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THE DISTRIBUTION OF PRODUCTS, QUOTIENTS AND POWERS OF INDEPENDENT *H*-FUNCTION VARIATES*

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Abstract. This paper introduces a new probability distribution based on the H-function of Fox. The distribution is shown to be a generalization of most common "nonnegative" ($\Pr[X<0]=0$) distributions. Furthermore, it is proved that products, quotients and powers of H-function variates are H-function variates. Several examples are given.

1. Introduction. The problem of products and quotients of random variates has been treated for some time (Craig [3], [4] and Huntington [8]) but it was not until 1948 that the first systematic approach was presented by Epstein [5] when he demonstrated that the Mellin transform is a natural analytic tool for analyzing problems of this type. Since that time, the application of the Mellin transform as a statistical tool has been promoted by Zolotarev [18], [19], Springer and Thompson [16], [17], Kotz and Srinivasan [9], Abraham and Prasad [1], Prasad [15] and others.

It is the purpose of this paper to introduce a new probability distribution which is (i) the general case of many common "nonnegative" probability distributions and (ii) easily "transformed" under the Mellin transformation. The probability density function (p.d.f.) of the new distribution is based on the H-function, a transcendental function first presented by Fox [7] in 1961.

The new distribution, called the H-function distribution, includes as special cases many of the more common classical distributions, e.g., the gamma, the beta, the Weibull, the chi-square, the exponential and the half-normal distributions as well as others. Hence, the H-function distribution can be considered as a generalization or characterization of these special cases and can serve as a basis for handling rational functions of "mixtures" of such variates.

Also, various combinations of products, powers and quotients of independent H-function variates are examined using the above mentioned Mellin transform procedures. Theorems are presented to show that the product of independent H-function variates is an H-function variate, the power of an H-function variate is an H-function variate, and the quotient or ratio of independent H-function variates is an H-function variate.

2. The *H*-function. The H-function was first introduced by Fox [7, p. 408] in 1961 as a symmetric Fourier kernel to the G-function of Meijer [6,I, pp. 206–222]. Furthermore, the H-function is recognized as a generalization of both the G-function and Wright's generalized hypergeometric function [6,I, p. 183]. More recently numerous papers related to the H-function and its properties have been presented and an extensive list of these is found in the bibliography of [2].

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2.1. Definition of $H(z)$. Although there are slight variations and generalizations in the definition of the H-function in the literature, this paper will use the definition

$$\mathbf{H}_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right]=H(z) \tag{2.1}$$
$$=\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^m\Gamma(b_j-\beta_j s)\prod_{j=1}^n\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^q\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}z^s\,ds$$

where

$$0\leqq m\leqq q,$$
$$0\leqq n\leqq p,$$
$$\alpha_j>0\quad\text{for } j=1,2,\cdots,p,$$
$$\beta_j>0\quad\text{for } j=1,2,\cdots,q,$$

and where a_j $(j=1,2,\cdots,p)$ and b_j $(j=1,2,\cdots,q)$ are complex numbers such that no pole of $\Gamma(b_j-\beta_j s)$ for $j=1,2,\cdots,m$ coincides with any pole of $\Gamma(1-a_j+\alpha_j s)$ for $j=1,2,\cdots,n$. Furthermore, C is a contour in the complex s-plane running from $\omega-i\infty$ to $\omega+i\infty$ such that the points

$$s=(b_j+k)/\beta_j$$

for $j=1,2,\cdots,m$ and $k=0,1,\cdots$ and the points

$$s=(a_j-1-k)/\alpha_j$$

for $j=1,2,\cdots,n$ and $k=0,1,\cdots$ lie to the right and left of C, respectively. In other words, (2.1) is a Mellin–Barnes integral [6,I, pp. 49–50].

2.2. Simple identities and special cases of $H(z)$. Variable substitution into (2.1) yields the following three identities which are very useful in manipulating H-function:

$$\mathbf{H}_{p,q}^{m,n}\left[\frac{1}{z}\,\middle|\,\begin{matrix}(a_1,\alpha_1,),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right]=\mathbf{H}_{q,p}^{n,m}\left[z\,\middle|\,\begin{matrix}(1-b_1,\beta_1),\cdots,(1-b_q,\beta_q)\\(1-a_1,\alpha_1),\cdots,(1-a_p,\alpha_p)\end{matrix}\right], \tag{2.2}$$

$$\mathbf{H}_{p,q}^{m,n}\left[z^c\,\middle|\,\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right] \tag{2.3}$$
$$=\frac{1}{c}\mathbf{H}_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_1,\alpha_1/c)\cdots,(a_p,\alpha_p/c)\\(b_1,\beta_1/c),\cdots,(b_q,\beta_q/c)\end{matrix}\right],\qquad c>0,$$

and

$$z^c\mathbf{H}_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right] \tag{2.4}$$
$$=\mathbf{H}_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_1+\alpha_1c,\alpha_1),\cdots,(a_p+\alpha_pc,\alpha_p)\\(b_1+\beta_1c,\beta_1),\cdots,(b_q,\beta_qc,\beta_q)\end{matrix}\right].$$

Many of the so-called "special functions" are found to be special cases of the H-function, including Gauss' hypergeometric function, the confluent hypergeometric function, Wright's generalized hypergeometric function, MacRobert's E-function, Meijer's G-function and Bessel functions. An excellent discussion of many of these and others related functions is given by Erdélyi [6,I] and Luke [11, I]. More important among these are:

(i) the exponential function,

$$\exp(x) = \mathbf{H}_{0,1}^{1,0}[-x|(0, 1)]; \tag{2.5}$$

(ii) the generalized, hypergeometric function,

$$\begin{aligned} &{}_p\mathbf{F}_q\left[\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\middle| x\right] \\ &= \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} \mathbf{H}_{p,q+1}^{1,p}\left[-x\middle|\begin{matrix}(1-a_1, 1), \cdots, (1-a_p, 1) \\ (0, 1), (1-b_1, 1), \cdots, (1-b_q, 1)\end{matrix}\right], \end{aligned} \tag{2.6}$$

(iii) Wright's generalized hypergeometric function,

$$\begin{aligned} &{}_p\mathbf{\Psi}_q\left[\begin{matrix} (a_1, \alpha_1), \cdots, (a_p, \alpha_p) \\ (b_1, \beta_1), \cdots, (b_q, \beta_q) \end{matrix}\middle| x\right] \\ &= \mathbf{H}_{p,q+1}^{1,p}\left[-x\middle|\begin{matrix}(1-a_1, \alpha_1), \cdots, (1-a_p, \alpha_p) \\ (0, 1), (1-b_1, \beta_1), \cdots, (1-b_q, \beta_q)\end{matrix}\right]; \end{aligned} \tag{2.7}$$

(iv) Meijer's G-function,

$$\mathbf{G}_{p,q}^{m,n}\left[x\middle|\begin{matrix}a_1, \cdots, a_p \\ b_1, \cdots, b_q\end{matrix}\right] = \mathbf{H}_{p,q}^{m,n}\left[x\middle|\begin{matrix}(a_1, 1), \cdots, (a_p, 1) \\ (b_1, 1), \cdots, (b_q, 1)\end{matrix}\right]. \tag{2.8}$$

It should be noted that Luke also gives an extensive list of special cases and identities for the generalized hypergeometric function and for Meijer's G-function and, with the use of (2.6) and (2.8), these results can be extended to the H-function.

2.3. The Mellin and Laplace transforms of $H(cz)$. Under the previous definition of the H-function and assuming convergence of the integral in the definition, the Mellin transform[1] can be found by interpreting the H-function of the coefficient on x^{-s} where (2.1) is written as

$$H(cz) = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^m \Gamma(b_j+\beta_j s)\prod_{j=1}^n \Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^p \Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^p \Gamma(a_j+\alpha_j s)}(cz)^{-s}\, ds.$$

Using the definition of the Mellin transform, one can express $H(cz)$ in the form

$$H(cz) = \mathcal{M}^{-1}\left[\frac{\prod_{j=1}^m \Gamma(b_j+\beta_j s)\prod_{j=1}^n \Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^q \Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^p \Gamma(a_j+\alpha_j s)}c^{-s}\right]$$

[1] See § 4.1.

from which it follows that

$$
\begin{aligned}
(2.9)\qquad &\mathcal{M}_s\left\{\mathbf{H}_{p,q}^{m,n}\left[cz\,\middle|\,\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right]\right\}\\
&=\mathcal{M}_s\{H(cz)\}=\frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_js)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_js)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_js)}c^{-s}
\end{aligned}
$$

is the Mellin transform of the H function with argument cz.

From the definition of the Laplace transform, one has

$$
\begin{aligned}
\mathcal{L}_r\{H(cz)\}&=\int_0^\infty e^{-rz}H(cz)\,dz\\
&=\int_0^\infty e^{-rz}\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_js)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_js)\prod_{j=n+1}^{q}\Gamma(a_j-\alpha_js)}(cz)^s\,ds\,dz.
\end{aligned}
$$

The contour integral in the s-plane converges absolutely under the conditions given by Erdélyi [6,I, pp. 49–50] so that when these conditions are satisfied (as they usually are), the Laplace integral will converge absolutely. Hence, the order of integration in the above equation can be changed giving

$$
\begin{aligned}
\mathcal{L}_r\{H(cz)\}&=\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_js)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_js)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_js)}c^s\left(\int_0^\infty e^{-rs}z^s\,dz\right)ds\\
&=\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_js)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_js)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_js)}c^s\left(\frac{\Gamma(s+1)}{r^{s+1}}\right)ds\\
&=\frac{1}{r}\mathbf{H}_{p+1,q}^{m,n+1}\left[\frac{c}{r}\,\middle|\,\begin{matrix}(0,1),(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right]
\end{aligned}
$$

and, from (2.1), it follows that

$$
\mathcal{L}_r\{H(cz)\}=\frac{1}{c}\mathbf{H}_{p,q+1}^{m+1,n}\left[\frac{c}{r}\,\middle|\,\begin{matrix}(1,1),(a_1+\alpha_1,\alpha_1),\cdots,(a_p+\alpha_p,\alpha_p)\\(b_1+\beta_1,\beta_1),\cdots,(b_q+\beta_q,\beta_q)\end{matrix}\right].
$$

Then, from (2.2), the Laplace transform is expressible as

$$
\begin{aligned}
(2.10)\qquad &\mathcal{L}_r\left\{\mathbf{H}_{p,q}^{m,n}\left[cx\,\middle|\,\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right]\right\}\\
&=\mathcal{L}_r\{H(cz)\}=\frac{1}{c}\mathbf{H}_{q,p+1}^{n+1,m}\left[\frac{1}{c}r\,\middle|\,\begin{matrix}(1-b_1-\beta_1,\beta_1),\cdots,(1-b_q-\beta_q,\beta_q)\\(0,1),(1-a_1-\alpha_1,\alpha_1),\cdots,(1-a_p-\alpha_p,\alpha_p)\end{matrix}\right].
\end{aligned}
$$

3. The H-function distribution.

3.1. Definition. Consider a random variable X which follows a probability law such that its probability density function is given by

$$
(3.1)\qquad f(x)=\begin{cases}k\,\mathbf{H}_{p,q}^{m,n}\left[cx\,\middle|\,\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right], & x>0,\\ 0 & \text{otherwise,}\end{cases}
$$

where the symbol H represents the H-function as defined in (2.1) and where k, c, a_j $(j=1,\cdots,p)$, α_j $(j=1,\cdots,p)$, b_j $(j=1,\cdots,q)$, and β_j $(j=1,\cdots,q)$ are the parameters of the distribution with values such that

$$\int_{-\infty}^{\infty} f(x)\,dx=\int_0^{\infty} f(x)\,dx=1$$

with $f(x)\geqq 0$ for $0x<\infty$. Furthermore, the values of a_j $(j=1,\cdots,p)$, α_j $(j=1,\cdots,p)$, b_j $(j=1,\cdots,q)$, and β_j $(j=1,\cdots,q)$ must conform to those restrictions in the definition of the H-function (equation (2.1)). The random variable X will then be called an H-function variate which follows an H-function probability law or H-function distribution.[2]

3.2. The characteristic function. The characteristic function (or Fourier transform) of $f(x)$ is given as

$$\begin{aligned}\phi(t)=\int_{-\infty}^{\infty} e^{itx}f(x)\,dx&=\int_0^{\infty} e^{itx}k\,\mathbf{H}_{p,q}^{m,n}\left[cx\left|\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right.\right]dx\\&=\mathscr{L}_{-it}\left\{k\,\mathbf{H}_{p,q}^{m,n}\left[cx\left|\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right.\right]\right\}.\end{aligned}$$

From (2.10), assuming absolute convergence of the integral in the definition of the H-function, the characteristic function distribution can be given as

$$\text{(3.2)}\quad \phi(t)=\frac{k}{c}\mathbf{H}_{q,p+1}^{n+1,m}\left[-\frac{i}{c}t\left|\begin{matrix}(1-b_1-\beta_1,\beta_1),\cdots,(1-b_q-\beta_q,\beta_q)\\(0,1),(1-a_1-\alpha_1,\alpha_1),\cdots,(1-a_p-\alpha_p,\alpha_p)\end{matrix}\right.\right].$$

3.3. Moments. Since the derivatives of the H-function exist, the moments of the H-distribution can be found by taking the derivatives of (3.2). However, there is a simpler method of finding the general expression for the rth moment about the origin which capitalizes on the ease with which the Mellin transform of the probability density function may be obtained. In this connection, note that the rth moment about the origin is defined as

$$\mu_r'=E\{x^r\}=\int_{-\infty}^{\infty} x^r f(x)\,dx$$

where E is the expected value operator. From the definition of the Mellin transform, it is clear that $\mathscr{M}_s\{f(x)\}=E\{x^{s-1}\}$ for distributions where $\Pr[x<0]=0$ so that the rth moment may be obtained from the Mellin transform of the relevant probability density function. Specifically,

$$\begin{aligned}\mu_r'&=\mathscr{M}_{r+1}\{f(x)\}\\&=\mathscr{M}_{r+1}\left\{k\,\mathbf{H}_{p,q}^{m,n}\left[cx\left|\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right.\right]\right\}.\end{aligned}$$

[2] Distributions and distributional structures based on the H-function have recently been introduced by Mathai and Saxena [12], [13].

Then, from (2.9),

$$\mu_r' = k\frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=n+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}c^{-s}\bigg|_{s=r+1}$$
$$(3.3)\qquad =\frac{k}{c^{r+1}}\frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j+\beta_j r)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j-\alpha_j r)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j-\beta_j r)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j+\alpha_j r)}.$$

3.4. Special cases of the *H*-function distribution. As indicated at the beginning of this paper, one of the most important assets of the H-function distribution is that many of the classical nonnegative distributions are special cases and can be expressed in the form of (3.1). In this section, some of the more common of these special cases are given and their respective probability density functions are shown in the form of (3.1). Although the mathematical development from the "common" form of the probability density function to the "H-function" form is given without explanation, the reader should easily follow the development with the use of (2.3), (2.4) and (2.9).

(i) The gamma distribution.

$$f(x)=\frac{x^{\theta-1}\exp(-x/\phi)}{\phi^{\theta}\Gamma(\theta)} \qquad x>0;\quad \theta,\phi>0$$
$$(3.4)\qquad =\frac{1}{\phi^{\theta}\Gamma(\theta)}x^{\theta-1}\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{\phi}x\,\middle|\,(0,1)\right]$$
$$=\frac{1}{\phi\Gamma(\theta)}\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{\phi}x\,\middle|\,(\theta-1,1)\right].$$

(ii) The Weibull distribution.

$$f(x)=\theta\phi x^{\theta-1}\exp(-\theta x^{\phi}) \qquad x>0$$
$$(3.5)\qquad =\theta\phi x^{\phi-1}\mathbf{H}_{0,1}^{1,0}[\theta x^{\phi}|(0,1)]$$
$$=\theta x^{\phi-1}\mathbf{H}_{0,1}^{1,0}[\theta^{1/\phi}x|(0,1/\phi)]$$
$$=\theta^{1/\phi}\mathbf{H}_{0,1}^{1,0}[\theta^{1/\phi}x|(1-1/\phi,1/\phi)].$$

(iii) The Maxwell distribution.

$$f(x)=\frac{4x^2\exp(-x^2/\theta^2)}{\theta^3\sqrt{\pi}} \qquad x>0$$
$$(3.6)\qquad =\frac{4}{\theta^3\sqrt{\pi}}x^2\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{\theta^2}x^2|(0,1)\right]$$
$$=\frac{2}{\theta^3\sqrt{\pi}}x^2\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{\theta}x\,\middle|\,(0,1/2)\right]$$
$$=\frac{2}{\theta\sqrt{\pi}}\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{\theta}x\,|(1,1/2)\right].$$

(iv) The beta distribution.

$$f(x)=\begin{cases}\dfrac{x^{\theta-1}(1-x)^{\phi-1}}{B(\theta,\phi)}, & 0<x\leqq 1;\ \theta,\phi>0\\ 0, & x>1\end{cases}$$

(3.7)
$$\begin{aligned}
&=\frac{1}{2\pi i}\int_C \mathcal{M}_s\{f(x)\}x^{-s}\,ds\\
&=\frac{1}{2\pi i}\int_C\left[\int_0^1\frac{1}{B(\theta,\phi)}x^{\theta-1}(1-x)^{\phi-1}x^{s-1}\,dx\right]x^{-s}\,ds\\
&=\frac{1}{2\pi i}\int_C\frac{\Gamma(\phi)\Gamma(\theta-1+s)}{B(\theta,\phi)\Gamma(\theta+\phi-1+s)}x^{-s}\,ds\\
&=\frac{\Gamma(\theta+\phi)}{\Gamma(\theta)}\left[\frac{1}{2\pi i}\int_C\frac{\Gamma(\theta-1-s)}{\Gamma(\theta+\phi-1-s)}x^{s}\,ds\right]\\
&=\frac{\Gamma(\theta+\phi)}{\Gamma(\theta)}\mathbf{H}_{0,1}^{1,0}\left[x\,\middle|\,\begin{matrix}(\theta+\phi-1,1)\\(\theta-1,1)\end{matrix}\right].
\end{aligned}$$

(v) The half-normal distribution.

(3.8)
$$\begin{aligned}
f(x)&=\frac{2\exp(-x^2/(2\theta^2))}{\theta\sqrt{2\pi}} \qquad x>0;\quad \theta>0\\
&=\frac{2}{\theta\sqrt{2\pi}}\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{2\theta^2}x^2\,\middle|\,(0,1)\right]\\
&=\frac{1}{\theta\sqrt{2\pi}}\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{\theta\sqrt{2}}x\,\middle|\,(0,1/2)\right].
\end{aligned}$$

(vi) The exponential distribution. Let $\theta=1$ in (3.4).

(3.9)
$$\begin{aligned}
f(x)&=\frac{\exp(-x/\phi)}{\phi} \qquad x>0;\quad \phi>0\\
&=\frac{1}{\phi}\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{\phi}x\,\middle|\,(0,1)\right].
\end{aligned}$$

(vii) The chi-square distribution. Let $\theta=v/2$ and $\phi=2$ in (3.4).

(3.10)
$$\begin{aligned}
f(x)&=\frac{x^{v/2-1}\exp(-x/2)}{2^{v/2}\Gamma(v/2)} \qquad x>0;\quad v>0\\
&=\frac{1}{2\Gamma(v/2)}\mathbf{H}_{0,1}^{1,0}\left[\frac{1}{2}x\,\middle|\,\left(\frac{v}{2}-1,1\right)\right].
\end{aligned}$$

(viii) The Rayleigh distribution. Let $\theta = 1/(2a^2)$ and $\phi = 2$ in (3.5).

$$
\begin{aligned}
f(x) &= \frac{x \exp(-x^2/(2a^2))}{2a^2} \qquad\qquad x>0 \\
&= \frac{1}{a\sqrt{2}} \mathbf{H}_{0,1}^{1,0}\left[\frac{1}{a\sqrt{2}}x \,\middle|\, (1/2,\, 1/2)\right].
\end{aligned} \tag{3.11}
$$

(ix) The general hypergeometric distribution [14].

$$
\begin{aligned}
f(x) &= \frac{da^{c/d}\Gamma(\beta)\Gamma(r-c/d)}{\Gamma(c/d)\Gamma(r)\Gamma(\beta-c/d)} x^{c-1}\, {}_1\mathbf{F}_1\left[\begin{matrix}\beta \\ r\end{matrix} \,\middle|\, -ax^d\right] \qquad x>0 \\
&= \frac{da^{c/d}\Gamma(\beta)\Gamma(r-c/d)}{\Gamma(c/d)\Gamma(r)\Gamma(\beta-c/d)} x^{c-1} \\
&\quad \cdot \left(\frac{\Gamma(r)}{\Gamma(\beta)} \mathbf{H}_{1,2}^{1,1}\left[ax^d \,\middle|\, \begin{matrix}(1-\beta,\, 1) \\ (0,\, 1),\, (1-r,\, 1)\end{matrix}\right]\right) \\
&= \frac{a^{c/d}\Gamma(r-c/d)}{\Gamma(c/d)\Gamma(\beta-c/d)} x^{c-1} \mathbf{H}_{1,2}^{1,1}\left[a^{1/d}x \,\middle|\, \begin{matrix}(1-\beta,\, 1/d) \\ (0,\, 1/d),\, (1-r,\, 1/d)\end{matrix}\right] \\
&= \frac{a^{1/d}\Gamma(r-c/d)}{\Gamma(c/d)\Gamma(\beta-c/d)} \mathbf{H}_{1,2}^{1,1}\left[a^{1/d}x \,\middle|\, \begin{matrix}(1-\beta+(c-1)/d,\, 1/d) \\ ((c-1)/d,\, 1/d),\, (1-r+(c-1)/d,\, 1/d)\end{matrix}\right].
\end{aligned} \tag{3.12}
$$

(x) The half-Cauchy distribution.

$$
\begin{aligned}
f(x) &= \frac{2\theta}{\pi(\theta^2+x^2)} \qquad\qquad x>0; \quad \theta>0 \\
&= \frac{1}{2\pi i}\int_C \mathcal{M}_s\{f(x)\} x^{-s}\, ds \\
&= \frac{1}{2\pi i}\int_C \left[\int_0^\infty \frac{2\theta x^{-s}}{\pi(\theta^2+x^2)}\, dx\right] x^{-s}\, ds \\
&= \frac{1}{2\pi i}\int_C \frac{\Gamma(\frac{1}{2}s)\Gamma(1-\frac{1}{2}s)\theta^{s-1}}{\pi} x^{-s}\, ds \\
&= \frac{1}{\theta\pi}\left[\frac{1}{2\pi i}\int_c \Gamma\left(-\frac{1}{2}s\right)\Gamma\left(1+\frac{1}{2}s\right)\left(\frac{1}{\theta}x\right)^s ds\right] \\
&= \frac{1}{\theta\pi}\mathbf{H}_{1,1}^{1,1}\left[\frac{1}{\theta}x \,\middle|\, \begin{matrix}(0,\, 1/2) \\ (0,\, 1/2)\end{matrix}\right].
\end{aligned} \tag{3.13}
$$

4. Products, quotients and powers of H-function variates.

4.1. Certain properties of Mellin transforms. If $f(x)$ is a probability density function for the distribution of random variables x, then the Mellin transform of the p.d.f. is given by

$$
\mathcal{M}_s\{f(x)\} = \int_0^\infty f(x) x^{s-1}\, dx. \tag{4.1}
$$

Furthermore, if $\mathcal{M}_s\{f(x)\}$ is an analytic function of the complex variable s for $c_1 \leqq \mathrm{Re}\,(s) \leqq c_2$ where c_1 and c_2 are real, then the integral

$$\mathcal{M}^{-1}[\mathcal{M}_s\{f(x)\}] = \frac{1}{2\pi} \lim_{\beta \to \infty} \int_{\omega - i\beta}^{\omega + i\beta} x^{-s} \mathcal{M}_s\{f(x)\}\, ds \tag{4.2}$$

along any line $c_1 \leqq \mathrm{Re}\,(s) = \omega \leqq c_2$ converges to the function $f(x)$ which is independent of ω and whose Mellin transform is $\mathcal{M}_s\{f(x)\}$, $c_1 < \mathrm{Re}\,(s) < c_2$.

If $X_1, X_2, \cdots, X_n$ are continuous independent random variables with probability density function $f_1(x_1), f_2(x_2), \cdots, f_n(x_n)$, respectively, where

$$\Pr[X_j \leqq 0] = 0, \qquad j = 1, 2, \cdots, n,$$

then Epstein and others have proved the following properties:

(i) the probability density function of the random variable

$$Y = \prod_{j=1}^{n} X_j$$

is given by

$$h(y) = \begin{cases} \mathcal{M}^{-1}\left[\prod_{j=1}^{n} \mathcal{M}_s\{f_j(x_j)\}\right], & 0 < y < \infty, \\ 0 & \text{otherwise;} \end{cases} \tag{4.3}$$

(ii) the probability density function of the random variable

$$Y = X_j^a$$

is given by

$$h(y) = \begin{cases} \mathcal{M}^{-1}[\mathcal{M}_{as-a+1}\{f_j(x_j)\}], & 0 < y < \infty, \\ 0 & \text{otherwise;} \end{cases} \tag{4.4}$$

(iii) the probability density function of the random variable

$$Y = X_j / X_k, \qquad j \neq k,$$

is given by

$$h(y) = \begin{cases} \mathcal{M}^{-1}[\mathcal{M}_s\{f_j(x_j)\} \mathcal{M}_{2-s}\{f_k(x_k)\}], & 0 < y < \infty, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

4.2. The distribution of products of *H*-function variates. As is stated in the following theorem one of the most significant properties of the *H*-function distribution is that the probabiltiy distribution of products of independent *H*-function variates is also an *H*-function distribution. It is well known that such a property is not common among the "named" probability distributions. Furthermore, since the beta, the gamma, the Weibull, the Maxwell, etc. are special cases of the *H*-function distribution, then mixed products of variates from these distributions will also follow the *H*-function probability law.

THEOREM 4.1. *If $X_1, X_2, \cdots, X_N$ are independent H-function variates with probability density functions $f_1(x_1), f_2(x_2), \cdots, f_N(x_N)$, respectively, where*

$$f_j(x_j) = \begin{cases} k_j \mathbf{H}_{p_j,q_j}^{m_j,n_j}\left[c_j x_j \middle| \begin{matrix} (a_{j1}, \alpha_{j1}), \cdots, (a_{jp_j}, \alpha_{jp_j}) \\ (b_{j1}, \beta_{j1}), \cdots, (b_{jq_j}, \beta_{jq_j}) \end{matrix} \right], & x_j > 0, \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1, 2, \cdots, N$, then the probability density function of the variate

$$Y = \prod_{j=1}^{N} X_j$$

is given by

$$h(y)$$

$$= \begin{cases} \left(\prod_{j=1}^{N} k_j\right) \mathbf{H}_{\sum_{j=1}^{N} p_j, \sum_{j=1}^{N} q_j}^{\sum_{j=1}^{N} m_j, \sum_{j=1}^{N} n_j} \left[\prod_{j=1}^{N} c_j y_j \middle| \begin{matrix} (a_{11}, \alpha_{11}), \cdots, (a_{Np_N}, \alpha_{Np_N}) \\ (b_{11}, \beta_{11}), \cdots, (b_{Nq_N}, \beta_{Nq_N}) \end{matrix} \right], & y > 0, \\ 0 & \text{otherwise} \end{cases} \tag{4.6}$$

where the sequence of the parameters (a_{jv}, α_{jv}) is

$$v = 1, 2, \cdots, n_j \quad \text{for } j = 1, 2, \cdots, N$$

followed by

$$v = n_j + 1, n_j + 2, \cdots, p_j \quad \text{for } j = 1, 2, \cdots, N$$

and the sequence of the parameters (b_{jv}, β_{jv}) is

$$v = 1, 2, \cdots, m_j \quad \text{for } j = 1, 2, \cdots, N$$

followed by

$$v = m_j + 1, m_j + 2, \cdots, q_j \quad \text{for } j = 1, 2, \cdots, N.$$

Proof of Theorem 4.1. From (2.9), the Mellin transform of $f_j(x_j)$ is

$$\mathcal{M}_s\{f_j(x_j)\} = \frac{k_j}{c_j^s} \frac{\prod_{v=1}^{m_j} \Gamma(b_{jv} + \beta_{jv}s) \prod_{v=1}^{n_j} \Gamma(1 - a_{jv} - \alpha_{jv}s)}{\prod_{v=m_j+1}^{q_j} \Gamma(1 - b_{jv} - \beta_{jv}s) \prod_{v=n_j+1}^{p_j} \Gamma(a_{jv} + \alpha_{jv}s)} \tag{4.7}$$

and, using (4.3), it follows that

$$h^+(y) = \mathcal{M}^{-1}\left[\prod_{j=1}^{N} \mathcal{M}_s\{f_j(x_j)\}\right]$$

$$= \mathcal{M}^{-1}\left[\prod_{j=1}^{N} \left(\frac{k_j}{c_j^s} \frac{\prod_{v=1}^{m_j} \Gamma(b_{jv} + \beta_{jv}s) \prod_{v=1}^{n_j} \Gamma(1 - a_{jv} - \alpha_{jv}s)}{\prod_{v=m_j+1}^{q_j} \Gamma(1 - b_{jv} - \beta_{jv}s) \prod_{v=n_j+1}^{p_j} \Gamma(a_{jv} + \alpha_{jv}s)}\right)\right].$$

Hence, due to the definition of the inverse Mellin transform (equation (4.2)), the above equation can be written as

$$h^+(y)=\frac{\prod_{j=1}^N k_j}{2\pi i}$$

$$\cdot\int\frac{\prod_{j=1}^N\prod_{v=1}^{m_j}\Gamma(b_{jv}+\beta_{jv}s)\prod_{j=1}^N\prod_{v=1}^{m_j}\Gamma(1-a_{jv}-\alpha_{jv}s)}{\prod_{j=1}^N\prod_{v=m_j+1}^{q_j}\Gamma(1-b_{jv}-\beta_{jv}s)\prod_{j=1}^N\prod_{v=n_j+1}^{p_j}\Gamma(a_{jv}+\alpha_{jv}s)}\cdot$$

$$\cdot\left(\left[\prod_{j=1}^N c_j\right]y\right)^{-s} ds$$

$$=\left(\prod_{j=1}^N k_j\right)\mathbf{H}^{\sum_{j=1}^N m_j,\,\sum_{j=1}^N n_j}_{\sum_{j=1}^N p_j,\,\sum_{j=1}^N q_j}\left[\prod_{j=1}^N c_j y_j\left|\begin{matrix}(a_{11},\alpha_{11}),\cdots,(a_{Np_N},\alpha_{Np_N})\\(b_{11},\beta_{11}),\cdots,(b_{Nq_N},\beta_{Nq_N})\end{matrix}\right.\right]$$

which completes the proof of Theorem 4.1.

Example 1. The product of N beta variables. Suppose that, in Theorem 4.1, $X_1, X_2, \cdots, X_N$ are all beta variables having the probability density function shown in (3.7), where, when written in terms of (4.6),

$$k_j=\frac{\Gamma(\theta_j+\phi_j)}{\Gamma(\theta_j)},$$
$$a_{j1}=\theta_j+\phi_j-1,$$
$$b_{j1}=\theta_j-1,$$
$$\alpha_{j1}=\beta_{j1}=1,$$
$$c_j=1,$$

and

$$m_j=1,\quad n_j=0,\quad p_j=1,\quad q_j=1$$

for $j=1, 2, \cdots, N$. Then, substituting into (4.6) of Theorem 4.1, one has

$$h(y)=\begin{cases}\left(\prod_{j=1}^N\frac{\Gamma(\theta_j+\phi_j)}{\Gamma(\theta_j)}\right)\mathbf{H}^{N,0}_{N,N}\left[y\left|\begin{matrix}(\theta_1+\phi_1-1,1),\cdots,(\theta_N+\phi_N-1,1)\\(\theta_1-1,1),\cdots,(\theta_N-1,1)\end{matrix}\right.\right], & y>0,\\ 0 & \text{otherwise.}\end{cases}$$

Application of the identity (2.8) now gives

$$\text{(4.8)}\quad h(y)=\begin{cases}\prod_{j=1}^N\frac{\Gamma(\theta_j+\phi_j)}{\Gamma(\theta_j)}\mathbf{G}^{N,0}_{N,N}\left[y\left|\begin{matrix}\theta_1+\phi_1-1,\cdots,\theta_N+\phi_N-1\\\theta_1-1,\cdots,\theta_N-1\end{matrix}\right.\right], & y>0,\\ 0 & \text{otherwise.}\end{cases}$$

That is, the probability density function of the product of N independent beta variates is given by (4.8). This result agrees with that of Lomnicki [10] and that of Springer and Thompson [17, p. 731].

Example 2. The product of N gamma variables. Now suppose that, in Theorem 4.1, $X_1, X_2, \cdots, X_N$ are all gamma variates having the probability density function given in (3.4) where, when written in terms of (4.6),

$$k_j = \frac{1}{\phi_j \Gamma(\theta_j)},$$
$$a_{j1} = \theta_j - 1,$$
$$\alpha_{j1} = 1,$$
$$c_{j1} = \phi_j^{-1},$$

and

$$m_j = 1, \quad n_j = 0, \quad p_j = 0, \quad q_j = 1$$

for $j = 1, 2, \cdots, N$. Then from (4.6),

$$h(y) = \begin{cases} \left(\prod_{j=1}^{N} \frac{1}{\phi_j \Gamma(\theta_j)}\right) \mathbf{H}_{0,N}^{N,0}\left[\left(\prod_{j=1}^{N} \phi_j^{-1}\right) y \,\middle|\, (\theta_1 - 1, 1), \cdots, (\theta_N - 1, 1)\right], & y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

which, upon application of (2.8), becomes

$$(4.9) \quad h(y) = \begin{cases} \left(\prod_{j=1}^{N} \frac{1}{\phi_j \Gamma(\theta_j)}\right) \mathbf{G}_{0,N}^{N,0}\left[\left(\prod_{j=1}^{N} \phi_j^{-1}\right) y \,\middle|\, \theta_1 - 1, \cdots, \theta_N - 1\right], & y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, (4.9) expresses the probability density function of the product of N independent gamma variables in terms of Meijer G-functions. Equation (4.9) agrees with the result obtained by Springer and Thompson [17, p. 722].

4.3. The distribution of rational powers of *H*-function variates. Another important property of the H-function distribution is the fact that a rational power of an H-function variate also follows an H-function distribution, as the following theorem shows.

THEOREM 4.2. *If X is an H-function variate with probability density function*

$$f(x) = \begin{cases} k \mathbf{H}_{p,q}^{m,n}\left[cx \,\middle|\, \begin{matrix} (a_1, \alpha_1), \cdots, (a_p, \alpha_p) \\ (b_1, \beta_1), \cdots, (b_q, \beta_q) \end{matrix}\right], & x > 0, \\ 0 & \textit{otherwise,} \end{cases}$$

then the probability density function of the variate

$$Y = X^P$$

is given by

$$h(y)$$

$$(4.10) = \begin{cases} kc^{P-1} \mathbf{H}_{p,q}^{m,n}\left[c^P y \,\middle|\, \begin{matrix} (a_1 - \alpha_1 P + \alpha_1, \alpha_1 P), \cdots, (a_p - \alpha_p P + \alpha_p, \alpha_p P) \\ (b_1 - \beta_1 P + \beta_1, \beta_1 P), \cdots, (b_q - \beta_q P + \beta_q, \beta_q P) \end{matrix}\right], & y > 0, \\ 0 & \textit{otherwise} \end{cases}$$

when $P>0$, *and*

(4.11)
$$h(y) = \begin{cases} kc^{P-1}\mathbf{H}_{q,p}^{m,n}\left[c^P y \left| \begin{matrix} (1-b_1+\beta_1 P-\beta_1, -\beta_1 P), \cdots, (1-b_p+\beta_p P-\beta_p, -\beta_p P) \\ (1-a_1+\alpha_1 P-\alpha_1, -\alpha_1 P), \cdots, (1-a_p+\alpha_p P-\alpha_p, -\alpha_p P) \end{matrix} \right.\right], & y>0, \\ 0 & \textit{otherwise} \end{cases}$$

when $P<0$.

Proof of Theorem 4.2. Equation (4.4) shows that the positive part, $h^+(y)$, of the probability density function of Y is given by

$$h^+(y) = \mathcal{M}^{-1}\left[\mathcal{M}_{Ps-P+1}\left(k\cdot \mathbf{H}_{p,q}^{m,n}\left[cx \left| \begin{matrix} (a_1, \alpha_1), \cdots, (a_p, \alpha_p) \\ (b_1, \beta_1), \cdots, (b_q, \beta_q) \end{matrix} \right.\right]\right)\right],$$

whilst from (2.9) and (4.7),

$$h^+(y) = \mathcal{M}^{-1}\left[\frac{k}{c^t}\frac{\prod_{v=1}^m \Gamma(b_v+\beta_v t)\prod_{j=1}^n \Gamma(1-a_v-\alpha_v t)}{\prod_{v=m+1}^q \Gamma(1-b_v-\beta_v t)\prod_{v=n+1}^p \Gamma(a_v+\alpha_v t)}\bigg|_{t=Ps-P+1}\right]$$

$$= \mathcal{M}^{-1}\left[\frac{k}{c^{Ps-P+1}}\frac{\prod_{v=1}^m \Gamma(b_v+\beta_v P+\beta_v+\beta_v Ps)}{\prod_{v=M+1}^q \Gamma(1-b_v+\beta_v P-\beta_v-\beta_v Ps)} \cdot \frac{\prod_{v=1}^n \Gamma(1-a_v+\alpha_v P-\alpha_v-\alpha_v Ps)}{\prod_{v=m+1}^p \Gamma(a_v-\alpha_v P+\alpha_v-\alpha_v Ps)}\right].$$

Application of (4.2) then yields

$$h^+(y) = \frac{kc^{P-1}}{2\pi i}\int_C \frac{\prod_{v=1}^m \Gamma(b_v-\beta_v P+\beta_v+\beta_v Ps)}{\prod_{v=M+1}^q \Gamma(1-b_v+\beta_v P-\beta_v-\beta_v Ps)} \cdot \frac{\prod_{v=1}^n \Gamma(1-a_v+\alpha_v P-\alpha_v-\alpha_v Ps)}{\prod_{v=n+1}^p \Gamma(a_v-\alpha_v P+\alpha_v+\alpha_v Ps)}(c^P y)^{-s}\,ds$$

and from the definition of the H-function, it follows that

$$h^+(y) = kc^{P-1}\mathbf{H}_{p,q}^{m,n}\left[c^P y \left| \begin{matrix} (a_1-\alpha_1 P+\alpha_1, \alpha_1 P), \cdots, (a_p-\alpha_p P+\alpha_p, \alpha_p P) \\ (b_1, \beta_1 P+\beta_1 P+\beta_1, \beta_1 P), \cdots, (b_q-\beta_q P+\beta_q, \beta_q P) \end{matrix} \right.\right]$$

when $P>0$, and

$$h^+(y) = kc^{P-1}\mathbf{H}_{q,p}^{n,m}\left[c^P y \left| \begin{matrix} (1-b_1+\beta_1 P-\beta_1, -\beta_1 P), \cdots, (1-b_q+\beta_q P-\beta_q, -\beta_q P) \\ (1-a_1+\alpha_1 P-\alpha_1, -\alpha_1 P), \cdots, (1-a_p+\alpha_p P-\alpha_p, -\alpha_p P \end{matrix} \right.\right]$$

when $P<0$.

Example 3. The square of a standard half-normal variable. Suppose that, in Theorem 4.2, the variate X has a standard half-normal distribution with the probability density function given in (3.8) with $\theta = 1$ where, when written in the form of (4.10),

$$k = \frac{1}{\sqrt{2\pi}},$$

$$b_1 = 0,$$

$$\beta_1 = 1/2,$$

$$c = 1/\sqrt{2},$$

and

$$m = 1, \quad n = 0, \quad p = 0, \quad q = 1.$$

Then, from Theorem 4.2, the probability density function of

$$Y = X^2$$

is given by

$$h(y) = \begin{cases} \dfrac{1}{2\sqrt{\pi}} \mathbf{H}_{0,1}^{1,0}\left[\dfrac{1}{2}y \middle| \left(-\dfrac{1}{2}, 1\right)\right], & y > 0, \\ 0 & \text{otherwise}. \end{cases} \tag{4.12}$$

or, with the use of (2.8),

$$h(y) \doteq \begin{cases} \dfrac{1}{2\sqrt{\pi}} \mathbf{G}_{0,1}^{1,0}\left[\dfrac{1}{2}y \middle| -\dfrac{1}{2}\right], & y > 0, \\ 0 & \text{otherwise}. \end{cases}$$

By examining (3.10), one can readily see that (4.12) is the probability density function for the chi-square distribution with the parameter v set to 1. This result agrees with the well-known fact that the square of either a standard normal variate or a standard half-normal variate follows a chi-square distribution.

4.4. The distribution of quotients of H-function variates. From (4.5) and Theorems 4.1 and 4.2, one obtains yet another important property of the H-function distribution, namely, that quotients of independent H-function variates also follow an H-function distribution. This result is stated formally in the following theorem.

THEOREM 4.3. *If X_1 and X_2 are independent H-function variates with probability density functions $f_1(x_1)$ and $f_2(x_2)$, respectively, where*

$$f_j(x_j) = \begin{cases} k_j \mathbf{H}_{p_j,q_j}^{m_j,n_j}\left[c_j x_j \middle| \begin{matrix} (a_{j1}, \alpha_{j1}), \cdots, (a_{jp_j}, \alpha_{jp_j}) \\ (b_{j1}, \beta_{j1}), \cdots, (b_{jq_j}, \beta_{jq_j}) \end{matrix}\right], & x_j > 0, \\ 0 & \textit{otherwise} \end{cases}$$

for $j = 1, 2$, then the probability density function of the variate

$$Y = X_1/X_2$$

is given by

$$(4.13)\qquad h(y)=\begin{cases}\dfrac{k_1k_2}{c_2^2}\mathbf{H}_{p_1+q_2,q_1+p_2}^{m_1+n_2,n_1+m_2}\left[\dfrac{c_1}{c_2}y\,\middle|\,\begin{matrix}(a_{11},\alpha_{11}),\cdots\\(b_{11},\beta_{11}),\cdots\end{matrix}\right], & y>0,\\ 0 & \textit{otherwise},\end{cases}$$

where the sequence of the H-function parameters is

$$(a_{11},\alpha_{11}),\cdots,(a_{1n_1},\alpha_{1n_1}),(1-b_{21}-2\beta_{21},\beta_{21}),\cdots,$$
$$(1-b_{2m_2}-2\beta_{2m_2},\beta_{2m_2}),(a_{1,n_1+1},\alpha_{1,n_1+1}),\cdots,(a_{1p_1},\alpha_{1p_1}),$$
$$(1-b_{2,m_2+1}-2\beta_{2,m_2+1},\beta_{2,m_2+1}),\cdots,(1-b_{2q_2}-2\beta_{2q_2},\beta_{2q_2}),$$

and

$$(b_{11},\beta_{11}),\cdots,(b_{1m_1},\beta_{1m_1}),(1-a_{21}-2\alpha_{21},\alpha_{21}),\cdots,$$
$$(1-a_{2n_2}-2\alpha_{2n_2},\alpha_{2n_2}),(b_{1m_1+1},\beta_{1,m_1+1}),\cdots,(b_{1q_1},\beta_{1q_1}),$$
$$(1-a_{2,n_2+1}-2\alpha_{2,n_2+1},\alpha_{2,n_2+1}),\cdots,(1-a_{2p_2}-2\alpha_{2p_2},\alpha_{2p_2}).$$

Proof of Theorem 4.3. From (4.5), the component of the probability density function of Y, which is obtained for nonnegative values of Y is given by

$$h^+(y)=\mathcal{M}^{-1}[\mathcal{M}_s\{f_1(x_1)\}\mathcal{M}_{2-s}\{f_2(x_2)\}]$$

or, from (4.7),

$$h^+(y)=\mathcal{M}^{-1}\Bigg[\frac{k_1}{c_1^s}\,\frac{\prod_{v=1}^{m_1}\Gamma(b_{1v}+\beta_{1v}s)\prod_{v=1}^{n_1}\Gamma(1-a_{1v}-\alpha_{1v}s)}{\prod_{v=m_1+1}^{q_1}\Gamma(1-b_{1v}-\beta_{1v}s)\prod_{v=n_1+1}^{p_1}\Gamma(a_{1v}+\alpha_{1v}s)}$$
$$\cdot\frac{k_2}{c_2^{2-s}}\,\frac{\prod_{v=1}^{m_2}\Gamma(b_{2v}+2\beta_{2v}-\beta_{2v}s)\prod_{v=1}^{n_2}\Gamma(1-a_{2v}-2\alpha_{2v}+\alpha_{2v}s)}{\prod_{v=m_2+1}^{q_2}\Gamma(1-b_{2v}-2\beta_{2v}+\beta_{2v}s)\prod_{v=n_2+1}^{p_2}\Gamma(a_{2v}+2\alpha_{2v}-\alpha_{2v}s)}\Bigg].$$

Rearranging and writing in terms of the Mellin inversion integral (equation (4.2)) yields

$$h^+(y)=$$
$$\frac{k_1k_2}{c_2^2}\cdot\frac{1}{2\pi i}\int_C\frac{\prod_{v=1}^{n_1}\Gamma(1-a_{1v}-\alpha_{1v}s)\prod_{v=1}^{m_2}\Gamma(b_{2v}+2\beta_{2v}-\beta_{2v}s)}{\prod_{v=n_1+1}^{p_1}\Gamma(a_{1v}+\alpha_{1v}s)\prod_{v=m_2+1}^{q_2}\Gamma(1-b_{2v}-2\beta_{2v}+\beta_{2v}s)}$$
$$\cdot\frac{\prod_{v=1}^{m_1}\Gamma(b_{1v}+\beta_{1v}s)\prod_{v=1}^{n_2}\Gamma(1-a_{2v}-2\alpha_{2v}-\alpha_{2v}s)}{\prod_{v=m_1+1}^{q_1}\Gamma(1-b_{1v}-\beta_{1v}s)\prod_{v=n_2+1}^{p_2}\Gamma(a_{2v}+2\alpha_{2v}-\alpha_{2v}s)}\left(\frac{c_1}{c_2}y\right)^{-s}ds$$
$$=\frac{k_1k_2}{c_2^2}\mathbf{H}_{p_1+q_2,q_1+p_2}^{m_1+n_2,n_1+m_2}\left[\frac{c_2}{c_1}y\,\middle|\,\begin{matrix}(a_{11},\alpha_{11}),\cdots\\(b_{11},\beta_{11}),\cdots\end{matrix}\right]$$

where the sequence of the parameters of the H-function is that given in Theorem 4.3.

Example 4. The quotient of two half-normal variables. Suppose that, in Theorem 4.3, X_1 and X_2 are half-normal variates having the probability density function given in (3.8) where, when written in the form of (4.13),

$$k_j = \frac{1}{\theta_j\sqrt{2\pi}},$$

$$a_{j1} = 0,$$

$$\alpha_{j1} = 1/2,$$

$$c_j = \frac{1}{\theta_j\sqrt{2}}$$

and

$$m_j = 1, \quad n_j = 0, \quad p_j = 0, \quad q_j = 1,$$

for $j = 1, 2$. Then substituting into (4.13) of Theorem 4.3, we get

$$h(y) = \begin{cases} \dfrac{\theta_2}{\theta_1\pi} \cdot \mathbf{H}_{1,1}^{1,1}\left[\dfrac{\theta_2}{\theta_1}y \,\middle|\, \begin{matrix}(0, 1/2)\\(0, 1/2)\end{matrix}\right], & y > 0, \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

which, when compared to (3.12), is recognized to be the probability density function of the half-Cauchy distribution.

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