On portfolio risk diversification

Hellinton H. Takada and Julio M. Stern

Citation: AIP Conference Proceedings **1853**, 070002 (2017); doi: 10.1063/1.4985363
View online: http://dx.doi.org/10.1063/1.4985363
View Table of Contents: http://aip.scitation.org/toc/apc/1853/1
Published by the American Institute of Physics

On Portfolio Risk Diversification

Hellinton H. Takada[1,2,a)] and Julio M. Stern[2]

[1]*Quantitative Research, Itaú Asset Management, São Paulo, Brazil*
[2]*Institute of Mathematics and Statistics, University of São Paulo, Brazil*

[a)]Corresponding author: hellistaka@yahoo.com.br

Abstract. The first portfolio risk diversification strategy was put into practice by the *All Weather* fund in 1996. The idea of risk diversification is related to the risk contribution of each available asset class or investment factor to the total portfolio risk. The maximum diversification or the risk parity allocation is achieved when the set of risk contributions is given by a uniform distribution. Meucci (2009) introduced the maximization of the Rényi entropy as part of a leverage constrained optimization problem to achieve such diversified risk contributions when dealing with uncorrelated investment factors. A generalization of the risk parity is the risk budgeting when there is a *prior* for the distribution of the risk contributions. Our contribution is the generalization of the existent optimization frameworks to be able to solve the risk budgeting problem. In addition, our framework does not possess any leverage constraint.

INTRODUCTION

Conceptually, asset allocation is an investment strategy and consists in determining the proportion of each asset class or investment factor in the portfolio. Particularly, in the portfolio risk diversification methodologies, the idea of diversification is related to the risk contribution of each available asset class or investment factor to the total portfolio risk. The maximum diversification or the risk parity allocation is achieved when the set of risk contributions is given by a uniform distribution. Meucci (2009) [1] introduced and Deguest et al. (2013) [2] advanced in the maximization of the Rényi entropy [3] as part of a leverage constrained optimization problem to achieve such diversified risk contributions when dealing with uncorrelated investment factors. A generalization of the risk parity is the risk budgeting when there is a *prior* for the distribution of the risk contributions. In this paper, we generalize the existent optimization frameworks to be able to solve the risk budgeting problem. In addition, we do not impose any leverage constraint.

The paper is organized as follows. First, we give an overview about risk-based asset allocation methodologies and, then, we introduce the idea of risk diversification. After presenting the main concepts involved, we review the definition of the restricted factor risk parity portfolios and propose the generalized factor risk parity portfolios with the correspondent optimization problem. Evolving the presented ideas, we extend further the concepts and propose the generalized factor risk budgeting portfolios with the correspondent optimization problem. Finally, we present some conclusions at the end.

RISK-BASED ASSET ALLOCATION

After the subprime crisis, the risk tolerance of investors decreased and risk-based allocation methodologies have arisen with the idea of combining both risk management and asset allocation. A risk-based allocation seeks for risk diversification and does not use performance forecasts of assets as inputs of the methodologies. Risk parity is the most widely used risk-based allocation methodology and has been used by several fund managers. Starting in 1996, one of the pioneers to use risk parity was the *All Weather* hedge fund from Bridgewater. The risk contribution of each asset to the total portfolio risk is made equal in the risk parity portfolio (or equal risk contribution portfolio). Consequently, risk parity is a risk-based allocation such that the distribution of the risk contribution of each asset to the total portfolio risk is uniform. The risk parity portfolio has been extensively studied in terms of its properties [4] and compared to other asset allocation heuristics [5]. For a complete review of the literature concerning risk parity see [6].

Bayesian Inference and Maximum Entropy Methods in Science and Engineering
AIP Conf. Proc. 1853, 070002-1–070002-6; doi: 10.1063/1.4985363
Published by AIP Publishing. 978-0-7354-1527-0/$30.00

On the other hand, the investor does not need to manage the risk contributions in a uniform way as the risk parity requires. Risk budgeting or targeting is the risk-based allocation such that the risk contributions of each asset class or investment factor are not equal. An example of risk budgeting was investigated in [7] where they have studied a portfolio in which the risk contribution of each sovereign bond from a set of countries is done proportional to the gross domestic product (GDP) of the respective country. A theoretical and applied study about risk budgeting techniques is presented in [8]. Since a risk parity portfolio is a risk budgeting portfolio when the risk budgets or targets are all equal, risk budgeting is a generalization of risk parity. In the literature, there are some proposed optimization frameworks to obtain the risk budgeting and, consequently, risk parity portfolios [8][9].

The existence and uniqueness of the solution for the optimization problems is proved for particular cases with the imposition of some restrictions such as long-only allocations [8][9], impossibility of leveraging the portfolio [8] and an upper bound for portfolio volatility [9]. In this paper, these restrictions will be relaxed. In addition, the objective functions used in the optimizations from [8][9] do not possess a strong theoretical basis related to the idea of maximizing risk diversification. Alternatively, the use of information theory was already addressed by Meucci (2009) [1] with the introduction of a measure called effective number of bets ENB_α based on Rényi entropy [3] with parameter α. The maximization of the objective function ENB_α brings the idea of maximizing risk diversification. Consequently, the approach from [1] solves the lack of a strong theoretical basis of the optimization problem inherent from previous approaches.

The use of asset classes instead of individual assets for risk budgeting has also been addressed in the literature [10]. Actually, the application of the existent results from individual assets to asset classes is straightforward. However, the use of investment factors instead of individual assets is not so direct and possesses several interesting implications [11]. In [2][12][13], they have applied principal component analysis (PCA) to extract uncorrelated factors and analyze the performance of the called factor risk parity (FRP) portfolios. A FRP portfolio is the application of risk parity to a portfolio of uncorrelated factors. In [2], they have developed an optimization framework for FRP portfolios using the concept of ENB_α. However, their approach has a leverage restriction and does not include the most general risk budgeting case. In this paper, we introduce the factor risk budgeting (FRB) portfolios without any leverage restriction generalizing the FRP portfolios. The FRB portfolio enables the inclusion of *priors* or beliefs related to the future performance of assets in the risk-based allocation process.

RISK DIVERSIFICATION

The risk-based allocations depend on some measure of each individual asset, asset class or investment factor risk contribution to the total portfolio risk. The relative marginal contribution of each asset to the portfolio risk is suitable for such a purpose. In this paper, asset or asset class are equivalent in terms of the results presented.

Definition 1. (*relative marginal contribution for assets*) Considering n available risky assets, the relative marginal contribution of each asset to the total portfolio risk is given by the following $n \times 1$ vector:

$$\mathbf{p} := \frac{\text{diag}(\mathbf{w})\Sigma\mathbf{w}}{\mathbf{w}'\Sigma\mathbf{w}}, \tag{1}$$

where Σ is the $n \times n$ covariance matrix of risky assets' excess returns, $\mathbf{w}$ is the vector of weights such that $\mathbf{w} \in \Re^{n\times 1}_{\neq \mathbf{0}_{n\times 1}}$ and $\text{diag}(\mathbf{w})$ is a diagonal matrix with $\mathbf{w}$ as its diagonal.

In the previous definition, the assets considered are risky while the risk free asset is a rate of return used to discount the risky assets' returns to result the excess returns. Additionally, $\mathbf{w} \neq \mathbf{0}_{n\times 1}$ because $\mathbf{w} = \mathbf{0}_{n\times 1}$ represents the absence of allocation in the risky assets and, then, it is not a desired solution. Finally, it is clear that $\sum_{i=1}^{n} p_i = 1$.

Usually, in the literature, a risk budgeting portfolio $\mathbf{w}_{RB}$ diversifies risk being the solution of the following optimization problem:

$$\mathbf{w}_{RB} = \arg\min_{\mathbf{p}} \sum_{i=1}^{n} (p_i - b_i)^2, \tag{2}$$

where $b_i, i = 1, \ldots, n$ are the risk budgets under the restrictions $b_i \geq 0, \forall i = 1, \ldots, n$ and $\sum_{i=1}^{n} b_i = 1$. The risk budgeting portfolio $\mathbf{w}_{RB}$ reduces to the risk parity portfolio $\mathbf{w}_{RP}$ when $b_i = 1/n, \forall i = 1, 2, \ldots, n$, i.e. risk budgeting portfolio definition generalizes risk parity portfolio definition. The optimization problem (2) was addressed in several papers and under certain constraints is guaranteed to possess a unique solution [8][9].

As we have already mentioned, Meucci (2009) [1] proposed the use of the Rényi entropy with parameter α as a measure of diversification and the measure was called the effective number of bets ENB_α. Instead of using assets or asset classes, ENB_α uses the distribution of the relative marginal risk contribution of each uncorrelated factor as a measure of risk diversification. In practical terms, the uncorrelated factors can be obtained using principal component analysis (PCA). Considering T excess returns of the risky assets represented by the $n \times T$ matrix $\mathbf{r}$, the $n \times T$ matrix of uncorrelated factors is given by

$$\mathbf{r}_F = \mathbf{A}'\mathbf{r} \tag{3}$$

and the corresponding covariance matrix is given by

$$\Sigma_F = \mathbf{A}'\Sigma\mathbf{A}, \tag{4}$$

where Σ_F is a positive definite $n \times n$ diagonal matrix and $\mathbf{A}$ is an invertible $n \times n$ matrix. Consequently, we define in the following the relative marginal contribution of each uncorrelated factor to the total portfolio risk analogously to (1).

Definition 2. (*relative marginal contribution for uncorrelated factor*) Considering n available risky assets, the relative marginal contribution of each uncorrelated factor to the total portfolio risk is given by the following $n \times 1$ vector:

$$\mathbf{p}_F := \frac{\text{diag}(\mathbf{w}_F)\Sigma_F\mathbf{w}_F}{\mathbf{w}_F'\Sigma_F\mathbf{w}_F}. \tag{5}$$

where Σ_F is the $n \times n$ diagonal covariance matrix (4), $\mathbf{w}_F$ is the vector of weights such that $\mathbf{w}_F \in \Re^{n\times 1}_{\neq \mathbf{0}_{n\times 1}}$ and $\text{diag}(\mathbf{w}_F)$ is a diagonal matrix with $\mathbf{w}_F$ as its diagonal.

Since $\sum_{i=1}^{n} p_{F,i} = 1$ and $p_{F,i} \geq 0, \forall i = 1, \ldots, n$, it is possible to notice that $\mathbf{p}_F$ represents a probability mass distribution. Using (5), the $\text{ENB}_\alpha(\cdot)$ is defined in the following.

Definition 3. (*effective number of bets*) The effective number of bets of portfolio $\mathbf{w}$ of n risky assets is given by

$$\text{ENB}_\alpha(\mathbf{w}) := \left\|\mathbf{p}_F\right\|_\alpha^{\frac{\alpha}{1-\alpha}}, \alpha \geq 0, \alpha \neq 1, \tag{6}$$

where the $\|\cdot\|_\alpha$ is the α-norm.

By definition,

$$\left\|\mathbf{p}_F\right\|_\alpha = \left(\sum_{i=1}^{n} p_{F,i}^{\alpha}\right)^{\frac{1}{\alpha}}. \tag{7}$$

Then,

$$\text{ENB}_\alpha(\mathbf{w}) = \left(\sum_{i=1}^{n} p_{F,i}^{\alpha}\right)^{\frac{1}{1-\alpha}}, \alpha \geq 0, \alpha \neq 1. \tag{8}$$

It is important to notice that $\log(\text{ENB}_\alpha(\mathbf{w}))$ is the Rényi entropy $H_\alpha(\mathbf{p}_F)$. The Rényi entropy generalizes the Hartley $H_0(\cdot)$ ($\alpha = 0$), Shannon $H_1(\cdot)$ ($\alpha \to 1$), collision $H_2(\cdot)$ ($\alpha = 2$) and min $H_\infty(\cdot)$ ($\alpha \to \infty$) entropy measures. It is interesting to notice that we write ENB_α as a function of $\mathbf{w}$ since $\mathbf{p}_F$ is a function of $\mathbf{w}_F$ (see (5)) and $\mathbf{w}_F$ is a function of $\mathbf{w}$ (see the next property).

Property 1. $\mathbf{w}_F = \mathbf{A}^{-1}\mathbf{w}$.

Proof of Property 1. The excess return of the portfolio in terms of assets or uncorrelated factors is the same: $\mathbf{w}'\mathbf{r} = \mathbf{w}_F'\mathbf{r}_F$. Using (3), $\mathbf{w}'\mathbf{r} = \mathbf{w}_F'\mathbf{A}'\mathbf{r} \Rightarrow \mathbf{w} = \mathbf{A}\mathbf{w}_F$. Finally, $\mathbf{w}_F = \mathbf{A}^{-1}\mathbf{w}$.

■

The ENB_α measure achieves its minimum equal to 1 when the portfolio is risk concentrated in only one factor. On the other hand, the ENB_α measure achieves its maximum when the portfolio is totally risk diversified with $p_{F,i} = 1/n, \forall i = 1, \dots, n$. The following property concerning ENB_α is presented because it will be useful in the next section.

Property 2. $\mathrm{ENB}_\alpha(\lambda\mathbf{w}) = \mathrm{ENB}_\alpha(\mathbf{w}), \forall\lambda \in \Re_{\neq 0}$.

Proof of Property 2. It is straightforward to state that $\mathrm{ENB}_\alpha(\lambda\mathbf{w}) = \mathrm{ENB}_\alpha(\mathbf{p}_F(\mathbf{w}_F(\lambda\mathbf{w})))$. Using **Property 1**, $\mathrm{ENB}_\alpha(\lambda\mathbf{w}) = \mathrm{ENB}_\alpha(\mathbf{p}_F(\lambda\mathbf{A}^{-1}\mathbf{w}))$. Additionally, using (5), it is trivial to see that $\mathbf{p}_F(\lambda\mathbf{w}_F) = \mathbf{p}_F(\mathbf{w}_F), \forall\lambda \in \Re_{\neq 0}$. Consequently, $\mathrm{ENB}_\alpha(\lambda\mathbf{w}) = \mathrm{ENB}_\alpha(\mathbf{p}_F(\mathbf{A}^{-1}\mathbf{w})) = \mathrm{ENB}_\alpha(\mathbf{p}_F(\mathbf{w}_F)) = \mathrm{ENB}_\alpha(\mathbf{p}_F(\mathbf{w}_F(\mathbf{w}))) = \mathrm{ENB}_\alpha(\mathbf{w}), \forall\lambda \in \Re_{\neq 0}$.

■

FACTOR RISK PARITY PORTFOLIOS

The factor risk parity (FRP) portfolios were defined in [2] using the $\mathrm{ENB}_\alpha(\cdot)$ measure. The weights of the uncorrelated factors of FRP portfolios are defined so as to equalize the relative contribution of each factor to the total portfolio variance. Since the FRP portfolios from [2] were developed under a leverage restriction, we are going to refer to them as restricted FRP (RFRP) portfolios. On the other hand, we are going to present a unrestricted family of FRP portfolios referred here as generalized FRP (GFRP) portfolios.

Restricted Factor Risk Parity Portfolios

Using the $\mathrm{ENB}_\alpha(\cdot)$ measure, the optimization problem from [2] to obtain the RFRP portfolios is

$$\max_{\mathbf{w}} \mathrm{ENB}_\alpha(\mathbf{w}), \text{ s.t. } \mathbf{1}_n'\mathbf{w} = 1, \tag{9}$$

where $\mathbf{1}_n$ is a $n \times 1$ vector of ones. It is important to notice that the restriction in (9) is a non-leverage constraint or budget condition and the optimization problem does not possess a unique solution.

Theorem 1. (*RFRP portfolios*) The family of RFRP portfolios is given by

$$\mathbf{w}_{RFRP} = \frac{\mathbf{A}\Sigma_F^{-\frac{1}{2}}\iota_n}{\mathbf{1}_n'\mathbf{A}\Sigma_F^{-\frac{1}{2}}\iota_n}, \tag{10}$$

where $\iota_n := \begin{pmatrix} \pm 1 & \cdots & \pm 1 \end{pmatrix}'$ is a vector of size $n \times 1$ representing all the combinations of ± 1.

Proof of Theorem 1. It was shown in [2] that the closed-form expression for the solutions of (9) is given by (10).

■

It is important to notice that (10) implies in 2^{n-1} possible solutions (for details, see [2]). Additionally, the portfolios with only positive signs coincide with the solutions provided by [4].

Generalized Factor Risk Parity Portfolios

As we have already mentioned, the non-leverage constraint in (9) is easily relaxed. Our optimization problem to obtain the GFRP portfolios is given by

$$\max_{\mathbf{w}} \mathrm{ENB}_\alpha(\mathbf{w}). \tag{11}$$

Theorem 2. (*GFRP portfolios*) The family of GFRP portfolios is given by

$$\mathbf{w}_{GFRP} = \lambda\frac{\mathbf{A}\Sigma_F^{-\frac{1}{2}}\iota_n}{\mathbf{1}_n'\mathbf{A}\Sigma_F^{-\frac{1}{2}}\iota_n}, \lambda \in \Re_{\neq 0}. \tag{12}$$

Proof of Theorem 2. It is straightforward to prove (12). Considering a modified version of the optimization problem (9): $\mathbf{w}_\lambda = \arg\max_{\mathbf{w}} \mathrm{ENB}_\alpha(\mathbf{w}), \text{ s.t. } \mathbf{1}_n'\mathbf{w} = \lambda, \lambda \in \Re_{\neq 0}$, the solution using **Property 2** is $\mathbf{w}_\lambda = \mathbf{w}_{RFRP}\lambda$. Consequently, it is necessary to solve the problem for each $\lambda \in \Re_{\neq 0}$ to obtain (12).

∎

Since the volatility of the portfolio is given by $\sigma_{RFRP} := \sqrt{\mathbf{w}'_{RFRP}\Sigma\mathbf{w}_{RFRP}}$, the volatility of $\mathbf{w}_{GFRP}(\lambda)$ is $\sigma_{GFRP}(\lambda) = |\lambda|\,\sigma_{RFRP}$. It is important to notice that unconstraining the problem, it is possible to obtain risk parity portfolios for any desired volatility or, in other words, risk level. Consequently, our GFRP portfolios are important in terms of asset allocation because it will adapt better to the investor's risk preference than the RFRP portfolios.

FACTOR RISK BUDGET PORTFOLIOS

From [7][8][9], it is clear that the risk budgeting portfolios are more flexible than the risk parity portfolios to adapt to investor's needs. In this section, the FRP is extended to factor risk budgeting (FRB). Analogously to the previous section, it would be possible to present first a leverage restricted FRB and, then, a generalized FRB. Due to the lack of space, we present only the non-restricted FRB that we call shortly by FRB. Our optimization problem to obtain the FRB portfolios is given by

$$\min_{\mathbf{w}} D_\alpha\left(\mathbf{p}_F \| \mathbf{b})\right), \tag{13}$$

where $\mathbf{b}$ is a $n \times 1$ vector containing the budgets or targets $b_i, i = 1, \ldots, n$ such that $\sum_{i=1}^n b_i = 1$ and $b_i \geq 0$, and $D_\alpha\left(\mathbf{p}_F \| \mathbf{b})\right)$ is the Rényi divergence or α-divergence given by

$$D_\alpha\left(\mathbf{p}_F \| \mathbf{b})\right) = \frac{1}{\alpha - 1} \log\left(\sum_{i=1}^n \frac{p_{F,i}^\alpha}{b_i^{\alpha-1}}\right), \tag{14}$$

where $0 < \alpha < \infty$ and $\alpha \neq 1$. Considering $\alpha \to 1$, the Rényi divergence becomes the Kullback-Leibler divergence.

Obviously, $\mathbf{b}$ works like a probability mass distribution and represents a *prior* to the risk allocation process. Additionally, it is trivial to prove that when $\mathbf{b}$ is uniformly distributed the optimization problem (13) reduces to $\max_{\mathbf{w}} \text{ENB}_\alpha(\mathbf{w})$ (12).

Theorem 3. (*FRB portfolios*) The family of FRB portfolios is given by

$$\mathbf{w}_{FRB} = \lambda \frac{\mathbf{A}\Sigma_F^{-\frac{1}{2}} \iota_n \odot \mathbf{b}^{\odot\frac{1}{2}}}{\mathbf{1}'_n\mathbf{A}\Sigma_F^{-\frac{1}{2}} \iota_n \odot \mathbf{b}^{\odot\frac{1}{2}}}, \lambda \in \Re_{\neq 0}, \tag{15}$$

where $\odot$ is the Hadamard product and $^{\odot}$ is the Hadamard power.

Proof of Theorem 3. Since $D_\alpha\left(\mathbf{p}_F \| \mathbf{b})\right)$ is a divergence, $D_\alpha\left(\mathbf{p}_F \| \mathbf{b})\right)$ is minimum and equal to zero when $\mathbf{p}_F = \mathbf{b}$. Consequently,

$$p_{F,k} = b_k \Leftrightarrow \frac{(\sigma_{F,k} w_{F,k})^2}{\mathbf{w}'_F \Sigma_F \mathbf{w}_F} = b_k, \forall k = 1, \ldots, n. \tag{16}$$

Then,

$$w_{F,k} = \pm \frac{\sqrt{\mathbf{w}'_F \Sigma_F \mathbf{w}_F}}{\sigma_{F,k}} \sqrt{b_k}, \forall k = 1, \ldots, n. \tag{17}$$

Using matrix notation, the factor weights are given by:

$$\mathbf{w}_F = \sqrt{\mathbf{w}'_F \Sigma_F \mathbf{w}_F}\,\Sigma_F^{-\frac{1}{2}} \iota_n \odot \mathbf{b}^{\odot\frac{1}{2}}. \tag{18}$$

Considering the relation $\mathbf{w} = \mathbf{A}\mathbf{w}_F$ (see **Property 1**) and the restriction $\mathbf{1}'_n\mathbf{w} = 1$, it is possible to obtain the following expression for leverage-restricted factor weights:

$$\mathbf{w}_{RFRB} = \frac{\mathbf{A}\Sigma_F^{-\frac{1}{2}} \iota_n \odot \mathbf{b}^{\odot\frac{1}{2}}}{\mathbf{1}'_n\mathbf{A}\Sigma_F^{-\frac{1}{2}} \iota_n \odot \mathbf{b}^{\odot\frac{1}{2}}}. \tag{19}$$

Using the same argument from **Proof of Theorem 2**, we obtain (15).

■

Considering that $\sigma_{RFRB} := \sqrt{\mathbf{w}'_{RFRB}\Sigma\mathbf{w}_{RFRB}}$, the volatility of $\mathbf{w}_{FRB}(\lambda)$ is given by $\sigma_{FRB}(\lambda) = |\lambda|\sigma_{FRB}$. It is important to point that the FRB portfolios achieve any desired volatility or, in other words, risk level. Consequently, our FRB portfolios are flexible because it will adapt to the investor's risk preference when varying λ.

CONCLUSIONS

In this paper, we review the literature related to the risk-based asset allocation methodologies and the use of entropy and divergence measures to ensure diversification. In terms of theoretical contributions, we generalize the existent optimization framework to obtain risk parity portfolios based on Rényi entropy taking out the leverage constraint. It is important to point that unconstraining the problem, we obtain risk parity portfolios for any desired volatility or, in other words, risk level. Consequently, our risk parity portfolios adapt better to the investor's risk preference. We give the analytical solutions to the risk parity unconstrained optimization problem. Finally, we generalize the risk parity optimization framework based on Rényi entropy introducing the risk budgeting optimization framework based on Rényi divergence. We also give the analytical solutions to the risk budgeting optimization problem.

Acknowledgments

The authors are grateful for the support of IME-USP, the Institute of Mathematics and Statistics of the University of São Paulo; FAPESP - the State of São Paulo Research Foundation (grants CEPID 2013/07375-0 and 2014/50279-4); and CNPq - the Brazilian National Counsel of Technological and Scientific Development (grant PQ 301206/2011-2).

REFERENCES

[1] A. Meucci, "Managing diversification," *Risk* **22** (5), 74–79 (2009).

[2] R. Deguest, L. Martellini, A. Meucci, *Risk parity and beyond - from asset allocation to risk allocation decisions*, EDHEC, 2013.

[3] A. Rényi, "On measures of entropy and information," *Proceedings of the 4th Berkeley Symposium on Mathematics, Statistics and Probability*, 547–561, Berkeley, CA, USA, 20 June–30 July 1960.

[4] S. Maillard, T. Roncalli, and J. Teïletche, "The properties of equally weighted risk contribution portfolios," *J. Portfolio Manage.* **36**, 60–70 (2010).

[5] D. Chaves, J. Hsu, F. Li, and O. Shakernia, "Risk parity portfolio vs. other asset allocation heuristic portfolios," *J. Invest.* **20**, 108–118 (2011).

[6] T. Roncalli, *Introduction to Risk Parity and Risk Budgeting*, 1st ed., Chapman & Hall: Boca Raton, USA, 2014.

[7] B. Bruder, P. Héreil, and T. Roncalli, "Managing sovereign credit risk," *J. Indexes Europe* **1**, 20–27 (2011).

[8] B. Bruder, and T. Roncalli, *Managing risk exposures using the risk parity approach*, Lyxor Asset Management, 2013.

[9] H. Kaya, and W. Lee, *Demystifying risk parity*, Neuberger Berman LLC, 2012.

[10] J. Bender, R. Briand, F. Nielsen, and D. Stefek, "Portfolio of risk premia: a new approach to diversification," *J. Portfolio Manage.* **36**, 17–25 (2010).

[11] S. Page, and M. Taborsky, "The myth of diversification: risk factors versus asset classes," *J. Portfolio Manage.* **37**, 1–2 (2011).

[12] H. Lohre, U. Neugebauer, and C. Zimmer, "Diversified risk parity strategies for equity portfolio selection," *J. Invest.* **21**, 111–128 (2012).

[13] T. Poddig, A. Unger, "On the robustness of risk-based asset allocations," *FMPM* **26** (3), 369–401 (2012).