Logical Hexagons of Statistical Modalities:
The Problem of Induction – Solved!
+Evolution of Science and its Logic

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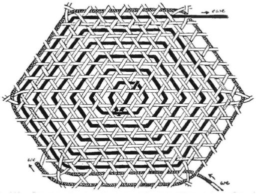
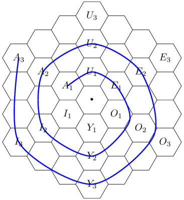




http://www.ime.usp.br/~jstern/miscellanea/jmsslide/hexa18.pdf
Square of Oppositon: Rapa Nui, 11-15/11/2016; Crete, 1-5/11/2018;
EBL, Pirenópolis, 8-12/05/2017; UniLog, Vichy, 16-26/06/2018;

This Presentation

I- Introduction;

II- Logical Hexagons of Opposing Modalities;

III- Testing (Accepting / Rejecting) Statistical Hypotheses,
> Desirable Logical Properties of Agnostic Tests,
> Failure of Probabilistic Statistical Tests;

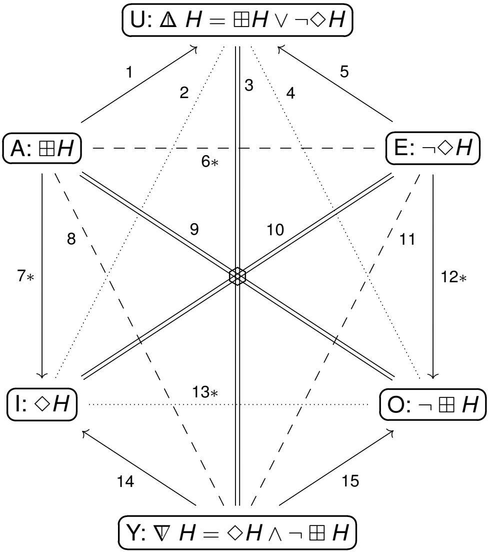
IV- Full Coherence $\Rightarrow$ Alethic/Possib. Calculus & Region Tests
$\Rightarrow$ GFBST – Generalized Full Bayesian Significance Test,
> GFBST's continuous mathematics under the hood;

V- Hybrid (Alethic / Probabilistic) Relations,
> Sharp Hypotheses: Importance and Paradoxes;
> The Problem of Induction: $\boxplus H$ or $\Diamond H \wedge \neg \Diamond\!\!\!+ H$?

VI- Pierre Gallais' Hexagonal Spirals and Science Evolution;

VII- How to model Pragmatic Acceptance of sharp H ?

VIII, XI- Bibliography & Goodbye.



Modal Operators:

□ - Necessity,
◇ - Possibility,
Δ - Contingency,
∇ - Non-Contingency;

Types:

∇ Δ ◇ □ - Alethic - by Possibilistic measure,
⍒̶ ⍋̶ ⟐ ⊞ - Probabilistic,
⍒ ⍋ - Hybrid;

Logical Operators:

¬ - Nega., → - Implic.,
∧ - Conjunction (and),
∨ - Disjunction (or);

Opposition relations:

═ Contradiction,
- - - Contrariety,
...... Sub-Contrariety.

⍋: Accept or Reject

⊞: Accept $H \Leftrightarrow \Pr(?H) \geq 1\text{-}\alpha$ $\qquad$ ¬⟐: Reject $H \Leftrightarrow \Pr(?H) < \beta$

⟐: Do not Reject $\qquad$ ¬⊞: Do not Accept

⍒: Agnostic $\Leftrightarrow$ Neither Accept nor Reject

Ideal world (wishful thinking), *not how it really works:*

Parameter space Θ, Posterior Probability $p_n(\theta) \propto p_0(\theta)p(X,\theta)$;

Hypotheses $H : \theta \in \Theta_H$ (relaxed notation: H for Θ_H);

Hypothesis $H \subset \Theta$ has known $\Pr(H) = \int_H p(\theta)d\theta$;

β = Pr(type II error = false negative);

$1 - \beta$ = Power = Pr(reject H if $\theta \notin H$);

α = Significance level = Pr(type I error = reject H if $\theta \in H$);

Choices for α or β:

Ronald Fisher: α = 0.05 (*), 0.02 (**), 0.01 (***);

Equal weight: Calibrate the test to minimize $\alpha + \beta$.

$\widetilde{H} = \Theta - H$, $\Pr(\widetilde{H}) = 1 - \Pr(H)$;

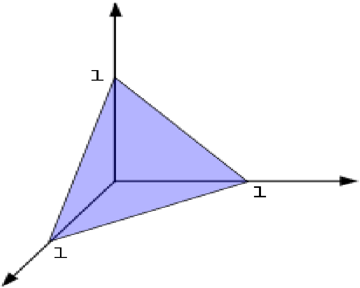
Slack and Sharp versions of Non-Cont. $\nabla = \Diamond \wedge \neg\Box$

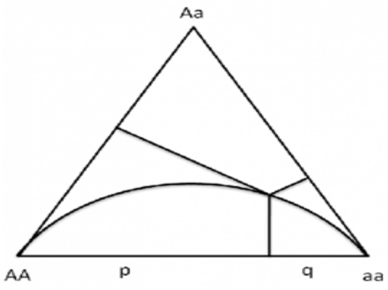
$\Box$: Mandatory $\Diamond$: Permitted	Δ: Ordained ∇: Indifferent	$\neg\Diamond$: Forbiden $\neg\Box$: Optional
$\Box$: Inclusion $\Diamond$: Inclu.or Intersct.	Δ: Inclu.or Exclu. ∇: Intersection	$\neg\Diamond$: Exclusion $\neg\Box$: Exclu.or Intersct.
$\Box$: $x < y$ $\Diamond$: $x \leq y$	Δ: $x \neq y$ ∇: $x = y$	$\neg\Diamond$: $x > y$ $\neg\Box$: $x \geq y$

- The interpretation of the ∇ modality can have a weak role (broad, vague, Slack) or a "reverse" strong role (equal, identical, Sharp)!

> Examples: Deontic relations from Gallais (1982); Order relations and set operations from Blanché (1966) & Béziau (2015).

Slack and Sharp Statistical Hypotheses



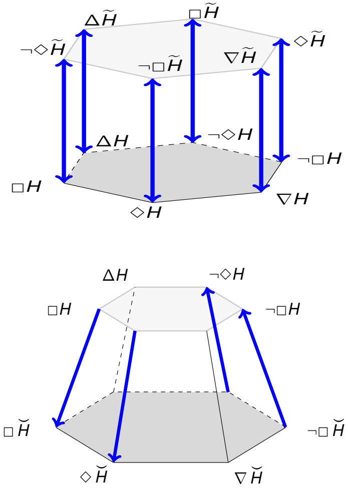


Simplex S1: $[\theta_A, \theta_a] \geq 0 \mid \theta_A + \theta_a = 1$

Simplex S2: $[\theta_{AA}, \theta_{Aa}, \theta_{aa}] \geq 0 \mid \theta_{AA} + \theta_{Aa} + \theta_{aa} = 1$

Hardy-Weinberg equilibrium law: Probababilities of genotypes are determined by independent probab. of alleles (A and a),

$\theta_{AA} = \theta_A^2$, $\theta_{Aa} = \theta_A \theta_a$, $\theta_{aa} = \theta_a^2$. (1d in S2)



Invertibility (for H complement):

$\Box H \iff \neg\Diamond \widetilde{H}$ and
$\nabla H \iff \nabla \widetilde{H}$

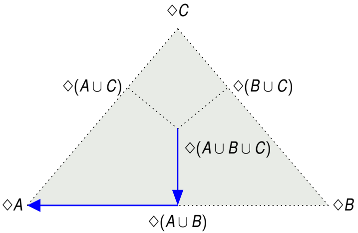
$\text{A} \leftrightarrow \widetilde{\text{E}}$, $\text{E} \leftrightarrow \widetilde{\text{A}}$,
$\text{I} \leftrightarrow \widetilde{\text{O}}$, $\text{O} \leftrightarrow \widetilde{\text{I}}$,
$\text{U} \leftrightarrow \widetilde{\text{U}}$, $\text{Y} \leftrightarrow \widetilde{\text{Y}}$;

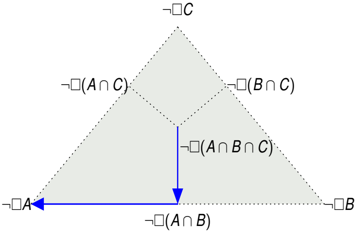
Monotonicity (for nested $H \subset \breve{H}$):

$$H \subseteq \breve{H} \Rightarrow \begin{cases} \Box H \Rightarrow \Box \breve{H} \\ \Diamond H \Rightarrow \Diamond \breve{H} \end{cases}$$

$\text{A} \rightarrow \breve{\text{A}}$, $\text{I} \rightarrow \breve{\text{I}}$,
$\breve{\text{O}} \rightarrow \text{O}$, $\breve{\text{E}} \rightarrow \text{E}$;

See Esteves et al. (2016),
(agnostic in statistics = ∇H)





Strong union consonance:
For every index set I,

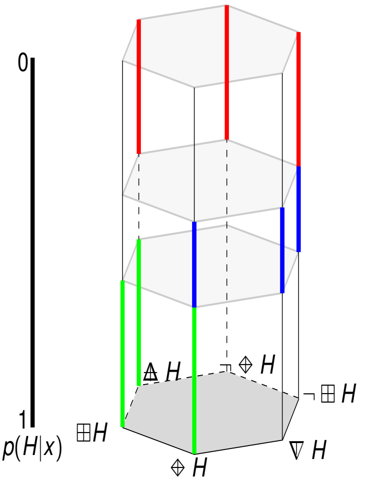
$$\Diamond(\cup_{i\in I} H_i) \Rightarrow \exists i \in I \,|\, \Diamond H_i \;;$$

Strong intersection consonance:
For every index set I

$$\neg\Box(\cap_{i\in I} H_i) \Rightarrow \exists i \in I \,|\, \neg\Box H_i \;;$$

Figures: Under strong consonance, there is at least one path from the center to a vertex of the polygon representing the indexed set of sub-hypotheses.

Failure of Decision Th. Posterior Probability Tests



Decis. \ Truth	H	$\widetilde{H}$
⊞H	0	1
▽ H	b	b
¬ ◈ H	a	0

Optim. Decis: $\min_D E_\theta Loss(\theta, D, H)$

$\Rightarrow$ Decide (choose) Prob. modality:

$$\begin{cases} ⊞H & \text{, if } p_n(H|x) > c_1, \\ \neg ◈ H & \text{, if } p_n(H|x) < c_2, \\ ▽ H & \text{, otherwise; where} \end{cases}$$

$c_1 = \max((1+a)^{-1}, b)$, and
$c_2 = \min((1+a)^{-1}, b/a)$.

These tests are logically incoherent: Can calibrate constants a and b s.t. tests are invertible & monotonic, but these tests are **not** consonant!

Failures of other Standard Statistical Tests

Property\Test	ALRT	Post.Pr.	GFBST
1- Invertibility (Logical)	X	✓	✓
2- Monotonicity (Logical)	X	✓	✓
3- Consonance (Logical)	X	X	✓
4- Consistency (Asympt.)	✓	?	✓
5- Invariance (Geometric)	✓	?	✓

> ALRT – Agnostic Likelihood Ratio Test: Slack or Sharp H;

> Posterior Probability: ?=✓ for Slack H, ?=X for Sharp* H;

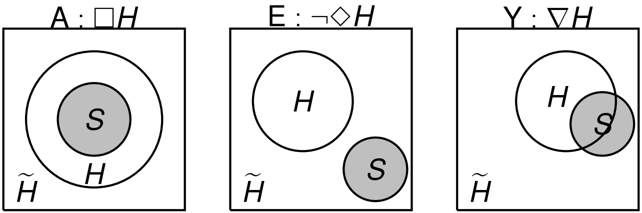
> Generalized Full Bayesian Significance Test ✓, Sharp ✓;

> *Logical* properties 1+2+3 ⇒ Test's *topological* properties:

———————————————————————-

*Sharp H: Posterior Probability/ Bayes Factor tests based on *ad hoc* atomic prior/posterior measures defined on H – Bad idea, leading to well known paradoxes. Situation fully acknowledged by orthodox (decision theoretic) Bayesian statistics, that regards sharp hypotheses as *ill formulated* ! See Izbicki & Esteves (2015); Esteves et al. (2016).

Fully Coherent (Alethic) Region Tests



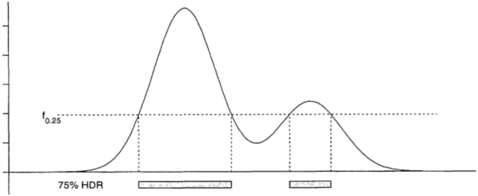
$$\text{Choose Alethic modality} \begin{cases} \Box H & \text{if } S \subseteq H \\ \neg\Diamond H & \text{if } S \subseteq \widetilde{H} \\ \nabla H & \text{if } S \cap H \neq \varnothing \;\&\; S \cap \widetilde{H} \neq \varnothing \end{cases}$$

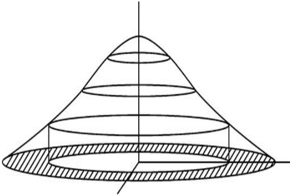
where S is a region estimator of the parameter θ, i.e., $S \subseteq \Theta$.

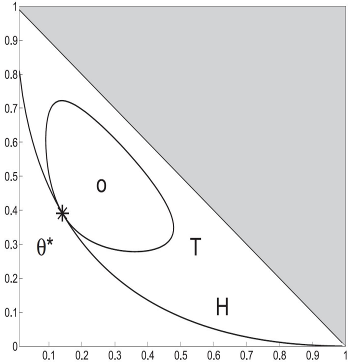
- Esteves (2016): Fully coherent tests *must be* region tests.
- ex: $S = \{\theta \in \Theta \,|\, p_n(\theta) > v\}$, Highest Probability Density Set.

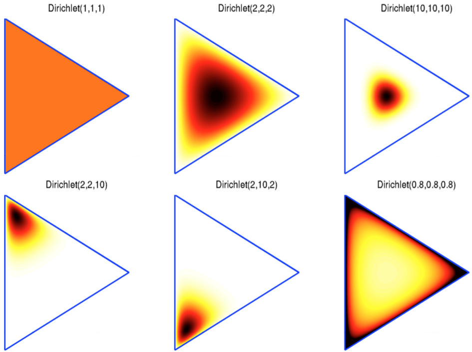
> S may not be path- or simply-connected.

The Miracle of Induction, Solved!









Generalized Full Bayesian Significance Test

- Surprise function $s(\theta) = p_n(\theta)/r(\theta)$;
- Reference density $r(\theta) \neq p_0(\theta)$, ex: Jeffreys invariant prior,

or representation of Fisher Information Metric, $dl^2 = d\theta' J(\theta) d\theta$;

$> J(\theta) \; = \; \mathbf{E}_{\mathcal{X}} \frac{\partial \log r(x \mid \theta)}{\partial \theta} \otimes \frac{\partial \log r(x \mid \theta)}{\partial \theta} \; = \; \mathbf{E}_{\mathcal{X}} \frac{\partial^2 \log r(x \mid \theta)}{\partial \theta^2}$;

- $T(v) = \{\theta \in \Theta \mid s(\theta) \geq v\}$, HSFS at level v.

> Highest Surprise Function Set, defining the region test.

Significance measure for hypothesis H:

- Wahrheit or truth function $W(v) = 1 - \int_{T(v)} p_n(\theta|x)\, d\theta$;
- *e*-value or Epistemic Value of H given observations X is

$\text{ev}\,(H \mid X) = W(s^*)$, where $s^* = \sup_{\theta \in H} s(\theta)$.

> $T(s^*)$ = Tangential Set = smallest HSFS $\mid \Diamond H$.

- GFBST: Alethic modality $\begin{cases} \Box H & \text{if } \text{ev}\,(\widetilde{H}) < c \\ \neg\Diamond H & \text{if } \text{ev}\,(H) < c \\ \nabla H & \text{otherwise.} \end{cases}$

- ev $(H \mid X)$ has good asymptotic properties;

> Sharp or precise hypotheses pose no special difficulties;

- ev $(H \mid X)$ is fully invariant by model reparameterization;
- ev $(H \mid X)$ can be logically computed for Coherent Structures, that is, for the series / parallel composition of statistical models and hypotheses, see Borges and Stern (2007).

Consistency and asymptotics:
Assume a “true” (vector) parameter θ^0 for the regular (ex. H is a differentiable algebraic sub-manifold of Θ) statistical model,

> $\text{sev}(H \mid X) = \text{Chi2}(t, \text{Chi2}^{-1}(t-h, \text{ev}\,(H \mid X)\,)$,

> $\text{Chi2}(k, x) = \gamma(\frac{k}{2}, \frac{x}{2}) / \gamma(\frac{k}{2}, \infty)$, $\gamma(c, x) = \int_0^x t^{c-1} e^{-t} dt$;

- If $\theta^0 \in H$, where H is sharp, $t = \dim(\Theta)$ & $h = \dim(H)$, then as $n \to \infty$ (increasing sample size) the Standarized *e*-value, sev$(H \mid X)$, converges in distribution to the Uniform in [0, 1]:

- If θ^0 is in the interior of H, ev $(H \mid X) \to 1$.

GFBST Invariance by Reparameterization of Θ

Consider a regular (bijective, integrable, a.s.cont. differentiable) reparameterization of the statistical model's parameter space, $\omega = \phi(\theta)$, $\Omega_H = \phi(\Theta_H)$, with Jacobian matrix

$$J(\omega) = \left[\frac{\partial\,\theta}{\partial\,\omega}\right] = \left[\frac{\partial\,\phi^{-1}(\omega)}{\partial\,\omega}\right] = \begin{bmatrix} \frac{\partial\,\theta_1}{\partial\,\omega_1} & \cdots & \frac{\partial\,\theta_1}{\partial\,\omega_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial\,\theta_n}{\partial\,\omega_1} & \cdots & \frac{\partial\,\theta_n}{\partial\,\omega_n} \end{bmatrix} .$$

$$\breve{s}(\omega) = \frac{\breve{p}_n(\omega)}{\breve{r}(\omega)} = \frac{p_n(\phi^{-1}(\omega))\,|J(\omega)|}{r(\phi^{-1}(\omega))\,|J(\omega)|} = s(\phi^{-1}(\omega))$$

and $\breve{s}^* = \sup_{\omega\in\Omega_H} \breve{s}(\omega) = \sup_{\theta\in\Theta_H} s(\theta) = s^*$. Hence,

$T(s^*) \mapsto \phi(T(s^*)) = \breve{T}(\breve{s}^*)$, making the significance measure

$$\breve{\text{ev}}(H) = 1 - \int_{\breve{T}(\breve{s}^*)} \breve{p}_n(\omega)d\omega = 1 - \int_{T(s^*)} p_n(\theta)d\theta = \text{ev}\,(H)$$

invariant by the reparameterization.

Disjunctive Normal Form for Coherent Structures

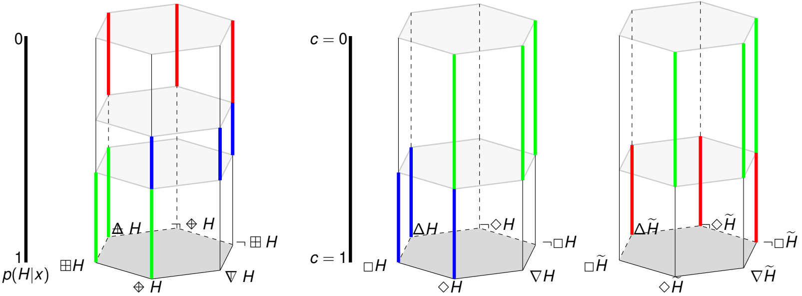
A Coherent Structure is a family, $M^{(i,j)} = \{\Theta^j, H^{(i,j)}, p_0^j, p_n^j, r^j\}$, of Independent Models, M^j, $j = 1 \dots k$, including, for each model M^j, a set of alternative hypotheses, $H^{(i,j)}$, $i = 1 \dots q$ (serial composition of models with parallel hypotheses).

$$\text{ev}\,(H) = \text{ev}\left(\bigvee_{i=1}^{q} \bigwedge_{j=1}^{k} H^{(i,j)}\right) = \max_{i=1}^{q} \text{ev}\left(\bigwedge_{j=1}^{k} H^{(i,j)}\right)$$

$$= W\left(\max_{i=1}^{q} \prod_{j=1}^{k} s^{*(i,j)}\right) \; ; \qquad W = \bigotimes_{1 \le j \le k} W^j \; .$$

- W is the Mellin Convolution of the models' truth functions, where $[f \otimes g](y) = \int_0^\infty (1/x) f(x) g(y/x) dx$;
- If all $s^* = 0 \vee \widehat{s}$, $\text{ev} = 0 \vee 1$, we get classical logic.

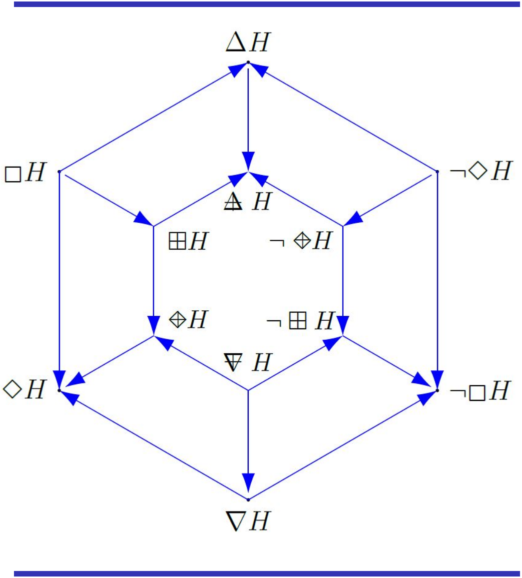
Hybrid (Alethic / Probabilistic) Relations



Setting constants $c_1 = 1 - c$ & $c_2 = c$, the modal operators defined by the GFBST and the agnostic probabilistic test obey:

- $\Box H \Rightarrow \Pr(H|X) \geq 1 - c \Rightarrow \boxplus H$;
- $\neg \Diamond H \Rightarrow \Pr(H|X) \leq c \Rightarrow \neg \diamondplus H$;

Nested (Alethic / Probabilistic) Hexagons



- Hence, setting consts. $c_1 = 1 - c$ and $c_2 = c$, $\neg\Diamond H \Rightarrow \neg\diamondplus H \Rightarrow \neg\boxplus H$, $\boxplus H \Rightarrow \diamondplus H \Rightarrow \Diamond H$; Nested implictns. hold!

- However, if H is sharp, $\Pr(H|X) = 0 \Rightarrow \neg\diamondplus H$ (trivial hybrid relations)

- Nevertheless, $\Diamond H$ is a consistent (s.14) outcome of the GFBST (FBST main motivation)

- Theoretical importance of sharp H; + Need for a meaningful Pragmatic (Acceptable) version of H!

Physical Laws & Inferencial Miracles

• Most important scientific hypotheses or Laws are Equations, naturally expressed as Sharp Hypotheses, see Stern (2011a);

• Sharp $H \Rightarrow \neg \diamond\!\!\!+ H$. However, if $\theta^0 \in H$ we will obtain $\diamond H$ with a given asymptotic frequency (sl.12); An inferential wonder!
> Indeed, corroborating an H that is almost surely false is a **Miracle!!!** (Infidels required to take Physics101-104+Lab.)

• As theories become the standard paradigm, ontologies get reified, and we would like to have a form of...

• *Pragmatic acceptance* of H, namely, $\Box \breve{H}$,
where the *Pragmatic hypotheses* $\breve{H}$ is a non-sharp version of the sharp hypothesis H.

• Methods of Construction of *Pragmatic hypotheses* are presented in section VII.

The Miracle of Induction, Solved!

The Problem of Induction:

$\boxplus H$ or $\Diamond H \wedge \neg \diamondplus H$?

- *He who wishes to solve the problem of induction must beware of trying to prove too much.*

Karl Popper, Replies to my Critics; in Schilpp (1974, Ch.32, p.1110), also quoted in Stern (2011b).

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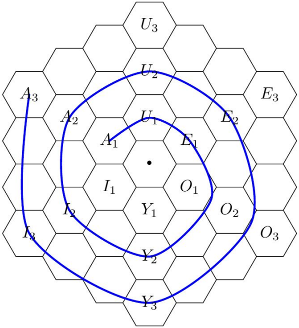


U: $\Delta H = \Box \breve{H} \vee \neg \Diamond H$

A: $\Box \breve{H}$ E: $\neg \Diamond H$

I': $\Diamond H'$ O': $\neg \Box \breve{H'}$

Y': $\nabla H' = \Diamond H' \wedge \neg \Box \breve{H'}$

A_1- Thesis: Current paradigm, well established concepts & theory;

U_1- Analysis: Is questioned vis-à-vis an alternative (maybe still vague or imprecise) class of models ;

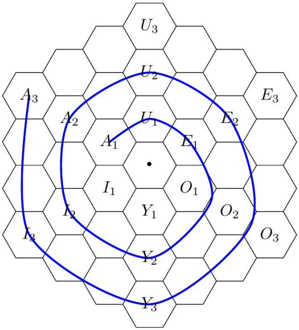
E_1- Antithesis: Old laws rejected;

O_2- Apothesis / Prosthesis: Alternative models considered (H'); Specific (precise) forms selected;

Y_2- Synthesis: New laws formulated!

I_2- Enthesis: Theory integration; Systematic empirical corroboration; Fundamental constants calibrated;

A_2- New standard paradigm: New laws accepted (well quantified imprecisions), reified ontology (correspond. principle).



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Y': $\nabla H' = \Diamond H' \wedge \neg \Box \breve{H'}$

A_1- Ptolemaic astronomy & system of circular cycles and epicycles;

U_1- Is put in question:
Circular or Oval orbits?

E_1- Orbits are Not circular;

O_2- Elliptical orbits (eureka);

Y_2- Kepler laws!

A_2- Vortex theories;

U_2- Tangential or Radial forces?

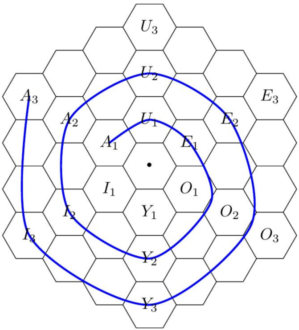
E_2- Forces are Not tangential;

O_3- Radial & inverse square!

Y_3- Newton laws!

A_3- Newtonian mechanics.

Gallais' Hexagonal Spirals & Science Evolution



U: $\Delta H = \Box\, \breve{H} \vee \neg \Diamond H$

A: $\Box\, \breve{H}$ E: $\neg \Diamond H$

I': $\Diamond H'$ O': $\neg \Box\, \breve{H'}$

Y': $\nabla H' = \Diamond H' \wedge \neg \Box\, \breve{H'}$

A_1- Geoffroy rules and tables as axioms of chemical affinity;

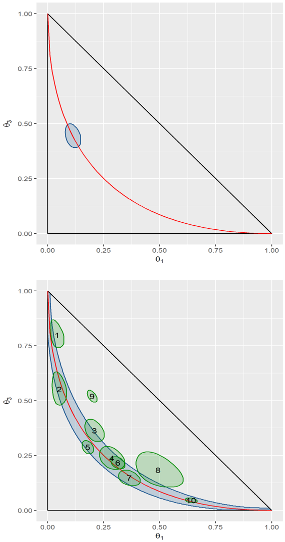
U_1- Ordinal or Numerical?
E_1- Not ordinal;
O_2- Integer affinity numbers;
Y_2- Morveau rules and tables!

A_2- Modern (1800) chemistry, Stoichiometry + Affinity rules & Substitution reactions;

U_2- Total or Parcial?
E_2- Not total substitution;
O_3- Reversible equilibria;
Y_3- Mass-Action kinetics!

A_3- Thermodynamic networks.

Pragmatic Hypotheses



Construction of Pragmatic Hypotheses based on *Predictive dissimilarities* concerning $\mathcal{Z}$, the possible outcomes of an idealized future experiment, **Z**.

Cross-Entropy Divergence:

$$\text{KL}_{\mathbf{Z}}(\theta_0, \theta^*) = \text{KL}(\mathbb{P}_{\theta^*}, \mathbb{P}_{\theta_0}) = \int_{\mathcal{Z}} \log\left(\frac{d\mathbb{P}_{\theta^*}}{d\mathbb{P}_{\theta_0}}\right) d\mathbb{P}_{\theta^*},$$

Classification Distance,
Total Variation or L_1 norm: :

$$\text{CD}(\theta_0, \theta^*) = 0.5\text{TV}(\mathbb{P}_{\theta_0}, \mathbb{P}_{\theta^*}) = 0.25\|\mathbb{P}_{\theta_0} - \mathbb{P}_{\theta^*}\|_1$$

Future Research: Metrology/ Fundamental Constants

- How to better calibrate the fundamental and empirical constants of a set of experiments testing hypotheses in a complex theory, and Accept, Reject or remain agnostic?
- ...
- ...
- Universal or case specific solutions?

- How to decide cut-off point?
- ...
- ...
- Well adapted basis?

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- The legendary Polynesian king and navigator *Hotu Matu'a* reached *Te pito te henua* (the navel of the world) some time arround 1000 CE, sailing a double hull catamaran from Mangareva, 2600 km, or the Marquesas, 3200 km away;
- Building of *Ariña ora ata tepuña*, face-living-image-idols or *moai* monoliths, lead to ecological devastation, famine, war, cultural breakdown & civilization collapse.

Goodbye! Adieu! Ko te pava kokorua! $\upsilon\gamma\epsilon\iota\alpha\ \sigma\alpha\varsigma$!

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