FBST for Cointegration Problems

M. Diniz, C. A. B. Pereira, and J. M. Stern

Citation: AIP Conference Proceedings **1073**, 157 (2008); doi: 10.1063/1.3038994
View online: http://dx.doi.org/10.1063/1.3038994
View Table of Contents:
http://scitation.aip.org/content/aip/proceeding/aipcp/1073?ver=pdfcov
Published by the AIP Publishing

FBST for Cointegration Problems

M.Diniz*,†, C.A.B.Pereira* and J.M.Stern*,†

*University of Sao Paulo, Institute of Mathematics and Statistics
†diniz_marcio@yahoo.com.br jstern@ime.usp.br

Abstract. In order to estimate causal relations, the time series econometrics has to be aware of spurious correlation, a problem first mentioned by Yule [21]. To solve the problem, one can work with differenced series or use multivariate models like VAR or VEC models. In this case, the analysed series are going to present a long run relation i.e. a cointegration relation. Even though the Bayesian literature about inference on VAR/VEC models is quite advanced, Bauwens et al. [2] highlight that *"the topic of selecting the cointegrating rank has not yet given very useful and convincing results"*.

This paper presents the Full Bayesian Significance Test applied to cointegration rank selection tests in multivariate (VAR/VEC) time series models and shows how to implement it using available in the literature and simulated data sets. A standard non-informative prior is assumed.

Keywords. VAR/VEC models, e-values, FBST, cointegration, reduced rank regression.

INTRODUCTION

One of the main goals of econometrics is to estimate causal relations. However, since the seminal work of Yule [21], it is known that spurious regression is a possible danger when one works with time series data. This problem might arise specially if the variables are integrated or, what is the same, have unit roots. One approach is to test the series for unit roots and then, if they are integrated, adopt the procedures to make them stationary, usually by differencing it. Another way is to estimate cointegration relations between them. The most used instrument to do this is the VAR (*Vector Autoregressive*) or VEC (*Vector Error Correction*) models.

In recent work the authors presented the FBST is applied to unit root tests [2]. Following this path, we demonstrate here how to apply the same procedure to cointegration problems. In the Bayesian literature there is a great number of works related to cointegration models, specially those concerned with the inference about the cointegration space parameters. These works usually assume that the cointegration rank[1] takes some value and all the inference is made conditionally on this conjecture.

On the other hand, the research about Bayesian cointegration tests is making advances just recently. Following the Bayesian orthodox approach, the proposed tests are based on Bayes factors and this way of treatment of sharp hypotheses has well document shortfalls. As highligthed by Bauwens et al. [2], using a posterior odds approach

[1] Testing, for instance, if three series are cointegrated we can find that there is none cointegration relation or that there is one or that there are, at most, two cointegration relations. The number of cointegration relations is given by the rank of the long term impact matrix, Π.

CP1073, *Bayesian Inference and Maximum Entropy Methods in Science and Engineering—28th International Workshop*
edited by M. de Souza Lauretto, C. A. de Bragança Pereira, and J. M. Stern

leads to rather heavy computations and requires definition of proper prior densities, see Kleibergen and Paap [11] Villani [20] and Sugita [19]. Since the hypotheses concerning the cointegration analysis, when one uses VAR/VEC models, are sharp hypotheses, our main contribution is to present a full Bayesian significance test for cointegration problems. We do not attach a prior probability to null measure sets, and suppose a standard non-informative prior, what is not always possible when working with Bayes factors.

THE VAR/VEC MODEL

Consider a n dimensional vector $Y_t = [y_{1t}, \ldots, y_{nt}]'$ whose data generating process is a VAR(p):

$$Y_t = \Phi_1 Y_{t-1} + \ldots + \Phi_p Y_{t-p} + E_t. \tag{1}$$

where $E_t \sim NI_n(0, \Sigma)$. It is possible to include deterministic terms in the model but, in this case, the results shown here are not invalid . Writing explicitly, we have:

$$\begin{bmatrix} y_{1t} \\ \vdots \\ y_{nt} \end{bmatrix} = \begin{bmatrix} \phi_{11}^{(1)} & \ldots & \phi_{1n}^{(1)} \\ \vdots & & \vdots \\ \phi_{n1}^{(1)} & \ldots & \phi_{nn}^{(1)} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ \vdots \\ y_{n,t-1} \end{bmatrix} + \ldots + \begin{bmatrix} \phi_{11}^{(p)} & \ldots & \phi_{1n}^{(p)} \\ \vdots & & \vdots \\ \phi_{n1}^{(p)} & \ldots & \phi_{nn}^{(p)} \end{bmatrix} \begin{bmatrix} y_{1,t-p} \\ \vdots \\ y_{n,t-p} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \vdots \\ \varepsilon_{n,t} \end{bmatrix}.$$

where $\Phi_i = \begin{bmatrix} \phi_{11}^{(i)} & \ldots & \phi_{1n}^{(i)} \\ \vdots & & \vdots \\ \phi_{n1}^{(i)} & \ldots & \phi_{nn}^{(i)} \end{bmatrix}$ e $E_t = \begin{bmatrix} \varepsilon_{1,t} \\ \vdots \\ \varepsilon_{n,t} \end{bmatrix}$ to $i = 1, \ldots, p$ e $t = 1, \ldots, T$.

The model (1) can be reformulated as an error correction model:

$$\Delta Y_t = \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \Pi Y_{t-1} + E_t. \tag{2}$$

where $\Delta Y_t = [\Delta y_{1t} \ldots \Delta y_{nt}]'$, $\Gamma_i = -(\Phi_{i+1} + \ldots \Phi_p)$ to $i = 1, 2, \ldots, p$ e $\Pi = -(I_n - \Phi_1 - \ldots - \Phi_p)$.

To know if the component series of Y_t are cointegrated we observe the rank of the matrix Π, $\rho(\Pi)$. If it is null, all the Π eigenvalues are null and this matrix is the null matrix. This means that $\Phi(1) = \Phi_1 + \ldots + \Phi_p$ has all the eigenvalues equal to one and, hence, the Y_t components are, at least, $I(1)$ and the valid representation is a VAR(p-1) in first differences[2]. If $\rho(\Pi) = n$, all the Π eigenvalues are not null, meaning that the Φ eigenvalues are less than one. In this case, the Y_t components are stationary and the valid representation is the VAR(p) for the level of the series, as in (1).

Finally, if $0 < \rho(\Pi) = r < n$, there is $n - r$ null eigenvalues and r not null. Therefore, the Y_t components are, at least, $I(1)$ and the valid representation is (2) with $\Pi = \alpha\beta'$ where α and β are matrices $n \times r$ with rank r. It is said that in the β matrix we find the r cointegrating vectors and the matrix α is load matrix. The FBST was specially designed

[2] It is possible that the component series of Y_t are $I(2)$ and, in this case, the error correction model would be written using $\Delta^2 Y_t$ as dependent variable.

to give an epistemic value, or value of evidence, supporting a sharp hypothesis H and - see above - the cointegration analysis using VAR/VEC models is made by the study of sharp hypotheses. Hence, to know if n series are cointegrated one should study the hypothesis $H: 0 \leq \rho(\Pi) = r \leq n$ and, accordingly to the e-values obtained, decide if they are cointegrated and, if they are, the number of existent cointegration relations.

COINTEGRATION TESTS

The tests developed by Johansen [9] and Johansen and Juselius [10] focus the attention on the rank of Π and try to determinate the number of non-null eigenvalues of this matrix. Once found, the n eigenvalues of Π, are ordered $\widehat{\lambda}_1 > \widehat{\lambda}_2 >, \ldots, > \widehat{\lambda}_n$. If they are all statistically equal to zero, the series are not cointegrated. To test these hypotheses, the authors developed the maximum eigenvalue and the trace tests. The first one is based on what follows: if one wants to test if $\rho(\Pi)$ is k, $0 < k < n$, against the alternative to be $k+1$, calculates:

$$\lambda_{max}(k) = -T \ln(1 - \widehat{\lambda}_{k+1})$$

since, if $\widehat{\lambda}_{k+1} \simeq 0$, the statistic will be close to zero and it will not be possible to reject H_0 once this result implies that $\widehat{\lambda}_1 > \widehat{\lambda}_2, \ldots, \widehat{\lambda}_k > 0$ and, therefore, $\rho(\Pi) = k$.

The trace test adopts as hypothesis that $\rho(\Pi)$ is less than or equal to k, k as before. The alternative implies that the rank is bigger than k. The statistic is, therefore:

$$\lambda_{tr}(k) = -T \sum_{i=k+1}^{n} \ln(1 - \widehat{\lambda}_i).$$

If this sum is statistically zero, the $n-k$ smallest eigenvalues are null and, therefore, it will not be possible to reject the hypothesis that $\rho(\Pi)$ is smaller than or equal to k. As shown by Hamilton [5], the asymptotic distribution of λ_{tr} is a multivariate generalization of the ADF test and depends on the $n-k$ dimension and on the deterministic terms used by the model being tested. The same occurs with the asymptotic distribution of λ_{max}. The critical values found by Osterwald-Lenum [16] and MacKinnon et al [15] are asymptotic.

NUMERICAL EXPERIMENTS AND RESULTS

Using matrix notation, the error correction model (2) can be written as:

$$\Delta Y = ZB + E \tag{3}$$

where $\Delta Y = \begin{bmatrix} \Delta Y'_{p+1} \\ \vdots \\ \Delta Y'_T \end{bmatrix}$, $Z = \begin{bmatrix} \Delta Y'_p & \ldots & \Delta Y'_2 & Y'_p \\ \Delta Y'_{p+1} & & \Delta Y'_3 & Y'_{p+1} \\ \vdots & & \vdots & \vdots \\ \Delta Y'_{T-1} & \ldots & \Delta Y'_{T-p+1} & Y'_{T-1} \end{bmatrix}$, $B = \begin{bmatrix} \Gamma_1 \\ \vdots \\ \Gamma_{p-1} \\ \Pi \end{bmatrix}$ and the error vector is given by $E \sim MN_{T \times n}(0, \Sigma \otimes I_T)$, denoting the matricvariate normal distribution[3].

Considering equation 3 and a non-informative prior

$$p(B,\Sigma) \propto |\Sigma|^{-(n+1)/2}$$

we have that the likelihood and the posterior are, respectively:

$$p(\Delta Y|B,\Sigma,Z) \propto |\Sigma|^{-T/2}\exp\left\{-\frac{1}{2}\mathrm{tr}[\Sigma^{-1}(\Delta Y - ZB)'(\Delta Y - ZB)]\right\}$$

$$\begin{aligned} p(B,\Sigma|\Delta Y,Z) &\propto |\Sigma|^{-(T+n+1)/2}\exp\left\{-\tfrac{1}{2}\mathrm{tr}[\Sigma^{-1}(\Delta Y - ZB)'(\Delta Y - ZB)]\right\} \\ &= |\Sigma|^{-(T+n+1)/2}\exp\left\{-\tfrac{1}{2}\mathrm{tr}\{\Sigma^{-1}[S + (B-\widehat{B})'Z'Z(B-\widehat{B})]\}\right\} \\ &\propto f_{MN}^{n\times k}(B|\widehat{B},\Sigma\otimes(Z'Z)^{-1})f_{IW}^{n}(\Sigma|S,T) \end{aligned} \tag{4}$$

where $\widehat{B} = (Z'Z)^{-1}Z'\Delta Y$ e $S = \Delta Y'\Delta Y - \Delta Y'Z(Z'Z)^{-1}Z'\Delta Y$. Besides, n represents the number of component series of Y_t and k the dimension of B, which is $k \times n$.

The posterior (4) is used by the FBST to test the rank of matrix Π. To exemplify, consider a bi-dimensional vector Y_t generated by a VAR(1):

$$Y_t = \Phi_1 Y_{t-1} + E_t \tag{5}$$

where $E_t \sim NI_2(0,\Sigma)$, and the same model written in the error correction form:

$$\Delta Y_t = \Pi Y_{t-1} + E_t. \tag{6}$$

We want to test, for instance, $H_0 : \rho(\Pi) = 1$. To implement the FBST we have to find the posterior maximum under the space of the hypothesis being tested and then to integrate the posterior over the tangential set[4]. In this case, we assume that Π has reduced rank and decompose it in two matrices of rank one and dimension 2×1, α e β, in accordance to the Granger representation theorem, implying that $\Pi = \alpha\beta'$. In order to define the restricted posterior (under H_0) in this situation, we write:

$$\Pi = \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \end{bmatrix} \begin{bmatrix} \beta_{11} & \beta_{12} \end{bmatrix}$$

[3] It is said that the vector $X \sim MN_{p\times q}(\mathrm{vec}M, Q\otimes P)$ if and only if $\mathrm{vec}(X)$ has a multivariate normal distribution, i.e., $\mathrm{vec}(X) \sim N_{pq}(\mathrm{vec}M, Q\otimes P)$.

[4] The tangential set is defined by $T(s^*) = \{\theta \in \Theta | s(\theta) \leq s^*\}$ where Θ is the parameter space, $p_n(\theta)$ the posterior, H the parameter space under the hypthesis being tested, $s(\theta) = \frac{p_n(\theta)}{r(\theta)}$ and $s^* = s(\theta^*) = \sup_{\theta\in H} s(\theta)$. The function $s(\theta)$ is known as the posterior surprise relative to a given reference density, $r(\theta)$. Its role in the FBST is to make $\mathrm{ev}(H)$ explicitly invariant under suitable transformations on the coordinate system of the parameter space. The tangential (to the hypothesis) set $\overline{T} = \overline{T}(s^*)$, is a Highest Relative Surprise Set (HRSS). It contains the points of the parameter space with higher surprise, relative to the reference density, than any point in the null set. When $r(\theta) \propto 1$, the possibly improper uniform density,$\overline{T}$ is the Posterior's Highest Density Probability Set (HDPS) tangential to the null set H. The e-value is just $\int_{T(s^*)} p_n(\theta)\, d\theta$.

and maximize the posterior under this restriction. To test $\rho(\Pi) = 0$ is enough to test if Π is the null matrix. When the vector Y_t has n components the approach is automatically extended. In the following examples we compare the results given by the FBST with the maximum eigenvalue test. The reported p-values are asymptotic.

Example 1: we simulated a two dimension VAR(1) with 50 observations, errors $NI_2(0,\Sigma)$ and the following parameters:

$$\Phi_1 = \begin{bmatrix} 0.8 & 0.1 \\ 1 & 0.5 \end{bmatrix} \text{ e } \Sigma = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}.$$

The Φ_1 matrix has eigenvalues equal to 1 and 0.3. Therefore, there is on cointegration vector. The test to $\rho(\Pi) = 0$ shows an e-value of 0.00428, taking us to reject the hypothesis. By testing $\rho(\Pi) = 1$ we obtain an e-value of 0.99686, and this confirms the existence of one cointegration vector. The maximum eigenvalue test to $H_0 : r = 0$ against $H_1 : r = 1$ reports a p-value close to zero and to $H_0 : r = 1$ against $H_1 : r = 2$, p-value of 0.4746, reaching the same conclusion obtained by the FBST, i.e. that there is one cointegration vector.

Example 2: we simulated another two dimension VAR(1) with 50 observations and the following parameters:

$$\Phi_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ e } \Sigma \text{ as above.}$$

The matrix Φ_1 has both eigenvalues equal to one. Therefore, the series are $I(1)$ and do not cointegrate. The FBST to test $\rho(\Pi) = 0$ gives an e-value equal to 0.4586, what is good evidence to not reject the hypothesis, as expected. The maximum eigenvalue test to $H_0 : r = 0$ against $H_1 : r = 1$ presents a p-value of 0.3889 and, therefore, the null is not rejected.

Example 3: now we present a three dimension VAR(1) with 50 observations and the following parameters:

$$\Phi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.3 \end{bmatrix} \text{ e } \Sigma = \begin{bmatrix} 1 & 0.2 & 0.2 \\ 0.2 & 0.5 & 0.2 \\ 0.2 & 0.2 & 1 \end{bmatrix}.$$

The Φ_1 matrix has eigenvalues equal to 1, 0.5 and 0.3. Therefore there are two cointegration vectors. The FBST to test $\rho(\Pi) = 0$ gives us an e-value of 0.0151, and is possible to reject it. After testing $\rho(\Pi) = 1$ we found an e-value of 0.0342, and also reject the hypothesis. Testing $\rho(\Pi) = 2$ the e-value is 0.9991, what confirms the existence of two cointegration vectors.

The maximum eigenvalue test reports a p-value close to zero for $H_0 : r = 0$ against $H_1 : r = 1$ and to $H_0 : r = 1$ against $H_1 : r = 2$. The test of $H_0 : r = 2$ against $H_1 : r = 3$ pesents a p-value of 0.11489. So, we do not reject this last hypothesis and conclude that there are two cointegration vectors.

Example 4: we simulated a two dimension VAR(2) with the following generating process:

$$Y_t = \begin{bmatrix} 0.45 & -0.2 \\ 1.1 & 0.3 \end{bmatrix} Y_{t-1} + \begin{bmatrix} 0.35 & 0.3 \\ -0.1 & 0.2 \end{bmatrix} Y_{t-2} + E_t.$$

where $\Sigma = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$. We know that there is one cointegration vector since the Π matrix has eigenvalues equal to 1 and 0.3.

The FBST reports an e-value of 0.0276 testing $H_0 : r = 0$, taking us to reject it. The maximum eigenvalue test, testing this null against $H_1 : r = 1$ presents a p-value of 0.0003. To test $H_0 : r = 1$, the FBST reaches an e-value of 0.9972 and the maximum eigenvalues test, against $H_1 : r = 2$, a p-value of 0.4392.

Example 5 (Johansen and Juselius, 1990): now we apply the FBST to the Finish data set used by Johansen and Juselius in their seminal work.

The authors used the series in natural logarithms of the $M1$ monetary aggregate, inflation rate, real income and the primary interest rate set by the Bank of Finland to model the money demand which, in theory, follows a long term relation. The sample has quarterly observations and starts at 1958:02 and goes until 1984:03. The chosen model was a VAR(2) with unrestricted constant and seasonal dummies for the first three quarters of the year. Writing the chosen model in the error correction form, we have:

$$\Delta Y_t = \mu + \Psi D_t + \Gamma_1 \Delta Y_{t-1} + \Pi Y_{t-1} + E_t \tag{7}$$

where $\Pi = \Phi_1 + \Phi_2 - I$, $\Gamma_1 = -\Phi_2$, μ is the constants vector and D_t has the seasonal dummies. This vector could also contain another deterministic variables. It is assumed that $E_t \sim NI_4(0,\Sigma)$ for all $t = 1, \ldots, T$.

To make the FBST implementation easier, given the great number of parameters to be defined at the maximization and integration steps, we use the Frisch-Waugh-Lovell theorem. We run the auxiliary regressions:

$$\Delta Y_t = \mu' + \Psi' D_t + \Gamma'_1 \Delta Y_{t-1} + R_{0,t}$$

$$Y_{t-1} = \mu^* + \Psi^* D_t + \Gamma^*_1 \Delta Y_{t-1} + R_{1,t}$$

and then we can work only with the residual vectors to study the rank of Π, since the theorem assures that

$$R_{0,t} = \Pi R_{1,t-1} + E_t,$$

being E_t the same of (7). The results are reported in the table below.

The authors concluded that there is, at least, two cointegration vectors, the same conclusion reached by the FBST.

Example 6 (Lucas, 2000): we apply, as a last example, the FBST to a US data set used by Lucas [13]. The observations have annual periodicity and go since 1900 until 1985. We

TABLE 1. FBST and maximum eigenvalue test applied to Finish data of Johansen and Juselius (1990)

H_0	FBST	λ_{max}	p-value
r=0	0.0006	38.489	0.0007
r=1	0.0505	26.642	0.0060
r=2	0.9726	7.8924	0.3983

TABLE 2. FBST and maximum eigenvalue test applied to US data of Lucas (2000)

H_0	FBST	λ_{max}	p-value
r=0	0.0145	25.334	0.0101
r=1	0.9299	4.2507	0.8271

tested for cointegration between real national income, $M1$ monetary aggregate deflated by the GDP deflator and the commercial papers return rate. We adjusted a VAR(1) with unrestricted constant. The data are used in natural logarithm and the results follow below.

These results show that the proposed procedure reaches the same conclusions that the Johansen and Juselius procedure. However, it is a full Bayesian test that can be implement even with improper priors and that obeys the likelihood principle.

CONCLUDING REMARKS

The FBST shows, once more, its versatility. The unit roots and cointegration frequentist tests need simulations to find their critical values since their statistics have distributions that can not be calculated analitically. If the researcher is working with samples that do not have critical values "tabulated", the asymptotical approximation or the closest sample size, whose critical values were calculated, are used. This can be a problem specially in small samples because the simulations assume that the data follow given distributions, usually the gaussian.

The FBST, however, does not need any hypothesis about the sample size and can be calculated assuming any parametric distribution for the data.

REFERENCES

[1] L.Bauwens and M.Lubrano (1996). Identification restrictions and posterior densities in cointegrated Gaussian VAR systems, in *Advances in Econometrics*, 11, Part B, (JAI Press, Greenwich) 3-28.

[2] L.Bauwens, M.Lubrano, J.F.Richard (1999). *Bayesian Inference in Dynamic Econometric Models.* Oxford University Press.

[3] E.G.Birgin, R.Castillo, J.M.Martinez (2004). Numerical Comparison of Augmented Lagrangian Algorithms for Nonconvex Problems. *Computational Optimization and Applications,* 31, 1-56. Software

and documentation available at *www.ime.usp.br/∼egbirgin/tango/* .

[4] D.DeJong (1992). Co-integration and trend-stationarity in macroeconomic time series. *Journal of Econometrics,* 52, 347-370.

[5] M. Diniz, C.A.B.Pereira, J.M.Stern (2007). FBST for Unit Root Problems. *AIP Conference Proceedings*, v. 954, p. 260-267, 2007.

[6] J.Geweke (1996). Bayesian reduced rank regression in econometrics. *Journal of Econometrics,* 75, 121-146.

[7] W.L.Goffe, G.D.Ferrier, J.Rogers (1994). Global Optimization of Statistical Functions with Simulated Annealing. *Journal of Econometrics*, 60, 65-99.

[8] J.D.Hamilton (1994). *Time Series Analysis.* Princeton University Press.

[9] S.Johansen (1988). Statistical analysis of cointegration vectors. *Journal of Economic Dynamics and Control*, 12, 231-254.

[10] S.Johansen and K.Juselius (1990). Maximum Likelihood estimations and inference on cointegration – With application to the demand for money. *Oxford Bulletin of Economics and Statistics*, 52, 169-210.

[11] F.Kleibergen and R.Paap (2002). Priors, posteriors and Bayes Factors for a Bayesian Analysis of cointegration. *Journal of Econometrics,* 111, 223-249.

[12] F.Kleibergen and H.K.van Dijk (1994). On the shape of the likelihood/posterior in cointegration models. *Econometric Theory,* 10, 514-551.

[13] R.Lucas (2000). Inflation and Welfare. *Econometrica*, 68, 247-274.

[14] M.Lauretto, C.A.B.Pereira, J.M.Stern, S.Zacks (2003). Full Bayesian Significance Test Applied to Multivariate Normal Structure Models. Comparing Parameters of Two Bivariate Normal Distributions Using the Invariant Full Bayesian Significance Test. *Brazilian J. Probability and Statistics,* 17, 147-168.

[15] J.G.MacKinnon, A.A.Haug, L.Michelis (1999). Numerical distribution functions of likelihood ratio tests for cointegration. *Journal of Applied Econometrics*, 14, 563-577.

[16] M.Osterwald-Lenum (1992). A note with quantiles of the aymptotic distribution of maximum likelihood cointegration rank statistics. *Oxford Bulletin of Economics and Statistics*, 54, 461-472.

[17] C.A.B.Pereira, J.M.Stern, (1999). Evidence and Credibility: Full Bayesian Significance Test for Precise Hypotheses. *Entropy Journal*, 1, 69–80.

[18] C.A.B.Pereira, J.M.Stern (2001). Model Selection: Full Bayesian Approach. *Environmetrics* 12, 6, 559-568.

[19] K.Sugita (2002). Testing For Cointegration Rank Using Bayes Factors. The Warwick Economics Research Paper Series, 654.

[20] M.Villani (2005). Bayesian reference analysis of cointegration. *Econometric Theory*, 21, 326-357.

[21] G.U.Yule (1926). Why do we sometimes get nonsense-correlations between time-series? A study in sampling and the nature of time series (with discussion). *Journal of the Royal Statistical Society*, 89,1-64.