(g(y)) = \prod_{i=1}^{n} K_i\delta_i \left[1 - \frac{\exp\{-(r+\delta_i)\phi_i\}}{r+\delta_i} \right]$$

in series form and evaluating each term of the corresponding Laplace inversion integral by contour integration or by using a standard table of Laplace transforms. (In this derivation, a lower bound of zero on C_i and x_i was chosen to avoid certain technical difficulties.)

When the i.r.v.'s X_i, $i=1,2,\ldots,n$ are identically distributed with $\theta_i=\theta$, $\alpha_i=\alpha$, and $C_i\equiv 1$ for all i, the p.d.f. above of the sum Y simplifies to

$$g(y) = \frac{(K\alpha)^n e^{-\alpha y}}{(n-1)!} \sum_{m=0}^{\nu} (-1)^m \binom{n}{m} (y-m\theta)^{n-1}, \qquad \nu\theta < y < (\nu+1)\theta.$$

9.11* DISTRIBUTION OF THE PRODUCT OF GENERALIZED F-VARIABLES

Recently, Shah and Rathie [331] have derived the exact distribution of the product of generalized F-variables, an r.v. X being defined as a generalized F-variable if it has the p.d.f.

$$f(x;p,m,\alpha,h) = \frac{kx^{p-1}}{(1+\alpha x^h)^m}, \qquad \alpha,m,p,h,x > 0,$$

where

$$k = \frac{h\alpha^{p/h}}{B(p/h, m-p/h)}, \qquad m > \frac{p}{h}$$

and B denotes the well-known beta function (see (4.4.42a, b)). (Hereafter, $f(x;p,m,\alpha,h)$ will be denoted by the simpler form $f(x)$.) Their result is stated below in the form of a theorem, and was obtained by using the Mellin transform and the residue theorem.

Theorem 9.11.1. Let $X_1, X_2, \ldots, X_n$ be n independent generalized F-variables with p.d.f.

$$f(x_i) = \frac{k_i x_i^{p_i-1}}{(1+\alpha_i x_i^{h_i})^{m_i}}, \qquad \alpha_i, m_i, p_i, h_i, x_i > 0; \qquad i=1,2,\ldots,n,$$

where

$$k_i = \frac{h_i \alpha_i^{p_i/h}}{B(p_i/h_i, m_i - (p_i/h_i))}, \qquad m_i > \frac{p_i}{h_i} \qquad \text{for all } i.$$

The p.d.f. $h(y)$ of the product $Y = \Pi_{i=1}^{n} X_i$ is the H-function

$$h(y) = C\,\mathbf{H}_{n,n}^{n,n}\left[y \prod_{1}^{n} \alpha_i^{1/h_i} \left| \begin{matrix} \left\{1-m_1+\dfrac{p_1-1}{h_1}, \dfrac{1}{h_1}\right\}, \ldots, \left\{1-m_n+\dfrac{p_n-1}{h_n}, \dfrac{1}{h_n}\right\} \\ \left\{\dfrac{p_1-1}{h_1}, \dfrac{1}{h_1}\right\}, \ldots, \left\{\left(\dfrac{p_n-1}{h_n}\right), \dfrac{1}{h_n}\right\} \end{matrix} \right. \right],$$

for $\alpha_i, m_i, p_i, h_i > 0$, and

$$C = \prod_{i=1}^{n} \frac{\alpha_i^{1/h_i}}{\{(p_i/h_i)\Gamma(m_i - p_i/h_i)\}}.$$

When all the h_i have identical values, say $h_i = h$, $i = 1, 2, \ldots, n$, the p.d.f. $h(y)$ reduces to a Meijer G-function:

$$h(y) = Ch\,\mathbf{G}_{n,n}^{n,n}\left[y^h \prod_{i=1}^{n} \alpha_i \left| \begin{matrix} 1-m_1+\dfrac{p_1-1}{h}, \ldots, 1-m_n-\dfrac{p_n-1}{h} \\ \dfrac{p_1-1}{h}, \ldots, \dfrac{p_n-1}{h} \end{matrix} \right. \right]$$

for $\alpha_i, m_i, p_i, h > 0$.

Shah and Rathie point out the following special cases of the generalized F-distribution:

1. When $h = 1, m = (m_1 + m_2)/2, \alpha = m_1/m_2, p = m_1/2$ with m_1, m_2 positive integers, the generalized F-function becomes the p.d.f. for the product of Snedecor F i.r.v.'s (see Section 9.2.4).
2. When $h = 1, \alpha = 1, p > 0, m > p$, the generalized F distribution reduces to the p.d.f. for the product of beta i.r.v.'s of the second kind (see Exercise 4.25).
3. When $\lambda = 2, \alpha = 1/n, p = 1, m = (n+1)/2$, one obtains the p.d.f. of the product of n independent folded Student i.r.v.'s (see (6.3.14)).
4. When $h = 2, p = 1, m = 1, \alpha = 1$, the generalized F p.d.f. becomes the p.d.f. of folded Cauchy i.r.v.'s (see [166, p. 163]).

9.12* THE DISTRIBUTION OF PRODUCTS OF POWERS OF GENERALIZED DIRICHLET COMPONENTS

The r.v. X has been defined by Stacy [363] as having a generalized gamma distribution if $P(X<0)=0$ and for $x \geqslant 0$, a,d,p positive constants, the p.d.f. is given by

$$f(x)=\frac{px^{d-1}e^{-(x/a)^p}}{a^d\Gamma(d/p)}.$$

Now let X_i be independent generalized gamma r.v.'s with corresponding parameters (a_i,d_i,p_i). Also, let

$$Y_i=\frac{X_i}{X_1+X_2+\ldots+X_{N+1}}, \qquad i=1,\ldots,N+1. \tag{9.12.1}$$

The vector $(Y_1,Y_2,\ldots,Y_N)$ is then said to have a generalized Dirichlet distribution [72, 168]. Rogers and Young [312] have proved the following theorem concerning the p.d.f. of $(Y_1,Y_2,\ldots,Y_N)$.

Theorem 9.12.1. In (9.12.1), $Y_1 \geqslant 0$, $Y_1+\cdots+Y_{N+1}=1$, and the p.d.f. of the vector $(Y_1+\cdots+Y_N)$ is

$$(a) \qquad \left[\frac{\Pi_{i=1}^{N+1} p_i y_i^{d_i-1}}{a_i^{d_i-1}\Gamma(d_i/p_i)}\right]\cdot I$$

where

$$I=\int_0^\infty z^{d_i-1}\exp\left[-\Sigma\left(\frac{y_iz}{a_i}\right)^{p_i}\right]dz$$

$$=\left(\frac{1}{p_{N+1}}\right)\sum_{j=0}^{\infty}(-1)^j\sum_{r_1+\ldots+r_N=j}\left[\frac{\Pi_{t=1}^N(y_t/a_t)^{p_tr_t}}{r_t!}\right]$$

$$\times\left(\frac{a_{N+1}}{y_{N+1}}\right)^{\Sigma_i d_i+\sum_{t=1}^N p_tr_t}\Gamma\left[\frac{(\Sigma_i d+\Sigma_{t=1}^N p_tr_t)}{p_{N+1}}\right],$$

$$(b) \qquad \frac{p^N\Gamma(\Sigma_i d_i/p)\Pi_i y_i^{d_i-1}}{\Pi_i a_i^{d_i}\Gamma(d_i/p)\{\Sigma_i(y_i/a_i)^p\}^{\Sigma_i d_i/p}}$$

when $p_1 = \ldots = p_{N+1} = p$;

$$(c) \qquad \frac{p^N \Gamma(\Sigma_i d_i/p) \Pi_i y_i^{d_i - 1}}{\Pi_i \Gamma(d_i/p)(\Sigma_i y_i^p)^{\sum_i d_i/p}}$$

when $p_1 = \cdots = p_{N+1} = p$ and $a_1 = \cdots = a_{N+1} = a$;

$$(d) \qquad \frac{\Pi_i y_i^{d_i - 1}}{\Gamma(d_i)} \Gamma\left(\sum_i d_i\right),$$

when

$$p_1 = \cdots = p_{N+1} = 1 \qquad \text{and} \qquad a_1 = \cdots = a_{N+1} = a.$$

In the case when the p_i are all equal to p, the independent variables involved have the same shape parameter and when this p has the value 1, the X_i are just two parameter gamma variables. The density in (d) then reduces to that of the ordinary Dirichlet distribution, and its derivation from gamma variables is well known [168].

Rogers and Young have also proved the theorem stated below concerning the distribution of products of powers of generalized Dirichlet components [312].

Theorem 9.12.2. If $(Y_1, \ldots, Y_N)$ is a Dirichlet r.v. with parameters $d_1, \ldots, d_{N+1}$, and $k_1, \ldots, k_{N+1}$ are nonnegative, then the density of $\Pi_i Y_i^{k_i}$, where $Y_{N+1} = 1 - Y_1 - \cdots - Y_N$, is given in terms of an H-function as

$$\frac{\Gamma(\Sigma_i d_i)}{\Pi_i \Gamma(d_i)} \mathbf{H}_{1,N+1}^{N+1,0}\left[x \,\middle|\, \begin{matrix} \left(\sum_i d_i - \Sigma k_i, \sum_i k_i\right) \\ (d_1 - k_1, k_1), \ldots, (d_{N+1} - k_{N+1}, k_{N+1}) \end{matrix} \right].$$

The proof, which depends on an auxiliary theorem, is not given here.

9.13* ON GROUPS OF *n* INDEPENDENT RANDOM VARIABLES WHOSE PRODUCT FOLLOWS THE BETA DISTRIBUTION

It is well known that if there are n i.r.v.'s $X_{p_0, p_1 - p_0}, \ldots, X_{p_{n-1}, p_n - p_{n-1}}$ having beta distributions with p.d.f.

$$\begin{aligned} f_{p,q}(x) &= \frac{1}{B(p,q)} x^{p-1}(1-x)^{q-1}, \qquad 0 \leqslant x \leqslant 1 \\ &= 0, \qquad \text{otherwise}, \end{aligned} \tag{9.13.1}$$

where all the numbers $p_0,p_1,\ldots,p_n$ satisfy the condition

$$0<p_0<p_1<\ldots<p_n=p \tag{9.13.2}$$

then their product

$$U=X_{p_0,p_1-p_0}\cdots X_{p_{n-1}\,p_n-p_{n-1}} \tag{9.13.3}$$

has a beta distribution with p.d.f. $f_{p_0,p-p_0}(x)$ [160].

The following question arises: if the product

$$U=X_1\cdot X_2\cdots X_n \tag{9.13.4}$$

of n i.r.v.'s has a beta distribution with p.d.f. $f_{p_0,p-p_0}(x)$, must the factors X_k also have beta distributions? Kotlarski [183] has shown that the answer is no; that is, there exist groups of n i.r.v.'s that do not have beta distributions though their product has a beta distribution. Specifically, he has determined the set of groups $(X_1,X_2,\ldots,X_n)$ of i.r.v.'s, denoted by $\Omega_{p_0,p}$, whose product follows the beta distribution $X_{p_0,p}$. He established that for the enumeration of the set Ω_{p_0,p_n-p_0}, it is sufficient to solve the equation

$$M_s(f(x_1))M_s(f(x_2))\cdots M_s(f(x_n))=\frac{B(p_0+s,p_n-p_0)}{B(p_0,p_n-p_0)}, \qquad Re(s)>-p_0, \tag{9.13.5}$$

in terms of Mellin transforms

$$M_s(f(x))=\int_0^\infty x^s f(x)\,dx \tag{9.13.6}$$

of positive r.v.'s. His results are stated in Theorem 9.13.1.

Theorem 9.13.1. For a group of n positive i.r.v.'s $(X_1,X_2,\ldots,X_n)$ to belong to the set Ω_{p_0,p_n-p_0} it is necessary and sufficient that their Mellin transforms (9.13.6) are given for $s=it$ (t real) in the form

$$M_{it}(f(x))=\frac{B(p_{k-1}+it,p_k-p_{k-1})}{B(p_{k-1},p_k-p_{k-1})}e^{\alpha_k(t)+i\beta_k(t)}, \tag{9.13.7}$$

where $p_1,p_2,\ldots,p_{n-1}$ are arbitrary positive numbers satisfying conditions (9.13.2), the functions $\alpha_k(t),\beta_k(t)$ are real and continuous on the whole axis

t satisfying conditions

$$\sum_{k=1}^{n} \alpha_k(t)=0, \qquad \sum_{k=1}^{n} \beta_k(t)=0,$$

$$\alpha_k(0)=0, \qquad \beta_k(0)=0,$$

$$\alpha_k(-t)=\alpha_k(t), \qquad \beta_k(-t)=-\beta_k(t),$$

$$\alpha_k(t) \leqslant \ln\left|\frac{B(p_{k-1},p_k-p_{k-1})}{B(p_{k-1}+it,p_k-p_{k-1})}\right|, \tag{9.13.8}$$

and the functions (9.13.7) are positive definite functions.

The following example by Kotlarski is illustrative.

Example 9.13.1. Determine a group $(X_1,X_2,\ldots,X_n)$ belonging to $\Omega_{p_0,p-p_0}$. Using the formulas

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \qquad \operatorname{Re}(p)>0, \qquad \operatorname{Re}(q)>0 \tag{9.13.9}$$

and

$$\Gamma(mw)=\frac{m^{mw-(1/2)}}{(2\pi)^{(m-1)/2}}\prod_{r=1}^{m}\Gamma\left(w+\frac{r-1}{m}\right), \qquad \operatorname{Re}(w)>0, \tag{9.13.10}$$

one can write (9.13.5) in the form

$$M_s(f(x_1))\cdots M_s(f(x_n))=\prod_{r=1}^{m}\frac{B\left(\dfrac{p_0+r-1+s}{m},\dfrac{p_n-p_0}{m}\right)}{B\left(\dfrac{p_0+r-1}{m},\dfrac{p_n-p_0}{m}\right)}. \tag{9.13.11}$$

Dividing the set of positive integers $R=\{1,2,\ldots,m\}$, $m>n$, into n mutually exclusive and exhaustive subsets $R_1,R_2,\ldots,R_n$ one can write

$$M_s(h(x_k))=\prod_{r\in R_k}\frac{B\left(\dfrac{p_0+r-1+s}{m},\dfrac{p-p_0}{m}\right)}{B\left(\dfrac{p_0+r-1}{m},\dfrac{p-p_0}{m}\right)}, \qquad k=1,2,\ldots,n. \tag{9.13.12}$$

The variables X_k corresponding to the Mellin transforms (9.13.12) are

$$X_k = \prod_{r \in R_k} Y_r, \qquad k = 1, 2, \ldots, n, \tag{9.13.13}$$

where the Y_r are positive i.r.v.'s having p.d.f.'s

$$g_r(y) = \frac{m}{B\left(\frac{p_0 + r - 1}{m}, \frac{p - p_0}{m}\right)} y^{p_0 + r - 1}(1 - y^m), \exp\left(\frac{p - p_0}{m} - 1\right), \quad 0 \leqslant y \leqslant 1$$

$$= 0, \qquad \text{otherwise.} \tag{9.13.14}$$

9.14* ON BIVARIATE RANDOM VARIABLES WHERE THE QUOTIENT OF THEIR COORDINATES FOLLOWS SOME KNOWN DISTRIBUTION

Let X_1, X_2 be a pair of i.r.v.'s, symmetrical about the origin, having the same distribution function $F(x)$, and for which the p.d.f. $f(v)$ of the quotient

$$V = \frac{X_1}{X_2} \tag{9.14.1}$$

is Cauchy. It is, of course, well known that V has the Cauchy distribution when $F(x)$ is normal with zero mean. The question arises whether the normal distribution can be characterized by this property. Various authors —Mauldon [247], Laha [196], and Steck [368]—have shown this supposition to be false. That is, there exist distribution functions $F(x)$ differing from the normal for which the quotient (9.14.1) follows the Cauchy law. Laha [196], for instance, cites the counter example

$$f(x) = \frac{\sqrt{2}}{\pi} \cdot \frac{1}{1 + x^4}, \qquad -\infty < x < \infty, \tag{9.14.2}$$

which also has the property that the quotient (9.14.1) follows the Cauchy law. Later Laha [197] and Kotlarski [182] carried out independent studies in which they characterized the set Λ of r.v.'s whose quotient follows the Cauchy law by the properties of their Mellin transform

$$M_s(f(x)) = \int_{-\infty}^{\infty} |x|^s \, dF(x).$$

Kotlarski [185] then extended his study to the case of a bivariate r.v. (X,Y) having distribution function $F(x,y)$ in which the coordinates (not necessarily independent) have identical marginal distributions $F(x,\infty)=F(\infty,x), -\infty<x<\infty$, and the quotient follows the Cauchy law. He discribed the set Λ of such distribution functions by using their two-dimensional Mellin transforms

$$\begin{aligned} M_{r,s}(f(x,y)) &= E[X^rY^s] \\ &= \int_0^\infty \int_0^\infty x^r y^s \, dF(x,y), \qquad x>0, \qquad y>0, \end{aligned} \tag{9.14.3}$$

The p.d.f. $f(x,y)$ is always defined in the set of pairs (r,s) of complex variables

$$\xi=\{(r,s)\ a_1<\operatorname{Re}(r)<a_2, \qquad b_1<\operatorname{Re}(s)<b_2\},$$

where $a_1<0<a_2, b_1<0<b_2$.

Similarly, using the same method, Kotlarski delineates the set Φ of distribution functions $F(x,y)$ of bivariate r.v.'s (X,Y) having positive coordinates (not necessarily independent or identically distributed) in which the quotient (9.14.1) follows Snedecor's law. Mauldon [247] also considered this problem.

In particular, Kotlarski showed that the quotient (9.14.1) belongs to the set Λ if the Mellin transform (9.14.3) satisfies the conditions

$$M_{r,s}(f(x,y))=M_{s,r}(f(x,y)) \tag{9.14.4a}$$

and

$$M_{r,-r}(f(x,y))=\frac{1}{\cos(\pi r/2)}, \qquad -1<\operatorname{Re}(r)<1. \tag{9.14.4b}$$

His complete results relative to the set Λ are summarized in Theorem 9.14.1.

Theorem 9.14.1. For a distribution function $F(x,y)$ satisfying conditions (9.14.5a, b) to belong to the set Λ, it is necessary and sufficient that its Mellin transform (9.14.3) be represented in the form (9.14.6), where $\alpha(r,s)$ and $\beta(r,s)$ satisfy conditions (9.14.7) and (9.14.8) and the function (9.14.6) is positive definite.

The conditions referred to in Theorem 9.14.1 are stated below:

$$F(0,y)=F(x,0)=0, \tag{9.14.5a}$$

$$F(x,y)=F(y,x) \tag{9.14.5b}$$

$$M_{ir,is}(f(x,y))=\left[\operatorname{ch}\left(\frac{\pi r}{2}\right)\cdot\operatorname{ch}\left(\frac{\pi s}{2}\right)\right]^{1/2}\exp\left[\alpha(r,s)+i\beta(r,s)\right], \tag{9.14.6}$$

$$\alpha(r,-r)=0, \qquad \beta(s,-s)=0, \tag{9.14.7a}$$

$$\alpha(r,s)=\alpha(s,r), \qquad \beta(r,s)=\beta(s,r), \tag{9.14.7b}$$

$$\alpha(-r,-s)=\alpha(s,r), \qquad \beta(-r,-s)=-\beta(r,s) \tag{9.14.7c}$$

$$\alpha(r,s)\leqslant\frac{1}{2}\log\left[\operatorname{ch}(\pi r/2)\cdot\operatorname{ch}(\pi s/2)\right] \tag{9.14.8}$$

$\alpha(r,s), \beta(r,s)$ should be real and continuous on the whole plane (r,s). The following example, given by Kotlarski, is illustrative.

Example 9.14.1. Let the distribution function $F(x,y)$ be given by the density

$$\begin{aligned} f(x,y)&=f_0(x^2+y^2), \qquad x>0, y>0\\ &=0, \qquad \text{otherwise,} \end{aligned} \tag{9.14.9}$$

in which the function $f_0(z)$ is in such a form that (9.14.9) is a p.d.f. Then the Mellin transform of the p.d.f. (9.14.9)

$$\begin{aligned} M_{r,s}(f(x,y))&=\int_0^\infty\int_0^\infty x^r y^s f_0(x^2+y^2)\,dx\,dy\\ &=\int_0^\infty u^{r+s+1}f_0(u^2)\,du\int_0^{\pi/2}\cos^r\phi\sin^s\phi\,d\phi\\ &=\frac{1}{2}\int_0^\infty u^{(r+s)/2}f_0(u)\,du\cdot\frac{1}{2}B\left(\frac{1+r}{2},\frac{1+s}{2}\right)\\ &=H(r+s)\cdot\Gamma\left(\frac{1+r}{2}\right)\Gamma\left(\frac{1+s}{2}\right), \end{aligned} \tag{9.14.10}$$

where

$$H(w)=\frac{\pi\int_0^\infty u^{w/2}f_0(u)\,du}{4\Gamma(1+w/2)},\qquad H(0)=\frac{1}{\pi}. \tag{9.14.11}$$

One notes that the Mellin transform (9.14.10) satisfies the conditions (9.14.4a, b), from which it follows that the distribution (9.14.9) belongs to Λ.

Kotlarski establishes a similar result with regard to Snedecor's F distribution. Thus let X_1 and X_2 be two i.r.v.'s having gamma distributions with densities

$$\begin{aligned} f_k(x)&=\left[\frac{a^{p_k}}{\Gamma(p_k)}\right]x^{p_k-1}e^{-ax}, \qquad x>0\\ &=0, \qquad x\leqslant 0, \end{aligned} \tag{9.14.12}$$

$k=1,2$, where the constants a, p_k are positive. It is well known that the quotient (9.14.1) has the Snedecor F distribution whose p.d.f. is

$$\begin{aligned} g(v)&=[B(p_1,p_2)]^{-1}\cdot\frac{v^{p_2-1}}{(1+v)^{p_1+p_2}}, \qquad v>0\\ &=0, \qquad \text{otherwise}. \end{aligned} \tag{9.14.13}$$

Kotlarski delineates the set of bivariate distribution functions $F(x,y)$—denoted by ϕ—whose coordinates take only positive values and are not necessarily independent, and whose quotient (9.14.2) follows the Snedecor distribution (9.14.13). The functions in the set ϕ are characterized by their Mellin transforms, which must satisfy the condition

$$M_{r,-r}(f(x,y))=\frac{\Gamma(p_1+r)}{\Gamma(p_1)}\cdot\frac{\Gamma(p_2-r)}{\Gamma(p_2)},\qquad -p_1<\operatorname{Re}(r)<-p_2. \tag{9.14.14}$$

More specifically, the conditions are summarized in Theorem 9.14.2.

Theorem 9.14.2. For a distribution function $F(x,y)$ to belong to the set Φ, it is necessary and sufficient that its Mellin transform (9.14.3) be represented in the form (9.14.15) where $\alpha(r,s)$ and $\beta(r,s)$ satisfy conditions (9.14.16) and (9.14.17), and the function (9.14.18) is positive definite.

The conditions referred to in Theorem 9.14.2 are stated below:

$$M_{ir,is}(f(x,y)) = \left[\frac{\Gamma(p_1+ir)}{\Gamma(p_1)}\right]\left[\frac{\Gamma(p_2+is)}{\Gamma(p_2)}\right]e^{\alpha(r,s)+i\beta(r,s)}, \quad (9.14.15)$$

$$\alpha(r,-r)=0, \qquad \beta(s,-s)=0, \quad (9.14.16)$$

$$\alpha(-r,-s)=\alpha(r,s)$$

$$\beta(-r,-s)=-\beta(r,s),$$

$$\alpha(r,s)\leqslant \log\left|\left[\frac{\Gamma(p_1)}{\Gamma(p_1+ir)}\right]\left[\frac{\Gamma(p_2)}{\Gamma(p_2+is)}\right]\right|, \quad (9.14.17)$$

$\alpha(r,s)$, $\beta(r,s)$ should be real and continuous on the whole plane (r,s).

The reader may find the following example, due to Kotlarski, illuminating.

Example 9.14.2. Let the distribution function $F(x,y)$ have the density

$$f(x,y) = x^{p_1-1}y^{p_2-1}f_0(x+y), \qquad x>0, y>0$$

$$=0, \qquad \text{otherwise}, \quad (9.14.18)$$

where the constant in the function $f_0(z)$ is chosen in such a way that the function (9.14.18) is a density function. Clearly, the Mellin transform of (9.14.18) is

$$M_{r,s}(f(x,y)) = \int_0^\infty\int_0^\infty x^{p_1+r-1}y^{p_2+s-1}f_0(x+y)\,dx\,dy. \quad (9.14.19)$$

Note that if one substitutes

$$x=\rho\cos^2\theta, y=\rho\sin^2\theta \quad (9.14.20)$$

the area of integration is changed into

$$0<\rho<\infty, \qquad 0<\theta<\frac{\pi}{2} \quad (9.14.21)$$

and the Jacobian is

$$J=2\rho\sin\theta\cos\theta. \quad (9.14.22)$$

The Mellin transform (9.14.19) then becomes

$$M_{r,s}(f(x,y)) = \frac{\int_0^\infty \rho^{p_1+p_2+r+s-1} f_0(\rho)\,d\rho}{2\Gamma(p_1+p_2+r+s)}\Gamma(p_1+r)\Gamma(p_2+s)$$

$$= H(r+s)\Gamma(p_1+r)\Gamma(p_2+s), \qquad (9.14.23)$$

where

$$H(w) = \frac{\int_0^\infty \rho^{p_1+p_2+w-1} f_0(\rho)\,d\rho}{2\Gamma(p_1+p_2+w)},$$

$$H(0) = \left[\Gamma(p_1)\Gamma(p_2)\right]^{-1}.$$

Note that the Mellin transform (9.14.19) satisfies (9.14.14), from which it follows that the density function (9.14.18) belongs to Φ.

9.15* INTEGRAL TRANSFORMS AND MULTIVARIATE STATISTICAL ANALYSIS

Many of the problems of univariate statistical analysis have direct counterparts in multivariate statistical analysis. For example, Hotelling's T^2 distribution is a generalization of the Student t distribution to the multivariate domain. Likewise, the multiple correlation coefficient is a multivariate counterpart of the simple correlation coefficient, the distributions of both having been derived by Fisher [104, 105, 108], using the geometric method. Again, Mahalanabois's D^2 statistic is used for measuring the generalized distance in multivariate analysis [33,34]. Similarly, the variance as a measure of dispersion for a univariate distribution extends to the determinant of the covariance in a multivariate statistical population. It is not surprising, therefore, that integral transforms have considerable potential in multivariate statistical analysis, even as they do in the statistical analysis of univariate problems.

In recent years, integral transforms have been used to a considerable extent in solving multivariate distribution problems. One of the earlier applications of integral transforms to multivariate statistical analysis was the derivation of the exact distribution of Votaw's criteria [389] for testing compound symmetry of a covariance matrix [61]. More recently, in 1971, Pillai and Young [289] determined the exact null density and distribution of $U^{(p)}$, a constant times Hotelling's generalized T^2 statistic, for integral values of $m=(n_1-p-1)/2$ by employing Laplace transforms. They gave explicit results for $p=3$ and 4, and small values of m. Also in 1971 Mathai and Rathai [245] utilized the Mellin transform, the calculus of residues, and the properties of the ψ-function (digamma function) to obtain the exact distribution, in a computable form, of Wilks's generalized variance [405] in the noncentral linear case for a general Wishart matrix of order p. Shortly thereafter, Pillai and Nagarsenker [290] derived the noncentral distributions of

$$Y=\prod_{i=1}^{p}\theta_i^a(1-\theta_i)^b,$$

where a and b are known real numbers and the θ_i's stand for the latent roots of a matrix arising in each of three situations in multivariate normal theory—specifically, the test of equality of two covariance matrices, MANOVA, and canonical correlation. The study is also extended to the complex case. The distributions are derived in terms of H-functions by means of inverse Mellin transforms.

Krishnaiah and Schuurmann [192] point out that there are several situations in which an experimenter is interested in testing for the equality of the latent roots of the matrices of real and complex multivariate normal populations. They have investigated the evaluation of the probability integrals of the following distribution functions:

1. The distribution functions of the ratios of the intermediate roots to the trace of the real Wishart matrix.
2. The distribution functions of the ratios of the individual roots to the trace of the complex Wishart matrix.
3. The distribution functions of the ratios of the extreme roots of the Wishart matrix in the real and complex cases.

In the field of multivariate analysis, the multivariate characteristic function plays a dominant role. In this connection, Lukacs [223] has made an important contribution (among many others) by giving a description of the elementary properties of multivariate characteristic functions. More recently, Wolfe [413] has investigated the finite series expansion of multivariate characteristic functions. In particular, he has established three theorems that relate the asymptotic behavior of a distribution function to the behavior of its characteristic function at the origin.

It is not the purpose of this section to provide a complete coverage of the use of integral transforms in multivariate statistical analysis. However the foregoing examples and the related bibliography should suffice to impress the reader with the utility of integral transforms in the multivariate domain. Section 9.15.1 illustrates somewhat the power and utility of the integral transform method, as well as its actual implementation, in the area of multivariate analysis. For further details concerning the application of the transform method to multivariate problems, the reader is referred to the relevant references cited in this section.

9.15.1 The Exact Distribution of Votaw's Criteria for Testing Compound Symmetry of a Covariance Matrix

One of the important problems arising in multivariate analysis is that of testing the hypothesis H that the covariance matrix is of the bipolar form

$$\begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2' & \Sigma_3 \end{bmatrix}, \tag{9.15.1}$$

where Σ_1 is a $p\times p$ matrix with diagonal elements equal to σ_{aa} and other elements to $\sigma_{aa'}$, Σ_2 is a $p X q$ matrix with all elements equal to σ_{ab}, and Σ_3 is a $q\times q$ matrix with diagonal elements σ_{bb} and other elements $\sigma_{bb'}$ [61]. More specifically, let x_{ij}, $i=1,2,\dots,n$ be n independent observations on $p+q$ stochastic variables X_j, $j=1,2,3,\dots,(p+q)$, which are distributed normally. Also, let

$$\bar{x}_j=n^{-1}\sum_{i=1}^{n} x_{ij}, \qquad s_{jj}=n^{-1}\sum_{i=j}^{n}(x_{ij}-\bar{x}_j)(x_{ij'}-\bar{x}_{j'}) \tag{9.15.2}$$

and let $S=((s_{ij'}))$ be the sum of products (SP) matrix for X's and

$$S_{aa}=p^{-1}\sum_{j=1}^{p} S_{jj}, \qquad S_{aa'}=2(p^2-p)^{-1}\sum_{j>j'=1}^{p} S_{jj'},$$

$$S_{bb}=q^{-1}\sum_{j=p+1}^{p+q} S_{jj}, \qquad S_{bb'}=2(q^2-q)^{-1}\sum_{j>j'=p+1}^{p+q} S_{jj'},$$

$$S_{ab}=(pq)^{-1}\sum_{j=1}^{p}\sum_{j'=p+1}^{p+q} S_{jj'}. \tag{9.15.3}$$

Then the likelihood ratio statistic for testing the aforementioned hypothesis

H can be defined by

$$L = \frac{|S|}{[S_{aa}+(p-1)S_{aa'}][S_{bb}+(q-1)S_{bb'}]-pqS_{ab}^2} \times \frac{1}{(S_{aa}-S_{aa'})^{p-1}(S_{bb}-S_{bb'})^{q-1}} \tag{9.15.4}$$

Votaw [389] used Wilks's [407] moment-generating operator and derived the expected value $E[L^t|H]$ of the tth moment of the p.d.f. of L when the hypothesis H was true. By orthogonal transformation and by integrating over the range of different variates, Roy [314] proved that the expected value $E[L^t|H]$ can be expressed in the form

$$E[L^t|H] = \{(p-1)^{p-1}(q-1)^{q-1}\}^t \cdot \left[\frac{\Gamma\{\frac{1}{2}(q-1)(n-1)\}\Gamma\{\frac{1}{2}(p-1)(n-1)\}}{\Gamma\{(p-1)(t+\frac{1}{2}(n-1))\}\Gamma\{(q-1)(t+\frac{1}{2}(n-1))\}}\right] \cdot \prod_{r=0}^{p+q-3} \Gamma\{t+\tfrac{1}{2}(n-3)-\tfrac{1}{2}r\}\left[\Gamma\{\tfrac{1}{2}(n-3)-\tfrac{1}{2}r\}\right]^{-1} \tag{9.15.5}$$

and obtained the distribution of L in the form of an infinite series. Roy [314] further modified his series to get a better approximation of the distribution by taking a few terms. However it was Consul [59–62] who first pointed out that the Mellin inversion integral could be used to determine the exact distribution of L. He noted that since the moments determine a distribution uniquely for likelihood criteria, and since the tth moment is precisely the Mellin transform $M_{t+1}(f(L))$, one can apply Mellin's inversion theorem to obtain the exact p.d.f. of L, namely,

$$f(L) = T(n)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} L^{-t-1}\{(p-1)^{p-1}(q-1)^{q-1}\}^t \cdot \prod_{r=0}^{p+q-3} \Gamma\left[t+\tfrac{1}{2}(n-3)-\tfrac{1}{2}r\right] \cdot \left\{\Gamma\left[(p-1)\left(t+\tfrac{1}{2}(n-1)\right)\right]\Gamma\left[(q-1)\left(t+\tfrac{1}{2}(n-1)\right)\right]\right\}^{-1} dt, \tag{9.15.6}$$

where

$$T(n)=\Gamma\left[\tfrac{1}{2}(p-1)(n-1)\right]\Gamma\left[\tfrac{1}{2}(q-1)(n-1)\right]$$
$$\cdot\left[\prod_{r=0}^{p+q-3}\Gamma\left\{\tfrac{1}{2}(n-3)-\tfrac{1}{2}r\right\}\right]^{-1}. \tag{9.15.7}$$

By setting the transformation $t+\frac{1}{2}(n-1)=s$ and on further simplification, he obtained the exact density function

$$f(L)=K(n)L^{(1/2)(n-3)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}L^{-s}\left[(p-1)^{p-1}(q-1)^{q-1}\right]^{s}$$
$$\cdot\prod_{r=0}^{p+q-3}\Gamma\left(s-1-\tfrac{1}{2}r\right)\left\{\Gamma\left[(p-1)s\right]\Gamma\left[(q-1)s\right]\right\}^{-1}ds, \tag{9.15.8}$$

where

$$K(n)=\frac{T(n)}{\left[(p-1)^{p-1}(q-1)^{q-1}\right]^{(1/2)(n-1)}}. \tag{9.15.9}$$

Consul observed that the expression for the distribution $f(L)$ splits up into a factor $K(n)$, depending upon n, and an integral, which is independent of n, and used this fact to obtain the exact p.d.f.'s $f(L)$ and distribution functions (d.f.) for some specific values of p and q, namely, $p=q=2$. Consul [61] also gives the exact p.d.f.'s and d.f.'s for $p=q=3$, $p=3$, $q=2$, $p=5$, $q=2$, and $p=5$, $q=3$. In particular, for $p=q=2$, if one puts these values of p and q in (9.15.8) and (9.15.9), one obtains, on simplification,

$$f(L)=K_1(n)\cdot L^{(n-3)/2}\cdot\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}L^{-s}\Gamma(s-1)\Gamma\left(s-\tfrac{3}{2}\right)\left[\Gamma(s)\right]^{-2}ds,$$

where

$$K_1(n)=\frac{\left(\Gamma\left[\frac{n-1}{2}\right]\right)^2}{\left[\Gamma\left\{\frac{n-3}{2}\right\}\Gamma\left\{\frac{n-4}{2}\right\}\right]}.$$

Using a result derived by Consul [59, pp. 553–558], namely,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{-s}\Gamma(ps+a)\Gamma(qs+b)}{\{\Gamma(ps+a+m)\Gamma(qs+b+n)\}}\,ds$$

$$=x^{a/p}(1-x^{1/p})^{m+n-1}\left[2\Gamma(m+n)\right]^{-1}{}_2F_1(n,a-b+m;m+n;1-x^{1/p})$$

$$\text{for}\quad p=q,$$

the expression above for $f(L)$ reduces to

$$f(L)=\left\{\frac{K_1(n)}{\Gamma(\frac{5}{2})}\right\}\cdot L^{(1/2)(n-3)}$$

$$(1-L)^{3/2}F\left(1,1;\tfrac{5}{2};1-L\right),\qquad 0\leqslant L\leqslant 1.\qquad (9.15.10)$$

To obtain the cumulative distribution $\Pr(L\leqslant x)$ for different values of n, one can integrate $f(L)$ with respect to L between the limits 0 to $x\leqslant 1$. Since the integral is not directly available, it is convenient to change $F(1,1;\frac{5}{2};1-L)$ in (9.15.10) to $L^{1/2}F(\frac{3}{2},\frac{3}{2};\frac{5}{2};1-L)$ and then to integrate $f(L)$ by parts, treating $L^{(n-5)/2}$ as a second function and the rest of the expression as a first function, using the formula [95, (22), p. 102]

$$\left(\frac{d^n}{dz^n}\right)\left[z^{c-1}F(a,b;c;z)\right]=(c-n)_n z^{c-1-n}F(a,b;c-n;z),$$

where $(c-n)_n=(c-n)(c-n+1)\cdots(c-2)(c-1)$. The result is the cumulative distribution

$$\Pr[L\leqslant x]=F(x)=I_x\left(\tfrac{1}{2}n-2,\tfrac{3}{2}\right)+2K_1(n)\left[(n-3)\Gamma(\tfrac{5}{2})\right]^{-1}$$

$$\cdot x^{(1/2)(n-3)}(1-x)^{3/2}F\left(\tfrac{3}{2},\tfrac{3}{2};\tfrac{5}{2};1-x\right),\qquad (9.15.11)$$

where $F(a,b,c,z)$ is Gauss hypergeometric function defined in Appendix D.1 and $I_x(a,b)$ is the incomplete beta function defined by (9.2.20) and tabulated by Pearson [282].

By using a number of relations between the hypergeometric functions, Consul [60] has, by a complicated process, transformed the exact p.d.f.'s and d.f.'s into elementary functions. For $p=q=2$, the results are

$$f(L)=4K_1(n)\pi^{-1/2}L^{(1/2)(n-3)}$$
$$\times\left[(1-L)^{1/2}-L^{1/2}\sin^{-1}(1-L)^{1/2}\right],\qquad 0\leqslant L\leqslant 1\quad (9.15.12)$$

and

$$F(x)=I_x\left(\tfrac{1}{2}n-2,\tfrac{3}{2}\right)+8\pi^{-1/2}(n-3)^{-1}K_1(n)$$
$$\cdot x^{(1/2)(n-2)}\left[(1-x)^{1/2}-x^{1/2}\sin^{-1}(1-x)^{1/2}\right]$$
$$=\Pr(L\leqslant x).\qquad (9.15.13)$$

The interested reader is referred to Consul's paper [60] for the relevant elementary functions involved in $f(L)$ and $F(x)=\Pr(L\leqslant x)$ for the other pairs of values p,q listed above.

EXERCISES

9.1 Derive the p.d.f. $g(\bar{x})$ of the mean $\bar{X}=1/n\Sigma_{j=1}^{n}X_j$ of n identically distributed exponential i.r.v.'s each having p.d.f.

$$f(x_i)=\frac{1}{\theta}e^{-x_i/\theta},\qquad 0\leqslant x_i<\infty,\qquad i=1,2,\ldots,n.$$

9.2 Derive the p.d.f. $g(\bar{x})$ of the mean $\bar{X}=(X_1+X_2)/2$ of two i.r.v.'s X_1 and X_2, each having p.d.f.

$$f(x_i)=\tfrac{3}{2}x_i^2,\qquad -1\leqslant x_i\leqslant 1$$
$$=0,\qquad \text{otherwise.}$$

9.3 Derive the p.d.f. $h(\bar{\bar{x}})$ of the mean $\bar{\bar{X}}=(\bar{X}_1+\bar{X}_2)/2$, where $\bar{X}_1$ and $\bar{X}_2$ are the sample means in Exercise 9.2.

9.4 Prove that the mean $\bar{X}=1/n\Sigma_{i=1}^{n}X_i$ of n Pearson type III i.r.v.'s each having p.d.f.

$$f(x)\frac{1}{a\Gamma(b)}\left(\frac{x}{a}\right)^{b-1}e^{-x/a},\qquad a>0,\qquad 0\leqslant x<\infty$$

has p.d.f.

$$g(\bar{x})=\frac{n}{a\Gamma(bn)}\left(\frac{n\bar{x}}{a}\right)^{nb-1}e^{-n\bar{x}/a},\qquad 0\leqslant\bar{x}\leqslant\infty.$$

9.5 Use the method of integral transforms to prove that the mean of n identically distributed Cauchy i.r.v.'s has a Cauchy distribution.

9.6 Use the method of integral transforms to derive the distribution of the mean of n identically distributed Maxwell i.r.v.'s each having the p.d.f.

$$f(x_i)=\frac{4}{\sqrt{\pi}}\frac{1}{a^3}x^2e^{-x_i^2/a^2}, \qquad x_i\geqslant 0, \qquad i=1,2,\ldots,n$$

$$=0, \qquad \text{otherwise.}$$

9.7 Use the method of integral transforms to derive the distribution of the mean of n identically distributed Rayleigh i.r.v.'s each having p.d.f.

$$f(x_i)=\frac{1}{a^2}x_i\exp\left[-\frac{1}{2}\left(\frac{x_i}{a_i}\right)^2\right], \qquad x_i\geqslant 0, \qquad i=1,2,\ldots,n.$$

9.8 Show that the mean of a sample $X_1,X_2,\ldots,X_n$ of n i.r.v.'s drawn from a gamma population

$$f(x)=\alpha^p x^{p-1}e^{-\alpha x}, \qquad 0\leqslant x<\infty$$

is

$$g(\bar{x})=\frac{(n\alpha)^{n-1}}{\Gamma(np)}\bar{x}^{np-1}e^{-n\alpha\bar{x}}, \qquad 0\leqslant\bar{x}<\infty.$$

(Irwin, 1927)

9.9* Consider the order statistics $X_{(r)},X_{(s)}$ $(r<s)$ based on a random sample of size n from a negative exponential distribution.

(a) Prove that the p.d.f. $h(y)$ of the product $Y=X_{(r)}X_{(s)}$ is

$$h(y)=\frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$

$$\times\int_0^1\frac{1}{u}\exp\left(-y^{1/2}u-\frac{y^{1/2}(n-s+1)}{u}\right)$$

$$\cdot\left[1-\exp(-y^{1/2}u)\right]^{r-1}\left[\exp(-y^{1/2}u)\right.$$

$$\left.-\exp\left(-\frac{y^{1/2}}{u}\right)\right]^{s-r-1} du, \qquad 0<y<\infty.$$

(*b*) Prove that the p.d.f. $h(y)$ of the product Y of the extreme order statistics is

$$h(y)=n(n-1)\int_0^1 \frac{1}{u}\exp\left(-y^{1/2}u-\frac{y^{1/2}}{u}\right)$$

$$\cdot\left[\exp(-y^{1/2}u)-\exp\left(\frac{-y^{1/2}}{u}\right)\right]^{n-2} du, \qquad 0<y<\infty.$$

(Subrahmanian, 1970)

9.10* Let $Y=X_{(r)}/X_{(s)}$ where $X_{(r)}, X_{(s)}$ are the order statistics defined in Exercise 9.9.

(*a*) Prove that Y has the p.d.f.

$$h(y)=\frac{n!}{(r-1)!(s-r-1)!(n-s)!}\sum_{j_1=0}^{r-1}\sum_{j_2=0}^{s-r-1}(-1)^{j_1+j_2}$$

$$\times\binom{r-1}{j_1}\binom{s-r-1}{j_2}\cdot(\tau_2+\tau_1 y)^{-2}, \qquad 0<y<1$$

$$=0, \qquad \text{otherwise,}$$

where

$$\tau_1=(j_1-j_2)+s-r,$$

$$\tau_2=n-s+1+j_2.$$

(*b*) Prove that the p.d.f. $g(y)$ of the ratio Y of extreme order statistics is

$$h(y)=n(n-1)\sum_{j=0}^{n-2}(-1)^j\binom{n-2}{j}\left[(j+1)\right.$$

$$\left.-\{n-(j+1)\}y\right]^{-2}, \qquad 0<y<1$$

$$=0, \qquad \text{elsewhere.}$$

(Subrahmanian, 1970)

9.11 Show that the geometric mean $Y=[X_1 \cdot X_2 \cdots X_n]^{1/n}$ of n Pearson type III i.r.v.'s with p.d.f.'s

$$\frac{x_1^{p-1}e^{-x_1}}{\Gamma(p)}, \frac{x_2^{p+(1/n)-1}e^{-x_2}}{\Gamma(p+1/n)}, \ldots,$$

$$\frac{x_n^{p+(n-1)/(n)-1}e^{-x_n}}{\Gamma[p+(n-1)/n]}, \qquad 0 \leqslant x_j < \infty$$

is the same as the distribution of the arithmetic mean of n i.r.v.'s each distributed as x_1.

(Kullback, 1934)

9.12 Use the Mellin transform to find the p.d.f. $h(y)$ of the geometric mean

$$Y = \left[\prod_{j=1}^{3} X_j\right]^{1/3}$$

of the three beta i.r.v.'s in Exercise 4.11.

9.13* *Prove.* If X_1 and X_2 are two normal i.r.v.'s with p.d.f.'s

$$f_1(x_1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x_1^2/2\sigma_1^2}, \qquad -\infty < x_1 < \infty,$$

$$f_2(x_2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2\right], \qquad -\infty < x_2 < \infty,$$

then the p.d.f. $h(y)$ of the product $Y = X_1X_2$ is given by

$$h(y) = \frac{e^{-\mu_2^2/2\sigma_2^2}}{\pi\sigma_1\sigma_2} \sum_{k=0}^{\infty} \left(\frac{\mu_2}{\sigma_2}\right)^{2k} \frac{1}{(2k)!}$$

$$\cdot \left(\frac{y}{\sigma_1\sigma_2}\right)^k K_{k/2}\left(\frac{y}{\sigma_1\sigma_2}\right), \qquad -\infty < y < \infty,$$

where $K_{k/2}$ is Bessel's function of the second kind with purely imaginary argument.

Hint. (*a*) Invert $M_s(h(y)) = M_s(f_1(x_1))M_s(f_2(x_2))$, using $M_s(f_1^{\pm}(\pm x_1))$ and $M_s(f_2^{\pm}(\pm x_2))$ as given, respectively, by (4.5.25) and (4.6.39), (4.6.42).

(*b*) Or use Laha's method [195, p. 66] based on the fact that the product X_1X_2 is expressible as the difference of two Bessel i.r.v.'s i.e., $X_1X_2 = C(v^2 - w^2)$, where v^2 and w^2 are type I Bessel i.r.v.'s and C is a constant).

9.14 Let X_1 and X_2 be two uniform i.r.v.'s with arbitrary means u_i, $i = 1, 2$ and p.d.f.'s

$$
\begin{aligned}
f_1(x_1) &= 1, \qquad |x_1 - u_1| \leqslant \frac{1}{2}\\
&= 0, \qquad \text{otherwise,}\\
f_2(x_2) &= 1, \qquad |x_2 - u_2| \leqslant \frac{1}{2}\\
&= 0, \qquad \text{otherwise.}
\end{aligned}
$$

Prove that the p.d.f. $h(y)$ of the product $Y = X_1X_2$ is

$$
\begin{aligned}
h(y) &= 0, \qquad y > \left(u_1 + \frac{1}{2}\right)\left(u_2 + \frac{1}{2}\right)\\
&= -\ln\left[\frac{y}{(u_1 + 1/2)(u_2 + 1/2)}\right], \qquad \left(u_1 - \frac{1}{2}\right)\left(u_2 + \frac{1}{2}\right) < y\\
&\quad < \left(u_1 + \frac{1}{2}\right)\left(u_2 + \frac{1}{2}\right)\\
&= -\ln\left[\frac{u_1 - 1/2}{u_1 + 1/2}\right], \qquad \left(u_1 + \frac{1}{2}\right)\left(u_2 - \frac{1}{2}\right) < y < \left(u_1 - \frac{1}{2}\right)\left(u_2 + \frac{1}{2}\right)\\
&= -\ln\left[\frac{(u_1 - 1/2)(u_2 - 1/2)}{y}\right], \qquad \left(u_1 - \frac{1}{2}\right)\left(u_2 - \frac{1}{2}\right) < y\\
&\quad < \left(u_1 + \frac{1}{2}\right)\left(u_2 - \frac{1}{2}\right)\\
&= 0, \qquad 0 < y < \left(u_1 - \frac{1}{2}\right)\left(u_2 - \frac{1}{2}\right)
\end{aligned}
$$

and sketch the density function for: (1) $u_1 = -1, u_2 = 1$ (2) $u_1 = 4.5$, $u_2 = 0.5$ (3) $u_1 = 0.5$, $u_2 = 0.5$ (4) arbitrary values of u_1 and u_2.

9.15 *Prove.* If X_1 and X_2 are two independent beta r.v.'s of the first kind with parameters (a, b) and $(a + b, c)$, the product $Y = X_1X_2$ is a beta r.v. of the first kind with parameters $(a, b+c)$ (see (4.4.11) for the definition of a beta r.v. of the first kind).

(Jambunathan, 1954)

9.16 *Prove.* If $X_1, X_2, \ldots, X_p$ are p independent beta r.v.'s of the first kind with parameters (a_i, b_i) for $i=1,2,\ldots,p$, and if $a_{i+1}=a_i+b_i$ for $i=1,2,\ldots,p-1$, the product $Y=X_1X_2\cdots X_p$ is a beta r.v. of the first kind with parameters a_1 and $b=\Sigma_{i=1}^p b_i$.

9.17 *Prove.* If X_i and Y_i for $i=1,2,\ldots,p$ are independent gamma r.v.'s with parameters a_i, b_i that are connected by the relation $a_{i+1}=a_i+b_i$ for $i=1,2,\ldots,p-1$, then the quotient

$$V=\prod_{i=1}^{p}\frac{X_i}{X_i+Y_i}$$

is a beta variate of the first kind with parameters (a_1, b) where $b=\Sigma_{i=1}^p b_i$.

(Jambunathan, 1954)

9.18* If $u=(1+y)/(1+x)$, and if x is a beta r.v. of the first kind with parameters $(b-d, d)$ while y is a beta r.v. of the second kind (Exercise 4.25) with parameters (a,b), then u is a beta r.v. of the second kind with parameters $B_2(a+d, b-d)$, provided u and y are independent.

(Jambunathan, 1954)

9.19 Let X_1, X_2 be a sample of two drawn at random from the exponential population $f(x)=e^{-x}$, $0\leqslant x\leqslant\infty$.

(a) Show that the joint p.d.f. of the sample mean $\bar{X}=(X_1+X_2)/2$ and standard deviation

$$S=\left[\frac{1}{n}\sum_{i=1}^{2}\left(X_i-\bar{X}\right)^2\right]^{1/2}=\frac{X_1-X_2}{2}$$

is

$$f(\bar{x}, s)=4e^{-2\bar{x}}, \qquad s\leqslant\bar{x}\leqslant\infty, \qquad 0\leqslant s<\bar{x}.$$

(b) Derive the p.d.f. $h(y)$ of the product $Y=\bar{X}S$. Note that the $\bar{X}$ and S are not independent.

9.20* Derive the p.d.f.'s of the quotients $W=\bar{X}/S$ and $Y=S/\bar{X}$ for the sample mean and standard deviation in Exercise 9.19. Again, note that $\bar{X}$ and S are not independent.

9.21 Let X_1 and X_2 be triangular i.r.v.'s with p.d.f.'s

$$\begin{aligned} f_1(x_1) &= a_1x_1 - a_2, & b_1 \leqslant x_1 \leqslant b_2 \\ &= a_3 - a_1x_1, & b_2 \leqslant x_1 \leqslant b_3 \\ &= 0, & \text{otherwise,} \\ f_2(x_2) &= c_1x_2 - c_2, & d_1 \leqslant x_2 \leqslant d_2 \\ &= c_3 - c_1x_2, & d_2 \leqslant x_2 \leqslant d_3 \\ &= 0, & \text{elsewhere.} \end{aligned}$$

Prove that the p.d.f. $h(y)$ of the product $Y=X_1X_2$ is

$$\begin{aligned} h(y) &= \left(\frac{a_1c_2}{d_1}+\frac{a_2c_1}{b_1}\right)y-(a_1c_1y+a_2c_2)\ln\left(\frac{y}{b_1d_1}\right) \\ &\quad -(a_1c_2b_1+a_2c_1d_1), \qquad y \leqslant b_1d_1 \\ &= \{2a_1c_1y+a_2(c_2+c_3)\}\ln\left(\frac{y}{b_1d_2}\right) \\ &\quad -\left\{\frac{a_1(c_2+c_3)}{d_2}+\frac{2a_2c_1}{b_1}\right\}y \\ &\quad +a_1(c_2+c_3)b_1+2a_2c_1d_2, \qquad y \leqslant b_1d_2 \\ &= \left(\frac{a_1c_3}{d_3}+\frac{a_2c_1}{b_1}\right)y-(a_1c_1y+a_2c_3)\ln\left(\frac{y}{b_1d_3}\right) \\ &\quad -(a_1c_3b_1+a_2c_1d_3), \qquad y \leqslant b_1d_3 \\ &= \{2a_1c_1y+c_2(a_2+a_3)\}\ln\left(\frac{y}{b_2d_1}\right)-\left\{\frac{2a_1c_2}{d_1}+\frac{c_1(a_2+a_3)}{b_2}\right\}y \\ &\quad +\{2a_1c_2b_2+c_1d_1(a_2+a_3)\}, \qquad y \leqslant b_2d_1 \end{aligned}$$

$$= \left\{ \frac{2a_1(c_2+c_3)}{d_2} + \frac{2c_1(a_2+a_3)}{b_2} \right\} y$$
$$- \{4a_1c_1y + (a_2+a_3)(c_2+c_3)\} \ln\left(\frac{y}{b_2d_2}\right)$$
$$- \{2a_1b_2(c_2+c_3) + 2c_1d_2(a_2+a_3)\}, \qquad y \leqslant b_2d_2$$

$$= \{2a_1c_1y + c_3(a_2+a_3)\} \ln\left(\frac{y}{b_2d_3}\right)$$
$$- \left\{ \frac{2a_1c_3}{d_3} + \frac{c_1(a_2+a_3)}{b_2} \right\} y$$
$$+ 2a_1b_2c_3 + c_1d_3(a_2+a_3), \qquad y \leqslant b_2d_3$$

$$= \left(\frac{c_1a_3}{b_3} + \frac{a_1c_2}{d_1} \right) y - (a_1c_1y + a_3c_2) \ln\left(\frac{y}{b_3d_1}\right)$$
$$- (c_1a_3d_1 + a_1c_2b_3), \qquad y \leqslant b_3d_1$$

$$= \{2a_1c_1y + a_3(c_2+c_3)\} \ln\left(\frac{y}{b_3d_2}\right)$$
$$- \left\{ \frac{2c_1a_3}{b_3} + \frac{a_1(c_2+c_3)}{d_2} \right\} y$$
$$+ 2c_1a_3d_2 + a_1(c_2+c_3)b_3, \qquad y \leqslant b_3d_2$$

$$= \left(\frac{a_3c_1}{b_3} + \frac{a_1c_3}{d_3} \right) y - \{a_1c_1y + a_3c_3\} \ln\left(\frac{y}{b_3d_3}\right)$$
$$- (a_3c_1d_3 + a_1c_3b_3), \qquad y \leqslant b_3d_3.$$

9.22 Verify that $\int_{b_1d_1}^{b_3d_3} h(y)\,dy = 1$ in Exercise 9.21, assuming that $\min(b_id_j) = b_1d_1$, $\max(b_id_j) = b_3d_3$.

9.23 Verify that $E[y] = E[x_1]E[x_2]$ in Exercise 9.21.

9.24 Let X_1 and X_2 be triangular i.r.v.'s with p.d.f.'s

$$\begin{aligned} f_1(x_1) &= x_1, && 0 < x_1 < 1 \\ &= 2 - x_1, && 1 < x_1 < 2 \\ &= 0, && \text{elsewhere,} \end{aligned}$$

$$f_2(x_2) = x_2 - 1, \qquad 1 < x_2 < 2$$
$$= 3 - x_2, \qquad 2 < x_2 < 3$$
$$= 0, \qquad \text{elsewhere.}$$

Prove that the p.d.f. $h(y)$ of $Y = X_1 X_2$ is

$$h(y) = 4 - 4y + (2 + 2y)\ln y, \qquad 0 < y < 1$$
$$= 10y - 20 - (5y + 10)\ln\left(\frac{y}{2}\right), \qquad 0 < y < 2$$
$$= 12 - 4y + (2y + 6)\ln\left(\frac{y}{3}\right), \qquad 0 < y < 3$$
$$= 16 - 4y + (2y + 8)\ln\left(\frac{y}{4}\right), \qquad 0 < y < 4$$
$$= 2y - 12 - (y + 6)\ln\left(\frac{y}{6}\right), \qquad 0 < y < 6.$$

Show that $h(y)$ is expressible in the equivalent form

$$h(y) = (2 + 2y)\ln y - (52 + 10)\ln\left(\frac{y}{2}\right) + (2y + 6)\ln\left(\frac{y}{3}\right)$$
$$+ (2Y + 8)\ln\left(\frac{y}{4}\right) - (y + 6)\ln\left(\frac{y}{6}\right), \qquad 0 \leqslant y \leqslant 1$$
$$= 4y - 4 - (5y + 10)\ln\left(\frac{y}{2}\right) + (2y + 6)\ln\left(\frac{y}{3}\right)$$
$$+ (2y + 8)\ln\left(\frac{y}{4}\right) - (y + 6)\ln\left(\frac{y}{6}\right), \qquad 1 \leqslant y \leqslant 2$$
$$= 16 - 6y + (2y + 6)\ln\left(\frac{y}{3}\right) + (2y + 8)\ln\left(\frac{y}{4}\right)$$
$$- (y + 6)\ln\left(\frac{y}{6}\right), \qquad 2 \leqslant y \leqslant 3$$
$$= 4 - 2y + (2y + 8)\ln\left(\frac{y}{4}\right) - (y + 6)\ln\left(\frac{y}{6}\right), \qquad 3 \leqslant y \leqslant 4$$
$$= 2y - 12 - (y + 6)\ln\left(\frac{y}{6}\right), \qquad 4 \leqslant y \leqslant 6.$$

9.25 Show that $\int_0^6 h(y)\,dy = 1$ and $E[Y] = E[X_1]E[X_2]$ for both forms of $h(y)$ in Exercise 9.24.

9.26 Let X_1 and X_2 be identically distributed triangular i.r.v.'s with p.d.f.

$$\begin{aligned} f(x_i) &= 4x_i, && 0 \leqslant x_i \leqslant \tfrac{1}{2}, && i=1,2 \\ &= 4(1-x_i), && \tfrac{1}{2} \leqslant x_i \leqslant 1, && i=1,2 \\ &= 0, && \text{elsewhere}, && i=1,2. \end{aligned}$$

Show that the $Y = X_1 X_2$ has the p.d.f.

$$\begin{aligned} h(y) &= 16\{2y - y\ln y + 4y\ln 2y - 4y\ln 4y\}, && 0 \leqslant y \leqslant \tfrac{1}{4} \\ &= 16\{2 - 6y - y\ln y + 4y\ln 2y + \ln 4y\}, && \tfrac{1}{4} \leqslant y \leqslant \tfrac{1}{2} \\ &= 16\{-2 + 2y - y\ln y - 2\ln 2y + \ln 4y\}, && \tfrac{1}{2} \leqslant y \leqslant 1 \\ &= 0, && \text{elsewhere.} \end{aligned}$$

9.27 Verify that $\int_0^1 h(y)\,dy = 1$ and $E[Y] = E[X_1]E[X_2]$ in Exercise 9.26.

9.28 Show that the quotient $Y = X_1/X_2$ of two i.r.v.'s each having the p.d.f.

$$f(X) = \frac{\sqrt{2}}{\pi}\,\frac{1}{1+X^4}, \qquad -\infty < X < \infty$$

has a Cauchy distribution (Laha [196], 1958). This proves that if Ω denotes the set of r.v.'s for which the quotient of two such identically distributed r.v.'s has a Cauchy distribution, the normal r.v. $N(0,1)$ is not the only r.v. belonging to Ω, as had previously been conjectured.

9.29 Show that the inversion of the Laplace transform

$$L_r(g(w)) = (1+2r)^{-\sum_{j=1}^{n} \frac{n_j}{2}}$$

yields the chi-square distribution

$$g(w) = \frac{1}{2^{N/2}\Gamma(N/2)} w^{(N/2)-1} e^{-w}, \quad 0 \leqslant w < \infty$$

with $N = \sum_{j=1}^{n} n_j$ degrees of freedom. Note the presence of a branch point at the origin when $N/2$ is not an integer. Follow the general approach used in Section 9.2.7 for dealing with a branch point when deriving the noncentral chi-square distribution.

9.30* Verify that the distribution of the r.v.

$$L_1 = \frac{Y^{1/N}}{V}$$

is an H-function r.v. with p.d.f. (9.1.13), where

$$Y = \frac{\prod_{j=1}^{k} n_j s_j^2}{\sigma_j^2}, \qquad V = \frac{1}{N}\sum \frac{n_j s_j^2}{\sigma_j^2}.$$

9.31* Show that the distribution $W = S_1/S_2$, where S_1 and S_2 are the standard deviations of two samples of size n from a bivariate normal population (see (3.4.12)) with correlation coefficient ρ and standard deviation σ for both variates, is

$$f(w) = \frac{2\Gamma(n-1)}{[\Gamma(n-1)]^2}(1-\rho^2)^{(n-2)/2}$$

$$\times \frac{w^{n-2}}{(1+w^2)^{n-1}}\left[1 - \frac{4\rho^2 w^2}{(1+w^2)^2}\right]^{-n/2}, \qquad 0 \leqslant w < \infty.$$

(Bose, 1935 and Finney, 1938).

Hint. Bose's method consisted of using the joint distribution of S_1, S_2 and the correlation coefficient r, which was found by Fisher [104] to be

$$f(s_1, s_2, r) = C_0 \exp\left(\frac{-s_1^2}{2g_1^2}\right) \cdot \exp\left(\frac{-s_2^2}{2g_2^2}\right)$$

$$\times \left(\frac{s_1}{g_1} \cdot \frac{s_2}{g_2}\right)^{n-2} e^{hr}(1-r^2)^{(n-4)/2},$$

where

$$g_1^2 = \frac{\sigma_1^2(1-\rho^2)}{n}, g_2^2 = \frac{\sigma_2^2(1-\rho^2)}{n}, h = \frac{\rho s_1 s_2}{g_1 g_2},$$

and ρ, σ_1, σ_2 are the population values of the coefficient of correlation and standard deviations, n is the size of sample, and C_0 is a constant. Integrate $f(s_1,s_2,r)$ with respect to r to obtain $f(s_1,s_2)$. Then apply the transformation $s_1 = ws_2, s_2, s_2 = s_2$ to obtain the joint distribution $f(w,s_2)$, which when integrated with respect to s_2 gives the p.d.f. $f(w)$ of the quotient $w = s_1/s_2$.

The distribution $f(w)$ can also be obtained using Mellin transforms. When the joint distribution $f(s_1 s_2)$ has been determined, apply the transformation $u = 1/s_2$, $s_1 = s_1$, to obtain the joint distribution $f(u,s_1)$. Determine the Mellin transform of the joint distribution $f(u,s_1)$ and invert via (4.8.14) to obtain $f(w)$, where $w = us_1 = s_1/s_2$.

9.32* Show that the distribution of Fisher's z variate

$$Z = \frac{1}{2}\ln\left(\frac{S_1^2}{S_2^2}\right),$$

where S_1^2 and S_2^2 are the variances of samples drawn from the correlated bivariate normal population (see (3.4.12)) with correlation coefficient ρ and $n_1 = n_2 = n$, $\sigma_1 = \sigma_2 = \sigma$, is

$$f(z) = \frac{2(1-\rho^2)^{(n-1)/2}}{B\left(\frac{n-1}{2}, \frac{n-1}{2}\right)} \frac{e^{(n-1)z}}{(1+e^{2z})^{n-1}} \left[1 - \frac{4\rho^2 e^{2z}}{(1+e^{2z})^2}\right], \qquad z \geqslant 0.$$

(Bose, 1935)

9.33* Let

$$Y = \frac{n_1 S_1^2}{n_1 S_1^2 + n_2 S_2^2}$$

for samples of size $n_1 = n_2 = n$ drawn at random from the bivariate normal population (see (3.4.12)) with correlation coefficient $\rho, n_1 = n_2 = n$, $\sigma_1 = \sigma_2 = \sigma$. Show that the p.d.f. of Y is

$$f(y) = \frac{(1-\rho^2)^{(n-1)/2}}{B\left(\frac{n-1}{2}, \frac{n-1}{2}\right)} (1-y^2)^{(n-3)/2} Y^{(n-3)/2}$$

$$\times\left[1 - 4\rho^2 y(1-y)\right]^{-n/2}, \qquad 0 \leqslant y < \infty.$$

(Bose, 1935).

Bose points out that if $\sigma_1 \neq \sigma_2$ in Exercises 9.31, 9.32, and 9.33, the distributions of W, Z, and Y remain unchanged except for a slight modification, due to constant terms.

9.34 Find the p.d.f. of the difference of two Student t i.r.v.'s with 5 degrees of freedom.[120]

9.35 Use the characteristic function to find the p.d.f. $h(v)$ of the classification statistic

$$V = \frac{|X|\cdot|Y|}{Z},$$

where X and Y each have the standardized normal distribution $N(0,1)$ and Z is distributed as χ^2/n with n degrees of freedom.

(Harter, 1951)

9.36 Find the p.d.f. $h(v)$ in Exercise 9.35 by using Mellin transforms.

9.37 Show that if X and Y are i.r.v.'s having chi-square distributions with m and n degrees of freedom, respectively, then the r.v. V defined by

$$V = \frac{X}{X+Y}$$

has the beta p.d.f.

$$f(v) = \frac{1}{B(m,n)} v^{m-1}(1-v)^{n-1}, \qquad 0 \leqslant v \leqslant 1.$$

9.38* Show that the difference $W = X_1 - X_2$ of two Pearson type III i.r.v.'s with p.d.f.'s

$$f(x_j) = \frac{e^{-x_j} x_j^{p-1}}{\Gamma(p)}, \qquad p>0, \qquad 0 \leqslant x_j < \infty, \qquad j=1,2,\ldots,n$$

has the p.d.f.

$$g(w) = \frac{w^{p-(1/2)} K_{p-(1/2)}(w)}{2^{p-1}\Gamma(p)\Gamma\left(\frac{1}{2}\right)},$$

where $K_r(x)$ is the Bessel function of second kind of order r and imaginary argument, defined in Appendix D.1.

APPENDIX A[47]

Jordan's Lemma

As Section 2.9 pointed out, it is particularly convenient to use the residue theorem, when applicable, to evaluate the integral along the Bromwich path. In particular, if the integral along the Bromwich path is equal to the integral along the Bromwich contour (closed circular arc whose diameter or chord is the Bromwich path), then the residue theorem may be used. The integral along the Bromwich contour will be equal to the integral along the Bromwich path if and only if the integral evaluated over the circular arc of the Bromwich contour approaches zero as the length of the radius approaches infinity. The conditions under which this occurs are given by Jordan's lemma, which, because of its importance, will now be proved.

Before proceeding with the proof of this lemma, it is important to point out that in using the residue theorem to evaluate the inversion integral over the Bromwich path, it is sometimes necessary to use the left Bromwich contour $C_L = QKLMPQ$ (Fig. 2.9.1a) and sometimes the right contour $C_R = QPTQ$ (Fig. 2.9.1b), depending on whether the poles are located to the left or to the right, respectively, of the Bromwich path. In either case, Jordan's lemma finds useful application.

A.1 JORDAN'S LEMMA AND ITS PROOF

Jordan's Lemma

If $f(s) \to 0$ uniformly with regard to $\arg s$ as $|s| \to \infty$ when $\pi/2 \leqslant \arg s \leqslant \frac{3}{2}$ and if $f(s)$ is analytic when both $|s| \to k$ and $\pi/2 \leqslant \arg s \leqslant 3\pi/2$ then

$$\lim_{a \to \infty} \int_{C_L} e^{ms} f(s)\, ds = 0,$$

where k and m are positive real constants.

[47]I acknowledge Dr. W. E. Thompson's contributions to the proofs and derivations in Appendix A, which are essentially equivalent to those appearing in an earlier document by Springer and Thompson [353].

The lemma as stated applies to the left-hand Bromwich contour in Fig. 2.9.1a. If in the statement of the lemma, the inequality $\pi/2 \leqslant \arg s \leqslant 3\pi/2$ is replaced by the inequality $-\pi/2 \leqslant \arg s \leqslant \pi/2$, the limits of integration changed accordingly from $(\pi/2, 3\pi/2)$ to $(-\pi/2, \pi/2)$, and the negative number $-m$ replaces m in the kernel e^{ms}, the lemma remains valid and the proof remains unchanged for the right-hand Bromwich contour C_R.

PROOF. In establishing the proof of this lemma for the left-hand Bromwich contour, note that since $f(s)\to 0$ uniformly as $|s|\to\infty$ and $\pi/2 \leqslant \arg s \leqslant 3\pi/2$, one can, given ε, choose a ρ_0 such that $|f(s)| < \varepsilon/\pi$ when $|s| > \rho_0$ and $\pi/2 \leqslant \arg s \leqslant 3\pi/2$. Then if $\rho > \rho_0$

$$\left|\int_{C_L} e^{ms} f(s)\,ds\right| = \left|\int_{\pi/2}^{3\pi/2} \exp[m\rho(\cos\theta + i\sin\theta)]\, f(\rho e^{i\theta})\rho e^{i\theta} i\, d\theta\right|$$
$$\leqslant \int_{\pi/2}^{3\pi/2} |e^{m\rho\cos\theta}|\,|e^{m\rho i\sin\theta}|\,|f(\rho e^{i\theta})|\,|\rho|\,|e^{i\theta}|\,|i|\, d\theta,$$

and since

$$|e^{m\rho i\sin\theta}| = |e^{i(m\rho\sin\theta)}|$$
$$= |\cos(m\rho\sin\theta) + i\sin(m\rho\sin\theta)|$$
$$= \cos^2(m\rho\sin\theta) + \sin^2(m\rho\sin\theta)$$
$$= 1,$$

it follows that

$$\left|\int_{C_L} e^{ms} f(s)\,ds\right| \leqslant \int_{\pi/2}^{3\pi/2} |e^{m\rho\cos\theta}|\,|f(\rho l^{i\theta})|\,|\rho|\, d\theta$$
$$\leqslant \frac{\varepsilon}{\pi} \int_{\pi/2}^{3\pi/2} \rho e^{m\rho\cos\theta}\, d\theta$$
$$\leqslant \frac{2\varepsilon}{\pi} \int_0^{\pi/2} e^{-m\rho\cos\theta}\, d\theta.$$

Furthermore, since

$$\cos\theta \geqslant 1 - \frac{2\theta}{\pi}, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2},$$

it follows that when $\rho > \rho_0$,

$$\left|\int_{C_L} e^{ms} f(s)\,ds\right| \leqslant \frac{\varepsilon}{\pi}\int_0^{\pi/2} 2\rho e^{-m\rho(1-(2\theta)/\pi)}\,d\theta$$

$$\leqslant \frac{\varepsilon}{m} e^{-m\rho(1-(2\theta)/\pi)}\Big|_0^{\pi/2}$$

$$\leqslant \frac{\varepsilon}{m}(1-e^{-m\rho})$$

$$< \frac{\varepsilon}{m},$$

hence

$$\int_{C_L} e^{ms} f(s)\,ds \to 0$$

as $\rho_0 \to \infty$.

As previously stated, the condition that $f(s) \to 0$ uniformly with respect to $\arg s$ will be satisfied if one can find constants $M > 0, K > 0$ such that on the relevant semicircular arc (C_L or C_R, where $s = Re^{i\theta}$)

$$|f(s)| < \frac{M}{R^K}. \tag{A.1.1}$$

For if two constants $M > 0, K > 0$ can be found such that inequality (A.1.1) is satisfied, then for any $\varepsilon > 0$ one can find a value $R_0 > 0$ depending on ε but independent of $\arg s$ such that if $R > R_0$, then $M/R^K < \varepsilon$, so that $|f(s)| \to 0$ independently of $\arg s$; hence $f(s)$ is uniformly convergent with respect to $\arg s$. It is also shown in textbooks on advanced calculus that the condition (A.1.1) holds if

$$f(s) = \frac{P(s)}{Q(s)}, \tag{A.1.2}$$

where $P(s)$ and $Q(s)$ are polynomials and the degree of $P(s)$ is less than the degree of $Q(s)$.

Jordan's lemma implies, but does not specifically state, that the value of the integral over each of the arcs QK and MP (Fig. 2.9.1a) approaches zero as R goes to infinity. This can be shown in the following way. Along the arc $\Gamma_1 = QK$, one has, since $s = Re^{i\theta}$, $\theta_0 \leqslant \theta \leqslant \pi/2$, $\theta_0 = QK/R$,

$$I_1 = \int_{\Gamma_1} e^{ms} f(s)\, ds = \int_{\theta_0}^{\pi/2} \exp(Re^{i\theta} m) f(Re^{i\theta}) \cdot iRe^{i\theta}\, d\theta. \tag{A.1.3}$$

Since $f(s) \to 0$ uniformly on Γ_1 as $|s| \to \infty$ when $\theta_0 <$ arg $s \leqslant \pi/2$, one can, given $\epsilon > 0$ choose an R' such that for $|s| > R'$, $|f(s)| < \dfrac{\epsilon}{e^{cm}\phi_0}$; so

$$|I_1| \leqslant \int_{\theta_0}^{\pi 2} |e^{(R\cos\theta)m}||e^{i(R\sin\theta)m}||f(Re^{i\theta})||iRe^{i\theta}|\, d\theta \tag{A.1.4}$$

$$\leqslant \int_{\theta_0}^{\pi/2} e^{(R\cos\theta)m} |f(Re^{i\theta})|\, R d\theta$$

$$\leqslant \frac{\epsilon}{e^{cm}\theta_0} \int_{\theta_0}^{\pi/2} e^{(R\cos\theta)m}\, d\theta, R > R'$$

$$= \frac{\epsilon}{e^{cm}\phi_0} \int_0^{\phi_0} e^{(R\sin\phi)m}\, d\phi, R > R', \tag{A.1.4a}$$

having utilized the transformation $\theta = \pi/2 - \phi$. Moreover, since $\sin\phi = c/R$ it follows that

$$\int_0^{\phi_0} e^{(R\sin\phi)m}\, d\phi = \int_0^{\phi_0} e^{cm}\, d\phi$$

$$= e^{cm}\phi_0 \tag{A.1.4b}$$

Thus, from (A.1.4a), one has

$$|I_1| < \frac{\epsilon}{e^{cm}\phi_0} \int_0^{\phi_0} e^{(R\sin\phi)m} d\phi,$$

Which in conjunction with (A.1.4b) yields the result

$$|I_1| < \epsilon, R > R'.$$

Hence,

$$\lim_{R\to\infty} \int_{\Gamma_1} e^{ms} f(s)\, ds = 0. \tag{A.1.4c}$$

In a similar manner, one can establish that

$$\lim_{R\to\infty}\int_{\Gamma_2} e^{ms}f(s)\,ds=0,$$

where Γ_2 denotes the arc MP (Fig. 2.9.1a).

It follows, therefore, that if the conditions of Jordan's lemma are satisfied, the value of the integral over the Bromwich path is equal to its value over the Bromwich contour, since the value of the integral over the circular arc approaches zero as R goes to infinity.

It should also be pointed out that the condition that $f(s)$ be analytic when both $|s|\to K$ and $\pi/2\leqslant \arg s\leqslant 3\pi/2$ where K is a positive constant must be satisfied. This condition requires all poles of $f(s)$ to be a finite distance from the origin. This condition may be removed if, for example, the poles of $f(s)$ are countable and are spaced at intervals along the real axis. For then one can, given ε, choose $\rho=\rho_i>0$ such that arcs of radius a_i lie between the poles of $f(s)$, $i=1,2,\ldots,\infty$, and the proof is carried out as before. Appendix F gives a procedure for selecting the sequence of radii a_i, $i=1,2,3,\ldots$, such that the arcs do not pass through any poles, for the case of an H-function inversion integral whose integrand contains an infinite number of poles.

APPENDIX B

The Verification of the Conditions of Jordan's Lemma for Certain Specified Integrals

In the derivation of several product distributions in Chapter 4, it was necessary to invoke Jordan's lemma, which required that $f(s)$ in (2.9.5) and (2.9.6) be uniformly convergent. The proof of this uniform convergence was deferred to this appendix, which covers the case of products and quotients of uniform i.r.v.'s, and to Appendix F, which covers the other cases, since they involve products and quotients of independent H-function r.v.'s (namely, beta and gamma r.v.'s).

B.1 THE PRODUCT AND QUOTIENT OF UNIFORM INDEPENDENT RANDOM VARIABLES

In the derivation of the product of n uniform i.r.v.'s in Section 4.4.1, it was necessary to evaluate the inversion integral

$$h(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^{-s}}{s^n}\,dy, \qquad c>0 \tag{4.4.4}$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{\ln y^{-s}}}{s^n}\,dy,$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{-s\ln y}}{s^n}\,ds, \tag{B.1.1}$$

the latter forms of the integral resulting because any real variable (say, w) is expressible in the form $w=e^{\ln w}$. Clearly,

$$\int_{c-i\infty}^{c+i\infty}\frac{e^{(-\ln y)s}}{s^n}\,ds=\int_{c-i\infty}^{c+i\infty}e^{ms}f(s)\,ds, \qquad m=-\ln y>0, \tag{B.1.2}$$

where $f(s)=s^{-n}$ and where the poles are to the left of the vertical line $(c-i\infty, c+i\infty)$. If $f(s)=s^{-n}$ is uniformly convergent with respect to $\arg s$, the conditions of Jordan's lemma are satisfied and the integral (4.4.4) will be equal to the integral over the (closed) Bromwich contour C_L (Fig. 2.9.1a). The integral can therefore be evaluated by the residue theorem in the manner indicated in Section 4.4.1.

Section 2.9 demonstrated that $f(s)$ is uniformly convergent if

$$|f(s)| \leqslant \frac{M}{R^K}, \tag{B.1.3}$$

where $M>0, K>0$. In the present problem,

$$|f(s)| = \frac{1}{|s^n|} < \frac{M}{R^K}, \tag{B.1.4}$$

where $M=1$ and $K=n-1, n \geqslant 2$; hence the uniform convergence of $f(s)=s^{-n}$ with respect to $\arg s$ has been established.

Similarly, the components of the p.d.f. of the quotient $Y=X_1/X_2$ of two uniform i.r.v.'s stemming from integration over the arcs $QKLMP$ and PTQ, respectively, are given (Fig. 2.9.1) by

$$h_1(y) = \frac{1}{2\pi i}\int_{QKLMP} \frac{y^{-s}}{s(-s+2)}\,ds, \qquad 0 \leqslant y \leqslant 1, \qquad 0<c<2 \tag{4.4.7$'$}$$

$$h_2(y) = \frac{1}{2\pi i}\int_{PTQ} \frac{y^{-s}}{s(-s+2)}\,ds, \qquad 1<y<\infty, \qquad 0<c<2. \tag{4.4.8$'$}$$

Rewriting (4.4.7′) and (4.4.8′), respectively, in the forms

$$\begin{aligned} h_1(y) &= \frac{1}{2\pi i}\int_{QKLMP} \frac{e^{(-\ln y)s}}{s(-s+2)}\,ds, \qquad 0 \leqslant y \leqslant 1 \\ &= \frac{1}{2\pi i}\int_{QKLMP} e^{ms} f(s)\,ds, \qquad m=-\ln y \geqslant 0 \end{aligned}$$

and

$$\begin{aligned} h_2(y) &= \frac{1}{2\pi i}\int_{PTQ} \frac{y^{-s}}{s(-s+2)}\,ds \\ &= \frac{1}{2\pi i}\int_{PTQ} f(s) e^{-ms}\,ds, \qquad m=\ln y>0, -ms \end{aligned}$$

one notes that

$$
\begin{aligned}
|f(s)| &= \left|\frac{1}{s(-s+2)}\right| \\
&= \frac{1}{|s||-s+2|} \\
&\leqslant \frac{1}{R}\cdot\frac{1}{R-2} \\
&\leqslant \frac{2}{R^2}, \qquad R>4.
\end{aligned}
$$

Consequently,

$$\lim_{|s|\to\infty} |f(s)|=0$$

independently of $\arg s$. Thus $f(s)\to 0$ uniformly with respect to $\arg s$, so that the integrals (4.4.7′) and (4.4.8′) approach zero in the limit as $|s|$ approaches infinity.

APPENDIX C*

Proofs of Theorems 5.1.1 and 5.1.2.

Theorem 5.1.1. Let $L_r(f(t))$ be the Laplace transform of $f(t)$, $t \geqslant 0$, such that $L_r(f(t))$ is analytic and of order $O(r^{-k})$ where $K>1$ for all r in $\mathrm{Re}(r)>\varepsilon<0$. Then the Mellin transform of $f(t)$ is given by

$$M_\alpha(f(t)) = \frac{\Gamma(\alpha)}{2\pi i}\int_{c-i\infty}^{c+i\infty} L_r(f(t))(-r)^{-\alpha}\,dr,$$

$$\mathrm{Re}(\alpha)>0, \qquad \varepsilon<c<0. \quad \text{(C.1)}$$

In the special case, when the singularities of $L_r(f(t))$ are poles in $\mathrm{Re}(r)\leqslant\varepsilon$ and $|L_r(f(t))|$ is bounded in $\mathrm{Re}(r)\leqslant\varepsilon$ then

$$M_\alpha(f(t)) = \Gamma(\alpha)\Big[\sum \mathrm{Res}\, L_r(f(t))(-r)^{-\alpha} \text{ at poles of } L_r(f(t))\Big] \quad \text{(C.2)}$$

(where Res stands for "residue of"). Also, when $L_r(f(t))$ is an entire function,

$$M_\alpha(f(t)) = \frac{M_{1-\alpha}(L_r(f(t)))}{\Gamma(1-\alpha)} \quad \text{(C.3)}$$

and $M_\alpha(f(t))$ is analytic in $\mathrm{Re}(\alpha)>0$.

PROOF. Under the stated conditions on $L_r(f(t))$, the inversion integral of $L_r(f(t))$ along any line $\mathrm{Re}(r)=c$ converges to a real-valued function $f(t)$, independent of c; that is,

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{rt}L_r(f(t))\,dr. \quad \text{(C.4)}$$

[48]I am indebted to Dr. Ram Prasad [295] for providing the proofs of these theorems.

The Mellin transform of $f(t)$, $t \geqslant 0$ is defined by

$$M_\alpha(f(t)) = \int_0^\infty f(t) t^{\alpha-1}\, dt. \tag{C.5}$$

Substituting for $f(t)$ in (C.5), one has

$$M_\alpha(f(t)) = \int_0^\infty t^{\alpha-1}\, dt \left[\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L_r(f(t)) e^{rt}\, dr \right]. \tag{C.6}$$

Consider the existence of the following integral:

$$\begin{aligned} I &= \int_0^\infty |t^{\alpha-1}|\, dt \int_{c-i\infty}^{c+i\infty} |L_r(f(t)) e^{rt}|\, dr \\ &= \int_0^\infty t^{\zeta-1}\, dt \int_{-\infty}^{\infty} |L_{c+i\omega}(f(t))| e^{ct}\, d\omega, \qquad \zeta = \mathrm{Re}(\alpha). \end{aligned} \tag{C.7}$$

From the order condition on $L_r(f(t))$, it follows that there exists a positive number κ such that

$$|L_{c+i\omega}(f(t))| \leqslant \kappa (c^2 + \omega^2)^{-\kappa/2}.$$

Hence

$$I \leqslant \int_0^\infty t^{\zeta-1} e^{ct}\, dt \int_0^\infty 2\kappa (c^2 + \omega^2)^{-\kappa/2}\, d\omega. \tag{C.8}$$

Now $\int_0^\infty 2\kappa (c^2 + \omega^2)^{-\kappa/2}$ converges (since $\kappa > 1$) and is equal to $R(c)$, say, and therefore

$$I \leqslant R(c) \int_0^\infty t^{\zeta-1} e^{ct}\, dt = \frac{R(c)\Gamma(\zeta)}{(-c)^\zeta}.$$

Thus I exists for $\zeta > 0$ and $c < 0$; therefore the order of integration can be interchanged in (C.6); that is,

$$\begin{aligned} M_\alpha(f(t)) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L_r(f(t))\, dr \int_0^\infty t^{\alpha-1} e^{rt}\, dt \\ &= \frac{\Gamma(\alpha)}{2\pi i} \int_{c-i\infty}^{c+i\infty} L_r(f(t)) (-r)^{-\alpha}\, dr. \end{aligned} \tag{C.9}$$

In the special case when the singularities of $L_r(f(t))$ are poles in $\mathrm{Re}(r)<\varepsilon$ and $|L_r(f(t))|$ is bounded for all r in $\mathrm{Re}(r)<\varepsilon$, the contour $c\pm i\infty$ can be replaced by a finite circle γ enclosing the singularities of $L_r(f(t))$. Thus

$$M_\alpha(f(t))=\int_0^\infty t^{\alpha-1}dt\cdot\frac{1}{2\pi i}\oint_\gamma L_r(f(t))e^{rt}\,dr. \qquad \text{(C.10)}$$

Now $\oint_r L_r(f(t))e^{rt}\,dr$ converges uniformly in every closed interval $0\leqslant t\leqslant h$; therefore the order of integration can be interchanged in (C.10), that is,

$$M_\alpha(f(t))=\frac{1}{2\pi i}\oint_\gamma L_r(f(t))\,dr\int_0^\infty t^{\alpha-1}e^{rt}\,dt$$

$$=\frac{\Gamma(\alpha)}{2\pi i}\oint_\gamma L_r(f(t))(-r)^{-\alpha}\,dr$$

or

$$M_\alpha(f(t))=\Gamma(\alpha)\sum \mathrm{Res}\,L_r(f(t))\cdot(-r)^{-\alpha} \quad \text{at poles of } L_r(f(t)). \qquad \text{(C.11)}$$

When $L_r(f(t))$ is entire, the only singularity of the integrand of (C.9) is a branch point in $\mathrm{Re}(r)>0$; therefore the contour of integration $c\pm i\infty$ can be replaced by the one in Fig. C.1. On the contour ab, $\arg(-r)=\pi$, so that $(-r)^{-\alpha}=\exp(-i\pi\alpha)x^{-\alpha}$. On the contour cd, $\arg(-r)=-\pi$; so that $(-r)^{-\alpha}=\exp(i\pi\alpha)x^{-\alpha}$. On the circle bc, $-r=\varepsilon e^{i\theta}$. Thus,

$$\begin{aligned}M_\alpha(f(t))=\lim_{R\to\infty}\frac{\Gamma(\alpha)}{2\pi i}\Big[&\int_R^\varepsilon L_x(f(t))e^{-i\pi\alpha}x^{-\alpha}\,dx\\&+\int_{-\pi}^{\pi}L_{\varepsilon e^{i\theta}}(f(t))\varepsilon^{-\alpha}e^{-i\alpha\theta}i\varepsilon e^{i\theta}\,d\theta\\&+\int_\varepsilon^R L_x(f(t))e^{i\pi\alpha}x^{-\alpha}\,dx\Big].\end{aligned} \qquad \text{(C.12)}$$

Let $\mathrm{Re}(\alpha)<1$; then

$$\lim_{\varepsilon\to0}\left|\int_{-\pi}^{\pi}L_{\varepsilon e^{i\theta}}(f(t))i\varepsilon^{1-\alpha}e^{i(1-\alpha)\theta}\,d\theta\right|\leqslant\lim_{\varepsilon\to0}\int_{-\pi}^{\pi}M\varepsilon^{1-\alpha}\,d\theta=0,$$

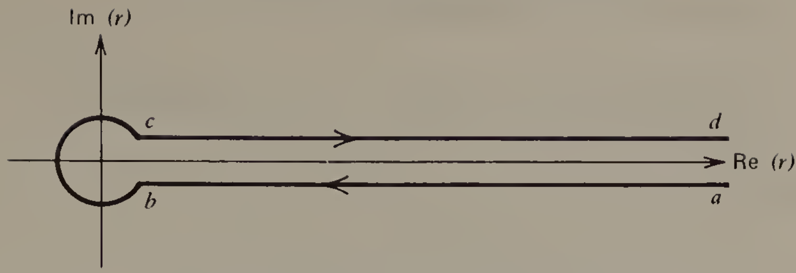


Fig. C.1 Contour of integration used in the proof of Theorem 5.1.1.

where

$$|L_0(f(t))| \leqslant M > 0.$$

Now let R go to infinity; then (C.12) reduces to

$$\begin{aligned} M_\alpha(f(t)) &= \frac{\Gamma(\alpha)}{2\pi i}\left[\int_0^\infty L_x(f(t))e^{i\pi\alpha}x^{-\alpha}\,dx\right. \\ &\quad \left. + \int_\infty^0 L_x(f(t))e^{-i\pi\alpha}x^{-\alpha}\,dx\right] \\ &= \frac{\Gamma(\alpha)}{2\pi i}(e^{i\pi\alpha}-e^{-i\pi\alpha})\int_0^\infty L_x(f(t))x^{-\alpha}\,dx \\ &= \Gamma(\alpha)\left(\frac{\sin\pi\alpha}{\pi}\right)\cdot M_{1-\alpha}(L_x(f(t))) \end{aligned}$$

or

$$M_\alpha(f(t)) = \frac{M_{1-\alpha}(L_x(f(t)))}{\Gamma(1-\alpha)}. \tag{C.13}$$

Since the contour *abcd* does not pass through the point $r=0$, there is no need to stipulate that $\mathrm{Re}(\alpha)<1$. Therefore $M_\alpha(f(t))$ is analytic in $\mathrm{Re}(\alpha)>0$.

Theorem 5.1.2. Let $L_r(f(t))$ and $M_\alpha(f(t))$ be the Laplace and Mellin transforms of $f(t)$, $t \geqslant 0$, respectively. If $f(t)$ is of bounded variation and $t^\kappa f(t) \in L^2$ on $(0,1)$ and $t^l f(t) \in L^2$ on $(1,\infty)$ with $\kappa < l$, then

$$L_r(f(t)) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M_\alpha(f(t))\Gamma(1-\alpha)r^{\alpha-1}\,d\alpha,$$

$$\kappa+\tfrac{1}{2} < c < \min\left(1, l+\tfrac{1}{2}\right). \tag{C.14}$$

PROOF. Under the stated conditions of $f(t)$,

$$M_\alpha(f(t))=\int_0^\infty t^{\alpha-1}f(t)\,dt, \qquad \kappa+\tfrac{1}{2}<\mathrm{Re}(\alpha)<l+\tfrac{1}{2}$$

is analytic, and by the inversion formula it follows that

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}M_\alpha(f(t))t^{-\alpha}\,d\alpha. \tag{C.15}$$

The Laplace transform of $f(t)$ is given by

$$\begin{aligned} L_r(f(t)) &= \int_0^\infty e^{-rt}f(t)\,dt \\ &= \int_0^\infty e^{-rt}\,dt\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}M_\alpha(f(t))t^{-\alpha}\,d\alpha. \end{aligned} \tag{C.16}$$

Now e^{-rt} is of bounded variation in every finite interval of t and $M_\alpha(f(t))$ is analytic in every interval $(c-iR, c+iR)$, $R>0$. Therefore, to prove the interchange of repeated integrals in (C.16), it is sufficient to show that there exists the integral

$$I=\frac{1}{2\pi}\int_0^\infty |e^{-rt}|\,dt\int_{c-i\infty}^{c+i\infty}|M_\alpha(f(t))t^{-\alpha}|\,d\alpha, \tag{C.17}$$

Also,

$$\begin{aligned} \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}|M_\alpha(f(t))t^{-\alpha}|\,d\alpha &\leqslant \frac{t^{-c}}{2\pi}\int_{-\infty}^{\infty}|M_{c+i\omega}(f(t))|\,d\omega \\ &\leqslant \frac{t^{-c}}{2\pi}\left[\int_{-\infty}^{\infty}|M_{c+i\omega}(f(t))|^2\,d\omega\right]^{1/2} \\ &= t^{-c}\left[\int_0^\infty \omega^{2c-1}|f(\omega)|^2\,d\omega\right]^{1/2}. \end{aligned} \tag{C.18}$$

The integral in (C.18) converges by assumption on $f(t)$ and is equal to $Q(c)$, say. Hence

$$\begin{aligned} I &\leqslant \sqrt{Q(c)}\int_0^\infty e^{-\zeta t}t^{-c}\,dt; \qquad \zeta=\mathrm{Re}(r) \\ &= \sqrt{Q(c)}\,\Gamma(1-c)\zeta^{c-1}; \qquad \zeta>0, c<1. \end{aligned}$$

That is, $I<\infty$, and (C.16) can be written as

$$L_r(f(t))=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M_\alpha(f(t))\,d\alpha\int_0^\infty e^{-rt}t^{-\alpha}\,dt$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M_\alpha(f(t))\Gamma(1-\alpha)r^{\alpha-1}\,d\alpha,$$

$$\left(\kappa+\tfrac{1}{2}\right)<c<\min\left(1,l+\tfrac{1}{2}\right) \tag{C.19}$$

which proves the theorem.

APPENDIX D

Special Functions and Transforms of Basic Probability Density Functions[49]

D.1 SPECIAL FUNCTIONS

There are certain special functions with which readers may at times be confronted. For their convenience, the more important of these special functions are defined here

1. Gauss' hypergeometric function or series [97, p. 56]

$$F(a,b;c,z)=\frac{\sum_{n=0}^{\infty}(a)_n(b)_n z^n}{[(c)_n n!]},$$

where $(a)_n = a(a+1)\cdots(a+n-1)$. When expressed in terms of the generalized hypergeometric function or series defined below, it has the form

$$_2F_1(a,b;c,z)=F\left[\begin{matrix} a,b;z \\ c \end{matrix}\right].$$

2. Generalized hypergeometric function or series [97, p. 182]

$$_pF_q\left[\begin{matrix} \alpha_1,\alpha_2,\ldots,\alpha_p;z \\ \rho_1,\rho_2,\ldots,\rho_q \end{matrix}\right]={}_pF_q(\alpha_r;\rho_t;z)$$

$$=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n z^n}{(\rho_1)_n\cdots(\rho_q)_n n!}.$$

[49]Since these special functions are valid for both real and complex variables, the notation is that of a complex variable z.

3. Kummer's confluent hypergeometric function [97, p. 248]

$$ {}_1F_1(a,c;z)=1+\frac{a}{c}\frac{z}{1!}+\frac{a(a+1)}{c(c+1)}\frac{z^2}{2!}+\ldots $$

(sometimes denoted by ϕ $(a,c;z)$, see, e.g., ref. 97, Chapter 6).

4. Bessel function of the first kind of order ν [98, (2), p. 4]

$$ J_\nu(z)=\sum_{m=0}^{\infty}\frac{(-1)^m(z/2)^{2m+\nu}}{[m!\Gamma(m+\nu+1)]} $$

$$ =\left(\frac{z}{2}\right)^{\nu}e^{-z}{}_1F_1\left(\nu+\tfrac{1}{2};2\nu+1;2iz\right)/\Gamma(\nu+1). $$

5. Modified Bessel function of the first kind of order ν [98 (12), p. 5]

$$ I_\nu(z)=\sum_{k=0}^{\infty}\frac{(z/2)^{2k+\nu}}{[k!(k+\nu)!]} $$

6. Bessel function of the second kind of order ν [98, (4), p. 4], where ν is neither zero nor a positive integer

$$ Y_\nu(z)=(\sin\nu\pi)^{-1}[J_\nu(z)\cos(\nu\pi)-J_{-\nu}(z)]. $$

7. Modified Bessel function of the second kind of order n, where n is zero or a positive integer [98, p. 9]

$$ K_n(z)=\frac{\pi}{2}i^{n+1}[J_n(iz)+iY_n(iz)]. $$

$$ =\lim_{\nu\to n}\frac{\pi}{2\sin(\nu\pi)}[I_{-\nu}(z)-I_\nu(z)] $$

8. Bessel function of the third kind of order ν, where ν is neither zero nor a positive integer [98, (5), (6), p. 4]

$$ H_\nu^{(1)}(z)=J_\nu(z)+iY_\nu(z)=[i\sin(\nu\pi)]^{-1}[J_{-\nu}(z)-J_\nu(z)e^{-i\nu\pi}] $$

$$ H_\nu^{(2)}(z)=J_\nu(z)-iY_\nu(z)=[i\sin(\nu\pi)]^{-1}[J_\nu(z)e^{i\nu\pi}-J_{-\nu}(z)]. $$

9. Modified Bessel function of the third kind of order ν, where ν is neither zero nor a positive integer [98, (13), p. 5]

$$ K_\nu(z)=\frac{\pi}{2\sin(\nu\pi)}[I_{-\nu}(z)-I_\nu(z)]. $$

10. Whittaker's function [95, p. 386]

$$W_{k,\mu}(z)=\frac{\Gamma(-2\mu)M_{k,\mu}(z)}{\Gamma(\frac{1}{2}-\mu-k)}+\frac{\Gamma(2\mu)M_{k,-\mu}(z)}{\Gamma(\frac{1}{2}+\mu-k)}$$

where

$$M_{k,\mu}(z)=z^{(1/2)+\mu}e^{-z/2}{}_1F_1\left(\tfrac{1}{2}+\mu-k;2\mu+1;z\right).$$

11. Parabolic cylinder function [98, p. 117, formula (3)]

$$D_\nu(z)=2^{1/2(\nu+1/2)}Z^{-1/2}W_{\frac{1}{2}\left(\nu+\frac{1}{2}\right),\frac{1}{4}}\left(\frac{z^2}{2}\right)$$

12. Dirac delta function (general a) [200, p. 21]

$$\delta(z-a)=0, \qquad z\neq a$$

$$\int_{-\infty}^{\infty}\delta(z-a)\,dz=1, \qquad z=a.$$

If the Dirac delta function is multiplied by any function $f(z)$, the product is zero everywhere except at $z=a$, so that

$$\int_{-\infty}^{\infty}(x-a)f(x)\,dx=f(a).$$

13. Error function [2, p. 297]

$$\operatorname{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-u^2}du, \qquad 0<z<\infty.$$

14. Sine integral [97, p. 267, formula (27)]

$$Si(z)=\int_0^z\frac{\sin u}{u}\,du$$

15. Cosine integral [97, p. 267, formula (28)]

$$Ci(z)=-\int_z^{\infty}\frac{\cos u}{u}\,du$$

16. Exponential integral [2, p. 228]

$$Ei(x)=\int_x^{\infty}\frac{e^{-u}}{u}\,du \qquad (x \text{ real and positive})$$

$$E_1(z)=\int_z^{\infty}\frac{e^{-t}}{t}\,dt, \qquad z \text{ complex}, |\arg z|<\pi.$$

Some basic p.d.f.'s are useful in both theoretical and applied areas of statistics and probability. These density functions and their integral transforms are given in Table D.2.

Table D.2 Integral Transforms of Basic Probability Density Functions

Probability Law	p.d.f	$L_r(f(x))$	$F_t(f(x))^c$	$M_s(f(x))$
1. Nonstandardized				
Normal	$\frac{1}{\sigma\sqrt{2\pi}}\exp\left[\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$, $-\infty<x<\infty$	$e^{-r\mu+r^2\sigma^2/2}$†	$\exp\left(it\mu-\frac{t^2\sigma^2}{2}\right)$	$\frac{(2\sigma^2)(s-1)/2}{2\sqrt{\pi}}\exp\left[-\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2\right]\cdot\Gamma\left(\frac{s}{2}\right){}_1F_1\left(\frac{s}{2};\frac{1}{2};\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2\right)$ $+\frac{\mu\sqrt{2}}{\sigma}\Gamma\left(\frac{s+1}{2}\right){}_1F_1\left(\frac{s+1}{2};\frac{3}{2};\frac{1}{2}\left(\frac{\mu}{\alpha}\right)^2\right)$; Re$(s)>0$, where ${}_1F_1(a,b,z)$ is defined in Appendix D.1
2. Standardized				
Normal	$\frac{1}{\sqrt{2\pi}}\exp\left[\frac{-x^2}{2}\right]$, $-\infty<x<\infty$	$e^{r^2/2}$†	$e^{-t^2/2}$	$\frac{(2)^{(s-3)/2}}{\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)$, Re$(s)>0$
3. Uniform	$f(x)=\frac{1}{b-a}$, $a<x<b$	$\frac{e^{-ra}-e^{-rb}}{r(b-a)}$	$\frac{e^{itb}-e^{ita}}{it(b-a)}$	$\frac{1}{b-a}\left(\frac{b^s-a^s}{s}\right)$, Re$(s)>0$
4. Gamma[b]	$f(x)=\frac{\lambda}{\Gamma(a)}(\lambda x)^{a-1}e^{-\lambda x}$, $a<x<\infty$, $a>0,\lambda>0$	$\left(1+\frac{r}{\lambda}\right)^{-a}$	$\left(1-\frac{it}{\lambda}\right)^{-a}$	$\lambda^{-s+1}\frac{\Gamma(s+a-1)}{\Gamma(a)}$, Re$(s)>-\alpha+1$
5. Chi-square with m degrees of freedom	$f(x)=\frac{1}{\Gamma(m/2)2^{m/2}}x^{(m/2)-1}e^{-x/2}$, $x>0$	$\frac{1}{(1+2r)^{m/2}}$	$\frac{1}{(1-2it)^{m/2}}$	$\frac{2^{s-1}}{\Gamma(m/2)}\Gamma\left(s+\frac{m}{2}-1\right)$, Re$(s)>1-\frac{m}{2}$
6. Chi	$f(x)=\frac{x^{m-1}e^{-x^2/2}}{\Gamma\left(\frac{m}{2}\right)2^{\frac{m}{2}-1}}$, $x>0$	$\frac{\Gamma(m)e^{r^2/4}}{2^{\frac{m}{2}-1}\Gamma\left(\frac{m}{2}\right)}D_{-m}(r)$, [95, p. 146, formula (24)]	$\frac{\Gamma(m)e^{-t^2/4}}{2^{\frac{m}{2}-1}\Gamma\left(\frac{m}{2}\right)}D_{-m}(-it)$	$\left(\frac{2}{m}\right)^{(s-1)/2}\frac{\Gamma\left(\frac{s+m-1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}$, Re$(s)>1-m$
7. Beta	$f(x)=\frac{x^{a-1}}{B(a,b)}(1-x)^{b-1}$, $0<x<1$	${}_1F_1(a,a+b,-r)=$ $\frac{\Gamma(a+b)}{\Gamma(a)}\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(a+j)r^j}{\Gamma(a+b+j)(j+1)}$, [225, p. 18]	${}_1F_1(a,a+b,it)=$ $\frac{\Gamma(a+b)}{\Gamma(a)}\sum_{j=0}^{\infty}\frac{\Gamma(a+b)}{\Gamma(a+b+j)\Gamma(j+1)}(it)^j$	$\frac{\Gamma(a+b)}{\Gamma(a)}\frac{\Gamma(s+a-1)}{\Gamma(s+a-1+b)}$, Re$(s)>-a+1$
8. Cauchy	$f(x)=\frac{1}{\pi(c^2+x^2)}$, $-\infty<x<\infty$	$\frac{1}{\pi c}\left[\sin cr\,Ci(cr)-\cos cr\left(Si(cr)-\frac{\pi}{2}\right)\right]$, $c>0$, $r>0$ [225]	$e^{-c\lvert t\rvert}$ [225, p. 18]	$\frac{c^{s-1}}{2}\csc\left(\frac{\pi s}{2}\right)$, $0<$ Re$(s)<2$
9. Rayleigh[d]	$f(x)=\frac{x}{\alpha^2}\exp\left[\frac{-1}{2}\left(\frac{x}{\alpha}\right)^2\right]$, $\alpha>0, x>0$	$1-\sqrt{\frac{\pi}{2}}\,\alpha r e^{\alpha^2r^2/2}\mathrm{erf}\left(\frac{\alpha r}{\sqrt{2}}\right)$ [95, p. 146, formula (22)]	$1-\sqrt{\frac{\pi}{2}}\,\alpha(-it)e^{-\alpha^2t^2/2}\mathrm{erf}\left(\frac{-\alpha it}{\sqrt{2}}\right)$	$(2\alpha^2)^{(s-1)/2}\Gamma\left(\frac{s+1}{2}\right)$, Re$(s)>-1$
10. Laplace	$f(x)=\frac{\lambda}{2}e^{-\lambda\lvert x\rvert}$, $-\infty<x<\infty$	$\left(1+\frac{r}{\lambda}\right)^{-1}$[a]	$\left(1-\frac{it}{\lambda}\right)^{-1}$	$\lambda^{-s+1}\Gamma(s)$

[a]Bilateral Laplace transform indicated by †.

[b]Special cases: Erlang, a = integer; exponential, $a=1$.

[c]$F[e^{i-x}]$

APPENDIX E[50]

The Complex Fourier or Bilateral Laplace Transform

Chapter 2 indicated that under certain conditions the Fourier transform could be replaced with the complex Fourier or the bilateral Laplace transform. The validity of this assertion is now established.

Consider first the question of the existence of the bilateral Laplace transform. From (2.8.7b) it is apparent that the bilateral Laplace transform exists if

$$\int_{-\infty}^{\infty} e^{rx} f(x)\,dx \tag{E.1}$$

is finite. That is, since

$$|f(x)e^{rx}| = |f(x)|e^{cx}, \qquad c = \mathrm{Re}(r), \tag{E.2}$$

the existence of $\mathfrak{F}_r(f(x))$ is ensured if

$$\int_{-\infty}^{\infty} |f(x)|e^{cx}\,dx \tag{E.3}$$

is finite. Now, if there exist real finite numbers M, α, and β such that

$$f(x) \leqslant \begin{matrix} Me^{-\alpha x}, & x<0 \\ Me^{-\beta x}, & x>0, \end{matrix} \tag{E.4}$$

then the integral (E.3) is finite for any value c greater than α but less than β; that is, the integral $\int f(x)e^{rx}\,dx$ is absolutely convergent for values of

[50]The approach used to establish the results in Appendix E, including Figs. E.1 and E.2, is essentially that of B. P. Lathi (*An Introduction to Random Signals and Communications Theory*, Dun-Donnelley Publishing Company, New York, 1968), and is used with the permission of the publisher.

$r = c + ix$ that satisfy the condition

$$\alpha < c < \beta. \tag{E.5}$$

This is readily apparent if one partitions the integral into two components:

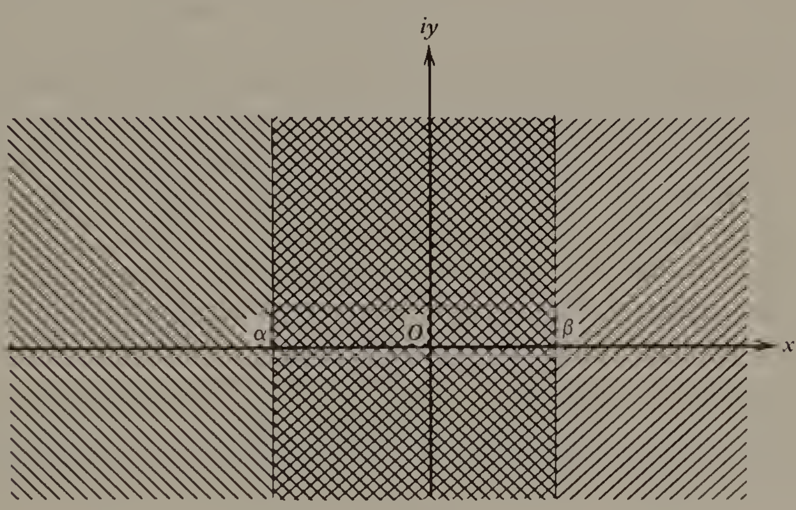
$$\mathfrak{F}_r(f(x)) = \int_{-\infty}^{0} e^{rx} f(x)\,dx + \int_{0}^{\infty} e^{rx} f(x)\,dx. \tag{E.6}$$

Use of the inequality (E.4) in conjunction with (E.6) yields

$$|\mathfrak{F}_r(f(x))| \leqslant \int_{-\infty}^{0} Me^{(r-\alpha)x}\,dx + \int_{0}^{\infty} Me^{(r-\beta)x}\,dx \tag{E.7}$$

$$\leqslant M\left(\frac{1}{r-\alpha} e^{(r-\alpha)x} \Big|_{-\infty}^{0} + \frac{1}{r-\beta} e^{(r-\beta)x} \Big|_{0}^{\infty}\right). \tag{E.8}$$

It is clear that the first integral of the inequality (E.8) converges for $\mathrm{Re}(r) > \alpha$ and the second integral converges when $\mathrm{Re}(r) < \beta$. The two regions are depicted in Fig. E.1. Both integrals converge in the common region determined by $\alpha < c < \beta$.



Region of convergence of $\mathfrak{F}_r(f^-(x)) = \int_{-\infty}^{0} e^{rx} f^-(x)\,dx$, where $f^-(x)$ denotes the part of $f(x)$ corresponding to $-\infty < x \leqslant 0$.

Region of convergence of $\mathfrak{F}_r(f^+(x)) = \int_0^\infty e^{rx} f^+(x)\,dx$, where $f(x)$ denotes the part of $f(x)$ corresponding to $0 \leqslant x < \infty$.

Region of convergence of $\mathfrak{F}_r(f(x))$ corresponding to the entire function $f(x)$ for $-\infty < x < \infty$.

Fig. E.1 Regions of absolute convergence for the bilateral Laplace transform.

Clearly, for all values of r lying in the region of convergence, $\mathfrak{F}_r(f(x))$ is finite; thus any singularities (poles) of $\mathfrak{F}_r(f(x))$ must lie outside the region of convergence. Thus if $f(x)$ is defined on the interval $(0, \infty)$ only, the poles of $\mathfrak{F}_r(f(x))$ must lie to the right of the region of convergence. Consequently, when $f(x)$ is determined from the inversion integral (2.8.7a) by the method of residues, the integration is performed over the right-hand Bromwich contour C_R (Fig. 2.9.1b), since this contour encloses the poles. Similarly, if $f(x)$ is defined on the interval $(-\infty, 0)$ only, the poles of $\mathfrak{F}_r(f(x))$ must lie to the left of the region of convergence. For a function $f(x)$ that is defined on the interval $(-\infty, \infty)$, $\mathfrak{F}_r(f(x))$ may have some poles lying to the left and some lying to the right of the region of convergence. From the foregoing discussion, it is clear that the poles to the right of the region arise because of the part of the function $f(x)$ corresponding to the positive domain of x (i.e., $0 \leqslant x < \infty$), whereas those to the left of the region arise from the portion of $f(x)$ associated with the negative domain of x. This is extremely important in determining the inverse transform.

The regions of convergence of the complex Fourier transform are indicated by Fig. E.1. The ordinary Fourier transform is the special case for which the region of convergence that is utilized is limited to the imaginary axis $r = it$. If the region of convergence of the bilateral Laplace transform of a function $f(x)$ includes the imaginary axis, the ordinary Fourier transform of $f(x)$ necessarily exists and can be obtained by a direct substitution of it for r in $\mathfrak{F}_r(f(x))$. On the other hand, if the region of convergence of $\mathfrak{F}_r(f(x))$ does not include the imaginary axis, the function $f(x)$ does not satisfy the condition of absolute integrability and cannot possess the ordinary Fourier transform [200, p. 310].

It bears stating at this point that if a function $f(x)$ is absolutely integrable, its ordinary Fourier transform exists, from which it follows that the region of convergence of $\mathfrak{F}_r(f(x))$ must include the imaginary axis. In such cases, the poles of $\mathfrak{F}_r(f(x))$ lying to the left of the region of convergence are necessarily located in the LHP, and all the poles lying to the right of the region are necessarily located in the RHP. From the previous discussion regarding the positive and negative domains of x, it is clear that if $f(x)$ is defined over the range $(-\infty, \infty)$ and is absolutely integrable—that is, if

$$\int_{-\infty}^{\infty} |f(x)|\, dx \tag{E.9}$$

is finite—all the terms of $\mathfrak{F}_r(f(x))$ represented by LHP poles correspond to the negative domain of x, and the terms represented by the RHP poles correspond to the positive domain of x. On the other hand, if $f(x)$ is absolutely integrable and is defined only for values of x in the interval $(0, \infty)$ (i.e., if $f(x)=0$ for $-\infty<x<0$), all the poles of $\mathfrak{F}_r(f(x))$ must lie in the RHP. Similarly, if $f(x)$ is absolutely integrable and is defined only for the values of x in the interval $(-\infty, 0)$, all the poles of $\mathfrak{F}_r(f(x))$ must lie in the LHP.

As has already been emphasized, the transform pair (2.8.5a, b) was originally selected so that the characteristic function would be identical with the Fourier transform. The extension of the ordinary Fourier transform to the complex Fourier (bilateral Laplace) transform then led one to the regions of convergence and relative positions of the poles as indicated in Fig. E.1.

If one were only concerned with Fourier and bilateral Laplace transforms, it would be preferable to stop at this point and not consider the other transform pair (2.8.6a, b). For problems involving unilateral Laplace transforms, however, the definition (2.8.2a) is universally employed, in which case the unilateral Laplace transform pair is

$$L_r(f(x)) = \int_0^{\infty} e^{-rx} f(x)\, dx,$$

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{rx} L_r(f(x))\, dx.$$

If now one uses the unilateral and bilateral Laplace transforms in conjunction with each other—as, for example in evaluating the bilateral Laplace transform by means of unilateral Laplace transforms—it is preferable to use the bilateral Laplace transform pair

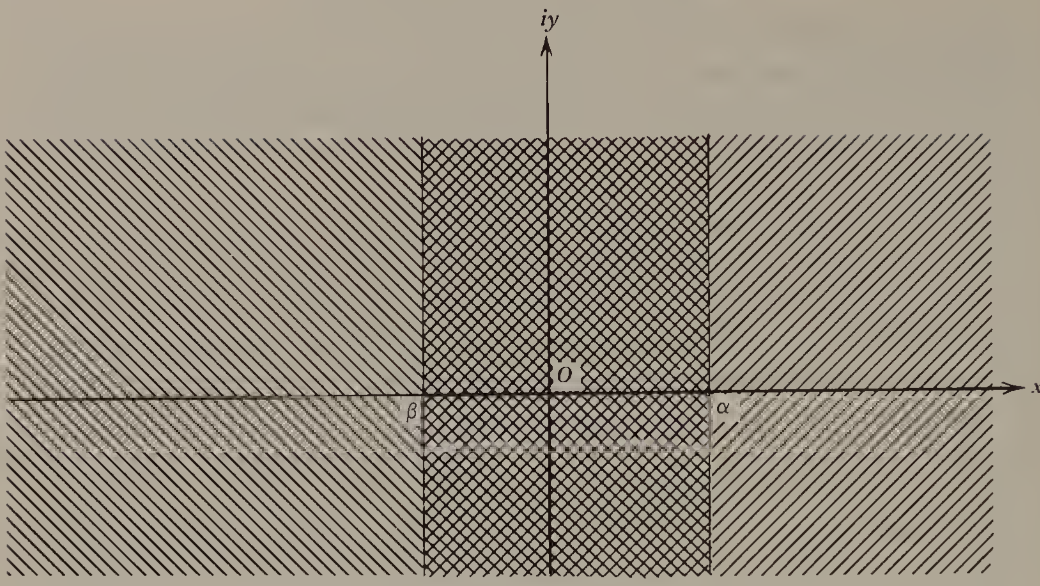
$$\mathfrak{F}_r(f(x)) = \int_{-\infty}^{\infty} e^{-rx} f(x)\, dx, \tag{E.10a}$$

$$f(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{rx} f(x)\, dr. \tag{E.10b}$$

Then the inequality (E.7) no longer obtains but is replaced by the inequality

$$|\mathfrak{F}_r(f(x))| \leqslant \int_{-\infty}^{0} M e^{(\alpha - r)x}\, dx + \int_0^{\infty} M e^{(\beta - r)x}\, dx. \tag{E.11}$$

Now the regions of convergence for $\mathfrak{F}_r(f(x))$ and the relative locations of the poles appear as in Fig. E.2. It is now true that if $f(x)$ is absolutely integrable, all the terms of $\mathfrak{F}_r(f(x))$ represented by LHP poles correspond to the positive domain of x, and the terms represented by the RHP poles



Region of convergence of $\mathfrak{F}_r(f^-(x)) = \int_{-\infty}^{0} e^{-rx} f^-(x)\,dx$, where $f^-(x)$ denotes that part of $f(x)$ corresponding to $-\infty < x \leqslant 0$.

Region of convergence of $\mathfrak{F}_r(f^+(x)) = \int_0^{\infty} e^{-rx} f^+(x)\,dx$, where $f^+(x)$ denotes that part of $f(x)$ corresponding to $0 \leqslant x < \infty$.

Region of convergence of $\mathfrak{F}_r(f(x))$ corresponding to the entire function $f(x)$ for $-\infty < x < \infty$.

Fig. E.2 Regions of absolute convergence for the bilateral Laplace transform.

correspond to the negative domain of x. Thus if $f(x)$ is absolutely integrable and is defined only for values of x on the nonnegative interval $(0,\infty)$, all the poles of $\mathfrak{F}_r(f(x))$ must lie in the LHP. Likewise, if $f(x)$ is defined only on the negative domain $(-\infty,0)$ of x, all the poles of $\mathfrak{F}_r(f(x))$ must lie on the RHP. Finally, if $f(x)$ is defined for both positive and negative values of x, all the terms of $\mathfrak{F}_r(f(x))$ represented by LHP poles correspond to the positive domain of x, and the terms represented by the RHP correspond to the negative domain of x.

An important property of the bilateral Laplace transform is that it can be expressed as a sum of two unilateral Laplace transforms, as we now show.

Consider, for example, the function $f(x)$ in Fig. E.3a, partitioned into two components corresponding to negative and positive values of x (Fig. E.3b,c):

$$f(x)=f_1(x), \qquad -\infty < x \leqslant 0 \tag{E.12a}$$

$$=f_2(x); \qquad 0 \leqslant x < \infty. \tag{E.12b}$$

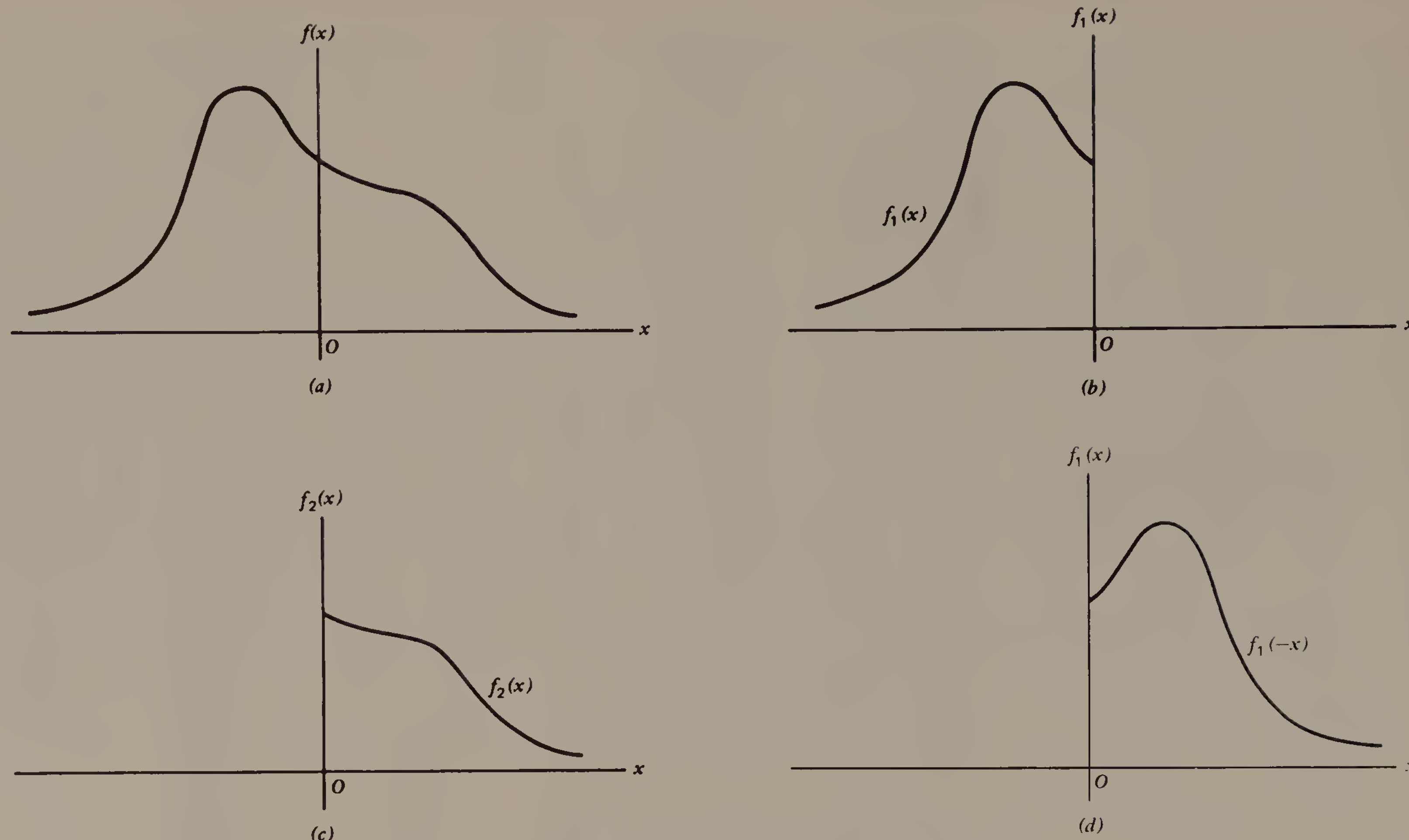


Fig. E.3 (a) Graphical representation of $f(x)$ as given by (E.10b). (b) Graphical representation of $f_1(x)$. (c) Graphical representation of $f_2(x)$. (d) Mirror image of $f_1(x)$.

Then

$$\mathfrak{F}(f(x))=\int_{-\infty}^{\infty} e^{-rx}f(x)\,dx$$

$$=\int_{-\infty}^{0} e^{-rx}f_1(x)\,dx+\int_{0}^{\infty} e^{-rx}f_2(x)\,dx$$

$$=\int_{0}^{\infty} e^{rx}f_1(-x)\,dx+\int_{0}^{\infty} e^{-rx}f_2(x)\,dx$$

$$=L_{-r}(f_1(-x))+L_r(f_2(x)),$$

where $L_{-r}(f_1(-x))$ and $L_r(f_2(x))$ are unilateral Laplace transforms. Since $f_1(-x)$ is a mirror image (Fig. E.3*d*) of the component $f_1(x)$ about the vertical axis $x=0$, it is clear that the contribution of this component to the total transform may be obtained by: (*a*) taking the mirror image of $f_1(x)$ about the vertical axis and finding its Laplace transform, and (*b*) replacing r and $-r$ in the transform so found.

Then the region of convergence of the transform of the complete function $f(x)$ is that common to both of the transforms $\mathfrak{F}_{-r}(f_1(x))$ and $\mathfrak{F}_r(f_2(x))$. The procedure is analogous to that given in Chapter 4 for finding the Mellin transform $M_s(f(x))$ for a function $f(x)$ defined over the doubly infinite range $-\infty<x<\infty$. The following example is illustrative.

Example E.1. Find the bilateral Laplace transform of the p.d.f.

$$f(x)=f_1(x), \qquad -\infty<x\leqslant 0$$

$$=f_2(x), \qquad 0\leqslant x<\infty,$$

where (Fig. E.4)

$$f_1(x)=\frac{6}{5}e^{2x}, \qquad -\infty<x\leqslant 0, \tag{E.13a}$$

$$f_2(x)=\frac{6}{5}e^{-3x}, \qquad 0\leqslant x<\infty. \tag{E.13b}$$

In view of the foregoing discussion,

$$f_1(-x)=\frac{6}{5}e^{-2x}, \qquad 0\leqslant x<\infty,$$

$$L_r(f_1(-x))=\frac{6}{5}\left(\frac{1}{r+2}\right),$$

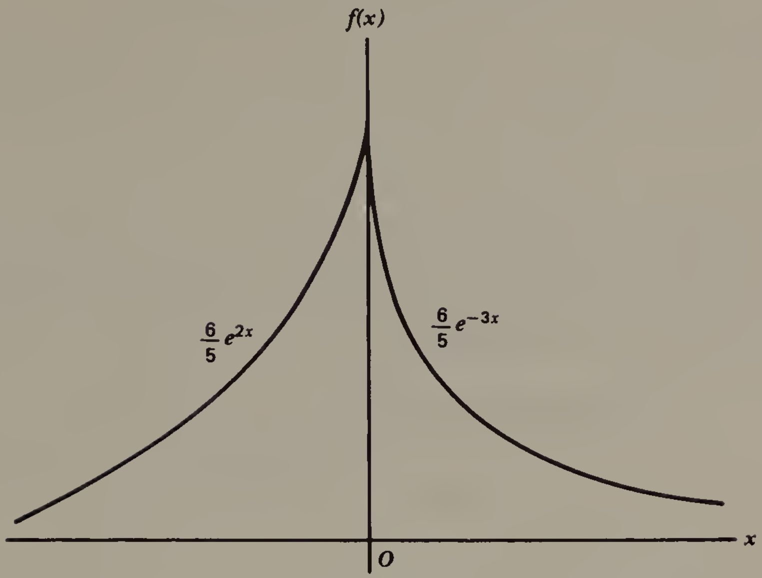


Fig. E.4 Graphical representation of $f(x)$ given in (E.13a, b).

and

$$L_{-r}(f_1(x)) = \frac{6}{5}\left(\frac{1}{-r+2}\right),$$

whereas for $f_2(x)$ one has directly

$$L_r(f_2(x)) = \frac{6}{5}\left(\frac{1}{r+3}\right).$$

Therefore

$$\begin{aligned}\mathfrak{F}_r(f(x)) &= \frac{6}{5}\left(\frac{1}{-r+2}\right) + \frac{6}{5}\left(\frac{1}{r+3}\right)\\ &= \frac{6}{(-r+2)(r+3)},\end{aligned}$$

where the region of convergence is $-3<r<2$.

The inverse problem of finding the p.d.f. $f(x)$, $-\infty<x<\infty$, whose bilateral Laplace (or complex Fourier) transform is known, is illustrated by Example E.2.

Example E.2. Find the inverse transform $f(x)$ if

$$\mathfrak{F}_r(f(x)) = \frac{3}{(-r+3)(r+1)}, \qquad -1<r<3.$$

When expressed in terms of partial fractions, $\mathfrak{F}_r(f(x))$ has the form

$$\mathfrak{F}_r(f(x)) = \frac{3}{4}\left[\frac{1}{(-r+3)} + \frac{1}{(r+1)}\right], \qquad -1<r<3.$$

Since the region of convergence of $F_r(f(x))$ is $-1<r<3$, the pole at $r=3$ lies to the right of the region of convergence, so that the term $1/(-r+3)$ stems from the component $f_1(x)$, $-\infty<x\leqslant 0$, and $1/(r+1)$ stems from the component $f_2(x)$, $0\leqslant x<\infty$. Thus

$$\begin{aligned} \mathfrak{F}_r(f(x)) &= \mathfrak{F}_r(f_1(x)), \qquad -\infty<x<0 \\ &= \mathfrak{F}_r(f_2nx), \qquad 0<x<\infty, \end{aligned}$$

where

$$\begin{aligned} \mathfrak{F}_r(f_1(x)) &= L_{-r}(f_1(-x)), \qquad 0\leqslant x<\infty, \\ &= \frac{3}{4}\frac{1}{(-r+3)} \end{aligned}$$

$$\begin{aligned} \mathfrak{F}_r(f_2(x)) &= L_r(f_2(x)) \\ &= \frac{3}{4}\left(\frac{1}{r+1}\right), \qquad 0\leqslant x<\infty. \end{aligned}$$

Then

$$f_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{3}{4}\right)\left(\frac{e^{rx}}{(-r+3)}\right)dr, \qquad c>3, \qquad -\infty<x\leqslant 0,$$

which, when evaluated by means of the residue theorem, becomes

$$f_1(x) = \frac{3}{4}e^{3x}, \qquad -\infty\leqslant x<0.$$

Similarly,

$$\begin{aligned} f_2(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{3e^{rx}}{4(r+1)}dr \\ &= \frac{3e^{-x}}{4}, \qquad 0\leqslant x<\infty. \end{aligned}$$

APPENDIX F

Proof of the Validity of the Residue Theorem in Evaluating the H-Function Inversion Integral

Application of the residue theorem to the evaluation of the H-function inversion integral introduced in Section 6.2.1 is valid if Jordan's lemma (Appendix A.1) applies. For if Jordan's lemma applies, the integral evaluated over the Bromwich path is equal to the integral evaluated over the relevant (closed) Bromwich contour C_{L_K} or C_{R_K} (Fig. F.1.), and this integral can be evaluated by the method of residues. It will be proved that the conditions of Jordan's lemma hold for all H-function inversion integrals. The proof is carried out in three parts, dealing with the following three cases, which are treated, respectively, in Sections F.1, F.2, and F.3.

Case 1. The integrand (i.e., the Mellin transform of $h(y)$) of the inversion integral contains poles in both the LHP and the RHP, hence contains both the factors

$$\prod_{i=1}^{M} \Gamma(b_i+\beta_i s) \qquad \text{and} \qquad \prod_{i=1}^{N} \Gamma(1-a_i-\alpha_i s).$$

The presence or absence of the remaining factors is immaterial, insofar as Jordan's lemma is concerned, as will be shown.

Case 2. The integrand of the inversion integral contains poles only in the LHP. That is, it contains the factor $\Pi_{i=1}^{M}\Gamma(b_i+\beta_i s)$ but not the factor $\Pi_{i=1}^{N}\Gamma(1-a_i-\alpha_i s)$.

Case 3. The integrand of the inversion integral contains poles only in the RHP. That is, it contains the factor $\Pi_{i=1}^{N}\Gamma(1-a_i-\alpha_i s)$ but not the factor $\Pi_{i=1}^{M}\Gamma(b_i+\beta_i s)$.

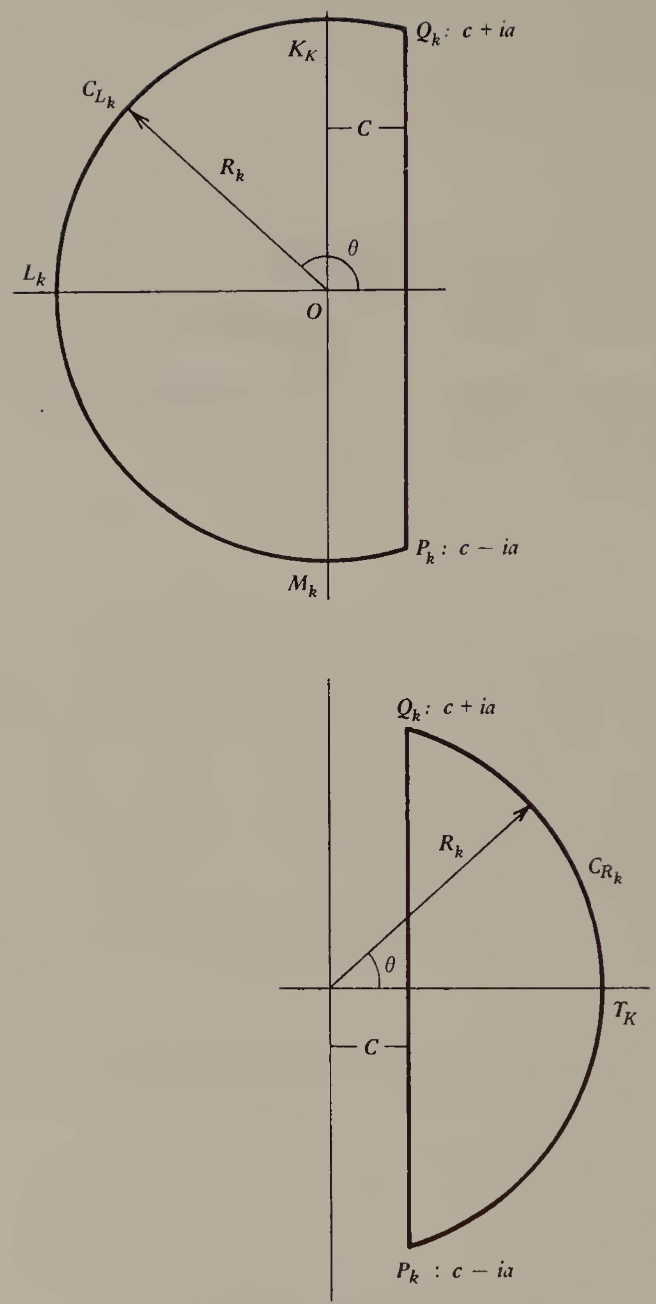


Fig. F.1 Bromwich contours of integration for infinite number of poles in the LHP and the RHP.

F.1 APPLICATION OF JORDAN'S LEMMA IN THE EVALUATION OF THE *H*-FUNCTION INVERSION INTEGRAL WHEN POLES OCCUR IN BOTH THE LEFT HALF-PLANE AND THE RIGHT HALF-PLANE

It is convenient, when considering the application of Jordan's lemma, to express the H-function inversion integral in the form

$$h(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)e^{-(\ln y)s}\,ds,$$

where $f(s)$ denotes the Mellin transform of the H-function. Recall that when $f(s)$ has an infinite number of poles, the three conditions that will ensure that Jordan's lemma is applicable are:

1. The kernel must be of the form e^{ms}, $m>0$, relative to the Bromwich contour C_L and e^{-ms}, $m>0$, relative to the Bromwich contour C_R.
2. The infinite number of poles must be countable and spaced at intervals along the real axis.
3. $\operatorname{Lim} f(s)\to 0$ uniformly with respect to $\arg s$ as $s\to\infty$.

Consider the derivation of the component $h_1(y)$, $0<y\leqslant 1$, utilizing the poles in the LHP ($h_1(y)$ is defined in Section 7.1). Arrange the distinct values of $B_i=-b_i/\beta_i$, $i=1,2,\ldots,M$, in algebraically increasing order of magnitude as follows:

$$B_1',B_2',\ldots,B_m',\qquad m\leqslant M.$$

Similarly, arrange the distinct values of $A_i=(1-a_i)/\alpha_i$, $i=1,2,\ldots,N$, in algebraically increasing order of magnitude as follows:

$$A_1',A_2',\ldots,A_n',\qquad n\leqslant N.$$

The Bromwich path can now be taken as any line $(c-i\infty,c+i\infty)$ for which $B_m'<c<A_1'$. (This is always possible, since from the definition of the H-function inversion integral, no pole of $\Gamma(b_i+\beta_i s)$ for $i=1,2,\ldots,M$ coincides with any pole of $\Gamma(1-a_i-\alpha_i s)$ for $i=1,2,\ldots,N$.) Suppose one takes the line segment P_kQ_k (Fig. F.1) corresponding to $c=\lambda$, where

$$\lambda=\min\left\{\frac{|B_m'|}{2},\frac{|A_1'|}{2}\right\},$$

and choose the kth closed contour C_{L_k} (Fig. F.1) whose arc $Q_kK_kL_kM_kP_k$ is that of a circle with center at $(B_m',0)$ and radius

$$|s|=R_k=|B_m'|+k+\varepsilon, \qquad k=0,1,2,\ldots,$$

where ε is chosen such that $0<\varepsilon<1$ and such that for any s on the arc, $(b_i+\beta_i s)$, $i=1,2,\ldots,M$; $(1-a_i-\alpha_i s)$, $i=1,2,\ldots,N$; $(1-b_i-\beta_i s)$, $i=M+1,M+2,\ldots,q$; and $(a_i+\alpha_i s)$, $i=N+1,N+2,\ldots,p$ are not integers. (The second constraint is not used at this point, but it is imposed for use in subsequently proving that the third condition of Jordan's lemma is satisfied. Note also that the gamma functions $\Gamma(1-b_i-\beta_i s)$ and $\Gamma(a_i+\alpha_i s)$ in the denominator of the H-function inversion integral are no cause for concern, since they are bounded from below.) Because B_m' and A_1' never coincide, it is always possible to find such an ε value, for M and N are finite, and the points on any line segment, however small, are everywhere dense. These choices of c and R_k ensure that no closed contour of the sequence of closed contours C_{L_k}, $k=0,1,2,\ldots,$ passes through any of the poles, and that the poles are countable, having been placed in a one-to-one correspondence with the set of positive integers.

It remains to show that $\lim_{s\to\infty} f(s)$ approaches zero uniformly with respect to $\arg s$, where

$$f(s)=\frac{\prod_{i=1}^{M}\Gamma(b_i+\beta_i s)\prod_{i=1}^{N}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=M+1}^{q}\Gamma(1-b_i-\beta_i s)\prod_{i=N+1}^{p}\Gamma(a_i+\alpha_i s)}.$$

Recall that for any s on C_{L_k}, R_k was chosen such that $(b_i+\beta_i s)$, $i=1,2,\ldots,M$ and $(1-a_i-\alpha_i s)$, $i=1,2,\ldots,N$ were not integers. Then, as is well known, [18, p. 27]

$$\Gamma(b_i+\beta_i s)\Gamma\left[1-(b_i+\beta_i s)\right]=\frac{\pi}{\sin\left[(b_i+\beta_i s)\pi\right]}$$

and

$$\Gamma(a_i+\alpha_i s)\Gamma\left[1-(a_i+\alpha_i s)\right]=\frac{\pi}{\sin\left[(a_i+\alpha_i s)\pi\right]},$$

or equivalently,

$$\Gamma\left[1-(a_i+\alpha_i s)\right]=\frac{\pi}{\Gamma(a_i+\alpha_i s)\sin\left[(a_i+\alpha_i s)\pi\right]},$$

and since $\Gamma[-(b_i+\beta_i s)+1]=-(b_i+\beta_i s)\Gamma[-(b_i+\beta_i s)]$,

$$\Gamma(b_i+\beta_i s)=-\frac{\pi}{\left[\sin(b_i+\beta_i s)\right](b_i+\beta_i s)\Gamma\left[-(b_i+\beta_i s)\right]}.$$

Recall also that none of the contours C_{L_k} passes through any of the poles. Hence for any s on C_{L_k}, $|\sin[(b_i+\beta_i s)\pi]|$, $i=1,2,\ldots,M$, and $|\sin[(a_i+\alpha_i s)\pi]|$, $i=1,2,\ldots,N$ are bounded from below; that is, one can find values $c_i>0$ and $l_i>0$ such that $|\sin[(b_i+\beta_i s)\pi]|>c_i$ and $|\sin[(a_i+\alpha_i s)\pi]|>l_i$.

Also, for any s on C_{L_k}, $|b_i+\beta_i s|$, $|\Gamma[-(b_i+\beta_i s)]|$, $i=1,2,\ldots,M$ are bounded from above, whereas $|\Gamma(a_i+\alpha_i s)|$, $i=N+1,N+2,\ldots,p$ and $|\Gamma(1-b_i-\beta_i s)|$, $i=M+1,M+2,\ldots,q$ are bounded from below. In other words, there exist positive numbers d_i, δ_i, Δ_i, and γ_i such that

$$|b_i+\beta_i s|=d_i R_k,$$

$$|\Gamma[-(b_i+\beta_i s)]|>\delta_i, \tag{F.1.1}$$

$$|\Gamma(a_i+\alpha_i s)|>\Delta_i, \tag{F.1.2}$$

$$|\Gamma[1-(b_i+\beta_i s)]|>\gamma_i. \tag{F.1.3}$$

Consequently,

$$\begin{aligned}
|f(s)|&=\left|\frac{\Pi_{i=1}^{M}\Gamma(b_i+\beta_i s)\Pi_{i=1}^{N}\Gamma(1-a_i-\alpha_i s)}{\Pi_{i=M+1}^{q}\Gamma(1-b_i-\beta_i s)\Pi_{i=N+1}^{p}\Gamma(a_i+\alpha_i s)}\right|\\
&=\frac{\Pi_{i=1}^{M}|\Gamma(b_i+\beta_i s)|\Pi_{i=1}^{N}|\Gamma(1-a_i-\alpha_i s)|}{\Pi_{i=M+1}^{q}|\Gamma(1-b_i-\beta_i s)|\Pi_{i=N+1}^{p}|\Gamma(a_i+\alpha_i s)|}\\
&\leqslant\frac{\pi^{M+N}}{\Pi_{i=1}^{M}\{\sin[(b_i+\beta_i s)\pi]\}(b_i+\beta_i s)\Gamma[-(b_i+\beta_i s)]}\\
&\quad\cdot\frac{1}{\Pi_{i=1}^{N}\Gamma(a_i+\alpha_i s)\Pi_{i=1}^{N}\sin[(a_i+\alpha_i s)\pi]\Pi_{i=M+1}^{q}\Gamma(1-b_i-\beta_i s)}\\
&\quad\cdot\frac{1}{\Pi_{i=N+1}^{p}\Gamma(a_i+\alpha_i s)}\\
&\leqslant\frac{\pi^{M+N}}{(\Pi_{i=1}^{M}c_i d_i\delta_i)R_k^{M}\Pi_{i=1}^{p}\Delta_i\Pi_{i=1}^{N}l_i\Pi_{i=M+1}^{q}\gamma_i}\\
&\leqslant\frac{B}{R_k^{M}},
\end{aligned}\tag{F.1.4}$$

where

$$B=\left[\prod_{i=1}^{M}(c_i d_i\delta_i)\prod_{i=1}^{p}\Delta_i\prod_{i=1}^{N}l_i\prod_{i=M+1}^{q}\gamma_i\right]^{-1}\cdot\Pi^{m+n}$$

It follows that

$$\lim_{R_k \to \infty} |f(s)| \to 0$$

uniformly with respect to $\arg s$, and the third condition of Jordan's lemma is satisfied.

Finally, when $y=1$ (which case is not covered by Jordan's lemma), the kernel reduces to the value 1; that is,

$$\begin{aligned} y^{-s} &= e^{-(\ln y)s} \\ &= 1. \end{aligned}$$

Hence the integrand $y^{-s}f(s)=f(s)$, which, as shown previously, approaches zero uniformly with respect to $\arg s$ as the radius R_k approaches infinity. Therefore, when $y=1$, the H-function inversion integral evaluated over the relevant circular arc approaches zero as R_k approaches infinity independently of $\arg s$. Thus when the conditions of Jordan's lemma are satisfied relative to either $\lim_{k\to\infty} C_{L_k}$ or $\lim_{k\to\infty} C_{R_k}$, the value of the H-function inversion integral evaluated over the Bromwich path $(c-i\infty, c+i\infty)$ is identical to that obtained by integrating over the relevant Bromwich contour. The derivation of the component $h_2(y)$, $1<y<\infty$, is achieved by a procedure analogous to that given earlier, utilizing the Bromwich contour C_{R_k}, $k=0,1,2,\ldots$ (Fig. F.1). The Bromwich path is identical to that utilized in the contour C_{L_k}, and is determined by $c=\lambda$, where λ is given by

$$\lambda = \min\left\{ \frac{|B'_m|}{2}, \frac{|A'_1|}{2} \right\},$$

as before. One now chooses the kth closed contour C_{R_k} as one whose arc $P_kT_kQ_k$ is that of a circle with center at the origin and radius

$$|s| = R_k = |A'_1| + k + \varepsilon, \qquad k=0,1,2,\ldots,$$

where again ε is chosen such that $0<\varepsilon<1$ and such that $(b_i+\beta_i s)$, $i=1,2,\ldots,m$, and $(1-a_i-\alpha_i s)$, $i=1,2,\ldots,n$, are not integers. Thus the second condition of Jordan's lemma is satisfied. Also, as before, some constant $d'_i>0$ exists such that

$$|b_i+\beta_i s| = d'_i R_k, \qquad k=i=1,2,\ldots,m.$$

Moreover, inequalities (F.1.1) through (F.1.4) hold (with d_i replaced by d'_i) and it follows that the third condition of Jordan's lemma is satisfied.

The use of Jordan's lemma requires one to express the relevant p.d.f. $h(y)$ as two components; that is,

$$h(y)=\begin{cases} h_1(y), & 0<y\leqslant 1 \\ h_2(y), & 1\leqslant y<\infty. \end{cases}$$

The component $h_1(y)$ requires utilization of the closed contour C_{L_k}, enclosing at least one pole in the LHP. Similarly, to obtain $h_2(y)$, one must utilize the closed contour C_{R_k}, which must enclose at least one pole in the RHP. Sections F.2 and F.3 provide the procedure for application of Jordan's lemma when poles occur in only one of these two strips or half-planes.

F.2 APPLICATION OF JORDAN'S LEMMA WHEN POLES OCCUR ONLY IN THE LEFT HALF-PLANE

Suppose, as is often the case, the H-function inversion integral of interest is of the form

$$h(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Pi_{i=1}^{M}\Gamma(b_i+\beta_i s)}{\Pi_{i=M+1}^{q}\Gamma(1-b_i-\beta_i s)\Pi_{i=N+1}^{p}\Gamma(a_i+\alpha_i s)}y^{-s}\,ds, \tag{F.2.1}$$

in which the factor $\Pi_{i=1}^{N}\Gamma(1-a_i-\alpha_i s)$ in the numerator of the general form is missing. Consequently, there are no poles in the RHP. (As mentioned previously, the presence or absence of the gamma products in the denominator poses no problem in the analysis.) Poles occur for

$$s=\frac{-(b_i+j)}{\beta_i}, \qquad j=0,1,\ldots.$$

These poles are located in the LHP and inside the Bromwich contour defined by $\lim_{k\to\infty}C_{L_k}$. The sum of the residues evaluated at these poles yields the value of the inversion integral over the Bromwich contour $\lim_{k\to\infty}C_{L_k}$. Since (as has been proved) the conditions of Jordan's lemma are here satisfied for $0<y\leqslant 1$, it follows that the integral evaluated over the Bromwich contour is identical to the integral evaluated over the Bromwich path $(c-i\infty,c+i\infty)$ for $0<y\leqslant 1$. That is, one obtains

$$h(y)=h_1(y), \qquad 0<y\leqslant 1.$$

To determine $h_2(y)$, $1 \leqslant y < \infty$, the inversion integral (F.2.1) must be evaluated over the same Bromwich path $(c-i\infty, c+i\infty)$ as before, but by way of the Bromwich contour $\lim_{k\to\infty} C_{R_k}$, in order for Jordan's lemma to be applicable. To simplify notation, let

$$f(s)=\frac{\Pi_{i=1}^{M}\Gamma(b_i+\beta_i s)}{\Pi_{i=M+1}^{q}\Gamma(1-b_i-\beta_i s)\Pi_{i=N+1}^{p}\Gamma(a_i+\alpha_i s)}. \tag{F.2.2}$$

Then the inversion integral (F.2.1) is expressible as

$$h_2(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s} f(s)\,ds, \qquad 1 \leqslant y < \infty. \tag{F.2.3}$$

Upon making the transformation $r=-s$, we note that the Bromwich contour C_{L_k} is reflected through the origin into the image $C'_{R_k} = P'_k Q'_k K'_k L'_k M'_k P'_k$ (Figure F.2). We also observe that the conditions of Jordan's lemma are satisfied for $1 \leqslant y < \infty$, since $\ln y \geqslant 0$ and since it follows from the preceding section that $f(r)$ approaches zero uniformly with respect to arg r as r approaches infinity. Hence, from the Residue

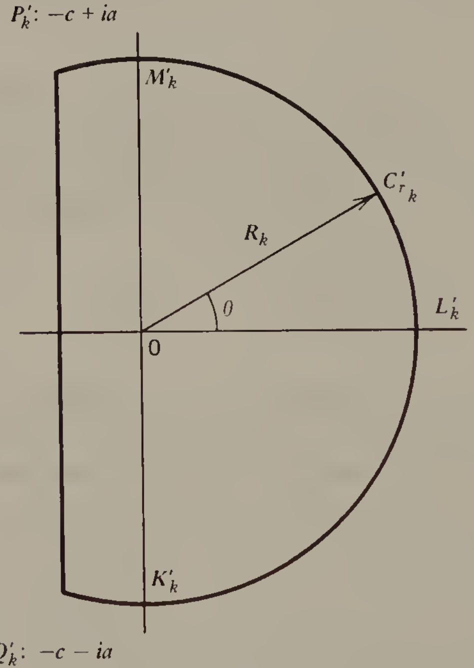


Fig. F.2 Deflected image of C'_{L_k}.

Theorem and Jordan's lemma, it follows that

$$h_2(y)=\lim_{r\to\infty}\frac{-1}{2\pi i}\int_{Q'_K P'_K} y^r f(-r)\,dr,\qquad 1\leqslant y<\infty$$

$$=\lim_{k\to\infty}\frac{-1}{2\pi i}\oint_{C'_{R_k}} y^r f(-r)\,dr,\qquad 1\leqslant y<\infty$$

which, in view of the left-hand rule, becomes

$$h_2(y)=\frac{-1}{2\pi i}\left(-2\pi i\sum_k R'_k\right)$$

$$=\sum_k R'_k$$

$$=\sum_j R_j,$$

where R_j and R'_k denote, respectively, the residues at the poles in the LHP and in the RHP. This result is an affirmation of the well-known fact that one obtains the same p.d.f. $h(y)$ by using either one of the two transform pairs

$$M_s(h(y))=\int_0^\infty y^{s-1}h(y)\,ds,$$

$$h(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}M_s(h(y))\,ds$$

or

$$M_{-s}(h(y))=\int_0^\infty y^{-(s-1)}h(y)\,ds,$$

$$h(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{s}M_{-s}(h(y))\,ds.$$

In summary, when poles are present in the LHP but absent in the RHP, the p.d.f. $h(y)$ consists of one component that is valid for the entire range $0<y<\infty$ and is obtained by summing the residues of the integrand of the relevant H-function inversion integral at the poles in the LHP.

F.3 APPLICATION OF JORDAN'S LEMMA WHEN POLES OCCUR ONLY IN THE RIGHT HALF-PLANE

In the event the H-function inversion integral is of the form

$$h(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Pi_{i=1}^{N}\Gamma(1-a_i-\alpha_i s)}{\Pi_{i=M+1}^{q}\Gamma(1-b_i-\beta_i s)\Pi_{i=N+1}^{p}\Gamma(a_i+\alpha_i s)}y^{-s}\,ds,$$

in which the factor $\Pi_{i=1}^{M}\Gamma(b_i+\beta_i s)$ in the numerator of the general form is missing, there are no poles in the LHP. Poles occur for

$$s=\frac{1-a_i+j}{\alpha_i},\qquad j=0,1,\ldots$$

and are located in the RHP inside the Bromwich contour $\lim_{k\to\infty}C_{R_k}$. Since the conditions of Jordan's lemma are satisfied for $1\leqslant y<\infty$, the component

$$h(y)=h_2(y),\qquad 1\leqslant y<\infty$$

is obtained by summing the residues evaluated at these poles. To determine the component $h_1(y)$, one applies the transformation $r=-s$, thereby shifting the poles to the LHP so that the left Bromwich contour $\lim_{k\to\infty}C_{L_k}$, containing the same Bromwich path as the right Bromwich contour $\lim_{k\to\infty}C_{R_k}$, can be utilized. Thus the inversion integral of interest becomes

$$h_1(y)=\frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Pi_{i=1}^{N}\Gamma(1-a_i+\alpha_i r)}{\Pi_{i=M+1}^{q}\Gamma(1-b_i+\beta_i r)\Pi_{i=N+1}^{p}\Gamma(b_i-\beta_i r)}y^{r}\,dr,$$

which can now be evaluated by the method of residues. Using the same approach as in the preceding section, one can readily show that $h_1(y)$, $0<y\leqslant 1$, so obtained is identical to $h_2(y)$, $1\leqslant y<\infty$.

Therefore when poles are present in the RHP but absent in the LHP, the p.d.f. $h(y)$ consists of one component that is valid for the entire range $0<y<\infty$ and is obtained by summing the residues of the integrand of the relevant H-function inversion integral at the poles in the RHP.

Note that since (7.1.1) and (6.2.1) are equivalent, the results obtained in Chapter 7 and Appendix F relative to (7.1.1) are valid for (6.2.1).

References

1. Abraham, C. T. and R. D. Prasad (1969), Stochastic Model for Manufacturing and Cost Estimating, *IBM Journal of Research and Development*, Vol. 13, pp. 343–350.
2. Abramowitz, M. and I. Stegun (1964–1971), *Handbook of Mathematical Functions*, (NBS AMS 55), Government Printing Office, Washington, D. C.
3. Akhiezer, N. I. (1966), *The Classical Moment Problem*, Hafner, New York.
4. Albert, G. E. and R. L. Tittle (1967), Non-normality of Linear Combinations of Normally Distributed Random Variables, *American Mathematical Monthly*, Vol. 74, pp. 583–585.
5. Anandani, P. (1967), Some Expansion Formulae for H-Function, *Ganita*, Vol. 18, pp. 89–101.
6. Anandani, P. (1968), Fourier Series for H-Function, *Proceedings of the Indian Academy of Sciences, Section A*, Vol. 58, pp. 291–295.
7. Anandani, P. (1969), On Finite Summation, Recurrence Relations and Identities of H-Functions, *Annales Polonici Mathematici*, Vol. 21, pp. 125–137.
8. Anandani, P. (1969), On Some Recurrence Formulae for the H-Function, *Annales Polonici Mathematici*, Vol. 21, pp. 113–17.
9. Anandani, P. (1970), Expansion of the H-Function Involving Generalized Legendre Associated Functions and H-Functions, *Kyungpook Mathematical Journal*, Vol. 10, pp. 53–57.
10. Anandani, P. (1970), On the Derivative of the H-Function, *Revue Roumaine de Mathematiques Pures et Appliqués*, Vol. 21, pp. 189–191.
11. Anandani, P. (1973), Expansion Theorem for the H-Function Involving Associated Legendre Functions, *Bulletin de la Société des Mathematiciens et des Physiciens de la Republique Populaire de Macedoine*, Vol. 24, pp. 39–43.
12. Anderson, R. L. (1942), Distribution of the Serial Correlation Coefficient, *Annals of Mathematical Statistics*, Vol. 13, pp. 1–13.
13. Anderson, R. L., J. W. Cell, and W. T. Wells (1962), The Distribution of the Product of Two Central or Non-Central Chi-Square Variates, *Annals of Mathematical Statistics*, Vol. 33, pp. 1016–1020.
14. Anderson, T. W. (1958), *An Introduction to Multivariate Statistical Analysis*, Wiley, New York, p.25.
15. Antelman, G. and I. R. Savage (1965), Characteristic Functions of Stochastic Integrals and Reliability Theory, *Naval Research Logistics Quarterly*, Vol. 12, pp. 199–222.
16. Aroian, L. A. (1944), Some Methods of Evaluation of a Sum, *Journal of the American Statistical Association*, Vol. 39, 511–515.
17. Aroian, L. A. (1947), The Probability Function of the Product of Two Normally Distributed Random Variables, *Annals of Mathematical Statistics*, Vol. 18, pp. 265–271.

18. Artin, Emil (1964), *The Gamma Function*, Holt, Rinehart & Winston, New York, pp. 25–27.

19. Bajpai, S. D. (1969), An Expansion for Fox's H-Function Involving Bessel Functions, *Labdev, Journal of Sciences and Technology, Part A*, Vol. 7, pp. 8–20.

20. Bajpai, S. D. (1969–1970), An Integral Involving Fox's H-Function and Its Application, *Universidade de Lisboã, Ciencias Matematicas*, Vol. 13, pp. 109–114.

21. Bajpai, S. D. (1970), Some Results Involving Fox's H-Function and Bessel Functions, *Proceedings of the Indian Academy of Sciences, Section A*, Vol. 72, pp. 42–46.

22. Baker, G. A. (1930), Distribution of the Means of Samples of n Drawn at Random from a Population Represented by a Gram-Charlier Series, *Annals of Mathematical Statistics*, Vol. 1, pp. 199–204.

23. Basu, A. P. and R. H. Lochner (1971), On the Distribution of the Ratio of Two Random Variables Having Generalized Life Distributions, *Technometrics*, Vol. 13, pp. 281–287.

24. Behboodian, J. (1972), A Simple Example on Some Properties of Normal Random Variables, *American Mathematical Monthly*, Vol. 79, pp. 632–634.

25. Behrens, W. U. (1929), Ein Beitrag zur Fehler-Berechnung bei wenigen Beobachtungen, *Landw. Jb.*, Vol. 68, pp. 807–837.

26. Beightler, C. S., L. G. Mitten, and G. L. Nemhauser (1961), A Short Table of Z-Transforms and Generating Functions, *Journal of Operations Research Society*, Vol. 9, pp. 574–578.

27. Berman, Simeon M. (1975), A New Characterization of Characteristic Functions of Absolutely Continuous Distributions, *Pacific Journal of Mathematics*, Vol. 58–2, pp. 323–331.

28. Birnbaum, Z. W. (1970), *On a Statistic Similar to Student's t, Nonparametric Techniques in Statistical Inference*, Cambridge University Press, Cambridge, pp. 427–433.

29. Birnbaum, Z. W. and I. Vincze (1973), Limiting Distributions of Statistics Similar to Student's t, *Annals of Statistics*, Vol. 1, pp. 958–963.

30. Blum, M. (1970), On the Sums of Independently Distributed Pareto Variates, *SIAM Journal of Applied Mathematics*, Vol. 19, pp. 191–198.

31. Bocher, M. (1906), Introduction to the Theory of Fourier Series, *Annals of Mathematics*, Vol. 7, pp. 81–134.

32. Bocher, M. (1938), *Introduction to Higher Algebra*, Macmillan, New York.

33. Bose, R. C. (1936), On the Exact Distribution and Moment-Coefficients of the D^2-Statistics, *Sankhya* (Calcutta), Vol. 2, pp. 143–154.

34. Bose, R. C. and S. N. Roy (1938), The Distribution of the Studentized D-Squared Statistics, *Sankhya* (Calcutta), Vol. 4, pp. 19–38.

35. Bose, S. (1935), On the Distribution of the Ratio of Variances of Two Samples Drawn from a Given Normal Bivariate Correlated Population, *Sankhya* (Calcutta), Vol. 2, Part 1, pp. 65–72.

36. Box, G. E. P. (1954), Some Theorems on Quadratic Forms Applied in the Study of Analysis of Variance Problems. 1. Effect of Inequality of Variance in the One-Way Classification, *Annals of Mathematical Statistics*, Vol. 25, pp. 290–302.

37. Braaksma, B. L. J. (1963), Asymptotic Expansions and Analytic Continuations for a Class of Barnes-Integrals, *Composito Mathematica*, Vol. 15, pp. 239–241.

38. Brender, D. M. (1968), The Prediction and Measurement of System Availability, *IEEE Transactions on Reliability*, Vol. R-17, pp. 127–138.

39. Brennan, L. E., I. S. Reed, and W. Solefrey (1968), A Comparison of Average Likelihood and Maximum Likelihood Ratio Tests for Detecting Radar Targets of Unknown Doppler Frequency, *IEEE Transactions on Information Theory*, Vol. IT-4, pp. 104–110.

40. Brock, A. D. and R. G. Krutchkoff (1970), Distributions of Z and Z^* for Complex Z With Results Applied to the Complex Normal Distribution, *Annals of Mathematical Statistics*, Vol. 41, pp. 1563–1567.

41. Brookner, R. J. and A. Wald (1941), On the Distribution of Wilks' Statistic for Testing the Independence of Several Groups of Variates, *Annals of Mathematical Statistics*, Vol. 12, pp. 137–152.

42. Buck, James R. and Thomas W. Hill, Jr. (1971), Laplace Transforms for the Economic Analysis of Deterministic Problems in Engineering, *Engineering Economist*, Vol. 16, No. 4, pp. 247–263.

43. Buchholz, H. (1969), *The Confluent Hypergeometric Function*, Springer-Verlag, New York.

44. Byers, J. K. (1970), "Application of GERT to Reliability Analysis," Doctoral dissertation, College of Engineering, University of Arkansas.

45. Camp, B. H. (1938), Notes on the Distribution of the Sample Mean, *Annals of Mathematical Statistics*, Vol. 9, pp. 221–226.

46. Carleman, T. (1925), *Les functions quasi-analytiques*, Gauthier-Villars, Paris.

47. Carter, B. D. (1970), "On the Probability Distribution of Rational Functions of Independent H-Function Variates," Ph.D. thesis, University of Arkansas.

48. Carter, B. D. and M. D. Springer (1977), The Distribution of Products, Quotients, and Powers of Independent H-Function Variates, *SIAM Journal of Applied Mathematics*, Vol. 33, pp. 542–558.

49. Chapman, D. G. (1950), *Some Two-Sample Tests*, Wiley-Interscience, Vol. 21, pp. 601–606.

50. Chew, V. (1961), Distribution of the Sum of Independent Uniform Random Variables with Unequal Ranges, *Virginia Journal of Science*, Vol. 12, pp. 45–51.

51. Chiang, C. L. (1966), On the Expectation of the Reciprocal of a Random Variable, *The American Statistician*, Vol. 20, No. 4, p. 28.

52. Churchill, R. V. (1958), *Operational Mathematics*, 2nd ed., McGraw-Hill, New York.

53. Clavier, P. A. (1929), The Frequency Function of X/Y. *Annals of Mathematical Statistics*, Vol. 30, pp. 471–486.

54. Clavier, P. A. (1936), On the Frequency Function of XY, *Annals of Mathematical Statistics*, Vol. 12, pp. 1–15.

55. Clavier, P. A. (1942), On the Frequency Distribution of the Quotient and of the Product of Two Statistical Variables, *The American Mathematical Monthly*, Vol. 44, pp. 24–32.

56. Clenshaw, C. W. (1955), A Note on the Summation of Chebyshev Series, *Math Tables and Aids to Computation*, Vol. 9, pp. 118–120.

57. Cochran, W. G. (1963), *Sampling Techniques*, Wiley, New York.

58. Constantine, A. G. (1963), Some Noncentral Distribution Problems in Multivariate Analysis, *Annals of Mathematical Statistics*, Vol. 34, pp. 1270–1285.

59. Consul, P. C. (1966), On Some Inverse Mellin Transforms, *Bulletin of Class. Science*, Vol. 52, pp. 547–561.

60. Consul, P. C. (1966), On Some Reduction Formulae for Hypergeometric Functions, *Bulletin of Class. Science*, Vol. 52, pp. 562–576.

61. Consul, P. C. (1969), On the Exact Distributions of Votaw's Criteria for Testing Compound Symmetry of a Covariance Matrix, *Annals of Mathematical Statistics*, Vol. 40, pp. 836–843.

62. Consul, P. C. (1969), On Some Integrals in Operational Calculus, *Bulletin of Class. Science*, Vol. 55.

63. Cornish, E. A. and R. A. Fisher (1937), Moments and Cumulants in the Specification of Distributions, *Revue de l'Institut International de Statistique*, Vol. 4, pp. 1–14.

64. Cox, D. R. (1962), *Renewal Theory* (Methuen Monograph), Wiley, New York.

65. Craig, A. T. (1938), On the Independence of Certain Estimates of Variance, *Annals of Mathematical Statistics*, Vol. 9, pp. 48–55.

66. Craig, C. C. (1929), The Frequency Function of X/Y, *Annals of Mathematical Statistics*, Vol. 30, pp. 471–486.

67. Craig, C. C. (1931), Note on the Distribution of Means of Samples of N Drawn from a Type A Population, *Annals of Mathematical Statistics*, Vol. 2, pp. 99–101.

68. Craig, C. C. (1936), A New Exposition and Chart for the Pearson System of Frequency Curves, *Annals of Mathematical Statistics*, Vol. 12, pp. 16–25.

69. Craig, C. C. (1936), On the Frequency Function of XY, *Annals of Mathematical Statistics*, Vol. 7, pp. 1–15.

70. Craig, C. C. (1941), Note on the Distribution of Noncentral t with an Application, *Annals of Mathematical Statistics*, Vol. 12, pp. 224–228.

71. Craig, C. C. (1942), On the Frequency Distributions of the Quotient and of the Product of Two Statistical Variables, *American Mathematical Monthly*, Vol. 49, pp. 26–32.

72. Craiu, M. and V. Craiu (1969), Repartitia Dirichlet Generalizata, *Analete Universitatti Bucuresti, Mathematica-Mecanica*, Vol. 18, No. 2, pp. 9–11.

73. Cramer, H. (1951), *Mathematical Methods of Statistics*, Princeton, N. J., Princeton University Press.

74. Cramer, H. (1962), *Random Variables and Probability Distributions*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 36, Cambridge, at the University Press.

75. Curtis, J. H. (1941), On the Distribution of the Quotient of Two Chance Variables, *Annals of Mathematical Statistics*, Vol. 29, pp. 563–578.

76. David, H. A. (1970), *Order Statistics*, Wiley, New York.

77. Davis, A. W. (1970), Exact Distributions of Hotelling's Generalized T_0^2 Statistic, *Annals of Mathematical Statistics*, Vol. 57, pp. 187–191.

78. Davis, A. W. (1970), On the Null Distribution of the Sum of the Roots of a Multivariate Beta Distribution, *Annals of Mathematical Statistics*, Vol. 41, pp. 1557–1562.

79. Ditkin, V. A. and A. P. Prudnikov (1965), *Integral Transforms and Operational Calculus*, translated by D. E. Brown, International Series of Monographs in Pure and Applied Mathematics, Vol. 78, Pergamon Press, Oxford.

80. Doetsch, G. (1961), *Guide to the Applications of Laplace Transforms*, Van Nostrand, London.

81. Dolan, B. A. (1964), The Mellin Transform for Moment Generation and for the Probability Density of Products and Quotients of Random Variables, *Proceedings of the Institute of Electrical and Electronics Engineers* (correspondence), Vol. 52, pp. 1745–1746.

82. Dolan, B. A. (1965), *A Generalized Mellin Transform for the Analysis of Learning Machines*, Technical Report No. AFAL-TR-65-281, Systems Techniques Laboratory, Stanford Electronics Laboratories, Stanford Univeristy, Stanford, Calif.

83. Donahue, J. D. (1964), *Products and Quotients of Random Variables and Their Applications*, Aerospace Research Laboratories Report No. ARL-64-115. Wright-Patterson Air Force Base, U. S. Air Force, Office of Aerospace Research.

84. Donahue, J. D. and R. S. DeZur (April 1965), *On Products and Quotients of Random Variables*, Aerospace Research Laboratories Report No. ARL 65-71, Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio.

85. Dubner, H. and J. Abati (January 1968), Numerical Inversion of Laplace Transforms by Relating them to the Fourier Cosine Transform, *Journal of the Association of Computational Machinery*, Vol. 48, pp. 41–55.

86. Durbin, J. (1961), Some Methods of Constructing Exact Tests, *Biometrika*, Vol. 48, pp. 41–55.

87. Dwivedi, T. D. (1966), The Distribution of Products of Independent Random Variables, *Progress in Mathematics* (Allahabad), Vol. 4, pp. 5–10.

88. Dwivedi, T. D. (1970), The Exact Distribution of the Quotient of Certain Independent Generalized Hypergeometric Variables, *South African Statistical Journal*, Vol. 4, pp. 77–80.

89. Ehrenfeld, S. (1963), Some Numerical Aspects of the Use of Transforms in Statistics, *Journal of the American Statistical Association*, Vol. 63, pp. 879–893.

90. Elderton, W. P. and N. L. Johnson (1969), *Systems of Frequency Curves, Cambridge University Press*, London-New York.

91. Eldred, B. S. (1978), "The Application of Integral Transform Theory to the Analysis of Probabilistic Cash Flows," Doctoral dissertation, College of Engineering, University of Texas.

92. Epstein, B. (1948), Some Applications of the Mellin Transform in Statistics, *Annals of Mathematical Statistics*, Vol. 19, pp. 370–379.

93. Epstein, B. and M. Sobel (1953), Life Testing, *Journal of the American Statistical Association*, Vol. 48, pp. 586–602.

94. Erdelyi, A. (1943), Inversion Formulae for the Laplace Transformation, *Philosophy Magazine*, Vol. 34, pp. 533–537.

95. Erdelyi, A. (1954), *Tables of Integral Transforms*, Vol. 1, McGraw-Hill, New York.

96. Erdelyi, A. (1954), *Tables of Integral Transforms*, Vol. 2, McGraw-Hill, New York.

97. Erdelyi, A. (1954), *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, New York.

98. Erdelyi, A. (1959), *Higher Transcendental Functions*, Vol. 2, McGraw-Hill, New York.

99. Feller, William (1957), *An Introduction to Probability Theory and Its Applications*, Vol. 1, 2nd ed., Wiley, New York.

100. Feller, William (1969), *An Introduction to Probability Theory and Its Applications*, Vol. 2, Wiley, New York.

101. Ferguson, Thomas S. (1967), *Mathematical Statistics, A decision Theoretic Approach*, Academic Press, New York.

102. Fieller, E. C. (1932), The Distribution of the Index in a Normal Bivariate Population, *Biometrika*, Vol. 24, pp. 428–440.

103. Finney, D. J. (1938), The Distribution of the Ratio of Estimates of Two Variances in a Sample from a Bivariate Population, *Biometrika*, Vol. 30, pp. 190–192.

104. Fisher, R. A. (1914–1915), Frequency Distribution of the Values of the Correlation Coefficient in Samples from an Indefinitely Large Population, *Biometrika*, Vol. 10, pp. 507–521.

105. Fisher, R. A. (1928), The General Sampling Distribution of the Multiple Correlation Coefficient, *Proceedings of the Royal Statistical Society, Series A*, Vol. 121, pp. 654–673.

106. Fisher, R. A. (1935), The Fiducial Argument in Statistical Inference, *Annals of Eugenics*, Vol. 6, pp. 391–398.

107. Fisher, R. A. (1941), The Asymptotic Approach to Behren's Integral with Further Tables for the Test of Significance, *Annals of Eugenics*, Vol. 11, pp. 141–172.

108. Fisher, R. A. (1950), *Contributions to Mathematical Statistics*, Wiley, New York.

109. Fisher, R. A. and E. A. Cornish (May 1960), The Percentile Points of Distributions Having Known Cumulants, *Technometrics*, Vol. 2, pp. 209–225.

110. Fleiss, J. L. (1966), A Note on the Expectation of the Reciprocal of a Random Variable, *The American Statistician*, Vol. 20, No. 1, p. 25.

111. Fox, C. (1957), Some Applications of Mellin Transforms to the Theory of Bivariate Statistical Distributions, *Proceedings of the Cambridge Philosophy Society*, Vol. 53, pp. 620–628.

112. Fox, C. (1961), The *G* and *H*-Functions as Symmetrical Fourier Kernels, *Transactions of the American Mathematical Society*, Vol. 98, pp. 395–429.

113. Franklin, Philip (1940), *A Treatise on Advanced Calculus*, Wiley, New York.

114. Friedman, H. J. and Z. W. Birnbaum (1974), Numerical Tabulations for a Statistic Similar to Student's t, *Statistischefte*. 15 Johrgang, Hefte 2/3, pp. 143–156.

115. Gaver, D. P. and M. Mazumdar (1967), Statistical Estimation in a Problem of System Reliability, *Naval Research Logistics Quarterly*, Vol. 14-4, pp. 473–488.

116. Gayen, A. K. (1951), The Distribution of the Product Moment Correlation Coefficient in Random Samples of any Size Drawn from Nonnormal Universes, *Biometrika*, Vol. 38, pp. 219–247.

117. Geary, R. C. (1930), The Frequency Distribution of the Quotient of Two Normal Variables, *Journal of the Royal Statistical Society*, Vol. 93, 442–446.

118. Geary, R. C. (1936), The Distribution of Student's Ratio for Non-normal Samples, *Supplement to the Journal of the Royal Statistical Society*, Vol. 3, pp. 178–184.

119. Gentlemen, W. W. and G. Sande (1966), "Fast Fourier Transform for Fun and Profit," 1966 Fall Joint Computer Conference, American Federation of Information Processing Societies, Vol. 29, 563–578.

120. Ghosh, B. K. (1975), On the Distribution of the Difference of Two *t*-Variables, *Journal of the American Statistical Association*, Vol. 70, pp. 463–467.

121. Giffin, W. C. (1976), *Transform Techniques for Probability Modeling*, Academic Press, New York.

122. Glazer, Ronald E. (1976), The Ratio of the Geometric Mean to the Arithmetic Mean for a Random Sample from a Gamma Distribution, *Journal of the American Statistical Association*, Vol. 71, pp. 480–487.

123. Goertzel, G. (1958), An Algorithm for the Evaluation of Finite Trigonometric Series, *American Mathematical Monthly*, Vol. 65, pp. 34–35.

124. Good, I. J. (1961), The Real Stable Characteristic Functions and Chaotic Acceleration, *Journal of the Royal Statistical Society*, Vol. 23, pp. 180–183.

125. Good, I. J. (1965), *Estimation of Probabilities* (research monograph), MIT Press, Cambridge, Mass.

126. Good, I. J. (1968), The Characteristic Functions of Functions, *Proceedings of the Royal Statistical Society, Series A*, Vol. 307, pp. 317–334.

127. Goyal, A. N. and G. K. Goyal (1967), Expansion Theorems of H-Functions, *Vijnan Parishad Anusandhar Patrika*, Vol. 10, pp. 205–217.

128. Gray, H. L. and P. L. Odell (1966), On Sums and Products of Rectangular Variates, *Biometrika*, Vol. 53, pp. 615–617.

129. Gumbel, E. J. (1960), *Statistics of Extremes*, Columbia University Press, New York–London.

130. Gumbel, E. J. and L. H. Herbach (1951), The Exact Distribution of the Extremal Quotient, *Annals of Mathematical Statistics*, Vol. 22, pp. 418–426.

131. Gumbel, E. J. (1962), Produits et Quotients de Deux Plus Petites Valeurs Indépendantes, *Publications de l'Institut de Statistique de l'Universite de Paris*, Vol. 11, pp. 191–193.

132. Gupta, S. S., K. C. S. Pillai, and G. P. Steck (1964), The Distribution of Linear Functions of Ordered Correlated Normal Random Variables with Emphasis on Range, *Biometrika*, Vol. 51, pp. 143–151.

133. Gupta, K. C. (1965), On the H-Function, *Annales de la Société Scientifique de Bruxelles*, Vol. T.79. II, pp. 97–106.

134. Gupta, K. C. and P. K. Mittai (1970), The H-Function Transform, *Journal of the Australian Mathematical Society*, Vol. 11, pp. 142–148.

135. Gupta, A. K. and Z. Govindarajulu (1975), Distribution of Quotient of Two Independent Hotelling T^2 Variates, *Communications in Statistics*, Vol. 4, pp. 449–453.

136. Gurland, John (1948), Inversion Formulae for the Distribution of Ratios, *Annals of Mathematical Statistics*, Vol. 19, pp. 228–237.

137. Gurland, John (1953), Distribution of Quadratic Forms and Ratios of Quadratic Forms, *Annals of Mathematical Statistics*, Vol. 24, pp. 416–427.

138. Gurland, John (1955), Distribution of the Maximum of the Arithmetic Mean of Correlated Random Variables, *Annals of Mathematical Statistics*, Vol. 26, pp. 294–300.

139. Gurland, John (1967), An Inequality Satisfied by the Expectation of the Reciprocal of a Random Variable, *The American Statistician*, Vol. 21, No. 2, p. 24.

140. Haldane, J. B. S. (1942), Moments of the Distribution of Powers and Products of Normal Variables, *Biometrika*, Vol. 31, pp. 226–242.

141. Hamedani, G. G. and M. N. Tata (1975), On the Determination of the Bivariate Normal Distribution from the Distributions of Linear Combinations of the Variables, *American Mathematical Monthly*, Vol. 82, pp. 913–915.

142. Han, Chien-pai (1975), Some Relationships Between Noncentral Chi-Squared and Normal Distributions, *Biometrika*, Vol. 62, pp. 213–214.

143. *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series 55, pp. 258–259.

144. Harper, W. M. and J. A. MacDonald (1960), The Use of Integral Tranforms to Determine Distribution Functions, *Biometrika*, Vol. 47, pp. 460–463.

145. Harper, H. L. (1951), On the Distribution of Wald's Classification Statistic, *Annals of Mathematical Statistics*, Vol. 22, pp. 58–67.

146. Hawkins, D. M. (1975), From the Noncentral t to the Normal Integral, *The American Statistician*, Vol. 29, pp. 42–43.

147. Hildebrand, F. B. (1948), *Advanced Calculus for Engineers*, Prentice-Hall, Englewood Cliffs, N. J.

148. Hill, T. W. (1969), On Determining a Distribution Function Known only by Its Moments and/or Moment Generating Function, Doctoral dissertation, Arizona State University.

149. Hill, T. W. and J. R. Buck (1974), Zeta Transforms, Present Value, and Economic Analysis, *AIIE Transactions*, Vol. 6, pp. 120–125.

150. Hille, Einar (1959), *Analytic Function Theory*, Vol. I, Blaisdell, New York.

151. Hinkley, D. V. (1969), On the Ratio of Two Correlated Normal Variables, *Biometrika*, Vol. 56, pp. 635–639.

152. Hirschfeld, H. Q. (1937), The Distribution of the Ratio of Covariance Estimates in Two Samples Drawn from Normal Bivariate Populations, *Biometrika*, Vol. 29, pp. 65–79.

153. Hogg, R. V. and A. T. Craig (1965), *Introduction to Mathematical Statistics*, 2nd ed., Macmillan, New York.

154. Hotelling, Harold (1931), The Generalization of Student's Ratio, *Annals of Mathematical Statistics*, Vol. 2, pp. 360–378.

155. Huntington, E. V. (1939), Frequency Distributions of Products and Quotients, *Annals of Mathematical Statistics*, Vol. 10, pp. 195–198.

156. *IEEE Transactions on Audio Electronics*, Vol. AE-15 (June 1967), No. 2. This entire issue is devoted to algorithms for fast Fourier transforms, as well as a historical survey.

157. *IEEE Transactions on Audio and Electroacoustics*, (1967–1968), pp. 15–16.

158. Irwin, J. O. (1927), On the Frequency Distribution of the Means of Samples from a Population Having any Law of Frequency with Finite Moments, with Special Reference to Pearson's Type II, *Biometrika*, Vol. 19, pp. 225–239.

159. Jain, N. C. (1971), Integrals that Contain Certain Hypergeometric Functions and the H-Function, *Republica de Venezuela Buletin de la Academia de Ciencias Fisicas, Matematicas y Naturales* (Caracas), Vol. 31, pp. 95–102.

160. Jambunathan, M. V. (1954), Some Properties of Beta and Gamma Distributions, *Annals of Mathematical Statistics*, Vol. 25, pp. 401–404.

161. James, Glenn and Robert C. James (1949), *Mathematics Dictionary*, Van Nostrand, New York.

162. James, A. T. (1955). The Noncentral Wishart Distribution, *Proceedings of the Royal Society of London, Series A*, Vol. 229, pp. 364–365.

163. James, I. R. (1972), Products of Independent Beta Variables with Application to Connor and Mosiman's Generalized Dirichlet Distribution, *Journal of the American Statistical Association*, Vol. 67, pp. 910–912.

164. Johnson, N. L. (1965), The Distribution of the Coefficient of Variation, *Biometrika*, Vol. 52, pp. 303.

165. Johnson, N. L. and S. Kotz (1969), *Distributions in Statistics, Discrete Distributions*, Houghton Mifflin, Boston.

166. Johnson, N. L. and S. Kotz (1970), *Distributions in Statistics, Continuous Univariate Distributions*, Vol. 1, Houghton Mifflin, Boston.

167. Johnson, N. L. and S. Kotz (1970), *Distributions in Statistics, Continuous Univariate Distributions*, Vol. 2, Houghton Mifflin, Boston.

168. Johnson, N. L. and S. Kotz (1972), *Distributions in Statistics, Continuous Multivariate Distributions*, Wiley, New York.

169. Jolley, L. B. W. (1961), *Summation of Series*, 2nd rev. ed., Dover, New York.

170. Jury, E. L. (1964), *Theory and Application of the Z-Transform Method*, Vol. 2, Wiley, New York.

171. Kabe, D. G. (1962), On the Exact Distribution of a Class of Multivariate Test Criteria, *Annals of Mathematical Statistics*, Vol. 33, pp. 1197–1200.

172. Kabe, D. G. (1965), On the Noncentral Distribution of Rao's U-Statistics, *Annals of the Institute of Statistical Mathematics*, Vol. 17, pp. 75–80.

173. Kabe, D. G. (1966), Dirichlet's Transformation and Distribution of Linear Functions of Ordered Gamma Variates, *Annals of the Institute of Statistical Mathematics*, Vol. 18, pp. 367–374.

174. Kailath, T. and J. Omura (1965), *Some Useful Probability Distributions*, Technical Report No. 7050-6, AD 474128, Stanford Electronics Laboratory, Stanford, Calif.

175. Kale, B. K. (1970), Normality of Linear Combinations of Non-normal Random Variables, *American Mathematical Monthly*, Vol. 77, pp. 992–995.

176. Kaneko, Toyohisa and Bede Liu (1969), *Accumulation of Round-off Error in Fast Fourier Transforms* (AD 698 769), Princeton University, Department of Electrical Engineering, Princeton, N.J.

177. Kapoor, V. K. and S. K. Gupta (1970), Fourier Series for H-Function, *Indian Journal of Pure and Applied Mathematics*, Vol. 2, pp. 433–437.

178. Kendall, M. G. and A. S. Stuart (1958), *The Advanced Theory of Statistics*, Vol. 1, Hafner, New York.

179. Kendall, M. G. and A. S. Stuart (1966), *The Advanced Theory of Statistics*, Vol. 2, Griffin, London.

180. Kesarwani, Roop Narain (1971), On an Integral Transform Involving G-Functions, *SIAM Journal of Applied Mathematics*, Vol. 20, pp. 93–98.

181. Khatri, C. G. (1962), Distributions of Order Statistics for Discrete Case, *Annals of the Institute of Statistical Mathematics*, Vol. 14, pp. 167–171.

182. Kotlarski, I. (1960), On Random Variables Whose Quotient Follows the Cauchy Law, *Colloquium Mathematicum*, Vol. 7, Fasc. 2, pp. 277–284.

183. Kotlarski, I. (1962), On Groups of n Independent Random Variables Whose Product Follows the Beta Distribution, *Colloquium Mathematicum*, Vol. 9, pp. 325–332.

184. Kotlarski, I. (1962), On Pairs of Independent Random Variables Whose Quotient Follows Some Known Distribution, *Colloquium Mathematicum*, Vol. 9, pp. 151–162.

185. Kotlarski, I. (1964), On Bivariate Random Variables Where the Quotient of Their Coordinates Follows Some Known Distribution, *Annals of Mathematical Statistics*, Vol. 35, pp. 1673–1684.

186. Kotlarski, I. (1965), On the Generalized Mellin Transform of a Complex Random Variable and Its Application, *Annals of Mathematical Statistics*, Vol. 36, 1459–1467.

187. Kotlarski, I. (1975), Some Deficiencies of Using Moment-Generating Functions, *The American Statistician*, Vol. 29, pp. 127–128.

188. Kotz, S. and J. Neumann (1963), On the Distribution of Precipitation Amounts for the Periods of Increasing Length, *Journal of Geophysical Research*, Vol. 68, pp. 3635–3641.

189. Kotz, S. and J. W. Adams (1964), Distribution of a Sum of Identically Distributed Exponentially Correlated Gamma-Variables, *Annals of Mathematical Statistics*, Vol. 35, pp. 277–283.

190. Kotz, S. and R. Srinivasan (1969), Distribution of Product and Quotient of Bessel Function Variates, *Annals of the Institute of Statistical Mathematics*, Vol. 21, pp. 201–210.

191. Kreyzig, E. (1962), *Advanced Engineering Mathematics*, Wiley, New York.

192. Krishnaiah, P. R. and F. J. Schuurmann (1974), On the Evaluation of Some Distributions That Arise in Simultaneous Tests for the Equality of the Latent Roots of the Covariance Matrix, *Journal of Multivariate Analysis*, Vol. 4, pp. 265–282.

193. Kullback, S. (1934), An Application of Characteristic Functions to the Distribution Problem of Statistics, *Annals of Mathematical Statistics*, Vol. 5, pp. 263–307.

194. Lachenbruch, Peter A. and Donna R. Brogan (1971), Some Distributions on the Positive Real Line Which Have No Moments, *The American Statistician*, Vol. 25, pp. 46–47.

195. Laha, R. G. (1954), On Some Properties of the Bessel Function Distributions, *Bulletin of the Calcutta Mathematical Society*, Vol. 46, pp. 59–71.

196. Laha, R. G. (1958), An Example of a Non-normal Distribution Where the Quotient Follows the Cauchy Law, *Proceedings of the National Academy of Science* (U.S.A.), Vol. 44, pp. 222–223.

197. Laha, R. G. (1959), On a Class of Distribution Functions Where the Quotient Follows the Cauchy Law, *Transactions of the American Mathematical Society*, Vol. 93, pp. 205–215.

198. Laha, R. G. (1959), On the Laws of Cauchy and Gauss, *Annals of Mathematical Statistics*, Vol. 30, pp. 205–215.

199. Lanczos, C. (1956), *Applied Analysis*, Prentice-Hall, Englewood Cliffs, N.J., Chapters 4 and 7.

200. Lathi, B. P. (1968), *An Introduction to Random Signals and Communication Theory*, Dun-Donnelley Publishing Company, New York.

201. Lauer, G. N. and C. P. Han, (1972), Joint Distribution Function of Certain Ratios of Chi-Square Variates, *Journal of Statistical Computation*, Vol. 1, pp. 255–260.

202. Lee, A. S. and John Gurland (1975), Size and Power of Tests for Equality of Means of Two Normal Populations with Unequal Variances, *Journal of American Statistical Association*, Vol. 70, pp. 933–941.

203. Levy, Paul (1935), Propriétés Asymptotiques des Sommes de Variables Aléatoires Indépendantes ou Enchainées, *Journal de Mathematiques Pures et Appliquées*, Vol. 14, pp. 347–402.

204. Levy, Paul (1937), L'Arithmetique des Lois de Probabilité de les Products Finis de Loi de Poisson, *Comptes Rendus de l'Academie des Sciences*, Vol. 204, pp. 944–946.

205. Levy, Paul (1954), *Théorie de l'Addition des Variables Aléatoires*, 2nd ed., Gauthier-Villars, Paris.

206. Levy, Paul (1959), Symetrie et Dissymetrie des Produits de Variables Aleatoires, *Comptes Rendus de l'Academie des Sciences*, Vol. 248, pp. 1920–1922.

207. Levy, Paul (1959), Équisse d'une Theorie de la Multiplication des Variables Aléatoires, *Annales Scientefiques de l'Ecole Normale Superieure*, pp. 59–82.

208. Lew, J. S. (1975), On Some Relations Between the Laplace and Mellin Transforms, *IBM Journal of Research and Development*, Vol. 19, pp. 582–584.

209. Likes, J. (1962), On the Distribution of Certain Linear Functions of Ordered Samples from Exponential Population, *Annals of the Institute of Statistical Mathematics*, Vol. 13, pp. 225–230.

210. Linnik, Yu. V. (1964), *Decomposition of Probability Distributions*, Oliver and Boyd, Edinburgh-London; originally published in Leningrad, 1960.

211. Locker, J. L. and N. C. Perry (1962), Probability Functions for Computations Involving More than One Operation, *Mathematics Magazine*, No. 351, pp. 87–89.

212. Lomnicki, Z. A. (1967), *On the Distribution of Products of Independent Beta Variables*, University of Washington Laboratory of Statistical Research, Technical Report No. 50, University of Washington, Seattle.

213. Lomnicki, Z. A. (1969), On the Distribution of Products of Independent Beta Variables, *Zastosowania Matematyki, Applicationes Mathematicae,* Vol. 10, pp. 155–169.

214. Lomnicki, Z A. (1969), On the Distribution of Products of Random Variables, *Journal of Royal Statistical Society, Series B*, Vol. 29-3, pp. 513–524.

215. Lord, R. D. (1954), The Distribution of Distance in a Hypersphere, *Annals of Mathematical Statistics*, Vol. 25, pp. 794–798.

216. Lord, R. D. (1954), The Use of the Hankel Tranform in Statistics: General Theory and Examples, *Biometrika*, Vol. 41, pp. 41–45.

217. Lord, R. D. (1954), The Use of the Hankel Transform: Methods of Computation, *Biometrika*, Vol. 41, pp. 344–350.

218. Lovett, John N. (1977), Evaluation of the *H*-Function Inversion Integral, Ph.D. thesis, University of Arkansas.

219. Lukacs, E. (1952), An Essential Property of the Fourier Transforms of Distribution Functions, *Proceedings of the American Mathematical Society*, Vol. 3, pp. 508–510.

220. Lukacs, E. and Otto Syasy (1952), On Analytic Characteristic Functions, *Pacific Journal of Mathematics*, Vol. 2, pp. 615–625.

221. Lukacs, E. (1963), Applications of Characteristic Functions in Statistics, *Sankhya, Series A* (Calcutta), Vol. 25, pp. 175–188.

222. Lukacs, E. and R. G. Laha (1964), *Applications of Characteristic Functions*, Griffins, London; Hafner, New York.

223. Lukacs, E. (1969), Some Recent Developments in the Theory of Multivariate Characteristic Functions, *Multivariate Analysis*, Vol. 2, P. R. Krishnaiah, Ed., Academic Press, New York, pp. 303–319.

224. Lukacs, E. (1969), Stable Distributions and Their Characteristic Functions, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Vol. 71, pp. 84–114.

225. Lukacs, E. (1970), *Characteristic Functions*, 2nd ed., Hafner, New York.

226. Luke, Y. L. (1969), *The Special Functions and Their Approximations*, Vol. 1, Academic Press, New York.

227. Luke, Y. L. (1969), *The Special Functions and Their Approximations*, Vol. 2, Academic Press, New York.

228. Luke, Y. L. (1975), *Mathematical Functions and Their Approximations*, Academic Press, New York.

229. MacRobert, T. M. (1954), *Functions of a Complex Variable*, 3rd. ed., Macmillan, London.

230. Malik, H. J. (1967), Exact Distribution of the Quotient of Independent Generalized Gamma Variables, *Canadian Mathematical Bulletin*, Vol. 10, pp. 463–465.

231. Malik, H. J. (1968), Exact Distribution of the Product of Two Independent Generalized Gamma Variables with the Same Shape Parameter, *Annals of Mathematical Statistics*, Vol. 29, pp. 1751–1752.

232. Malik, H. J. (1969), Distribution of the Sum of Truncated Binomial Variates, *Canadian Mathematical Bulletin*, Vol. 12, pp. 334–336.

233. Malik, H. J. (1970), Distribution of Product Statistics from a Pareto Population, *Metrika*, Vol. 16, pp. 120–128.

234. Malik, H. J. (1970), The Distribution of the Product of Two Noncentral Beta Variates, *Naval Research Logistics Quarterly*, Vol. 17, pp. 327–330.

235. Malik, H. J. (1976), Distribution of a Linear Function and the Ratio of Two Independent Linear Functions of Independent Generalized Gamma Variables, *Naval Research Logistics Quarterly*, Vol. 23, pp. 339–343.

236. Malik, H. and R. Trudel (1976), Distributions of Product and Quotient of Order Statistics, *University of Guelph Statistical Series 1975-30*, pp. 1–25.

237. Mandelbrot, B. (1961), Stable Paretian Random Functions and the Multiplicative Variation of Income, *Econometrica*, Vol. 29, pp. 517–543.

238. Mandelbrot, B. (1964), *On the Variation of Certain Speculative Prices, The Random Character of Stock Market Prices*, P. Cootner, Ed., MIT Press, Cambridge, Mass., pp. 307–337.

239. Mantel, Nathan (1973), A Characteristic Function Exercise, *The American Statistician*, Vol. 27, p. 31.

240. Mardo, J. G., P. V. Z. Cole, Capt. G. S. Seibel, and Capt. A. R. Stephenson (April 1972), *A Tri-service Bayesian Approach to Nuclear Weapons Assessment*, Technical Report No. 4359, Picatinny Arsenal, Dover, N.J.

241. Marsaglia, George (1965), Ratios of Normal Variables and Ratios of Sums of Uniform Variables, *Journal of the American Statistical Association*, Vol. 60, pp. 193–204.

242. Mathai, A. M. and R. K. Saxena (1966), On a Generalized Hypergeometric Distribution, *Metrika*, Vol. 11, pp. 127–132.

243. Mathai, A. M. and R. K. Saxena (1969), Applications of Special Functions in the Characterization of Probability Distributions, *South African Statistical Journal*, Vol. 3, Part 1.

244. Mathai, A. M. and R. K. Saxena (1969), Distribution of a Product and the Structural Setup of Densities, *Annals of Mathematical Statistics*, Vol. 40, pp. 1439–1448.

245. Mathai, A. M. and P. N. Rathie (1971), The Exact Distribution of Wilks' Generalized Variance in the Noncentral Linear Case, *Sankhya, Series A* (Calcutta), Vol. 33, pp. 45–60.

246. Mathai, A. M. (1972), The Exact Noncentral Distribution of the Generalized Variance, *Annals of the Institute of Statistical Mathematics*, Vol. 24, pp. 53–65.

247. Mauldon, J. G. (1956), Characterizing Properties of Statistical Distributions, *Quarterly Journal of Mathematics, Oxford Series*, Vol. 2.7, pp. 155–160.

248. McKenzie, R. D., M. D. Springer, R. S. Frenk, P. A. Lepley, and C. T. Lin (1968), System Effectiveness Demonstration and Validation Techniques, Report No. RADC-TR-68-225; Rome Air Development Center, Griffiss Air Force Base, New York.

249. McNolty, Frank (1962), A Contour Integral Derivation of the Noncentral Chi-Square Distribution, *Annals of Mathematical Statistics*, Vol. 33, pp. 796–800.

250. McNolty, Frank (1967), Applications of Bessel Function Distributions, *Sankhya, Series B* (Calcutta), Vol. 29, pp. 235–248.

251. McNolty, F. (1968), A Note on Radial Distributions, *Operations Research*, Vol. 16, pp. 211–216.

252. McNolty, Frank and Jack Tomsky (1972), Some Properties of Special Function Bivariate Distributions, *Sankhya, Series B* (Calcutta), Vol. 34, pp. 251–264.

253. McNolty, Frank (1972), Quadratic Form Distributions Associated with Special Functions, *Indian Journal of Statistics, Series B*, Vol. 34, Part 1, pp. 21–26.

254. McNolty, Frank (1973), Some Probability Density Functions and Their Characteristic Functions, *Mathematics of Computation*, Vol. 27, pp. 495–504.

255. McNolty, Frank, Eldon Hansen, and J. R. Huynen (1975), Component Distributions for Fluctuating Radar Targets, *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-11, pp. 1316–1330.

256. Milicer-Gruzewska, H. (1954), The Arithmetic of Random Variables, *Cahiers Rhodaniens*, Vol. 6, pp. 9–56.

257. Miller, M. K. and W. T. Guy (1966), Numerical Inversion of the Laplace Transform by Use of Jacobi Polynomials, *SIAM Journal of Numerical Analysis*, Vol. 3, pp. 624–635.

258. Mitra, S. K. (1970), On the Probability Distribution of the Sum of Uniformly Distributed Random Variables, *SIAM Journal of Applied Mathematics*, Vol. 19, pp. 195–198.

259. Mood, A. M. (1939), Note on the L_1 Test for Many Samples, *Annals of Mathematical Statistics*, Vol. 10, pp. 187–189.

260. Moran, P. A. P. (1950), The Distribution of the Multiple Correlation Coefficient, *Proceedings of the Cambridge Philosophical Society*, Vol. 46, pp. 521–522.

261. Morrison, Milton and Filbert Tobias (1965), Some Statistical Characteristics of a Peak to Average Ratio, *Technometrics*, Vol. 7, pp. 379–385.

262. Morrison, D. F. (1971), Distribution of Linear Functions of Independent F Variates, *Journal of American Statistical Association*, Vol. 66, pp. 383–385.

263. Moyal, J. E. (1964), Multiplicative Population Process, *Journal of Applied Probability*, Vol. 1, pp. 267–283.

264. Mullen, K. (1967), A Note on the Ratio of Two Independent Random Variables, *The American Statistician*, Vol. 21, pp. 30–31.

265. Murphy, B. P. (1967), Some Two-Sample Tests When the Variances Are Unequal: A Simulation Study, *Biometrika*, Vol. 54, pp. 679–683.

266. Murthy, M. N. and S. C. Pillai (1966), A Note on the Reciprocal and Square Root of the Reciprocal of a Random Variable, *The American Statistician*, Vol. 20, No. 5, p. 30.

267. Muth, E. J. (1977), *Transform Methods with Applications to Engineering and Operations Research*, Prentice-Hall, Englewood Cliffs, N.J.

268. Nair, A. K. (1941), Distributions of Student's t and the Correlation Coefficient in Samples From Nonnormal Populations, *Sankhya* (Calcutta), Vol. 5, pp. 383–400.

269. Nath, G. (1974), On Sums and Products of Truncated Exponential Variates, *Scandinavian Actuarial Journal*, Vol. 57, pp. 205–210.

270. National Bureau of Standards, *Table of the Bivariate Normal Distribution Function and Related Functions*, Applied Mathematics Series No. 50, Government Printing Office, Washington, D.C., 1959.

271. Neyman, J. and E. S. Pearson (1931), On the Problem of k Samples, *Bulletin de l'Academie Polonaise des Sciences et des Lettres, Series A*, No. 6, pp. 460–481.

272. Nicolich, M. J. (1976), Definition of the Moment Generating Function, *The American Statistician*, Vol. 30, p. 104.

273. Norris, Nilan (1940), The Standard Errors of the Geometric and Harmonic Means and Their Application to Index Numbers, *Annals of Mathematical Statistics*, Vol. 11, pp. 445–446.

274. Oppenheim, A. B., R. W. Schafer and T. G. Stockham (1968), Nonlinear Filtering of Multiplied and Convolved Signals, *IEEE Proceedings, Part II*, Vol. 56, pp. 1263–1291.

275. Papoulis, A. (1957), A New Method of Inversion of the Laplace Transform, *Quarterly of Applied Mathematics*, Vol. 14, pp. 405–414.

276. Papoulis, A. (1968), *Systems and Transforms with Applications in Optics*, McGraw-Hill, New York.

277. Parzen, E. (1960), *Modern Probability Theory and Its Applications*, Wiley, New York.

278. Patil, S. A. and S. H. Liao (April 1970), The Distribution of the Ratios of Means to the Square Root of the Sum of Variances, *Annals of Mathematical Statistics*, Vol. 41, pp. 723–728.

279. Pearl, J. (1971), Application of Walsh Transform to Statistical Analysis, *IEEE Transactions on Systems, Man, and Cybernetics*, Vol. SMC-1, pp. 111–119.

280. Pearl, J. (1972), Walsh Processing of Random Signals, UCLA Engineering Report, Computer Science Department, School of Engineering and Applied Science, Los Angeles.

281. Pearson, Karl, G. B. Jeffrey, and E. M. Elderton (1929), On the Distribution of the First Product Moment Coefficient in Samples Drawn from an Indefinitely Large Normal Population, *Biometrika*, Vol. 21, pp. 164–201.

282. Pearson, Karl (1932), *Tables of Incomplete Beta Functions*, Cambridge University Press, London.

283. Pearson, Karl (1934), *Tables of the Incomplete Gamma Function*, Cambridge University Press, London.

284. Pendse, Asha (1970), Integration of H-Function with Respect to Its Parameters, *Vijnana Parishad Anusandhan Patrika*, Vol. 13, pp. 129–138.

285. Perlo, V. (1933), On the Distribution of Student's Ratio for Samples of Three Drawn from a Rectangular Distribution, *Biometrika*, Vol. 25, pp. 203–204.

286. Pietz, Kenneth C. (1975), Cauchy Transforms and Characteristic Functions, *Pacific Journal of Mathematics*, Vol. 58-2, pp. 563–569.

287. Pillai, K. C. S., G. P. Steck, and S. S. Gupta (1964), The Distribution of Linear Functions of Ordered Correlated Normal Random Variables with Emphasis on Range, *Biometrika*, Vol. 51, pp. 143–151.

288. Pillai, K. C. S., S. Al-Ani, and G. M. Jouris (1969), On the Distributions of the Ratios of the Roots of a Covariance Matrix and Wilks' Criterion for Tests of Three Hypotheses, *Annals of Mathematical Statistics*, Vol. 40, pp. 2033–2040.

289. Pillai, K. C. S. and D. L. Young (1971), On the Exact Distribution of Hotelling's Generalized T_0^2, *Journal of Multivariate Analysis*, Vol. 1, pp. 90–107.

290. Pillai, K. C. S. and B. N. Nagarsenker (1972), On the Distributions of a Class of Statistics in Multivariate Analysis, *Journal of Multivariate Analysis*, Vol. 2, pp. 96–114.

291. Pistoia, A. (1958), Dirichlet Transforms Applied to Interest Functions, *Skandinavisk Aktuarietidsskrift*, Vol. 41, pp. 167–171.

292. Pitman, E. J. G. (1956), On the Derivation of a Characteristic Function at the Origin, *Annals of Mathematical Statistics*, Vol. 27, pp. 1156–1160.

293. Posten, H. O. and J. D. Woods (1968), *Fourier Series and Chebyshev Polynomials in Statistical Distribution Theory*, Technical Report, Department of Statistics, University of Connecticut, Storrs. Also available as Technical Report AD-672-346, National Technical Information Services, Springfield, Va. 22161.

294. Prasad, R. D. (1970), Probability Distributions of Algebraic Functions of Independent Random Variables, *SIAM Journal of Applied Mathematics*, Vol. 18, pp. 614–626.

295. Prasad, R. D. (May–June, 1971), Probability Distributions of Algebraic Functions of Independent Random Variables, *Mathematical Reviews*, Vol. 41-3, No. 7738, p. 1420.

296. Pratt, W. K., J. Kane, and H. C. Andrews (1969), Hadamard Transform Image Coding, *Proceedings of the IEEE*, Vol. 57, pp. 58–66.

297. Price, R. (1964), Some Noncentral F Distributions Expressed in Closed Form, *Biometrika*, Vol. 51, pp. 107–122.

298. Pruett, James (1972), *The Distribution of Products of Some Independent Nonstandardized Random Variables*, Doctoral dissertation, University of Arkansas.

299. Rahman, M. and A. K. Saleh (1974), Explicit Form of the Distribution of the Behrens-Fisher d-Statistics, *Journal of the Royal Statistical Society*, Vol. 36, pp. 54–60.

300. Raiffa, H. and R. Schlaifer (1961), *Applied Statistical Decision Theory*, Graduate School, Business Administration, Harvard University, Cambridge, Mass.

301. Ramachandran, B. (1967), *Advanced Theory of Characteristic Functions*, Statistical Publishing Society, Calcutta, India.

302. Rao, C. R. (1948), Tests of Significance in Multivariate Analysis, *Biometrika*, Vol. 35, pp. 58–79.

303. Ray, W. D. and A. E. N. T. Pitman (1961), An Exact Distribution of the Behrens-Fisher-Welch Statistics for Testing the Difference Between the Means of Two Normal Populations with Unknown Variances, *Journal of the Royal Statistical Society, Series B*, Vol. 23, pp. 377–384.

304. Reed, I. S. (1944), The Mellin Type of Double Integral, *Duke Mathematical Journal*, Vol. 11, pp. 565–572.

305. Rider, P. R. (1964), Distribution of Product and Quotient of Maximum Values in Samples from a Power-Function Population, *Journal of the American Statistical Association*, Vol. 59, pp. 877–880.

306. Rider, P. R. (1965), Distributions of Product and Quotient of Cauchy Variables, *The American Mathematical Monthly*, Vol. 72, pp. 303–305.

307. Rider, P. R. (1967), Products and Quotients of Generalized Cauchy Variables, *ARL: Blanch Anniversary Volume*, B. Mond., Ed., Wright-Patterson Air Force Base: U.S. Air Force, Office of Aerospace Research, Dayton, Ohio.

308. Rietz, H. L. (1936), On the Frequency Distribution of Certain Ratios, *Annals of Mathematical Statistics*, Vol. 7, pp. 145–158.

309. Roach, S. A. (1963), The Frequency Distribution of the Sample Mean, Where Each Member of the Sample is Drawn from a Different Rectangular Distribution, *Biometrika*, Vol. 50, pp. 508–513.

310. Robbins, H. and E. J. G. Pitman (1949), Application of the Method of Mixtures to Quadratic Forms in Normal Variates, *Annals of Mathematical Statistics*, Vol. 20, pp. 552–560.

311. Robinson, J. (1965), The Distribution of a General Quadratic Form in Normal Variables, *Australian Journal of Statistics*, Vol. 7, pp. 110–114.

312. Rogers, G. S. and D. L. Young (1973), On the Products of Powers of Generalized Dirichlet Components with an Application, *Canadian Journal of Statistics*, Vol. 1, pp. 159–179.

313. Rosenberg, Lloyd (1965), Non-normality of Linear Combinations of Normally Distributed Random Variables, *American Mathematical Monthly*, Vol. 72, pp. 888–890.

314. Roy, J. (1951), Distribution of Certain Likelihood Criteria Useful in Multivariate Analysis, *Proceedings of the International Statistical Conference*, pp. 219–230.

315. Ruben, H. (1959), A Theorem on the Cumulative Product of Independent Random Variables, *Proceedings of the Cambridge Philosophical Society*, Vol. 55, pp. 333–337.

316. Ruben, H. (1960), On the Distribution of the Weighted Difference of Two Independent Student Variables, *Journal of the Royal Statistical Society, Series B*, Vol 22, pp. 188–194.

317. Ruymgaart, F. H. (1973), Non-normal Bivariate Densities with Normal Marginals and Linear Regression Functions, *Statistica Neerlandica*, Vol. 27, pp. 11–17.

318. Sakamoto, H. (1943), On the Distributions of the Product and Quotient of the Independent and Uniformly Distributed Random Variables, *Tohoku Mathematical Journal*, Vol. 49, pp. 243–260.

319. Salzer, H. E. (1958), Tables for the Numerical Calculation of Inverse Laplace Transforms, *Journal of Mathematics and Physics*, Vol. 37, pp. 89–109.

320. Schatzoff, M. (1966), Exact Distributions of Wilks' Likelihood Ratio Criterion, *Biometrika*, Vol. 53, pp. 347–358.

321. Scheffe, H. (1943), On Solutions of the Behrens-Fisher Problem Based on the t-distribution, *Annals of Mathematical Statistics*, Vol. 14, pp. 33–44.

322. Scheffe, H. (December 1970), Practical Solutions to the Behrens-Fisher Problems, *Journal of the American Statistical Association*, Vol. 65, pp. 1501–1508.

323. Schulz-Arenstorf, R. and J. C. Morelock (1959), The Probability Distribution of the Product of n (uniform) Random Variables, *American Mathematical Monthly*, Vol. 66, pp. 95–99.

324. Sclove, S. L., G. Simons, and J. Van Ryzin (1967), Further Remarks on the Expectation of the Reciprocal of a Positive Random Variable, *American Statistician*, Vol. 21, No. 4, pp. 33–34.

325. Seale, K. C. (1956), Sums of Covariances of Order Statistics from Normal Populations, *Bulletin of the Calcutta Statistical Association*, Vol. 7, pp. 33–34.

326. Sethuraman, J. and B. V. Sukhatime (1959), Joint Asymptotic Distribution of U-Statistics and Order Statistics, *Sankhya* (Calcutta), Vol. 21, pp. 289–298.

327. Shah, M. (1969), On Applications of Mellin's and Laplace's Inversion Formulae to H-Functions, *Labdev, Journal of Sciences and Technology, Part A*, Vol. 7, pp. 10–17.

328. Shah, M. (1969), On Some Results of the H-Functions Involving Hermite Polynomials, *Journal of Natural Sciences and Mathematics*, Vol. 9, pp. 223–233.

329. Shah, M. (1969), Some Integrals Involving H-Functions, I, *Mathematics Education*, Vol. 3, pp. 87–88.

330. Shah, M. (1969), Some Results on the H-Function Involving the Generalized Laguerre Polynomial, *Proceedings of the Cambridge Philosophical Society*, Vol. 65, pp. 713–720.

331. Shah, M. C. and P. N. Rathie (1974), Exact Distribution of Product of Generalized F-Variates, *Canadian Journal of Statistics*, Vol. 2, pp. 13–25.

332. Shapiro, J. M. (1957), Sums of Independent Truncated Random Variables, *Annals of Mathematical Statistics*, Vol. 28, pp. 754–761.

333. Sharma, O. P. (1968), On the Hankel Transformations of the H-Function, *Journal of Mathematical Sciences*, Vol. 3, pp. 17–26.

334. Shirtliffe, C. J. and D. G. Stephenson (1961), Computer Oriented Adaption of Salzer's Mehtod for Inverting Laplace Transforms, *Journal of Mathematics and Physics*, Vol 40, pp. 135–141.

335. Shohat, J. A. and J. D. Tamarkin (1943), *The Problem of Moments*, American Mathematical Society, Providence, R.I.

336. Siddiqui, M. M. (1960), Distribution of Quantiles in Samples from a Bivariate Population, *Journal of Research of the National Bureau of Standards*, Vol. 643, pp. 145–150.

337. Siddiqui, M. M. (1964), Distribution of Student's t in Samples from a Rectangular Universe, *Revue de l'Institut International de Statistique*, Vol. 32, pp. 242–250.

338. Sinai, Y. G. (1957), On the Distribution of the First Positive Sum for a Sequence of Independent Random Variables, *Theory of Probability and Its Applications*, Vol. 2, pp. 122–129.

339. Singh, M. P. (1965), On the Estimation of Ratio and Product of the Population Parameters, *Sankhya, Series B* (Calcutta), Vol. 27, pp. 321–328.

340. Singh, R. (1968), On Some Results Involving H-Function of Fox, *Proceedings of the National Academy of Science* (India), *Section A*, Vol. 38, pp. 240–250.

341. Sinha, S. K. (1973), Distributions of Order Statistics and Estimation of Mean Life When an Outlier May Be Present, *Canadian Journal of Statistics*, Vol. 1, pp. 119–123.

342. Siotani, M. (1957), On the Distributions of the Hotelling's T-Square Statistics, *Annals of the Institute of Statistical Mathematics*, Vol. 8, pp. 1–14.

343. Skellam, J. G. (1946), The Frequency Distribution of the Difference Between Two Variates Belonging to Different Populations, *Journal of the Royal Statistical Society, Series A*, Vol. 109, p. 296.

344. Slater, L. J. (1966), Generalized Hypergeometric Functions, Cambridge University Press, p. 3.

345. Smith, D. R. (1973), Bayesian Approach to Generalized Reliability Networks Utilizing Distribution-Free Techniques, Doctoral dissertation, College of Engineering, University of Arkansas.

346. Smith, J. H. (1966), Correction to a Note on the Expectation of the Reciprocal of a Random Variable, *The American Statistician*, Vol. 20, no. 5, p. 37.

347. Sneddon, I. N. (1951), *Fourier Transforms*, McGraw-Hill, New York.

348. Snow, B. A. S. (1963), The Distribution of Kendall's Tau for Samples from a Normal Bivariate Population, *Biometrika*, Vol. 50, pp. 538–540.

349. Spiegel, M. R. (1964), *Theory and Problems of Complex Variables*, Schaum's Outline Series, McGraw-Hill, New York.

350. Spiegel, M. R. (1965), *Theory and Problems of Laplace Transforms*, Schaum's Outline Series, McGraw-Hill, New York.

351. Soni, S. L. (1973), An Expansion Formula for Fox's H-Function in Series of Geginbauer (Ultraspherical) Polynomials, *Ranchi University Mathematical Journal*, Vol. 4, pp. 15–20.

352. Springer, M. D. and J. K. Byers (1971), Bayesian Confidence Limits for the Reliability of Mixed Exponental and Distribution-Free Cascade Subsystems, *IEEE Transactions on Reliability*, Vol. R-18, pp. 721–737.

353. Springer, M. D. and W. E. Thompson (1964), *The Distribution of Products of Independent Random Variables*, G. M. Defense Research Laboratories Technical Report, No. TR 64-46, pp. 1–89.

354. Springer, M. D. and W. E. Thompson (1966), The Distribution of Products of Independent Random Variables, *SIAM Journal of Applied Mathematics*, Vol. 14, pp. 511–526.

355. Springer, M. D. and W. E. Thompson (1966), Bayesian Confidence Limits for the Product of N Binomial Parameters, *Biometrika*, Vol. 53, Nos. 3 and 4, pp. 611–613.

356. Springer, M. D. and W. E. Thompson (1967), Bayesian Confidence Limits for the Reliability of Cascade Exponential Subsystems, *IEEE Transactions on Reliability*, Vol. R-16, pp. 86–89.

357. Springer, M. D. and W. E. Thompson (1968), Bayesian Confidence Limits for Reliability of Redundant Systems When Tests Are Terminated at First Failure, *Technometrics*, Vol. 10, pp. 29–36.

358. Springer, M. D. and W. E. Thompson (1970), The Distribution of Products of Beta, Gamma and Gaussian Random Variables, *SIAM Journal of Applied Mathematics*, Vol. 18, pp. 721–737.

359. Springer, M. D. and W. E. Thompson (1972), A Bayesian Analysis of Availability for a System Consisting of Several Independent Subsystems, *IEEE Transactions on Reliability*, Vol. R-21, pp. 212–214.

360. Srivastava, H. M. and J. P. Singhal (1972), On a Class of Generalized Hypergeometric Distribution, *Jnanabha, Section A*, Vol. 2, pp. 1–9.

361. Srivastava, H. M. and R. G. Buschman (1976), Mellin Convolutions and *H*-Function Transformations, *Rocky Mountain Journal of Mathematics*, Vol. 6, pp. 331–343.

362. Srodka, T. (1967), On the Distribution of Product and Ratio of Powers of Two Independent Random Variables with the Generalized Gamma, Maxwell, and Weibull Distribution, *Annales Societatis Mathematici Polonae*, Vol. 11, pp. 78–85.

363. Stacy, E. W. (1962), A Generalization of the Gamma Distribution, *Annals of Mathematical Statistics*, Vol. 33, pp. 1187–1192.

364. Stark, R. M. (1969), Operational Techniques for Random Scalar and Vector Products, Unpublished technical report, University of Delaware.

365. Stark, R. M. and D. K. Shukla (1972), Statics of Random Beams, *Journal of the Engineering Mechanics Division ASCE*, Vol. 98, pp. 1487–1497.

366. Stark, R. M. and R. L. Nicholls (1972), *Mathematical Foundations for Design—Civil Engineering Systems*, McGraw-Hill, New York.

367. Stark, Robert M. (1977), On Zero Degree Stochastic Programs, *Journal of Optimization, Theory and Applications*, Vol. 23, pp. 167–182.

368. Steck, G. P. (1958), A Uniqueness Property Not Enjoyed by the Normal Distribution, *Annals of Mathematical Statistics*, Vol 29, pp. 604–606.

369. Stelson, H. E. (1956), Laplace Transforms Applied to Interest Functions, *Skandinavisk Aktuarietidsskrift*, Vol. 39, pp. 97–104.

370. Stuart, A. (1962), Gamma Distributed Products of Independent Random Variables, *Biometrika*, Vol. 49, pp. 561–564.

371. Subrahmanian, K. (1970), On Some Applications of Mellin Transforms to Statistics: Dependent Random Variables, *SIAM Journal of Applied Mathematics*, Vol. 19, pp. 658–662.

372. Sukhatme, P. V. (1938), On Fisher and Behrens' Test of Significance for the Difference in Means of Two Normal Distributions, *Sankhya* (Calcutta), Vol. 4, pp. 39–48.

373. Taneja, V. S., L. W. Cornwell, and L. A. Aroian (1977), Mathematical Forms of the Distribution of the Product of Two Normal Variables, Abstract No. 157-2, *IMS Bulletin*, May.

374. Tang, P. C. (1938), The Power Function of the Analysis of Variance Tests with Tables and Illustrations of Their Use, *Statistical Research Memoirs, University of London*, Vol. 2, pp. 126–149.

375. Taxak, R. L. (1971), Integration of Some *H*-Functions with Respect to Parameters, *Defense Science Journal* (Delhi), Vol. 21, pp. 111–118.

376. Thompson, W. E. and P. A. Palicio (1975), Bayesian Confidence Limits for the Availability of Systems, *IEEE Transactions on Reliability*, Vol. R-24, pp. 118–120.

377. Thompson, W. E. and E. Y. Chang (1976), Bayes Analysis of Reliability of Complex Systems, *Operations Research*, Vol. 24, pp. 156–168.

378. Tiku, M. L. and H. J. Malik (1972), On the Distribution of Order Statistics, *Australian Journal of Physics*, Vol. 14, pp. 103–108.

379. Tintner, G. (1944), An Application of the Variate Difference Method to Multiple Regression, *Econometrica*, Vol. 12, pp. 97–113.

380. Titchmarsh, E. C. (1937), *Theory of Functions*, Clarendon Press, Oxford.

381. Titchmarsh, E. C. (1948), *Introduction to the Theory of Fourier Integrals*, 2nd ed., Clarendon Press, Oxford.

382. Trantner, C. J. (1956), *Integral Transforms in Mathematical Physics*, 2nd ed., Methuen, London.

383. Traub, J. F. (1964), *Iterative Methods for the Solution of Equations*, Prentice-Hall, Englewood Cliffs, N.J.

384. Tretter, M. J. and Walster, G. W. (1975), Central and Noncentral Distributions of Wilks Statistics in MANOVA as Mixtures of Incomplete Beta Functions, *Annals of Statistics*, Vol 3, pp. 467–472.

385. Tukey, J. and S. S. Wilks (1946), Approximation of the Distribution of the Product of Beta Variables by a Single Beta Variable, *Annals of Mathematical Statistics*, Vol. 17, pp. 257–281.

386. Uranisi, H. (1950), The Distribution of Statistics Drawn from the Gram-Charlier Type A Population, *Bulletin of Mathematical Statistics* (Fukuoka), Vol. 4, pp. 1–14 (in English).

387. Van Klinken, J. (1961), A Method for Inquiring Whether the Distribution Represents the Frequency Distribution of Accident Costs, *Actuariele Studien*, Vol. 3, Nijhoff-The Hague.

388. Von Mises, Richard (1964), *A Mathematical Theory of Probability and Statistics*, Academic Press, New York, pp. 384–401.

389. Votaw, D. F. (1948), Testing Compound Symmetry in Normal Multivariate Distributions, *Annals of Mathematical Statistics*, Vol. 19, pp. 447–473.

390. Walsh, John E. (1949), On the Range-Midrange Test and Some Tests with Bounded Significance Levels, *Annals of Mathematical Statistics*, Vol. 20, pp. 257–267.

391. Watson, G. N. (1922), *A Treatise On the Theory of Bessel Functions*, University Press, London.

392. Watson, G. N. (1966), *Theory of Bessel Functions*, 2nd ed., Cambridge University Press, London.

393. Weapon Systems Effectiveness Industry Advisory Committee (WSEIAC), AF Systems Command, Andrews Air Force Base, Maryland:
   1. Final Report of Task Group I: Requirements Methodology: AFSC-TR-65-2, Vol. III, pp. 1–69, January 1965.
   2. Final Report of Task Group II
      (*a*) Prediction-Measurement (Concepts, Task Analysis, Principles of Model Construction), AFSC-TR-65-2, Vol. II, pp. 1–103, January 1965.
      (*b*) Prediction Measurement (Technical Supplement), Vol. III, pp. 1–416, January 1965.
      (*c*) Prediction-Measurement (Summary, Conclusions, Recommendations), pp. 1–15, January 1965.
   3. Final Report of Task Group III: Data Collection and Management Reports, AFSC-TR-65-3; pp. 1–259, January 1965.

4. Final Report of Task Group IV:
   (*a*) Cost-Effectiveness Optimization (Summary, Conclusions, and Recommendations), AFSC-TR-65-4, Vol. I, pp. 1–15, January 1965.
   (*b*) Cost-Effectiveness Optimization (Tasks and Analysis Methodology), AFSC-TR-65-4, Vol. II, pp. 1–160, January 1965.
   (*c*) Cost-Effectiveness Optimization (Technical Supplement), AFSC-TR-65-4, Vol. III, pp. 1–202, January 1965.
5. Final Report of Task Group V:
   (*a*) Management Systems (System Effectiveness Assurance—Summary Policy Issues, Recommendations), AFSC-TR-65-5, Vol. I, pp. 1–52, January 1965.
   (*b*) Management Systems (Elements of Effectiveness Assurance Management), AFSC-TR-65-5, Vol. II, pp. 1–158, January 1965.

394. Webb, E. L. R. (1956), On the Distribution of the Product of Diode Detector Waveforms, *Canadian Journal of Physics*, Vol. 34, pp. 679–691.

395. Weibull, M. (1950), The Distributions of the T and Z Variables in the Case of Stratified Sample with Individuals Taken from Normal Parent Populations with Varying Means, *Skandinarisk Aktuarietidsskrift*, Vol. 33, pp. 137–167.

396. Weibull, M. (1953), The Distributions of T and F Statistics and of Correlation and Regression Coefficients in Stratified Samples from Normal Populations with Different Means (Supplement to Vol. 35), *Skandinarisk Aktuarietidsskrift*, Vol. 36, pp. 9–106.

397. Weibull, C. (1958), The Distribution of the Student Ratio in the Case of Serially Correlated Normal Variables, *Skandinarisk Aktuarietidsskrift*, Vol. 41, pp. 172–176.

398. Weiler, H. (1959), Note on Harmonic and Geometric Means, *Australian Journal of Statistics*, Vol. 1, pp. 44–45.

399. Weiss, H. K. (1961), The Distribution of Urban Population and an Application to a Servicing Problem, *Operations Research*, Vol. 9, pp. 860–874.

400. Welch, B. L. (1938), The Significance of Difference Between Two Means When the Population Variances Are Unequal, *Biometrika*, Vol. 29, pp. 350–352.

401. Welch, B. L. (1947), The Generalization of Student's Problem When Several Different Population Variances Are Involved, *Biometrika*, Vol. 34, pp. 28–35.

402. Whittaker, E. T. and G. N. Watson (1915), *A Course of Modern Analysis*, Cambridge University Press, London.

403. Whittaker, E. T. and G. N. Watson (1940), *A Course of Modern Analysis*, 4th ed., Cambridge University Press, London.

404. Widder, D. V. (1941), *The Laplace Theorem*, Princeton Mathematical Series No. 6, Princeton University Press, Princeton, N. J.

405. Wilks, S. S. (1932), Certain Generalizations in the Analysis of Variance, *Biometrika*, Vol. 20, pp. 471–494.

406. Wilks, S. S. (1932), On the Sampling Distribution of the Multiple Correlation Coefficient, *Annals of Mathematical Statistics*, Vol. 3, pp. 196–203.

407. Wilks, S. S. (1934), Moment Generating Operators for Determinants of Product Moments in Samples from a Normal System, *Annals of Mathematical Statistics*, Vol. 35, pp. 312–340.

408. Wilks, S. S. (1937), The Sampling Distribution of the Criterion λ_H, When the Hypothesis Tested Is Not True, *Biometrika*, Vol. 29, pp. 124–132.

409. Wilks, S. S. (1962), *Mathematical Statistics*, Wiley, New York.

410. Williams, E. J. (1977), Some Representations of Stable Random Variables as Products, *Biometrika*, Vol. 64, pp. 167–169.

411. Wintner, A. (1934), On the Addition of Independent Distributions, *American Journal of Mathematics*, Vol. 56, pp. 8–16.

412. Wolfe, Stephen J. (1973), On the Local Behavior of Characteristic Functions, *The Annals of Probability*, Vol. 1, pp. 862–866.

413. Wolfe, Stephen J. (1973), On the Finite Series Expansion of Multivariate Characteristic Functions, *Journal of Multivariate Analysis*, Vol. 3, pp. 328–335.

414. Wolfe, Stephen J. (1975), On Derivatives of Characteristic Functions, *Annals of Probability*, Vol. 3–4, pp. 737–738.

415. Wylie, C. R., Jr. (1951), *Advanced Engineering Mathematics*, McGraw-Hill, New York.

416. Zeger, I. E. (1967), On the Ratio of Two Random Variables, *Proceedings of the IEEE*, Vol. 55, pp. 247–248.

417. Zelen, M. and Mary C. Dannemiller (1961), The Robustness of Life Testing Procedures Derived from the Exponential Distribution, *Technometrics*, Vol. 3, pp. 29–49.

418. Zemanian, A. H. (1968), *Generalized Integral Transforms, Pure and Applied Mathematics*, Vol. 18, Wiley-Interscience, New York.

419. Zolotarev, V. M. (1957), Mellin-Stieltjes Transforms in Probability Theory, *Theory of Probability and Its Applications*, Vol. 2, pp. 433–460.

420. Zolotarev, V. M. (1962), On a General Theory of Multiplication of Independent Random Variables, *Soviet Mathematics, Doklady*, (in Russian), Vol. 3, pp. 166–170, and *Doklady Akademii Nauk SSSR*, Vol. 142, pp. 788–791.

Index

DATE DUE

Applied Probability and Statistics (Continued)

FLEISS · Statistical Methods for Rates and Proportions
GALAMBOS · The Asymptotic Theory of Extreme Order Statistics
GIBBONS, OLKIN, and SOBEL · Selecting and Ordering Populations: A New Statistical Methodology
GNANADESIKAN · Methods for Statistical Data Analysis of Multivariate Observations
GOLDBERGER · Econometric Theory
GOLDSTEIN and DILLON · Discrete Discriminant Analysis
GROSS and CLARK · Survival Distributions: Reliability Applications in the Biomedical Sciences
GROSS and HARRIS · Fundamentals of Queueing Theory
GUTTMAN, WILKS, and HUNTER · Introductory Engineering Statistics, *Second Edition*
HAHN and SHAPIRO · Statistical Models in Engineering
HALD · Statistical Tables and Formulas
HALD · Statistical Theory with Engineering Applications
HARTIGAN · Clustering Algorithms
HILDEBRAND, LAING, and ROSENTHAL · Prediction Analysis of Cross Classifications
HOEL · Elementary Statistics, *Fourth Edition*
HOLLANDER and WOLFE · Nonparametric Statistical Methods
HUANG · Regression and Econometric Methods
JAGERS · Branching Processes with Biological Applications
JESSEN · Statistical Survey Techniques
JOHNSON and KOTZ · Distributions in Statistics
Discrete Distributions
Continuous Univariate Distributions-1
Continuous Univariate Distributions-2
Continuous Multivariate Distributions
JOHNSON and KOTZ · Urn Models and Their Application: An Approach to Modern Discrete Probability Theory
JOHNSON and LEONE · Statistics and Experimental Design in Engineering and the Physical Sciences, Volumes I and II, *Second Edition*
KEENEY and RAIFFA · Decisions with Multiple Objectives
LANCASTER · An Introduction to Medical Statistics
LEAMER · Specification Searches: Ad Hoc Inference with Nonexperimental Data
McNEIL · Interatcive Data Analysis
MANN, SCHAFER, and SINGPURWALLA · Methods for Statistical Analysis of Reliability and Life Data
MEYER · Data Analysis for Scientists and Engineers
OTNES and ENOCHSON · Applied Time Series Analysis: Volume I, Basic Techniques
OTNES and ENOCHSON · Digital Time Series Analysis
PRENTER · Splines and Variational Methods
RAO and MITRA · Generalized Inverse of Matrices and Its Applications
SARD and WEINTRAUB · A Book of Splines
SEARLE · Linear Models
SPRINGER · The Algebra of Random Variables
THOMAS · An Introduction to Applied Probability and Random Processes
UPTON · The Analysis of Cross-Tabulated Data
WHITTLE · Optimization under Constraints

Applied Probability and Statistics (Continued)

WILLIAMS · A Sampler on Sampling
WONNACOTT and WONNACOTT · Econometrics
WONNACOTT and WONNACOTT · Introductory Statistics, *Third Edition*
WONNACOTT and WONNACOTT · Introductory Statistics for Business and Economics, *Second Edition*
YOUDEN · Statistical Methods for Chemists
ZELLNER · An Introduction to Bayesian Inference in Econometrics

Tracts on Probability and Statistics

BARNDORFF-NIELSEN · Information and Exponential Families in Statistical Theory
BIBBY and TOUTENBERG · Prediction and Improved Estimation in Linear Models
BHATTACHARYA and RAO · Normal Approximation and Asymptotic Expansions
BILLINGSLEY · Convergence of Probability Measures
JARDINE and SIBSON · Mathematical Taxonomy
KINGMAN · Regenerative Phenomena
RIORDAN · Combinatorial Identities

ISBN 0-471-01406-0