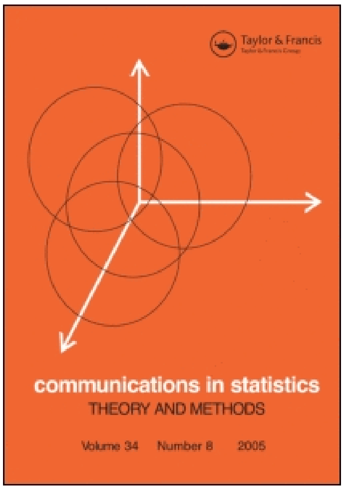


Communications in Statistics - Theory and Methods
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597238

Full Bayesian Analysis for a Class of Jump-Diffusion Models

Laura L. R. Rifo [a]; Soledad Torres [b]
[a] Institute of Mathematics, University of Campinas - UNICAMP, Campinas, Brazil [b] CIMFAV-DEUV, University of Valparaíso,

Online Publication Date: 01 January 2009

To cite this Article Rifo, Laura L. R. and Torres, Soledad(2009)'Full Bayesian Analysis for a Class of Jump-Diffusion Models',Communications in Statistics - Theory and Methods,38:8,1262 — 1271

To link to this Article: DOI: 10.1080/03610920802395694

URL: http://dx.doi.org/10.1080/03610920802395694

Communications in Statistics—Theory and Methods, 38: 1262–1271, 2009

ISSN: 0361-0926 print/1532-415X online
DOI: 10.1080/03610920802395694

Full Bayesian Analysis for a Class of Jump-Diffusion Models

LAURA L. R. RIFO[1] AND SOLEDAD TORRES[2]

[1]Institute of Mathematics, University of Campinas – UNICAMP, Campinas, Brazil
[2]CIMFAV–DEUV, University of Valparaíso

The Full Bayesian Significance Test (FBST) is adjusted for jump detection in a diffusion process. Under a natural parameterization, pure diffusion can be seen as a precise hypothesis. The evidence measure defined by FBST deals with absolutely continuous posterior distributions, when posterior rates for precise hypotheses are not appropriate. Applications to simulated and real data are shown.

Keywords Full Bayesian significance test; Jump-diffusion process.

Mathematics Subject Classification Primary 62F15; Secondary 60J60.

1. Introduction

In the context of financial data, some variables often present strong discontinuities, the so-called jumps. The well-known stochastic differential equation driven by a Brownian motion of Black and Scholes (1973) may then become unsuitable.

Decision making in the presence of jumps has been recently considered, in both theoretical and empirical work. Merton (1976) proposed diffusion models with jumps where the logarithm of jump sizes is assumed to be Gaussian. Kou (2002) suggested a double exponential law for that variable and a more general case, with the power exponential distribution, was considered by Galea et al. (2004).

Several inferential techniques have been developed in this area. Lee and Mykland (2007) presented a nonparametric approach; Chan (2005) suggested maximum likelihood estimation. Continuous time models face however a difficulty in detecting jumps, as available data are obviously discrete.

On the other hand, diffusion processes with jumps are inherently non identifiable models: trajectories are sums of diffusion and jump processes.

Received May 7, 2008; Accepted August 7, 2008
Address correspondence to Laura L. R. Rifo, Institute of Mathematics, University of Campinas – UNICAMP, P.O. Box 6065, Campinas 13083-970, SP – Brazil; E-mail: lramos@ime.unicamp.br

The usual methodology of classical statistical inference then become inappropriate (Khalaf et al., 2003; Luan and Xie, 2001). Finally, asymptotic sampling distributions are particularly inadequate for small samples in this context.

This article proposes a full Bayesian inference approach for the problem. The Full Bayesian Significance Test (FBST) was developed by Pereira and Stern (1999) for sharp hypothesis testing in parametric models. The FBST is also the optimal solution for the considered decision problem, as shown by Madruga et al. (2001), who obtained well-defined loss functions that make FBST a genuine Bayes test. Such loss functions are very useful in our context as the statistician may fix them to particular numerical descriptions of her world. A deep analysis and revision of FBST may be found in Pereira et al. (2008).

This article considers diffusion models with jumps driven by the following stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t + dJ_t = \mu\, dt + \sigma\, dW_t + d\left(\sum_{i=1}^{N_t}(V_i - 1)\right).$$

In the equation above, W_t is a standard Brownian motion, N_t is a Poisson Process, and V_i are non negative independent identically distributed random variables. The derivative μ represents expected return.

By making time discrete with unit steps, the equation above can be approximated by the difference equation

$$\frac{\triangle S_t}{S_t} \stackrel{d}{=} \mu + \sigma Z + B\,X.$$

Here, Z, B, and X are independent random quantities, Z having a standard Normal law, B and X, and therefore the product BX, a Bernoulli law.

A natural parameterization of the previous convolution allows us to frame the problem of jump detection as a test of hypothesis. In this test, the null hypothesis of no jumps is a sharp hypothesis.

Section 2 describes the formulation of the diffusion model with jumps and its discrete version. The parameterization of the model and the application of the Full Bayesian Significance Test (FBST) to it are seen in Sec. 3. Section 4 has numerical results for both real and simulated data, yielding also parameter estimates of maximum posterior density. Section 5 discusses possible generalizations and presents conclusions.

2. The Jump-Diffusion Model Formulation

In this section, we state the jump-diffusion model that motivated the statistics test that we are interested in. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ be a completed filtered probability space on which is defined a Brownian motion W and a compound Poisson process J, both adapted to the filtration $\{\mathcal{F}_t\}$. More precisely, we assume that the process J takes the following form:

$$J_t = \sum_{j=1}^{N_t}(V_j - 1), \quad t \geq 0,$$

where $N = \{N_t\}$ is a standard Poisson process with rate λ, and $\{V_j\}$ is a sequence of i.i.d. non negative random variables. We assume that:

1. for each j, $X_j = \log(V_j)$ has a given distribution;
2. the process W, N, and X_j's are independent;
3. $\mathcal{F}_t = \sigma\{W_s, J_s : 0 \leq s \leq t\}$, $t \geq 0$, augmented under P so that it satisfies the *usual hypothesis.*

With the previous notation, we consider the following stochastic differential equation (SDE):

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t + dJ_t = \mu\, dt + \sigma\, dW_t + d\left(\sum_{i=1}^{N_t}(V_i - 1)\right). \tag{1}$$

For notational simplicity the Brownian motion and jumps are assumed to be one dimensional. These assumptions, however, can be relaxed to develop a general theory.

2.1. *Discrete Model*

The goal of this section is to approximate Eq. (1) using the Euler method. From Protter (1990), the solution to the SDE (1), describing the dynamics of the process, is given by

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} \prod_{i=1}^{N_t} V_i.$$

From this,

$$\begin{aligned}\frac{\triangle S_t}{S_t} &= \frac{S_{t+1} - S_t}{S_t} \\ &= \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)\triangle t + \sigma(W_{t+\triangle t} - W_t) + \sum_{i=N_t+1}^{N_{t+\triangle t}} X_i\right] - 1.\end{aligned}$$

If $\triangle t$ is small enough, we can reject the terms of greatest order from Taylor expansion, obtaining

$$\begin{aligned}\frac{\triangle S_t}{S_t} &\sim \left(\mu - \frac{\sigma^2}{2}\right)\triangle t + \sigma(W_{t+\triangle t} - W_t) + \sum_{i=N_t+1}^{N_{t+\triangle t}} X_i + \frac{1}{2}\sigma^2(W_{t+\triangle t} - W_t)^2 \\ &\sim \mu\triangle t + \sigma Z\sqrt{\triangle t} + \sum_{i=N_t+1}^{N_{t+\triangle t}} X_i,\end{aligned}$$

where Z is a normal standard random variable. The unknown parameters are μ, which represents the expected return, σ, the volatility, and λ, the jump rate.

As it was shown in Kou (2002), for $\triangle t = \delta$ small enough we have:

$$\sum_{i=N_t+1}^{N_{t+\delta}} X_i = \begin{cases} X_{N_t+\delta}, & \text{w.p. } \lambda\delta; \\ 0, & \text{w.p. } 1 - \lambda\delta. \end{cases}$$

In other words, if δ is sufficiently small, the return can be approximated in distribution by

$$\frac{\triangle S_t}{S_t} \stackrel{d}{=} \mu\delta + \sigma Z\sqrt{\delta} + BX,$$

where B is a Bernoulli random variable with $P(B = 1) = \lambda\delta$, and $Z \sim N(0, 1)$. Hence, returns can be modeled as a convolution between a normal quantity and a random quantity with distribution F, defined below. The random quantity that defines the convolution is $\sigma\sqrt{\delta}Z + B\,X$.

Note that

$$\begin{aligned} P\big(\sigma\sqrt{\delta}Z + BX \le x\big) &= P\big(\sigma\sqrt{\delta}Z + X \le x\big)P(B = 1) + P\big(\sigma\sqrt{\delta}Z \le x\big)P(B = 0) \\ &= P\big(\sigma\sqrt{\delta}Z + X \le x\big)\lambda\delta + P\big(\sigma\sqrt{\delta}Z \le x\big)(1 - \lambda\delta). \end{aligned} \tag{2}$$

Let $\delta = 1$. Considering B and X independent Bernoulli quantities, with parameters p' and λ, respectively, then BX has a Bernoulli distribution with parameter $p = p'\lambda$.

There are many alternative models depending on the distribution of the jumps, see for example Merton (1976), for the normal distribution, Kou (2002) for double exponential, and Galea et al. (2004) for a generalization of the previous works.

From (2), the density function of $R = \mu + \sigma Z + BX$ has the following representation:

$$f(r \mid p, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\Big((1-p)e^{\frac{-1}{2\sigma^2}(r-\mu)^2} + p\,e^{\frac{-1}{2\sigma^2}(r-(\mu+1))^2}\Big). \tag{3}$$

Figure 1(a) shows a realization of the returns $R_t = \Delta S_t/S_t$ and the density (3), for $\delta = 1$ day, $\mu = 5$, $\sigma = 0.2$, $p = 0.025$. That is, there are $p = 2.5\%$ jumps with size one.

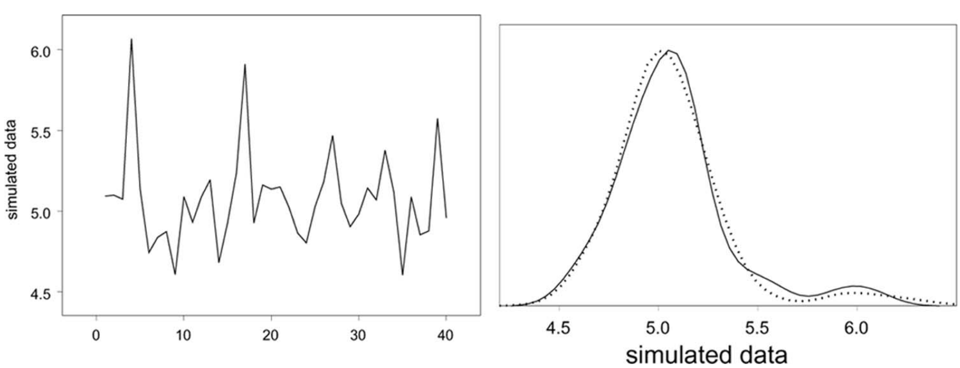


Figure 1. Time plot and fitted density for frequencies of simulated data, $p = 0.025$, $ev(\Theta_0) = 0.00442$.

3. Full Bayesian Significance Test for Bernoulli Jumps

For the problem presented in Sec. 2, let us consider the parametric space:

$$\Theta = \{\theta = (p, \mu, \sigma^2) \in R^3 : p \in [0, 1], \mu \in R, \sigma^2 > 0\}.$$

The hypothesis,

$$\Theta_0 = \{(p, \mu, \sigma^2) \in \Theta : p = 0\}, \tag{4}$$

that states that the process has no jumps, is a precise hypothesis in Θ.

Let $(\Theta, \mathscr{B}, \pi)$ be a probability measure structure on Θ such that π determines a priori probability distribution on $(\Theta, \mathscr{B})$, $\mathscr{B}$ the Borelian σ-algebra on Θ, as usual.

After observing data $d = (r_1, \ldots, r_n)$, according to Eq. (3), the likelihood for the parameter θ may be written as:

$$f(d \mid \theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \left[p\, e^{-\frac{1}{2\sigma^2}(r_i-(\mu+1))^2} + (1-p)e^{-\frac{1}{2\sigma^2}(r_i-\mu)^2}\right].$$

The information about θ can be updated by Bayes' theorem and quantified by the posterior probability law on Θ, π_d.

Full Bayesian Significance Test (FBST) can be applied when this posterior distribution has a density function with respect to Lebesgue measure, $f(\theta \mid d)$. In that case, the posterior probability of the precise hypothesis (4) is null, and therefore posterior ratio is not a good criterion to compare both models with and without jumps.

The *evidence* measure for the precise hypothesis (4) introduced by the FBST procedure is defined as the posterior probability

$$ev(\Theta_0) = 1 - \pi_d(T_0), \tag{5}$$

where T_0 is the so-called tangential set to Θ_0,

$$T_0 = \{\theta \in \Theta : f(\theta \mid d) > f_0\}, \quad \text{with } f_0 = \sup_{\Theta_0} f(\theta \mid d).$$

Hence, if tangential set has high posterior probability, the evidence in favor of Θ_0 is small; if it has low posterior probability, the evidence against Θ_0 is small.

In Madruga et al. (2001), the Bayesianity of the test of significance based on this evidence measure is shown, in the sense that there exists a loss function such that the decision for rejecting the null hypothesis is based on its posterior expected value minimization.

In order to compute evidence (5), we will define a prior distribution for $\theta = (p, \mu, \sigma^2)$, $f(\theta)$, assuming that $f(\theta)$ has a non null finite value at $p = 0$. This condition avoids $ev(\Theta_0)$ being trivially 0 or 1.

Under this formulation, the measure of evidence in favor of Θ_0 defined by (5) allows us to perform a significance test for Θ_0 without having to modify the posterior distribution, for instance, by assigning positive prior probability to a precise hypothesis.

We will assume, in this article, that p and (μ, σ^2) are independent, that p has a Beta($\alpha = 1, \beta$) distribution, σ^2 has a InverseGamma(a, b) distribution, and, given σ^2, μ has a uniform law on $(\mu_0 - c\sigma, \mu_0 + c\sigma)$.

The posterior density for $\theta = (p, \mu, \sigma^2)$ is then

$$\begin{aligned} f(\theta \mid d) &\propto f(\theta) f(d \mid \theta) \\ &\propto f(d \mid \theta)(1-p)^{\beta-1}(\sigma^2)^{-(b+3)/2} e^{-a/(2\sigma^2)} 1_{[0,1]\times(\mu_0 - c\sigma, \mu_0 + c\sigma)\times[0,\infty]}(p, \mu, \sigma^2), \end{aligned}$$

for hyperparameters β, a, b, μ_0, and c given.

4. Numerical Results

4.1. *Simulation*

We used S-Plus to perform FBST on simulated data from the convolution of a Normal$(5, 0.2^2)$ and a Bernoulli with parameter p having the values 0, 0.025, 0.10, 0.35, and 0.50.

Data were standardized

$$z_i = \frac{r_i - \bar{r}}{s_r}, \quad i = 1, \ldots, n,$$

where $\bar{r}$ and s_r are, respectively, the sample mean and sample standard deviation, with sample size $n = 40$.

The hyperparameters for the prior distribution were chosen to be $\beta = 1$, $a = 1.5$, $b = 0.5$, $\mu_0 = 0$, $c = 10$.

Table 1 presents the evidence values $ev(\Theta_0)$ for the distinct simulated rates, second column, and also maximum posterior estimates and mean posterior estimates for (p, μ, σ), third and fourth columns, respectively.

Figure 1 presents the simulated data on time for a process having $p = 0.025$ as jump rate. The right-hand graph shows the data empirical density (in full line), and the density adjusted by the posterior mode (in dotted line). Data suggest the possibility of jumps, quantified by the small evidence in favor of the null hypothesis, $ev(\Theta_0) = 0.00442$.

Table 1

In the second column, evidence values for several jump rates, p, and modal and mean posterior estimates for (p, μ, σ) at the third and fourth column, respectively ($\mu = 5$, $\sigma = 0.2$, sample size $n = 40$)

		Posterior estimates for (p, μ, σ)	
p	$ev(\Theta_0)$	Mode	Mean
0	1	(0.0000, 5.055, 0.2076)	(0.4761, 4.567, 0.2110)
0.025	0.00442	(0.0505, 5.012, 0.1999)	(0.0753, 5.016, 0.2060)
0.10	9.74e-8	(0.1293, 5.012, 0.1725)	(0.1366, 5.016, 0.1836)
0.35	1.66e-4	(0.2933, 5.004, 0.2000)	(0.3283, 5.004, 0.2094)
0.50	2.92e-5	(0.4837, 5.026, 0.2129)	(0.5122, 5.030, 0.2314)

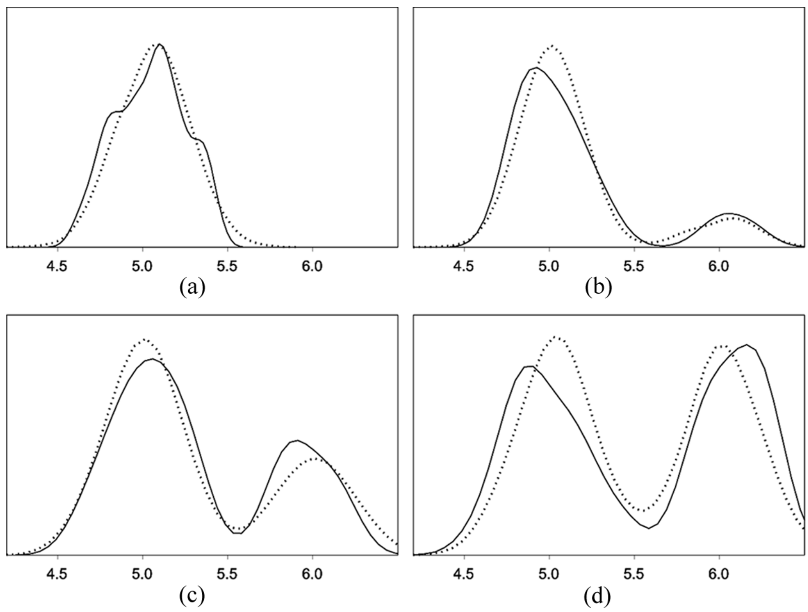


Figure 2. Simulated data empirical density (full line), density adjusted by posterior mode (dotted line), with jump rate and evidence, respectively: (a) $p = 0$, $ev(\Theta_0) = 1$; (b) $p = 0.1$, $ev(\Theta_0) = 9.74e - 8$; (c) $p = 0.35$, $ev(\Theta_0) = 1.66e - 4$; and (d) $p = 0.5$, $ev(\Theta_0) = 2.92e - 5$.

Figure 2 shows the simulated data empirical density (full line) in each case and the density determined by the posterior mode (dotted line), for the various rates considered.

It can be seen that a strong bimodality in the sample is well detected by the FBST procedure as expected. On the other hand, without a strong bimodality, the evidence in favor of Θ_0 is large, as seen in the graph of simulated data with $p = 0$, where $ev(\Theta_0) = 1$.

The computation of evidence values $ev(\Theta_0)$ was based on the simulation of 400,000 independent points uniformly distributed over the support of the posterior distribution, and having their non normalized values, $\pi(\theta \mid d)$, computed. Other 400,000 points were uniformly generated on Θ_0, in order to obtain the sample maximum of the non normalized posterior density on Θ_0, φ_0.

The posterior probability of the tangential set can be approximated by the ratio of the sum of values of $\pi(\theta \mid d)$ larger than φ_0 to the total sum of $\pi(\theta \mid d)$ for all generated points.

The mean time for running of this program was 600 s approximately, in a standard PC.

Other optimization numerical methods which could be more efficient for determining φ_0 and integrating on T_0, may be used. Our simple method, however, had a good performance for the problem of model choice between models having Bernoulli jumps or models without jumps. Furthermore, it can be generalized in a straightforward way for other classes of jumps.

4.2. *Venezuela Rainfall Data*

Figure 3 presents Annual Maximum Rainfall Data (Maiquetia station at Venezuela central coast) from 1951–1998.

Coles and Pericchi (2003) used these data to compare Bayesian and Classical methods on the prediction of the 1999 catastrophic maximum of 410 mm.

For this example, the jump size was considered random, so the parametric space had one more dimension added to it, k, having a uniform prior distribution on $[0, 30]$, independently from the other components,

$$\Theta = \{\theta = (p, \mu, \sigma^2, k) \in R^4 : p \in [0, 1], \mu \in R, \sigma^2 > 0, k \in [0, 30]\},$$

and

$$f(d \mid \theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \left[p\, e^{-\frac{1}{2\sigma^2}(r_i-(\mu+k))^2} + (1-p)e^{-\frac{1}{2\sigma^2}(r_i-\mu)^2}\right].$$

The analysis obtained an evidence value in favor of Θ_0, $ev(\Theta_0) = 0.05353$. The posterior mode and mean estimates for normalized data are

$$(\hat{p}, \hat{\mu}, \hat{\sigma}, \hat{k})_{\text{mode}} = (0.09064, -0.1258, 0.4869, 2.242),$$
$$(\hat{p}, \hat{\mu}, \hat{\sigma}, \hat{k})_{\text{mean}} = (0.1406, -0.2742, 0.7348, 2.398).$$

In original units, the posterior mode estimate represents the model

$$\widehat{R}_{\text{mode}} = \overline{R} - 0.13 sd_R + 0.49 sd_R Z + 2.2 sd_R B_1,$$

and the posterior mean estimate represents the model

$$\widehat{R}_{\text{mean}} = \overline{R} - 0.27 sd_R + 0.73 sd_R Z + 2.4 sd_R B_2,$$

where $\overline{R}$ and sd_R are the sample mean and the sample standard deviation of observed returns, Z has standard normal distribution, B_1 has a Bernoulli(0.09) distribution, and B_2 has a Bernoulli(0.14) law.

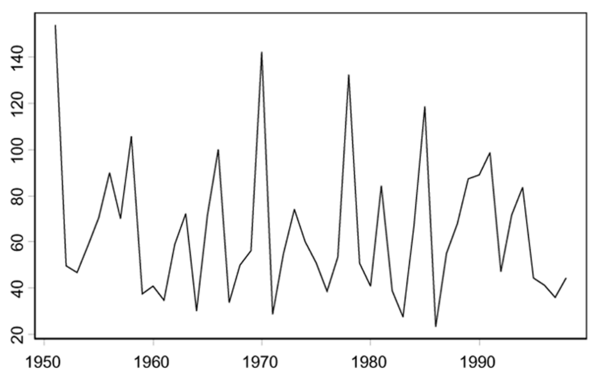
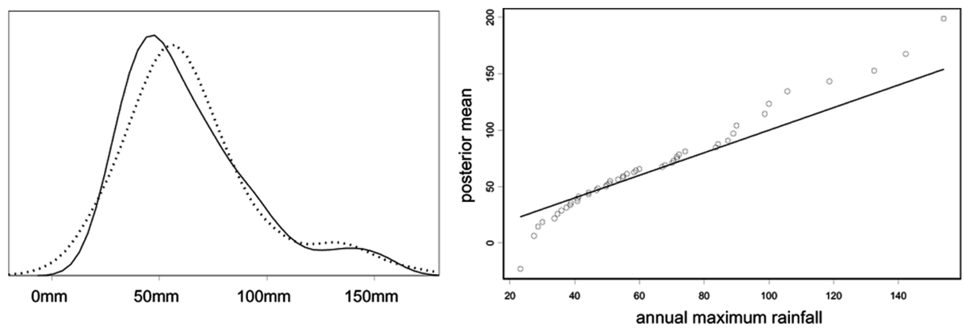


Figure 3. Venezuela annual maximum rainfall data, 1951–1998. Source: http://www.blackwellpublishing.com/rss/Volumes/Cv52p4.htm



(a) (a)

Figure 4. Real data empirical density (full line), posterior mean adjusted density (dotted line); qqplot for data and posterior mean model. Evidence in favor of non-existence of jumps is $ev(\Theta_0) = 0.05353$.

As seen in Fig. 4, those estimates fit the observed data well. The posterior mode estimate is here the maximum likelihood estimate, as the chosen prior is uniform with large support.

5. Conclusions

In this work, we are testing two kinds of models: diffusion processes vs. diffusion processes with Bernoulli jumps. We could also be interested in testing for more general jump families, which allow, for instance, heavy-tailed distributions or diffusion processes having non constant volatility against processes having constant volatility.

The proposed test procedure extends naturally for each of these families of models, as long as there is a sharp hypothesis to be tested. Modifications on the likelihood function are straightforward. The computational cost increases with the dimension of the parameter space, as the required probabilities are integrated directly on the latter.

As in those examples, strong likelihood may jeopardize mixture properties of the envolved Markov chains and therefore slow down the convergence rate of other Monte Carlo methodologies.

We stress the absolute continuity of the posterior distribution. The chosen priors are usually Lebesgue absolutely continuous, entailing absolute continuity of the posterior for the models in our context.

Any sharp hypothesis has therefore null posterior probability and posterior ratios are not a good criterion to compare models defined by sharp hypotheses.

The measure of evidence used in this article is associated to a decision problem (Sec. 1). In the real decision problem, we could—and should—consider the loss function the minimization of its expected value is tantamount to performing the FBST.

Acknowledgment

This work was partially supported by Project Fondecyt 1070918 and Project PBCT/ACT 13 Laboratorio ANESTOC.

References

Black, F., Scholes, M. (1973). The pricing of options and corporate liabilities. *J. Polit. Econ.* 81:637–654.

Chan, K. F. (2005). Modelling conditional heteroscedasticity and jumps in australian short-term interest rates. *Account. Fin.* 45:537–551.

Coles, S., Pericchi, L. (2003). Anticipating catastrophes through extreme value modelling. *J. Roy. Statist. Soc. Ser. C* (Applied Statistics) 52(4):405–416.

Galea, M., Ma, J., Torres, S. (2004). Price calculation for power exponential jump-diffusion models: a hermite series approach. *Contemp. Math.* 336:137–159.

Kadane, J. B. (1977). The role of identification in bayesian theory. In: Fienberg, S. E., Zellner, A. eds. *Studies in Bayesian Econometrics and Statistics, In Honor of Leonard J. Savage*, Vol. I: pp. 175–191.

Khalaf, L., Saphores, J.-D., Bilodeau, J.-F. (2003). Simulation based exact jump tests in models with conditional heteroskedasticity. *J. Econ. Dynam. Contr.* 28:531–553.

Kou, S. (2002). A jump diffusion model for option pricing. *Manag. Sci.* 48(8):1086–1101.

Lee, S., Mykland, P. A. (2007). Jumps in financial markets: a new nonparametric test and jump dynamics. *Rev. Fin. Stud.* To appear.

Luan, Y., Xie, Z. (2001). The wavelet identification for jump points of derivative in regression model. *Statist. Probab. Lett.* 53(2):167–180.

Madruga, M. R., Esteves, L. G., Wechsler, S. (2001). On the bayesianity of pereira-stern tests. *Test* 10(2):291–299.

Merton, R. (1976). Option pricing when underlying stock prices are discontinuous. *J. Finan. Econ.* 3:125–144.

Pereira, C. A. B., Stern, J. (1999). Evidence and credibility: full bayesian significance test for precise hypothesis. *Entropy* 1:99–110.

Pereira, C. A. B., Stern, J., Wechsler, S. (2008). Can a significance test be genuinely bayesian? *Bayesian Anal.* 3:15–36.

Protter, P. (1990). *Stochastic Integration and Stochastic Differential Equations: A New Approach.* Berlin: Springer.