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THE DISTRIBUTION OF PRODUCTS OF INDEPENDENT RANDOM VARIABLES*

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1. Summary. Fundamental methods are developed for the derivation of probability density functions (p.d.f.'s) of products of n independent random variables (i.r.v.'s), and are used to obtain particular results which, aside from the case $n = 2$, are believed to be new. The methods use the Mellin integral transform, and are a generalization to n variables of a method presented by Epstein [1]. P.d.f.'s are obtained in explicit form for products of n monomial[1], $n \leqq 10$ Cauchy, and $n \leqq 7$ Gaussian variables. Tables for products of $n = 2, 3, 6$ Gaussian[2] i.r.v.'s $N(0, 1)$ have been calculated using this method [12], abridged versions of which are included in this paper. Entries in the unabridged tables were obtained with accuracy to six decimal places and permit linear interpolation with four-digit accuracy in the area column. These tables offer a heretofore unavailable tool to the engineer and research scientist concerned with reliability analysis, communications theory, and other applications requiring consideration of products of i.r.v.'s.

2. Introduction. Unlike the distribution of sums of i.r.v.'s, the distribution of products of more than two has received relatively little attention, and results which are available supply little useful information to the applied scientist. Epstein [1] has suggested a systematic approach to the study of products of i.r.v.'s using the Mellin integral transform, but did not carry out the application for products of more than two. Levy [2] posed the question of a general theory of multiplication of i.r.v.'s and derived some results for products of two variables. In 1962, Zolotarev [3] began the construction of a general theory of multiplication of i.r.v.'s analogous to the theory of addition based on infinitely divisible distributions. His program was carried out in a sequence of theorems, stated without proof, which show both the similarity to, and difference from, the results for addition. Jambunathan [4] and Sakamoto [5], respectively, have derived the distribution of products of beta and rectangular i.r.v.'s. Other established results deal

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[1] $f(x) = (\alpha + 1)x^{\alpha}$, $0 \leqq x \leqq 1$. The rectangular random variable is the special case for which $\alpha = 0$.

[2] The notation $N(m, \sigma)$ denotes the Gaussian p.d.f. with mean m and standard deviation σ.

with isolated phases of the subject. (For a detailed bibliography, see [12].) It is the purpose of this paper to develop fundamental methods for the derivation of p.d.f.'s of products of i.r.v.'s, and to utilize these methods to determine explicit results for various important products heretofore not considered for $n > 2$, specifically for products of monomial, Cauchy, and Gaussian i.r.v.'s.

3. The role of the Mellin transform and convolution in the derivation of product distributions. The Mellin transform of $f(x)$, defined only for $x \geqq 0$, is

$$M(f(x) \mid s) = E[x^{s-1}] = \int_0^\infty x^{s-1} f(x)\, dx. \tag{1}$$

Under suitable restrictions [6] on $M(f(x) \mid s)$, considered as a function of the complex variable s, there is an inversion integral

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} M(f(x) \mid s)\, ds, \tag{2}$$

for which the identity relation

$$f(x) = \int_{c-i\infty}^{c+i\infty} x^{-s} \left\{ \int_0^\infty \xi^{s-1} f(\xi)\, d\xi \right\} ds \tag{3}$$

is valid almost everywhere. The Mellin convolution of two functions $f_1(x), f_2(x)$, $0 \leqq x < \infty$, is defined as

$$g(x) = \int_0^\infty \frac{1}{y} f_2\left(\frac{x}{y}\right) f_1(y)\, dy, \tag{4}$$

which is also the p.d.f. $h_2(x)$ of the product $x = x_1 x_2$ of two independent, positive, random variables with p.d.f.'s $f_1(x_1)$ and $f_2(x_2)$ [7]. Since $g(x) = h_2(x)$ and the Mellin transform of $h_2(x)$ is [1]

$$M(h_2(x) \mid s) = M(f_1(x_1) \mid s) \cdot M(f_2(x_2) \mid s), \tag{5}$$

it follows that the p.d.f. of a product of two independent random variables with p.d.f.'s $f_1(x_1)$ and $f_2(x_2)$ is the Mellin convolution whose transform is the product of the Mellin transforms of $f_1(x_1)$ and $f_2(x_2)$.

Again, by (4), the p.d.f. $h_3(\eta)$ of the product $\eta = x_1 x_2 x_3$ is

$$h_3(\eta) = \int_0^\infty \frac{1}{x} f_3\left(\frac{\eta}{x}\right) h_2(x)\, dx, \tag{6}$$

which in combination with (5) yields

$$M(h_3(\eta) \mid s) = \prod_{i=1}^{3} M(f_i(x_i) \mid s). \tag{7}$$

Then, $n - 1$ successive applications of (4) and (5) for nonnegative random variables lead to the general results

$$h_n(\eta) = \int_0^\infty \frac{1}{x} f_n\left(\frac{\eta}{x}\right) h_{n-1}(x)\, dx \tag{8}$$

and

$$M(h_n(\eta) \mid s) = \prod_{i=1}^{n} M(f_i(x_i) \mid s). \tag{9}$$

Thus, $h_n(\eta)$ can be obtained directly from the $M(f_i(x_i) \mid s)$ using

$$h_n(\eta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \prod_{i=1}^{n} M(f_i(x_i) \mid s)\, ds. \tag{10}$$

Therein lies the utility of the Mellin transform in the derivation of product distributions for nonnegative random variables.

To treat the more general problem of products of independent random variables which may assume both positive and negative values, a procedure developed by Epstein [1] for the case of two variables will now be extended to n variables. This extension is accomplished by decomposing a function $f_i(x_i)$, $-\infty < x_i < \infty$, $i = 1, 2, \cdots, n$, into two components, namely,

$$f_i(x_i) = f_i^-(x_i) + f_i^+(x_i),$$

in which $f_i^-(x_i)$ vanishes identically except on the interval $-\infty < x_i \leqq 0$, where $f_i^-(x_i) = f_i(x_i)$. Similarly, $f_i^+(x_i)$ is defined to be identically zero except over the interval $0 \leqq x_i < \infty$, where $f_i^+(x_i) = f_i(x_i)$. Since [8] $\eta = x_1 x_2$ has the p.d.f.

$$h_2(\eta) = \int_{-\infty}^\infty \frac{1}{|x_1|} f_2\left(\frac{\eta}{x_1}\right) f_1(x_1)\, dx_1, \tag{11}$$

it follows by direct substitution that

$$h_2(\eta) = \int_{-\infty}^\infty \frac{1}{|x_1|} \left\{ f_2^+\left(\frac{\eta}{x_1}\right) + f_2^-\left(\frac{\eta}{x_1}\right) \right\} \{f_1^+(x_1) + f_1^-(x_1)\}\, dx_1, \tag{12}$$

or

$$h_2(\eta) = \int_0^\infty \frac{1}{x_1} f_2^+\left(\frac{\eta}{x_1}\right) f_1^+(x_1)\, dx_1 + \int_0^\infty \frac{1}{x_1} f_2^+\left(\frac{-\eta}{x_1}\right) f_1^-(-x_1)\, dx_1$$
$$+ \int_0^\infty \frac{1}{x_1} f_2^-\left(\frac{\eta}{x_1}\right) f_1^+(x_1)\, dx_1 + \int_0^\infty \frac{1}{x_1} f_2^-\left(\frac{-\eta}{x_1}\right) f_1^-(-x_1)\, dx_1.$$

If one now defines

$$h_2(\eta) = h_2^-(\eta) + h_2^+(\eta),$$

where

$$h_2^-(\eta) = \begin{cases} h_2(\eta) & \text{if } -\infty < \eta \leqq 0, \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$h_2^+(\eta) = \begin{cases} h_2(\eta) & \text{if } 0 \leqq \eta < \infty, \\ 0 & \text{elsewhere,} \end{cases}$$

then for $0 \leqq \eta < \infty$,

$$\begin{aligned} h_2^-(-\eta) &= \int_0^\infty \frac{1}{x_1} f_2^+\left(\frac{\eta}{x_1}\right) f_1^-(-x_1)\, dx_1 \\ &\quad + \int_0^\infty \frac{1}{x_1} f_2^-\left(\frac{-\eta}{x_1}\right) f_1^+(x_1)\, dx_1, \\ h_2^+(\eta) &= \int_0^\infty \frac{1}{x_1} f_2^+\left(\frac{\eta}{x_1}\right) f_1^+(x_1)\, dx_1 \\ &\quad + \int_0^\infty \frac{1}{x_1} f_2^-\left(\frac{-\eta}{x_1}\right) f_1^-(-x_1)\, dx_1. \end{aligned} \tag{13}$$

Thus, $h_2^+(\eta)$ and $h_2^-(-\eta)$ have been expressed in terms of convolutions of pairs of functions defined over the interval $(0, \infty)$ whose Mellin transforms are well-defined by (1) and

$$\begin{aligned} M(h_2^-(-\eta) \mid s) &= M(f_2^+(x) \mid s) \cdot M(f_1^-(-x) \mid s) \\ &\quad + M(f_2^-(-x) \mid s) \cdot M(f_1^+(x) \mid s), \\ M(h_2^+(\eta) \mid s) &= M(f_2^+(x) \mid s) \cdot M(f_1^+(x) \mid s) \\ &\quad + M(f_2^-(-x) \mid s) \cdot M(f_1^-(-x) \mid s). \end{aligned} \tag{14}$$

The inversion integral (2) then yields $h_2^+(\eta)$ and $h_2^-(-\eta)$. In turn, $h_2^-(-\eta)$ defines $h_2^-(\eta)$.

Then $n - 1$ successive applications of this procedure lead to the p.d.f. of $\eta = \prod_1^n x_i$, namely,

$$h_n(\eta) = h_n^-(\eta) + h_n^+(\eta),$$

whose components are defined by inverting the Mellin transforms

$$\begin{aligned} M(h_n^-(-\eta) \mid s) &= M(f_n^+(x) \mid s) \cdot M(h_{n-1}^-(-x) \mid s) \\ &\quad + M(f_n^-(-x) \mid s) \cdot M(h_{n-1}^+(x) \mid s), \\ M(h_n^+(\eta) \mid s) &= M(f_n^+(x) \mid s) \cdot M(h_{n-1}^+(x) \mid s) \\ &\quad + M(f_n^-(-x) \mid s) \cdot M(h_{n-1}^-(-x) \mid s). \end{aligned} \tag{15}$$

The two products on the right in (15), when expanded into terms involving $f_i^+(x_i)$ and $f_i^-(-x_i)$, result in 2^{n-1} products of the form appearing on the right in (14).

In the special case where the $f_i(x_i)$ are identical even functions of x, i.e.,

$$(16) \qquad f_i^-(-x_i) = f_i^+(x_i), \qquad i = 1, 2, \cdots, n,$$

the p.d.f. of the product $\eta = x_1x_2 \cdots x_n$ is even, so that

$$(17) \qquad h_n^+(\eta) = h_n^-(-\eta)$$

and

$$(18) \qquad M(h_n^+(\eta) \mid s) = M(h_n^-(-\eta) \mid s) = 2^{n-1}\{M(f^+(x) \mid s)\}^n.$$

Equation (18) supplies a direct relation between the p.d.f. $h(\eta)$ of the product $\eta = x_1x_2 \cdots x_n$ and the common p.d.f. $f(x)$ of the i.r.v.'s x_i, $-\infty < x_i < \infty$, $i = 1, 2, \cdots, n$.

The quotient $\zeta = x_1/x_2$ of two i.r.v.'s x_1 and x_2 may be considered as the product of x_1 and $1/x_2$. As Epstein has pointed out [1], if x is a nonnegative i.r.v. and $y = 1/x$ has the p.d.f. $g(y)$, then

$$M(g(y) \mid s) = M(f(x) \mid -s + 2).$$

Applying this fact to the case of i.r.v.'s x_i with p.d.f.'s $f_i(x_i)$, $-\infty < x_i < \infty$, $i = 1, 2$, one obtains

$$q(\zeta) = q^-(\zeta) + q^+(\zeta),$$

in which $q^-(\zeta)$ and $q^+(\zeta)$ denote the components of $q(\zeta)$ which obtain over the negative and positive ranges of ζ, respectively, and where

$$(19) \qquad \begin{aligned} M(q^-(-\zeta) \mid s) &= M(f_2^+(x) \mid 2 - s)M(f_1^-(-x) \mid s) \\ &\quad + M(f_2^-(-x) \mid 2 - s)M(f_1^+(x) \mid s), \\ M(q^+(\zeta) \mid s) &= M(f_2^+(x) \mid 2 - s)M(f_1^+(x) \mid s) \\ &\quad + M(f_2^-(-x) \mid 2 - s)M(f_1^-(-x) \mid s). \end{aligned}$$

In particular, if the x_i are identical even i.r.v.'s with p.d.f. $f(x)$, then $q(\zeta)$ is even, and

$$(20) \quad q(\zeta) = 2\left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta^{-s}\{M(f^+(x) \mid s)M(f^+(x) \mid 2 - s)\}\, ds\right], \quad \zeta > 0,$$

which was obtained by Epstein in [1].

4. The distribution of monomial products and quotients. Consider the random variable with a monomial p.d.f. of the form

$$f(x) = \begin{cases} (\alpha + 1)x^\alpha & \text{if } 0 \leqq x \leqq 1, \\ 0 & \text{otherwise,} \end{cases} \tag{21}$$

which for $\alpha = 0$ becomes the standardized rectangular p.d.f. The Mellin transform of the p.d.f. of the product $\eta = \prod_{i=1}^n x_i$ is

$$M(h(\eta) \mid s) = \left[\frac{(\alpha + 1)}{s + \alpha}\right]^n,$$

which upon inversion yields the p.d.f. of monomial products, i.e.,

$$\begin{aligned} h(\eta) &= \frac{(\alpha + 1)^n}{2\pi i} \int_{c-i\infty}^{c+i\infty} \eta^{-s}(s + \alpha)^{-n}\, ds \\ &= \frac{(\alpha + 1)^n}{(n - 1)!}\, \eta^\alpha \left(\log \frac{1}{\eta}\right)^{n-1} \quad \text{if } 0 \leqq \eta \leqq 1, \\ h(\eta) &= 0 \quad \text{otherwise.} \end{aligned} \tag{22}$$

The quotient $\zeta = x_1/x_2$ has the Mellin transform

$$M(q(\zeta) \mid s) = \frac{(\alpha + 1)^2}{(s + \alpha)(-s + \alpha + 2)},$$

whose inversion yields the p.d.f.

$$q(\zeta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(\alpha + 1)^2(\zeta)^{-s}}{(s + \alpha)(-s + \alpha + 2)}\, ds, \qquad -\alpha < c < \alpha + 2. \tag{23}$$

To obtain $q(\zeta)$ for $\zeta \leqq 1$ one may evaluate integral (23) using a closed contour of the form C_{La} (Fig. 1). By the theorem of residues [10, p. 112] the integral is equal to $2\pi i$ times the sum of the residues at the poles enclosed by the contour. Since the conditions of Jordan's lemma are satisfied [10, p. 115], [12] and the residue at the pole $s = -\alpha$ is $\{(\alpha + 1)/2\}\zeta^\alpha$, it follows that

$$q(\zeta) = \left(\frac{\alpha + 1}{2}\right) \zeta^\alpha, \qquad 0 < \zeta \leqq 1.$$

For $\zeta > 1$ one evaluates integral (23) using the closed contour C_{Ra} of Fig. 1. The same reasoning shows that

$$q(\zeta) = \left(\frac{\alpha + 1}{2}\right) \zeta^{-\alpha-2}, \qquad 1 \leqq \zeta < \infty,$$

since the conditions of Jordan's lemma are satisfied with regard to the contour C_{Ra} when $\zeta > 1$. Thus, the p.d.f. of the quotient of two monomial variables, each having the p.d.f. (21), is

$$q(\zeta) = \begin{cases} \left(\dfrac{\alpha + 1}{2}\right) \zeta^\alpha & \text{if } 0 \leqq \zeta < 1, \\ \left(\dfrac{\alpha + 1}{2}\right) \zeta^{-\alpha-2} & \text{if } 1 \leqq \zeta < \infty. \end{cases} \tag{24}$$

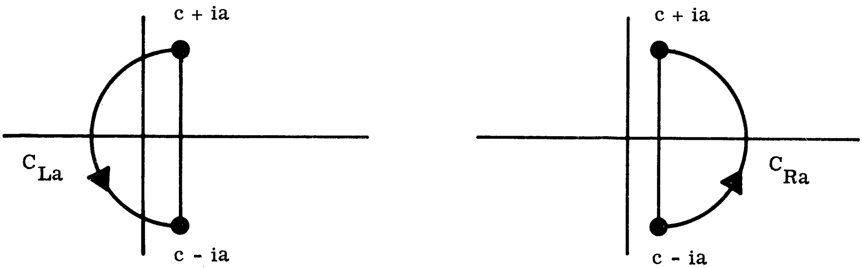

FIG. 1. *Contours utilized in evaluating integrals of the Bromwich type*

5. The distribution of Cauchy products and quotients. Consider the product of n Cauchy i.r.v.'s each having the p.d.f.

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \qquad -\infty < x < \infty.$$

Since $f(x)$ is an even function, $f^+(x) = f^-(-x)$, whose Mellin transform [9, p. 309, formula (11)] is

$$M(f^+(x) \mid s) = \frac{1}{2}\csc\left(\frac{\pi s}{2}\right), \qquad 0 < \text{Re}(s) < 2.$$

Thus, the p.d.f. $h(\eta)$ of the product $\eta = \prod_{i=1}^n x_i$ has the Mellin transform

$$M(h^+(\eta) \mid s) = 2^{n-1}\left[\frac{1}{2}\csc\left(\frac{\pi s}{2}\right)\right]^n, \qquad 0 < \text{Re}(s) < 2,$$

and the associated inversion integral is

$$(25) \qquad h(\eta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (\eta^{-2})^s \csc^n \pi s \, ds, \qquad 0 < c < 1.$$

To evaluate this integral, let $z = \eta^{-2}$ and note that

$$(26) \qquad \int_C z^s \csc^n \pi s \, ds = 2\pi i \sum_j R_j,$$

where C is taken to be C_{La} for $z > 1$ and C_{Ra} for $z < 1$, as $a \to \infty$, and where $\sum_j R_j$ denotes the sum of the residues at the poles enclosed by the contour C. From Jordan's lemma [10, p. 115], it follows [12] that integrals (25) and (26) are equivalent, so that the problem reduces to evaluation of the sum of these residues. It turns out that the functional form of $h(\eta)$ is the same for $\eta \geqq 1$ as for $\eta < 1$ which results from analytic continuation of the sum of the residues from either contour.

Applying Leibnitz's rule for the differentiation of products, one has

$$R(z,j,n) = z^s \sum_{k=0}^{n-1} \frac{1}{(n-1-k)!k!} (\log z)^{n-1-k} \frac{d^k}{ds^k} [(s-j) \csc \pi s]^n \Big|_{s=j}$$
$$= \frac{(-1)^{jn}}{\pi^n} z^{-j} \sum_{k=0}^{n-1} \frac{\pi^k}{(n-1-k)!k!} (\log z)^{n-1-k} \left\{ \frac{d^k}{ds^k} [s \csc s]^n \right\} \Big|_{s=0},$$
$$j = 0, 1, 2, \cdots,$$

so that summing the residues gives

$$(27) \quad h(z) = \frac{1}{\pi^n[1-(-1)^n z^{-1}]} \sum_{k=0}^{n-1} \frac{\pi^k}{(n-1-k)!k!} \cdot (\log z)^{n-1-k} \frac{d^k}{ds^k} [s \csc s]^n \Big|_{s=0}.$$

It can be shown [11, p. 144, formula (761)] that in terms of Bernoulli numbers

$$(28) \quad s \csc s = 1 + \sum_{m=0}^{\infty} \frac{2(2^{2m+1}-1)}{(2m+2)!} B_{m+1} s^{2m+2}, \qquad s^2 < \pi^2,$$

which when substituted in (27) gives the p.d.f. of η, namely,

$$(29) \quad h(\eta) = \frac{1}{\pi^n(1-(-1)^n\eta^2)} \sum_{k=0}^{n-1} \frac{\pi^k}{k!(n-1-k)!} \left[\log \frac{1}{\eta^2}\right]^{n-1-k} \cdot \frac{d^k}{ds^k} \left[1 + \sum_{m=0}^{\infty} \frac{2(2^{m+1}-1)}{(2m+2)!} B_{m+1} s^{2m+2}\right]^n \Big|_{s=0}.$$

Since [9, p. 307, formula (2)] $M(f_i(a_i s_i) \mid s) = a_i^{-s} M(f_i(x_i) \mid s)$, the p.d.f. of the product $\eta = \prod_1^n x_i$, where x_i has the nonstandard p.d.f. $f(x_i) = a_i/\pi(a_i^2 + x_i^2)$, $i = 1, 2, \cdots, n$, is obtained from (29) by replacing η by $\eta/\prod_{i=1}^n a_i$.

The various derivatives in (29) when evaluated give the following closed form expressions for the p.d.f.'s of products of n standard Cauchy variables, $n = 1, 2, \cdots, 10$. Aside from the case $n = 2$, these results are believed to be new.

$$n = 1: h(\eta) = \frac{1}{\pi(\eta^2+1)},$$

$$n = 2: h(\eta) = \frac{1}{\pi^2(\eta^2-1)} \log(\eta^2),$$

$$n = 3: h(\eta) = \frac{1}{2!\pi^3(\eta^2+1)} \{[\log \eta^2]^2 + \pi^2\},$$

$$n = 4: h(\eta) = \frac{1}{3!\pi^4(\eta^2-1)} \{[\log \eta^2]^3 + 4\pi^2 [\log \eta^2]\},$$

$$n = 5: h(\eta) = \frac{1}{4!\pi^5(\eta^2 + 1)} \{[\log \eta^2]^4 + 10\pi^2 [\log \eta^2]^2 + 9\pi^4\},$$

$$n = 6: h(\eta) = \frac{1}{5!\pi^6(\eta^2 - 1)} \{[\log \eta^2]^5 + 20\pi^2 [\log \eta^2]^3 + 64\pi^4 [\log \eta^2]\},$$

$$n = 7: h(\eta) = \frac{1}{6!\pi^7(\eta^2 + 1)} \{[\log \eta^2]^6 + 35\pi^2 [\log \eta^2]^4 + 259\pi^4 [\log \eta^2]^2 + 225\pi^6\},$$

$$n = 8: h(\eta) = \frac{1}{7!\pi^8(\eta^2 - 1)} \{[\log \eta^2]^7 + 56\pi^2 [\log \eta^2]^5 + 784\pi^4 [\log \eta^2]^3 + 2304\pi^6 [\log \eta^2]\},$$

$$n = 9: h(\eta) = \frac{1}{8!\pi^9(\eta^2 + 1)} \{[\log \eta^2]^8 + 84\pi^2 [\log \eta^2]^6 + 1974\pi^4 [\log \eta^2]^4 + 12916\pi^6 [\log \eta^2]^2 + 11025\pi^8\},$$

$$n = 10: h(\eta) = \frac{1}{9!\pi^{10}(\eta^2 - 1)} \{[\log \eta^2]^9 + 120\pi^2 [\log \eta^2]^7 + 4368\pi^4 [\log \eta^2]^5 + 52480\pi^6 [\log \eta^2]^3 + 147456\pi^8 [\log \eta^2]\}.$$

The p.d.f. of the quotient of two independent random Cauchy variables as obtained [9, p. 346, formula (20)] from (20) is

$$q(\zeta) = \frac{1}{\pi^2}\left(\frac{\log \zeta^2}{\zeta^2 - 1}\right)$$

and is identical to the p.d.f. of the product of two Cauchy variables.

6. The distribution of Gaussian products. In view of the fact that more statistical theory is focused upon the Gaussian distribution than any other, it seems rather surprising that the distribution of products of more than two independent random Gaussian variables has never been derived. For while the inversion of the Mellin transform of the p.d.f. of Gaussian products cannot be accomplished in closed form, it is quite amenable to evaluation by electronic computers, as will presently be shown.

Consider now the product of n Gaussian i.r.v.'s $\eta = \prod_1^n x_i$, where each x_i, $i = 1, 2, \cdots, n$, has the p.d.f.

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right),$$

whose Mellin transform [1] is

$$M(f^+(x) \mid s) = \frac{2^{(s-3)/2}}{\sqrt{\pi}} \Gamma\left(\frac{s}{2}\right).$$

The Mellin transform of the p.d.f. of η, $h(\eta)$, is therefore, $2^{n-1}\{M(f^+(x) \mid s)\}^n$, and the associated inversion integral is

$$h(\eta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \eta^{-s}\left\{\frac{2^{(s-1)/2}}{\sqrt{\pi}}\,\Gamma\left(\frac{s}{2}\right)\right\}^n ds, \qquad c > 0.$$

This is equivalent to

$$(30) \qquad h(\eta) = \frac{1}{(2\pi)^{n/2}}\cdot\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-s}\Gamma^n(s)\,ds, \qquad c > 0,$$

where $z = \eta^2 2^{-n}$ and the path of integration is a line parallel to the imaginary axis and to the right of the origin. The integral may be evaluated by contour integration to give

$$h(\eta) = \sum_{j=0}^{\infty}\frac{1}{(2\pi)^{n/2}}R(z, n, j), \qquad z > 0,$$

where $R(z, n, j)$ denotes the residue of $z^{-s}\Gamma^n(s)$ at the nth order pole $s = -j$, viz.,

$$R(z, n, j) = \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}\{(s+j)^n z^{-s}\Gamma^n(s)\}\Big|_{s=-j}, \qquad j = 0, 1, 2, 3, \cdots.$$

The problem, then, is to evaluate $\sum_j R(z, n, j)$. To accomplish this, it is convenient to apply Leibnitz's rule for the differentiation of products, which enables one to write

$$(31) \qquad \begin{aligned} R(z, n, j) = z^j \sum_{k=0}^{n-1}&\frac{1}{(n-1-k)!k!}(-\log z)^{n-1-k} \\ &\cdot\frac{d^k}{ds^k}\left\{\frac{\Gamma(s+j+1)}{s(s+1)\cdots(s+j-1)}\right\}^n\Big|_{s=-j}. \end{aligned}$$

The problem is thus reduced to the evaluation of the $n-1$ derivatives

$$\frac{d^k}{ds^k}\left\{\frac{\Gamma(s+j+1)}{s(s+1)\cdots(s+j-1)}\right\}^n, \qquad k = 0, 1, \cdots, n-1.$$

While a closed form expression has not been found for the kth derivative, each derivative when evaluated at $s = -j$ reduces to a closed form expression involving the Euler psi function and the Riemann zeta function, as will now be shown.

In evaluating the required derivatives, it will be convenient to utilize the following notation:

$$u(s, j, n) = \left[\frac{\Gamma(s+j+1)}{s(s+1)\cdots(s+j-1)}\right]^n;$$

$$u^{(k)}(s, n) = \frac{d^k}{ds^k}u(s, j, n);$$

$$v(s, j) = \psi(s+j+1) - \sum_{l=0}^{j-1}\frac{1}{s+l}, \quad \text{where} \quad v(-j, j) = \psi(j+1);$$

$\psi(s+j+1) = d \log \Gamma(s+j+1)/ds$ is the Euler psi function which for s an integer [11, pp. 206–208] becomes

$$\psi(s+j+1) = -C + \sum_{l=0}^{s+j-1} \frac{1}{s+j-l}$$
$$= -C - \sum_{\alpha=0}^{\infty} \left(\frac{1}{s+j+1+\alpha} - \frac{1}{\alpha+1} \right),$$

where C is Euler's constant;

$$v^{(k)}(s, n) = \frac{d^k}{ds^k} v(s, j), \quad \text{with}$$

$$v^{(k)}(-j, n) = \begin{cases} (-1)^{k+1} k! \zeta(k+1, j+1) & \text{if} \quad k > 0, k \text{ even}, \\ k! \zeta_1(k+1, j+1) & \text{if} \quad k > 0, k \text{ odd}, \end{cases}$$

where

$$\zeta(\beta, a) = \sum_{\alpha=0}^{\infty} \frac{1}{(a+\alpha)^\beta}$$

is the Riemann zeta function [11, p. 212, formula (1103)], and for convenience

$$\zeta_1(\beta, a) = \zeta(\beta, a) + 2 \sum_{\alpha=1}^{a-1} \frac{1}{\alpha^\beta}.$$

Finally,

$$\psi^{(k)}(s+j+1) = \frac{d^k}{ds^k} \psi(s+j+1)$$
$$= (-1)^{k+1} k! \zeta(k+1, s+j+1), \qquad k > 0$$

The derivatives $u^{(k)}(s, n), k = 1, 2, \cdots$, are expressible in terms of $u(s, j, n)$, powers of $v(s, j)$, and derivatives of $v(s, j)$, which upon evaluation at $s = -j$ give $u^{(k)}(-j, n)$ in terms of the Euler psi function and the Riemann zeta function. To indicate how these results develop, note that

$$u^{(1)}(s, n) = n\{(s+j)\Gamma(s)\}^n \frac{d}{ds} \log \left(\frac{\Gamma(s+j+1)}{s(s+1) \cdots (s+j-1)} \right)$$
$$= nu(s, j, n) \left\{ \psi(s+j+1) - \sum_{k=0}^{j-1} \frac{1}{s+k} \right\}$$
$$= nuv,$$

which when evaluated at $s = -j$ gives

$$u^{(1)}(-j, n) = \frac{(-1)^{jn}}{(j!)^n} \{n\psi(j+1)\}.$$

Similarly,

$$u^{(2)}(-j, n) = \frac{d}{ds}\{nuv\}\Big|_{s=-j}$$

$$= \frac{(-1)^{jn}}{(j!)^n}\{n^2\psi^2(j+1) + n\zeta_1(2, j+1)\}.$$

Continuing this procedure, one obtains, after some algebraic simplification [12], $u^{(k)}(-j, n)$, $k = 1, 2, \cdots$, in terms of the Euler psi function and the Riemann zeta function. Specifically,

$$u^{(3)}(-j, n) = \frac{(-1)^{jn}}{(j!)^n}\{n^3\psi^3(j+1) + 3n^2\psi(j+1)\zeta_1(2, j+1) - 2n\zeta(3, j+1)\},$$

$$u^{(4)}(-j, n) = \frac{(-1)^{jn}}{(j!)^n}\{n^4\psi^4(j+1) + 6n^3\psi^2(j+1)\zeta_1(2, j+1) - 8n^2\psi(j+1)\zeta(3, j+1) + 3n^2\zeta_1^2(2, j+1) + 6n\zeta_1(4, j+1)\},$$

$$u^{(5)}(-j, n) = \frac{(-1)^{jn}}{(j!)^n}\{n^5\psi^5(j+1) + 10n^4\psi^3(j+1)\zeta_1(2, j+1) - 20n^3\psi^2(j+1)\zeta(3, j+1) + 15n^3\psi(j+1)\zeta_1^2(2, j+1) + 30n^2\psi(j+1)\zeta_1(4, j+1) - 20n^2\zeta_1(2, j+1)\zeta(3, j+1) - 24n\zeta(5, j+1)\},$$

$$u^{(6)}(-j, n) = \frac{(-1)^{jn}}{(j!)^n}\{n^6\psi^6(j+1) + 15n^5\psi^4(j+1)\zeta_1(2, j+1) - 40n^4\psi^3(j+1)\zeta(3, j+1) + 45n^4\psi^2(j+1)\zeta_1^2(2, j+1) - 120n^3\psi(j+1)\zeta_1(2, j+1)\zeta(3, j+1) + 90n^3\psi^2(j+1)\zeta_1(4, j+1) + 15n^3\zeta_1^3(2, j+1) - 144n^2\psi(j+1)\zeta(5, j+1) + 90n^2\zeta_1(2, j+1)\zeta_1(4, j+1) + 40n^2\zeta^2(3, j+1) + 120n\zeta_1(6, j+1)\}.$$

It has now been shown that the derivatives $u^{(k)}(-j, n)$ required to evaluate the residues $R(z, n, j)$ can be expressed in terms of the Euler psi function and the Riemann zeta function. Explicit expressions have been given above for $k \leqq 6$, which when substituted in (31) enable one to

TABLE 1.

Distribution of Gaussian products, $n = 2$

η	$h(\eta)$	$\int_0^\eta h(\eta)\, d\eta$	η	$h(\eta)$	$\int_0^\eta h(\eta)\, d\eta$
10^{-20}	14.695614	.000000	.3600	.384370	.248024
.0001	2.968644	.000329	.3800	.369105	.255557
.0010	2.235710	.002554	.4000	.354766	.262794
.0050	1.723422	.010209	.4200	.341265	.269753
.0100	1.502819	.018211	.4400	.328526	.276450
.0150	1.373805	.025381	.4600	.316482	.282899
.0200	1.282298	.032010	.4800	.305074	.289114
.0250	1.211348	.038238	.5000	.294252	.295106
.0300	1.153406	.044145	.5200	.283969	.300887
.0350	1.104442	.049786	.5400	.274187	.306468
.0400	1.062054	.055201	.5600	.264867	.311858
.0500	.991292	.065454	.6000	.247493	.322100
.0600	.933565	.075069	.6400	.231624	.331677
.0700	.884844	.084155	.6800	.217077	.340647
.0800	.842722	.092788	.7200	.203699	.349059
.0900	.805648	.101026	.7600	.191361	.356957
.1000	.772560	.108914	.8000	.179956	.364380
.1100	.742702	.116488	.8400	.169387	.371364
.1200	.715515	.123777	.8800	.159576	.377941
.1300	.690574	.130806	.9200	.150450	.384140
.1400	.667549	.137595	.9600	.141948	.389986
.1500	.646178	.144163	1.0000	.134016	.395503
.1600	.626251	.150524	1.1000	.116375	.407998
.1700	.607589	.156692	1.2000	.101384	.418866
.1800	.590055	.162679	1.3000	.088569	.428347
.1900	.573527	.168496	1.4000	.077558	.436640
.2000	.557903	.174153	1.5000	.068057	.443909
.2100	.543096	.179657	1.6000	.059828	.450293
.2200	.529030	.185017	1.7500	.049459	.458463
.2300	.515642	.190240	2.0000	.036254	.469085
.2600	.479006	.205147	2.5000	.019846	.482676
.2800	.457092	.214505	3.0000	.011058	.490180
.3000	.436868	.223442	4.0000	.003552	.496770
.3200	.418118	.231990	6.0000	.000396	.499630
.3400	.400667	.240176	8.0000	.000044	.499954

TABLE 2.

Distribution of Gaussian products, $n = 3$

η	$h(\eta)$	$\int_0^\eta h(\eta)\,d\eta$	η	$h(\eta)$	$\int_0^\eta h(\eta)\,d\eta$
10^{-20}	271.50380	.000000	.30	.383119	.298732
10^{-10}	68.504805	.000000	.32	.359647	.306155
.000004	20.327126	.000095	.34	.338447	.313133
.000100	11.339660	.001398	.36	.319203	.319706
.0010	6.525049	.008578	.38	.301658	.325912
.0050	3.959295	.028017	.40	.285598	.331782
.0070	3.506025	.035452	.44	.257248	.342624
.0100	3.056922	.045248	.48	.233032	.352417
.0130	2.747179	.053930	.52	.212126	.361310
.0160	2.514422	.061807	.56	.193918	.369423
.0200	2.276474	.071367	.60	.177936	.376853
.0250	2.051142	.082160	.66	.157358	.386894
.0300	1.876390	.091962	.72	.140046	.395801
.0340	1.761291	.099231	.80	.120899	.406213
.0400	1.617745	.109353	.90	.101745	.417307
.0460	1.499629	.118695	1.00	.086523	.426692
.0520	1.400066	.127386	1.10	.074222	.434709
.0580	1.314570	.135524	1.20	.064144	.441611
.0640	1.240063	.143183	1.30	.055790	.447595
.0700	1.174349	.150422	1.40	.048797	.452814
.0800	1.080152	.161680	1.50	.042892	.457390
.0900	1.000754	.172074	1.70	.033566	.464988
.1000	.932649	.181733	2.00	.023855	.473499
.1100	.873407	.190757	2.50	.014231	.482770
.1200	.821278	.199225	3.50	.005782	.492040
.1300	.774966	.207201	4.50	.002634	.496014
.1400	.733484	.214740	5.00	.001831	.497116
.1500	.696067	.221885	5.50	.001294	.497889
.1600	.662111	.228673	6.00	.000927	.498438
.1800	.602729	.241304	6.50	.000672	.498834
.2000	.552423	.252842	7.00	.000492	.499123
.2200	.509184	.263448	8.00	.000271	.499493
.2400	.471572	.273247	10.00	.000090	.499819
.2600	.438528	.282341	12.00	.000033	.499931
.2800	.409250	.290813	14.00	.000015	.499974

TABLE 3.

Distribution of Gaussian products, $n = 6$

η	$h(\eta)$	$\int_0^\eta h(\eta)\,d\eta$	η	$h(\eta)$	$\int_0^\eta h(\eta)\,d\eta$
10^{-20}	233,817.16	.000000	.0200	3.627986	.221946
.0000050	385.97978	.002941	.0250	2.977812	.238356
.0000250	202.42522	.008202	.0300	2.521872	.252044
.0000500	149.33515	.012496	.0350	2.183538	.263768
.0001000	108.16438	.018771	.0400	1.922120	.274006
.00020	76.750149	.027764	.0450	1.713906	.283078
.00030	62.135358	.034638	.0500	1.544086	.291209
.00040	53.214044	.040375	.0600	1.283716	.305275
.00050	47.040591	.045371	.0700	1.093512	.317115
.00075	37.323983	.055797	.0800	.948589	.327295
.00100	31.481936	.064346	.0900	.834613	.336189
.00120	28.184727	.070297	.1100	.667160	.351098
.00140	25.623368	.075668	.1300	.550450	.363210
.00160	23.562278	.080580	.1500	.464786	.373321
.00180	21.859357	.085117	.1700	.399466	.381936
.00200	20.423149	.089341	.2000	.326496	.392764
.00250	17.636834	.098816	.2500	.245108	.406895
.00300	15.600026	.107101	.3000	.192210	.417742
.00350	14.033193	.114494	.3500	.155477	.426383
.00400	12.783225	.121187	.4000	.128730	.433455
.00500	10.900304	.132973	.5000	.092872	.444384
.00600	9.537884	.143159	.6000	.070363	.452466
.00700	8.499025	.152156	.7000	.055203	.458698
.00800	7.676717	.160229	.8000	.044462	.463653
.00900	7.007300	.167560	1.0000	.030555	.471027
.01000	6.450306	.174281	1.5000	.014750	.481670
.01200	5.573489	.186260	2.5000	.005352	.490638
.01400	4.911820	.196717	5.0000	.001104	.496922
.01600	4.392883	.206002	10.0000	.000171	.499238
.01800	3.973934	.214355	17.0000	.000033	.499793
			20.0000	.000019	.499868
			25.0000	.000008	.499931

evaluate the residues numerically for $n \leqq 7$. This allows the tabulation of the p.d.f. and the distribution function for products of up to seven independent Gaussian variables. Such tables have been constructed for $n = 2, 3, 6$ at the General Motors Defense Research Laboratories, and an abridged version is given in Tables 1, 2, 3.

In the case $n = 2$, the results give the modified Bessel function of order zero, as is known [1]. For values of $n > 2$ the results are believed to be new.

For completeness and convenience of application, the first six derivatives $u^{(k)}(s, n)$ are given explicitly above, and it is seen that they become quite involved for large k. In practice this tedious expansion need not be carried out explicitly, since the iterative procedure has been programmed for a digital computer and will supply the expansion for arbitrary k. This program may be incorporated directly into an overall computer mechanization [13].

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