Probabilistic Equilibrium: A Review on the Application of MAXENT to Macroeconomic Models

Paulo Hubert and Julio M. Stern

Abstract The concept of equilibrium is central to many macroeconomic models. However, after the 2008 crisis, many of the most used macroeconomic models have been subject to criticism, after their failure in predicting and explaining the crisis. Over the last years, a response to this situation has been the proposal of new approaches to the study of macroeconomical systems, in particular, with the introduction of thermodynamics and statistical physics methods. In this paper, we offer a brief review of the application of the maximum entropy framework in macroeconomics, centered around the different interpretations of the equlibrium concept.

Keywords Maximum entropy · Macroeconomy · Equilibrium

1 Introduction

The classical example of equilibrium in dynamical systems comes from mechanics: the pendular system, composed of a point mass suspended by a chord, attached to a fixed platform. Put on movement by an initial impulse, the system, in the absence of attrition, enters an equilibrium state of perpetual and periodical motion. When attrition enters the picture, the system dissipates energy to the surroundings, and the new equilibrium is one of rest: the point mass evolves to the least-energy state, and remains there unless some new action is imposed upon it.

Many states of equilibrium in mechanics are states in which the system repeats itself, or stays at rest. These are situations in which our description of the system dynamics can dispense of an infinite time axis: all possible configurations present

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themselves during a limited time interval. Therefore, it is much easier to make predictions about the future (or draw conclusions about the past) of a system in a state of equilibrium. Even further: once the system's singularities are known (for mechanical systems, these are the solutions to the equation $\dot{x} = 0$), the theory of differential equations allows one to consider the limiting behavior of a system (either forwards or backwards in time) in terms of its relation with these stationary points.

Macroeconomic theory has for a long-time drawn inspiration from mechanics and its methods [1]. Therefore, the concepts of equilibrium and limiting behavior of a system are central in many macroeconomical models. However, as we intend to argue, this is not a mere consequence of the use of rational mechanics' methods: equilibrium states are objects with an intrinsic epistemological interest. One important issue that arises, then, is the ontological interpretation one gives to the equilibrium and its attainment as a limiting behavior of the system's trajectories.

In this paper, we offer a brief review of the use of equilibrium concepts both in classical and contemporary macroeconomics, and rely on the relationship between rational mechanics and economy to discuss the recent application of statistical mechanical methods to macroeconomics. We argue, in the spirit of Kuhn, that macroeconomical science is in a state of exploration after a paradigm crisis, and analyze briefly some of the conceptual distinctions between classical and statistical equilibrium.

2 Equilibria in Classical Economics

Perhaps the most concrete example of the analogy between economical and mechanical systems is the Phillips' machine, or *MONIAC (Monetary National Income Analogue Computer)* (Fig. 1). It is an analogical computer that uses water flows to model the dynamic of income in an economy. The MONIAC, officially presented at the London School of Economics in 1949, was able to solve systems of up to nine simultaneous equations, with parametrizable coefficients [2, 3]. Once a dynamical equilibrium was reached the solution could be read in scales attached to the water tanks.

Regardless of the method of computation, however, economists are in general very fond of equilibria. We find mention of the term already in the seminal work of Debreu [4], in which he proposes a formalization of economical analysis based on the axiomatic method. There, he defines equilibrium in the following terms:

> *If the actions x_i, y_i satisfy the market equilibrium equality $x - y = w$, the economy is in equilibrium, i.e., every agent, given the price system and the actions of other agents, has no incentive to choose a different action, and the state of the economy is a market equilibrium. (Debreu* [4], *p. 79)*

Equilibrium is thus a state of affairs, a situation (described by a set of values for economic variables, plus a price system relating them) in which no agent has

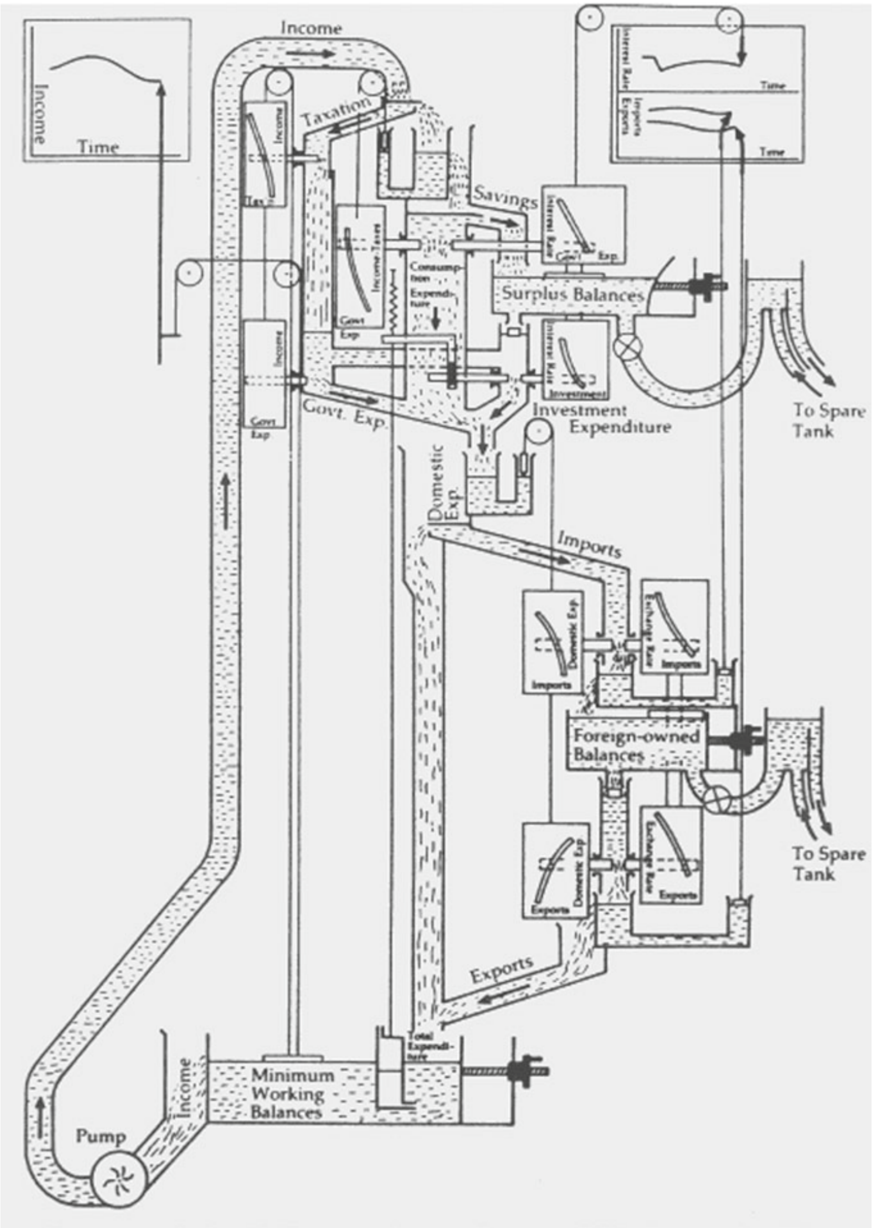

Diagram of the Phillips machine. Source: LSE Quarterly, Winter 1988, Nick Barr.

Fig. 1 Schematic drawing of the MONIAC [2] any incentive to alter its decisions and economical activities (i.e., a Pareto optimal situation). Economical agents here are optimizers: they act in the search of a certain maximum, and equilibrium arises when no individual can improve its cost function without leaving the feasible set (given by the constraints of common economical life).

Another aspect of Debreu's equilibrium is the market equality. This equation defines equilibrium as a condition about the aggregates of the economy, in a precise (sharp) form: it constrains the total values for the variables, and the individuals, in their search for optimal utility points, can only move over the hypersurface given by market equilibrium.

After Debreu, another influent writer in macroeconomy was Samuelson. He also proposes a formalization of economic analysis founded in the methods of mathematics, and uses the concept of equilibrium many times in his work. One of such uses is the following:

> *This, in brief, is the method of comparative statics, meaning by this the investigation of changes in a system from one position of equilibrium to another without regard to the transitional process involved in the adjustment. (Samuelson* [5]*, p. 8)*

Samuelson then adds:

> *By equilibrium is meant here only the values of variables determined by a set of conditions, and* ***no normative connotation attaches to the term****. (Samuelson* [5]*, p. 8, emphasis is ours)*

The caution exerted by Samuelson is notable. Even though he treats equilibrium states as objects of economical analysis (*investigation of changes (...) from one position of equilibrium to another*), he makes a conscient effort to avoid committing ontologically to this concept, treating equilibria as objects of an abstract (mathematical) nature. Besides avoiding ontological commitment, he also explicitly refuses any *normative connotation* to economic equilibrium.

This is, however, a difficult *desideratum* to be kept. Economical science is often burdened with the task of not only explaining reality, but also building it. Economists are perhaps the members of the scientific community that are most engaged in the administration of actual institutions, be it national states or companies. As such, and with the broad adoption of equilibrium models [6] in modern macroeconomics, it is tempting to assign a much more concrete status to equilibrium states than was desired by Samuelson. It is tempting, as well, to guide policy formulation in the direction of the fulfillment of the model's hypothesis (the case for deregulation of markets is perhaps one example of this temptation).

However, even though some of the recent criticisms of so-called classical models direct their attacks precisely to the equilibrium concept [7, 8], some of the alternative frameworks proposed in the current literature still have it as a central epistemic notion. The reason for this attachment can perhaps be found in the epistemological analysis of Foerster and Luhmann.

3 Equilibria as Objects of Knowledge

In the work of Foerster [9], we find the following famous quotation about *eigenvalues*:

> *Eigenvalues have been found ontologically to be discrete, stable, separable, and composable, while ontogenetically to* ***arise as equilibria*** *that determine themselves through circular processes. (Foerster* [9], *p. 266, emphasis is ours)*

In his paper, Foerster analysis the "organization of sensorimotor interactions" of a cognoscent being with its environment. He proposes a model consisting of the alternate and recursive application of the operators *observation* and *coordination*: observation of new data triggers a coordination (behavior), which leads to a new observation and so forth. His eigenvalues (arising as equilibria) are the very objects of knowledge, coming to be in the interaction between being and environment.

A radical application and enlargement of Foerster's ideas can be found in the work of Luhmann [10]. He considers hierarchical models of recursive systems in a ladder of growing complexity, and uses this framework to describe not only the relation of one individual with its environment, but the very organization of human societies. In this same spirit, he analyzes the scientific endeavor by describing the collective work of scientists as a subsystem, horizontally differentiated from the broader system of human society. Successful scientific theories, then, would emerge as *eigenvalues* of a self-referent, recursive, dynamic system; theories, as collective objects of knowledge, share the same nature of Foerster's objects of understanding: they are also equilibria, and as such stable, discrete, limit states of a recursive process.

Under the light of Foerster and Luhmann's interpretation of the knowledge-building process, it becomes clearer why equilibrium states are useful objects inside theories. As eigenvalues of the studied system, these states are discrete, stable, separable; they are therefore much easier to name, classify, and study than the whole dynamics of the system. Equilibrium methods, in this sense, work like traps with which we take hold of a system of interest, in order to be able to describe it, make predictions about its future and inferences about its past.

This description of scientific theories introduces a strong recursive, self-referent aspect in epistemology. Equilibrium states and their attainment are objects of many sciences (physics and economics in particular), but in this framework, they also describe the process of scientific enquiry in itself. In this sense, when speaking about the nature of equilibria and limiting behavior, science is also talking about itself and its objects. In the very words of Luhmann:

> *The concept of self-referential systems can and must subsume science and one's own research. This requires taking leave of ontological metaphysics and apriority. Systems with built-in reflection are forced to forgo absolutes. And if science discovers this fact in the domain of its objects, the fact holds irrefutably for science, too (Luhmann* [10], *p. 485).*

This self-reference can also be found in the interpretation of statistical methods in physics, according to Jaynes [11]. For him, the maximum entropy solution represented the subjective (i.e., the scientist's) solution to an inference problem. However, it agrees with all measurements made in actual, thermodynamical systems in states of equilibrium (i.e., with the objective solution). Entropy, besides being interpreted as a measure of a system's quality, can also be interpreted as a measure of the scientist's knowledge about the system. What holds for the object, holds for the scientist as well.

4 The Change from One State of Equilibrium to Another

According to Kuhn, science advances in a dual fashion: continuously, in the periods of what he calls *normal science*, and discontinuously, when it is subject to a *paradigm shift* [12]. A paradigm, in this epistemology, is a scientific realization with two properties: if offers sufficiently unprecedented results, in order to attract an enduring group of participants and form a delimited (discrete) and stable research group. At the same time, it is sufficiently open to contain many unsolved problems in which this group can work (recursively), feeding itself in its own questions.

When normal science is taking place, Kuhn identifies two kinds of events: inventions and discoveries. The inventions are identified with *puzzle-solving* activities: new applications are developed, empirical evidence is accumulated, minor problems are solved (the ones that are not urgent, for their lack of solution is not enough to provoke a crisis).

Discovery, on the other hand, is associated to more extreme movements, caused by the presence of an *anomaly*:

> *Discovery begins with the awareness of an anomaly, that is, with the recognition that nature has in some way violated the paradigmatic expectations governing normal science. What follows is a more or less broad exploration of the area in which the anomaly has occurred. This work only stops after the paradigm's theory has been adjusted, in such a way that the anomalous has now turn into the expected. The assimilation of a new fact demands more than an additive adjustment of the theory. Until such an adjustment is completed - until the scientist has learnt to see nature in a different way, the new fact will not be considered completely scientific (Kuhn* [12]*, p. 78, free translation from the Brazilian edition).*

In the realm of macroeconomy, the most recent and important anomaly was the financial crisis of 2007. The crisis, and subsequent depression, escaped entirely the models' predictions, and even their power of *post factum* explanation [13–16].

The economists reacted immediately: a thorough and deep theoretical exploration of monetary and financial systems (where it is consensual that the crisis originated) was launched. New approaches for economical modeling were proposed (Stock-flow consistent models, Agent-based models), ideas of less orthodox thinkers were revisited (maybe the most prominent example is the work of Minsky [17]), and methodological discussions were sharpened.

Olivier Blanchard, IMF's chief economist, makes a clear point about the necessity for theoretical exploration in postcrisis macroeconomics [18]:

> *Turning from policy to research, the message should be to let a hundred flowers bloom. Now that we are more aware of nonlinearities and the dangers they pose, we should explore them further theoretically and empirically—and in all sorts of models. This is happening*

> *already, and to judge from the flow of working papers since the beginning of the crisis, it is happening on a large scale. Finance and macroeconomics in particular are becoming much better integrated, which is very good news. (Blanchard 2014)*

It seems reasonable, then, to describe the current moment of macroeconomical thinking in Kuhn's terms, and to say that macroeconomics is now going through a paradigm crisis: the change from one state of equilibrium to another. In these moments, as Kuhn points out (and Blanchard apparently agrees), it is fruitful to explore new methodological possibilities.

The application of maximum entropy methods is one of these possible explorations, one that is becoming frequent in the macroeconomics literature. In the next section, we briefly review a few papers on the subject.

5 The Statistical Equilibrium in Macroeconomics

We begin by analyzing the paper by Foley [19], which presents a direct application of the maximum entropy principle. Foley defines a space of transactions, in an economy with a certain (finite) number of goods. A transaction is a point in the space of goods, where each coordinate can represent demand (if it has a negative value) or supply (if it has a positive value) for that particular good. Agents are divided into categories, each category defined by a supply set, representing the totality of transactions which are at the same time feasible and desirable for agents belonging to that group.

He then defines an average excess demand measure, and by constraining this quantity to be 0 (i.e., applying the idea of market equilibrium) he obtains the maximum entropy distribution of agents inside each supply set. The statistical equilibrium he obtains (i.e., his maximum entropy distribution) is thus associated with the usual market equilibrium (zero excess demand), but instead of constraining the *total* excess demand to be 0, he constrains the *expected* excess demand to be 0, where this expectation is taken with respect to the MAXENT distribution.

Foley also points that the statistical equilibrium usually is not Pareto efficient. In other words, in an economy in equilibrium it is possible to find transactions between two or more agents that can improve both of their utility values. In a pure exchange model, the entropy associated to the Pareto equilibrium would be null (assuming convex utility functions), because all agents will be concentrated at the minimum cost point over the hypersurface of constant utility. In the statistical equilibrium model, however, agents can spread all along this hypersurface, even though with greater concentration around the minimum cost. In other words, the statistical equilibrium allows for horizontal inequality, even between individuals from the same class (i.e., agents with the same supply set), and, even further, an endogenous inequality that arises even between agents with identical initial resource allocation.

About this new possibility of horizontal inequality, he says:

> *The statistical equilibrium theory of markets is methodologically less ambitious than Walrasian competitive equilibrium theory. Walrasian theory seeks to predict the actual market*

> *outcome for every individual agent, while the statistical approach seeks only to characterize the equilibrium distributions of agents over outcomes, without predicting the fate of specific agents. (Foley* [19] *pp.343–344).*

Another recent work applying thermodynamic methods to macroeconomics is the paper by Caticha and Golan [20]. In their model, there are a finite number of goods, and a finite number of agents. Goods can be seen as production and consumption goods simultaneously; an agent can be a producer and/or a consumer of any particular good. Each agent has an utility function that models its relative interest for different mixtures of goods. To each microstate of the economy (the specification of each agent's consumption and production functions, alongside with its utility function), there corresponds a macrostate: the total amount produced and consumed of each good, and the total utility of each agent. Again, as in Foley's work, the macrostates constraints are taken to be expected values, instead of exact, aggregate quantities.

By imposing an expected market closure constraint, a budget constraint on the agents, and fixating each agent's expected utility, they obtain the MAXENT distribution on the space of agents and goods, and use it to analyze the dynamics of an economy in state of (statistical) equilibrium.

Another line of application of the MAXENT framework has been the labor market. The paper published in Brazil by Soromenho [21], for one example, models the economy as composed of two kinds of agents: workers and firms. The goods of his economy are labor and currency, and the transactions considered are the exchange of a worker's labor for a firm's money. Workers can either be unemployed, or receive a wage for one unit of work. Firms have a wage budget that must be entirely spent. The wages are not fixed, meaning that, with a given fixed budget, each firm can hire more or less workers, paying a smaller or bigger wage respectively.

By imposing a viability condition (according to which the number of workers receiving wage i is the same number of workers employed by the firms with this same wage), the author obtains a MAXENT distribution. His method up to this point is quite similar to the methods we previously described. He differs, however, in the use he gives to the equilibrium distribution: he uses it to analyze a classical (nonstatistical) kaleckian model for a single good economy. He concludes that it is possible to replace the usual closure of the kaleckian model (exogenous markup for the wages, uniform distribution of employment between firms) by expected values for the same quantities (average wage, MAXENT distribution of employment) calculated with the MAXENT distribution, and still obtain the same results of the usual kaleckian model.

When applied to either the labor or the goods market, the maximum entropy framework is based on the idea of distributing a fixed quantity among individuals. The paper of Banerjee and Yakovenko [22] discuss the method precisely in this terms: in their paper, the economy is defined with three resources: money, income, and energy. The microstates of the economy are the sets of (either) goods in possession of each agent. Transactions occur between pairs of agents, and consist of the exchange of a constant amount of resources.

Agents are divided in classes, and the constraints are of a constant number of agents, and constant total amount of resources. With these constraints, the authors obtain the equilibrium distribution. Here, the statistical equilibrium receives a clearly ergodic interpretation:

> *After many transactions between different agents, we expect that a stationary probability distribution of money would develop in the system. (Banerjee and Yakovenko* [22], *p. 4)*

After obtaining the equilibrium distribution, the authors use it to analyze the situation in which two economies, with different equilibrium "temperatures" (which translates to different resource's stocks *per capita*), start to interact. By postulating the validity of the second law of thermodynamics, they conclude that the flow of resources must be from the richer to the poorer countries.

5.1 Conclusion

The papers (very) briefly reviewed in the last section allow us to draw in broad lines what is the statistical equilibrium method, as applied to macroeconomics.

First of all, it begins with the modeling of individuals: a set of phase variables is chosen, and a model for the interactions between agents (exchanging of values for the phase variables) is proposed. From the start, it is recognized that *individual variability* might exist, even in equilibrium, and even between individuals which have exactly equal initial conditions. This recognition is the natural consequence of having a *probabilistic* solution for the model: a probability distribution over the possible microstates.

After the model for microstates is developed, in order to obtain a solution it is necessary to incorporate knowledge about macrostates as well. In physics, this is done by postulating conservation laws; in macroeconomy, the analogous idea is that of *market closure* (equality of supply and demand levels). This kind of constraint was used since the early works of Debreu and Samuelson; the main difference is that, in the statistical equilibrium framework, markets close only in *expected values*. This opens the possibility that real economies working outside market closure can still be analyzed by equilibrium methods. In the classical theory, at least in principle, the closure is a logical necessity in equilibrium, and any deviance from that breaks the model altogether (the recognition of this fact might have been one of the reasons that motivated Samuelson to insist upon the abstract nature of equilibrium solutions).

But besides these two points, we believe that the adoption of statistical equilibrium methods in macroeconomics can have another epistemological impact. This might be the case because, in the words of Blanchard [18]:

> *The techniques we use affect our thinking in deep and not always conscious ways. This was very much the case in macroeconomics in the decades preceding the crisis. The techniques were best suited to a worldview in which economic fluctuations occurred but were regular, and essentially self-correcting. The problem is that we came to believe that this was indeed the way the world worked. (Blanchard, 2014)*

In most papers we analyzed, statistical equilibrium receives an interpretation which can be linked to the ergodic school of thought in statistical mechanics [23]. Equilibrium is a concrete state of affairs, one that ought to be reached by the real economy, given that the interactions modeled are allowed to continue for a sufficient long period of time, and as long as the model assumptions are satisfied. The ergodic interpretation is then a very normative one, in the sense that it precognizes that, if the transactions and elements of the economy are such and such, the *free course of the economy will lead* to this and this situation. Uncertainty only enters the picture because individuals are unpredictable and do not exactly behave as the model says they do; if the economy is free to evolve for a sufficient long time, this unpredictability will "cancel out" and the predictions will be fulfilled.

In the subjective interpretation, on the other hand, statistical equilibrium solutions are understood as the most conservative (maximum entropy) probabilistic models for an economy, given the macro constraints we expect to be satisfied. It is explicitly a *tool* used by the scientist to describe what he believes would be the case if his description of the microstates is accurate and if his expectations about aggregates hold at least approximately. The uncertainty, in this case, is assumed *by the scientist*; he will be talking (probabilistically) about the current state and nature of a system, and not about its potential infinite evolution. Equilibrium solutions lose their normative status, as Samuelson wanted; or rather they cease to have a normative status over the real economies, to receive a normative status over the scientist's work.

In this sense, the subjective interpretation can work as a safeguard against the epistemological risk that Blanchard points out: the probabilistic model, derived as an equilibrium, will still be useful (and used) as a scientific and decision-making tool. However, the uncertainty of its predictions will be associated with the lack of complete knowledge about the system. Equilibrium will not be an ideal situation the economy might reach in the long run, but a practical state of affairs reached by macroeconomical science in the investigation of real economies. Knowing that, if the scientist again comes to believe that "this is indeed the way the world works," he might be not very much distant from the truth.

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References

1. Mirowsky, P.: More Heat Than Light. Cambridge University Press, Cambridge (1995)
2. Swade, D.: Computer ressurrection: the bulletin of the Computer Conservation Society. The Phillips Economic Computer. http://www.cs.man.ac.uk/CCS/res/res12.htm. Accessed 13 Dec 2016
3. https://www.theguardian.com/business/2008/may/08/bankofenglandgovernor.economics. Accessed 13 Dec 2016
4. Debreu, G.: Theory of Value: An Axiomatic Analysis of Economic Equilibrium, 4th edn. Yale University Press, London (1971)

5. Samuelson, P.: Foundations of Economic Analysis, 9th edn. Cambridge Harvard University Press, Cambridge (1971)
6. Tovar, C.E.: BIS Working paper no. 258
7. Dixon, H., Creedy, J. (eds.): The foundations of economic thought. Equilibrium and Explanation. Blackwells, Oxford (1990)
8. Mirowsky, P.: Louvain Economic Review **55**(4), pp. 447–468
9. von Foerster, H.: Understanding Understanding: Essays on Cybernetics and Cognition. Springer, New York (2003)
10. Luhmann, N.: Social Systems. Stanford University Press, Stanford (1995)
11. Jaynes, E.T.: The Physical Review **106**(4), pp. 620–630
12. Kuhn, T.: A Estrutura das Revoluções Científicas, 9th edn. Perspectiva, São Paulo (2007)
13. https://www.gpo.gov/fdsys/pkg/CHRG-111hhrg57604/pdf/CHRG-111hhrg57604.pdf. Accessed 15 Dec 2016
14. Kocherlakota, N.: https://www.minneapolisfed.org/publications/the-region/modern-macroeconomic-models-as-tools-for-economic-policy. Accessed 15 Dec 2016
15. Blanchard, O.: https://piie.com/system/files/documents/pb16-11.pdf. Accessed 15 Dec 2016
16. Krugman, P.: http://krugman.blogs.nytimes.com/2016/08/12/the-state-of-macro-is-sad-wonkish/. Acessed 15 Dec 2016
17. Wray, L.R., Tymoigne, E.: The Levy Economics Institute, Working paper 543
18. Blanchard, O.: https://www.weforum.org/agenda/2014/10/olivier-blanchard-financial-crisis-macroeconomics/. Accessed 15 Dec 2016
19. Foley, D.K.: Journal of Economic Theory **62**, pp. 321–345
20. Caticha, A., Golan, A.: Physica A **408**, pp. 149–163
21. Soromenho, J.E.C.: Revista EconomiA **12**(3), pp. 407–425
22. Banerjee, A., Yakovenko, V.M.: New Journal of Physics **12**
23. Guttman, Y.M.: The Concept of Probability in Statistical Physics. Cambridge University Press, Cambrige (1999)