FBST: Compositionality

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Abstract. In this paper, the relationship between the credibility of a complex hypothesis, H, and those of its constituent elementary hypotheses, H^j, $j = 1 \ldots k$, is analyzed, in the independent setup, under the Full Bayesian Significance Testing (FBST) mathematical apparatus.
Key words: Bayesian models; Complex hypotheses; Compositionality; Mellin convolution; Possibilistic and probabilistic reasoning; Significance tests; Truth values, functions and operations.

INTRODUCTION

The Full Bayesian Significance Test (FBST) has been introduced by Pereira and Stern (1999), as a coherent Bayesian significance test for sharp hypotheses. For detailed definitions, interpretations, implementation and applications, see the authors' previous articles, including two papers in this conference series, [9], [17].

In this paper we analyze the relationship between the credibility, or truth value, of a complex hypothesis, H, and those of its elementary constituents, H^j, $j = 1 \ldots k$. This problem is known as the question of *Compositionality*, which plays a central role in analytical philosophy, see [3].

According to Wittgenstein [22], (2.0201, 5.0, 5.32):

- Every complex statement can be analyzed from its elementary constituents.

- Truth values of elementary statement are the results of those statements' truth-functions (Wahrheitsfunktionen).

- All truth-function are results of successive applications to elementary constituents of a finite number of truth-operations (Wahrheitsoperationen).

The compositionality question also plays a central role in far more concrete contexts, like that of reliability engineering, see [1] and [2], (1.4):

"One of the main purposes of a mathematical theory of reliability is to develop means by which one can evaluate the reliability of a structure when the reliability of its components are known. The present study will be concerned with this kind of mathematical development. It will be necessary for this purpose to rephrase our intuitive concepts of structure, component, reliability, etc. in more formal language, to restate carefully our assumptions, and to introduce an appropriate mathematical apparatus."

When brought into a parametric statistical hypothesis testing context, a complex hypothetical scenario or complex hypothesis is a statement, H, concerning $\theta = (\theta^1, \ldots, \theta^k) \in \Theta = (\Theta^1 \times \ldots \times \Theta^k)$ which is equivalent to a logical composition of statements, $H^1, \ldots, H^k$, concerning the elementary components, $\theta^1 \in \Theta^1, \ldots, \theta^k \in \Theta^k$, respectively. Within this setting, means to evaluate the credibility of H, as well as that of

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each of its elementary components, $H^1, \ldots, H^k$, is provided by the FBST mathematical apparatus introduced in [12]. Further general references on the subject include [8-13] and [16-20]. It is of interest, however, to know what can be said about the credibility of H, from the knowledge of the credibilities of its elementary components, and this is what the authors endeavor to explore in the present paper, in the case of independent elementary components.

FBST FORMAL STRUCTURES

By a FBST *Structure*, we mean a quintuple $M = \{\Theta, H, p_0, p_n, r\}$, where

- Θ is the parameter space of an underlying statistical model $(S, \Sigma(S), P_\theta)$;
- $H : \theta \in \Theta_H = \{\theta \in \Theta \,|\, g(\theta) \leq \mathbf{0} \wedge h(\theta) = \mathbf{0}\}$ is the *Hypothesis*, stating that the parameter lies in the (null) set Θ_H, defined by inequality and equality constraints given by vector functions g and h in the parameter space. We are particularly interested in sharp (precise) hypotheses, i.e., those in which $\dim(\Theta_H) < \dim(\Theta)$, with at least one equality constraint. In the sequel we often use a relaxed notation, writing the hypothesis, H, instead of the set Θ_H defining it.
- p_0, p_n and r are the *Prior*, the *Posterior* and the *Reference* probability densities on Θ, all with respect to the same σ-finite measure μ on a measurable space $(\Theta, \Sigma(\Theta))$.

Within a FBST structure, the following definitions are essential:

- The posterior *Surprise* function, $s(\theta)$, relative to the structure's reference density, $r(\theta)$, and its constrained and unconstraind suprema are defined as:

$$s(\theta) = \frac{p_n(\theta)}{r(\theta)} \ , \ \ s^* = s(\theta^*) = \sup_{\theta \in H} s(\theta) \ , \ \ \widehat{s} = s(\widehat{\theta}) = \sup_{\theta \in \Theta} s(\theta) \ .$$

- The *Highest Relative Surprise Set* (HRSS) at level v, $\overline{T}(v)$, and its complement, $T(v)$, are defined as:

$$T(v) = \{\theta \in \Theta \,|\, s(\theta) \leq v\} \ , \ \ \overline{T}(v) = \Theta - T(v) \ ,$$

- The *Truth Function* or cumulative surprise distribution, $W : R_+ \mapsto [0, 1]$, and the the *Untruth Function* of M, $\overline{W}(v)$, are defined as:

$$W(v) = \int_{T(v)} p_n(\theta) \mu(d\theta) \ , \ \ \overline{W}(v) = 1 - W(v) \ .$$

- The *Truth Value*, $\mathrm{ev}(H)$, or evidence value (e-value) supporting the hypothesis H in M, and the *Untruth Value*, $\overline{\mathrm{ev}}(H)$ or evidence-value against H, are defined as:

$$\mathrm{ev}(H) = W(s^*) \ , \ \ \overline{\mathrm{ev}}(H) = \overline{W}(s^*) = 1 - \mathrm{ev}(H) \ .$$

The role of the reference density in the FBST is to make $\mathrm{ev}(H)$ implicitly invariant under suitable transformations of the coordinate system of the parameter space. The natural choices for reference density are an uninformative prior, interpreted as a representation of no information in the parameter space, or the limit prior for no observations, or the

neutral ground state for the Bayesian operation. Standard (possibly improper) uninformative priors include the uniform and maximum entropy densities, see Dugdale (1996) and Kapur (1989) for a detailed discussion.

The *Tangential Set*, $\overline{T} = \overline{T}(s^*)$, contains the points of the parameter space with higher surprise, relative to the reference density, than any point in H. When $r(\theta) \propto 1$, the possibly improper uniform density, $\overline{T}$ is the Posterior's Highest Density Probability Set (HDPS) tangential to H. Small values of $\overline{\text{ev}}(H)$ indicate that the hypothesis traverses high density regions, favoring the hypothesis.

As we will see in the next sections, it is not possible to obtain the truth value of a complex hypothesis only from the truth values of its elementary constituents. It is possible, however, to obtain upper and lower bounds for the truth value of the complex hypothesis from the truth values of its elementary constituents. We will also see that it is possible to obtain the truth function, W, of a complex structure, from the truth functions, W^j, of its elementary constituents, and the constrained supremum, s^*, of the complex structures surprise function from the elementary suprema, s^{*j}.

Since $\text{ev}(H) = W(s^*)$, the pair (W, s^*) will be referred to as the *Truth Summary* of the structure M. Since we will be dealing in this paper, exclusively with complex hypotheses in an independent setup, we close this section by establishing the precise meaning of this framework. By an independent setup we mean that the FBST structures corresponding to the complex hypothesis H, $M = \{\Theta, H, p_0, p_n, r\}$, and to each of its elementary constituent hypotheses, H^j, $M^j = \{\Theta^j, H^j, p_0^j, p_n^j, r^j\}$, $j = 1, \ldots k$, bear the following relationships between their elements:

- the parameter space, Θ, of the underlying statistical model, $(S, \Sigma(S), P_\theta)$, is the product $\Theta^1 \times \Theta^2 \times \ldots \times \Theta^k$;

- H, is a logical composition (conjunctions and disjunctions) of H^1, H^2, ..., H^k;

- p_n and r, are probability densities with respect to the product measure $\mu = \mu^1 \times \mu^2 \times \ldots \times \mu^k$ on $(\Theta, \Sigma(\Theta))$, where μ^j denote the σ-finite measure on $(\Theta^j, \Sigma(\Theta^j))$ with respect to which p_0^j, p_n^j and r^j are densities ; and

- the probability densities p_n and r are such that

$$p_n(\theta) \;=\; \prod_{j=1}^{k} p_n^j(\theta^j) \;\text{ and }\; r(\theta) \;=\; \prod_{j=1}^{k} r^j(\theta^j)\,, \;\; \theta = (\theta^1, \ldots, \theta^k) \in \Theta\,.$$

TRUTH-VALUES INEQUALITIES FOR CONJUNCTIONS

In this section we shall investigate, within the independent setup, the question of whether the truth value of a complex hypothesis, H, can be obtained from the truth values of its elementary constituents, H^1, H^2, ..., H^k. We consider the case of a conjunctive composite hypothesis, that is, the case in which H is equivalent to $H^1 \wedge H^2 \wedge \ldots \wedge H^j$.

In this case only bounds can be obtained for the truth and untruth values of H, from the corresponding truth and untruth values of the elementary constituents, H^j:

Proposition 3.1: If H is equivalent to $H^1 \wedge H^2 \wedge \ldots \wedge H^k$, then

$$\prod_{j=1}^{k} \text{ev}(H^j) \leq \text{ev}(H^1 \wedge H^2 \wedge \ldots \wedge H^k) \;\text{ and}$$

$$\prod_{j=1}^{k} \overline{\text{ev}}(H^j) \leq \overline{\text{ev}}(H^1 \wedge H^2 \wedge \ldots \wedge H^k)\,.$$

In order to prove proposition 3.1, the following lemmas will be needed:

Lemma 3.2: For any conjunctive composite hypothesis H with elementary constituents $H^1, H^2, \ldots H^k$,

$$s^* = \sup_{\theta \in H} s(\theta) = \prod_{j=1}^{k} \sup_{\theta^j \in H^j} s^j(\theta^j) = \prod_{j=1}^{k} s^{*j} \ .$$

Proof: Since for $\theta \in H$, $s^j(\theta^j) \leq s^{*j}$, for $1 \leq j \leq k$, $s(\theta) = \prod_{j=1}^{k} s^j(\theta^j) \leq \prod_{j=1}^{k} s^{*j}$ so that $s^* \leq \prod_{j=1}^{k} s^{*j}$. On the other hand, if for $\varepsilon > 0$ and $s = \prod_{j=1}^{k}(s^{*j} - \varepsilon)$, there must exist $\theta \in \bigwedge_{j=1}^{k} H^j$ such that $s(\theta) = \prod_{j=1}^{k} s^j(\theta^j) > \prod_{j=1}^{k}(s^{*j} - \varepsilon)$. Consequently, $\sup_{\theta \in H} s(\theta) > \prod_{j=1}^{k} (s^{*j} - \varepsilon)$, and the result follows by making $\varepsilon \to 0$.

Lemma 3.3:

$$\prod_{j=1}^{k} W^j(v^j) \ \leq \ W(\prod_{j=1}^{k} v^j) \ ,$$

where W^j, $1 \leq j \leq k$, and W are the truth functions of M^j, $1 \leq j \leq k$, and M, respectively.

Proof: Let $G : R_+^k \mapsto [0,1]$ be defined as

$$G(v^1, \ldots, v^k) = \int_{\{s^1(\theta^1) \leq v^1, \ldots, s^k(\theta^k) \leq v^k\}} p_n(\theta) \mu(d\theta) \ .$$

Since $s = \prod_{j=1}^{k} s^j$, $\mu = \prod_{j=1}^{k} \mu^j$, and

$$\left\{ s^1(\theta^1) \leq v^1, \ldots, s^k(\theta^k) \leq v^k \right\} \subseteq \left\{ \prod_{j=1}^{k} s^j(\theta^j) \leq \prod_{j=1}^{k} v^j \right\} = \left\{ s(\theta) \leq \prod_{j=1}^{k} v^j \right\} ,$$

it follows that

$$\prod_{j=1}^{k} W^j(v^j) = G(v^1, \ldots, v^k) \leq W(\prod_{j=1}^{k} v^j) \ .$$

Proof of Proposition 3.1: In the inequality of Lemma 3.3, replacing each v^j by s^{*j}, $1 \leq j \leq k$, and then using Lemma 3.2, the first result in proposition 3.1 follows. The same argument proves the other assertion.

Consequently, if H is equivalent to $H^1 \wedge H^2 \wedge \ldots \wedge H^k$, the truth values of the elementary constituent hypotheses give us lower and upper bounds for the truth value of the complex hypothesis. More precisely,

Proposition 3.4: If H is equivalent to $H^1 \wedge H^2 \wedge \ldots \wedge H^k$, then

$$\prod_{j=1}^{k} \mathrm{ev}(H^j) \leq \mathrm{ev}(H^1 \wedge H^2 \wedge \ldots \wedge H^k) \leq 1 - \prod_{j=1}^{k} (1 - \mathrm{ev}(H^j)) \ , \text{ and}$$

$$\prod_{j=1}^{k} \overline{\mathrm{ev}}(H^j) \leq \overline{\mathrm{ev}}(H^1 \wedge H^2 \wedge \ldots \wedge H^k) \leq 1 - \prod_{j=1}^{k} (1 - \overline{\mathrm{ev}}(H^j)) \ .$$

In the null-or-full support case, that is, when, for $1 \leq j \leq k$, $s^{*j} = 0$ or $s^{*j} = \widehat{s}^j$, and the truth values of the simple constituent hypotheses are either 0 or 1, the bounds in proposition 3.4 are sharp. In fact, it is not hard to see that the composition rule of classical logic holds, that is,

$$\mathrm{ev}(H^1 \wedge \ldots \wedge H^k) \ = \begin{cases} 1 \ , & \text{if } s^{*1} = \widehat{s}^1 \ldots s^{*k} = \widehat{s}^k \ ; \\ 0 \ , & \text{if, for some } j = 1 \ldots k, \ s^{*j} = 0 \ . \end{cases}$$

In the example below, illustrated by Figure 1, we show that the inequality in proposition 3.4 can, in fact be strict. Figure 1 is followed by a Matlab program giving thr Mellin convolution of discretized (stepwise) distributions, used to generate all examples.

Example 3.5: In the third, first and second subplots of Figure 1, we have the graphs of truth functions corresponding, respectively, to the complex hypothesis $H^1 \wedge H^2$ and to its elementary constituents, H^1 and H^2. Note that while $\text{ev}(H^1) = 0.5$ and $\text{ev}(H^2) = 0.7$, $\text{ev}(H^1 \wedge H^2) = 0.64$, which is strictly grater than $\text{ev}(H^1)\text{ev}(H^2) = 0.35$.

THE TRUTH OPERATION FOR CONJUNCTIONS

In this section we shall investigate, also within the independent setup, the question of whether the truth function of the FBST structure corresponding to a complex hypothesis, H, can be obtained from the truth functions of the FBST structures corresponding to its elementary constituents, H^1, H^2, ..., H^k. As in section 3, we consider the case of a conjunctive composite hypothesis, that is, the case in which H is equivalent to $H^1 \wedge H^2 \wedge \ldots \wedge H^j$.

Definition 4.1: Given two probability distribution functions $G^1 : R_+ \mapsto [0,1]$ and $G^2 : R_+ \mapsto [0,1]$. Their Mellin convolution, $G^1 \otimes G^2$, is the distribution function defined by

$$G^1 \otimes G^2(v) \;=\; \int_0^\infty \int_0^{v/y} G^1(dx) G^2(dy) \;=\; \int_0^\infty G^1(v/y) G^2(dy) \;.$$

In probabilistic terms, the Mellin convolution $G^1 \otimes G^2$ gives us the distribution function of the product, of two independent random variables, X and Y, with distribution functions, G^1 and G^2, respectively, see [6], [15] and [21]. From this interpretation, commutativeness and associativeness of Mellin convolution, $\otimes$, follows immediately.

Lemma 4.2: For a conjunctive hypothesis H,

$$H = \bigwedge_{j=1}^{k} H^j \;\;,\;\; W = \bigotimes_{1 \le j \le k} W^j \;=\; W^1 \otimes W^2 \otimes \ldots \otimes W^k(v) \;.$$

Proof: 4.2 follows straight from the definition of W.

In view of the above result, we shall refer to the Mellin convolution, in the present context, as the *Truth Operation*.

The following proposition shows that, together with the truth operation, truth summaries, (W^j, s^{*j}), $1 \le j \le k$, efficiently synthetize the independent setup information, in the sense that the truth value of a complex hypothesis H can be obtained.

Proposition 4.3: If H is a complex hypothesis with elementary constituents $H^1, H^2, \ldots H^k$, and (W^j, s^{*j}), $1 \le j \le k$, are their corresponding truth summaries, the truth value of H is given by

$$\text{ev}(H) \;=\; W(s^*) \;=\; \bigotimes_{1 \le j \le k} W^j \left(\prod_{j=1}^{k} s^{*j} \right) \;.$$

Proof: Immediate, from Lemmas 3.1 and 4.2.

DISJUNCTIVE NORMAL FORM

Let us now consider the case where H is *Homogeneous* and expressed in *Disjunctive Normal Form*, that is:

$$H = \bigvee_{i=1}^{q} \bigwedge_{j=1}^{k} H^{(i,j)} , \quad M^{(i,j)} = \{\Theta^j, H^{(i,j)}, p_0^j, p_n^j, r^j\} .$$

Let us also define $s^{*(i,j)}$ and $\widehat{s}^{(i,j)}$ as the respective constrained and unconstrained suprema of $s(\theta^{(i,j)})$ on the elementary hypotheses $H^{(i,j)}$.

Proposition 5.1:

$$\mathrm{ev}(H) = \mathrm{ev}\left(\bigvee_{i=1}^{q} \bigwedge_{j=1}^{k} H^{(i,j)}\right) = W\left(\sup_{i=1}^{q} \prod_{j=1}^{k} s^{*(i,j)}\right) =$$

$$\max_{i=1}^{q} W\left(\prod_{j=1}^{k} s^{*(i,j)}\right) = \max_{i=1}^{q} \mathrm{ev}\left(\bigwedge_{j=1}^{k} H^{(i,j)}\right) .$$

Proof: Since the supremum of a function over the (finite) union of q sets, is the maximum of the suprema of the same function over each set, and W is non-decreasing, the result follows.

Proposition 5.1 discloses the *Possibilistic* nature of the FBST truth value, that is, the e-value of a disjunction is the maximum e-value of the disjuncts, see [16-19].

FINAL REMARKS

In this gives a theoretical framework for the compositionality problem is established in the context of parametric statistical hypothesis testing, based on the FBST evidence value, $\mathrm{ev}(H)$. Forthcoming papers will illustrate several applications, such as the simultaneous calibration of measurement procedures, and the psychometric analysis of learning experiments, see [5] and [14]. Forthcomming papers will also detail the implementation of computational procedures for estimating the truth function, $W(v), 0 \leq v \leq \widehat{s}$, by Markov Chain Monte Carlo (MCMC). Such procedures require only minor adaptations, with small computational overhead, of the MCMC procedures for estimating $\mathrm{ev}(H) = W(s^*)$, see [8].

It is worth mentioning that the present article does not abridge the most general composition cases of nested or heterogeneous (independent) structures, where composite hypotheses are simultaneously assessed in heterogeneous sub-structures of (possibly) different dimensions. The following example indicates that this is not a trivial matter:

Example 5.2: Let $m = \arg\max_{j=1,2} \mathrm{ev}(H^j)$ and H be equivalent to $(H^1 \vee H^2) \wedge H^3$. Is it true that $\mathrm{ev}(H) = \max\{\mathrm{ev}(H^1 \wedge H^3), \mathrm{ev}(H^2 \wedge H^3)\} = \mathrm{ev}(H^m \wedge H^3)$? Interestingly the answer is in the negative. In the third and forth subplots of Figure 1 we have the graphs of the Truth Functions corresponding, respectively, to the complex hypothesis $H^1 \wedge H^3$ and $H^2 \wedge H^3$, where the structure M^3 is an independent replica of M^2. Observe that $\mathrm{ev}(H^1) = 0.5 < \mathrm{ev}(H^2) = 0.7$, while $\mathrm{ev}(H^1 \wedge H^3) = 0.64 > \mathrm{ev}(H^2 \wedge H^3) = 0.49$.

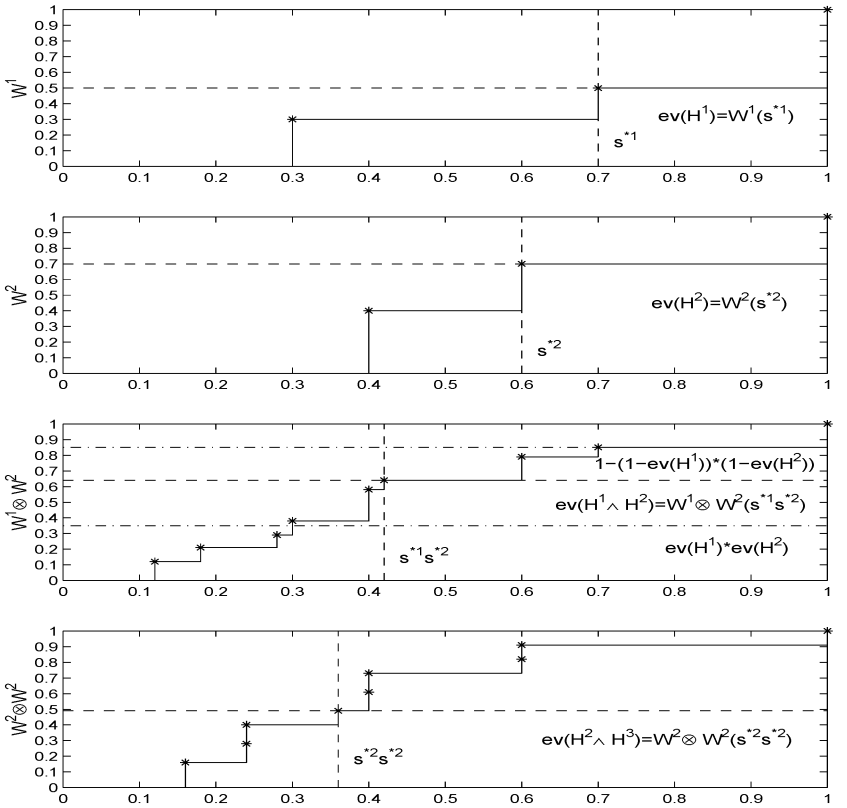

Figure 1. Truth functions $W(v),\ v \in [0, \hat{s}]$, normalized s.t. $\hat{s} = 1$
Subplots 1,2: W^J, s^{*J}, and $\mathrm{ev}(H^J)$, for $j = 1,2$;
Subplot 3: $W^1 \otimes W^2$, $s^{*1}s^{*2}$, $\mathrm{ev}(H^1 \wedge H^2)$ and bounds;
Subplot 4: Structure M^3 is an independent replica of M^2,
$\mathrm{ev}(H^1) < \mathrm{ev}(H^2)$, but $\mathrm{ev}(H^1 \wedge H^3) > \mathrm{ev}(H^2 \wedge H^3)$.

```
function [z,kk]= combine(x,y,ii,jj);
%z(1,j)= coord in [0,max_t s(t)]
%z(1,kk)= s* ,  max surprise over H
%z(2,j)= prob mass at z(1,j),     M
%z(3,j)= cumulative distribution, W
n= size(x,2); m=size(y,2); nm= n*m;
z= zeros(3,nm); k=0; skk=0;
for i=1:n   for j=1:m
  k= k+1;
  z(1,k)= x(1,i)*y(1,j);
  z(2,k)= x(2,i)*y(2,j);
*   if (i==ii & j==jj)
*       skk= z(1,k); end
* end  end  %for_i for_j
* z(3,:)= z(2,:);
* [s,ind]= sort(z(1,1:nm)');
* z= z(1:3,ind);  kk= 1;
* for k=2:nm
*   z(3,k)= z(3,k)+z(3,k-1);
*   if ( z(1,k)<=skk )
*      kk= k; end
* end %for_k
```

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