Residual analysis for accelerated failure time models with random effects

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Abstract

We adapt marginal, conditional and random effects residual plots developed for linear mixed models to assess the fit of accelerated failure time models with random effects in the presence of censored observations. We propose two imputation procedures to replace the unobserved failure times and perform a brief simulation study with the objective of studying the ability of residual plots to evaluate the model assumptions under increasing proportions of censoring. We illustrate the proposed residual analysis with a data set involving failure of oil wells.

Keywords: censored observations; diagnostics; imputation; linear mixed models; Weibull distribution.

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1 Introduction

We consider a retrospective study related to the operating times of oil wells during the period 2000 - 2006 designed to identify wells needing preventive maintenance based on some characteristics like production level, lifting method, pump depth, well age, region etc. Since each well may have recurrent failure times, we expect a dependence between the repeated observations on the same well. Moreover, as some wells are disabled from production and others are operating at the end of the study, it is necessary to take censoring into account. Standard parametric accelerated failure time (AFT) models are often used to model data with this nature when the observations are independent [see Lawless (2003), for example]. However, these models are not appropriate to fit correlated survival times. Some authors deal with correlated survival data in the context of reliability of repairable systems [see Ascher and Feingold (1984), Lawless and Thiagarajah (1996) and Percy and Alkali (2007), for example]. Keiding et al. (1997), Lambert et al. (2004), Bolivarine and Valença (2005), Santos and Valença (2012)] recommend the use of AFT models with the inclusion of (non-observable) random effects acting multiplicatively on the event times. Under this parametric setting, a common approach is to use Weibull regression models. Lambert et al. (2004) propose the use of empirical Bayes methods to fit such models, while Carvalho et al. (2014) consider a classical approach based on linear mixed models. These authors use empirical best linear unbiased predictors (EBLUP) where only the existence of the first two moments of the distribution of the random effects is required. Carvalho et al. (2014) use an adaptation of the non-parametric imputation methods proposed by Ageel (2002) to deal with censored data.

For any statistical model, the validity of the underlying assumptions need to be checked by diagnostic techniques such as residual analysis. Nevertheless, there are few proposals for residual analysis in the context of survival models with random effects. Dobson and Henderson (2003), Rizopoulos et al. (2009) and Rizopoulos (2010) define residuals for models that jointly consider the analysis of longitudinal and survival data. Their proposals, however, do not investigate assumptions on the distribution of the random effects.

In mixed models, there is more than one source of variability and consequently, more than one type of residuals. Hilden-Minton (1995), Verbeke and Lesaffre (1996a) or Pin-
heiro and Bates (2000), for example, define three types of residuals to accommodate the additional sources of variability. A summary of the available tools may be found in Nobre and Singer (2007), for example.

Our objective is to use residual analysis techniques originally developed for linear mixed models to assess the assumptions on the fixed and random effects employed in the analysis of correlated survival data. In this context, the presence of censored observations can result in distortions regarding the interpretation of the residual plots. To bypass this problem we consider imputation methods for the censored data.

In Section 2, we specify the AFT model with random effects and consider a brief summary on residual analysis for linear mixed models. In Section 3, we describe two imputation methods designed to replace the censored observations. In Section 4, we present the results of a brief simulation study conducted to investigate the impact of different percentages of censored observations on the ability of residual plots to assess the model assumptions. In Section 5, we analyze the oil well data and we conclude with a brief discussion in Section 6.

2 Accelerated failure time models with random effects

The AFT model with random effects may be written as

$$ \ln T_{ij} = b_i + x_{ij}^\top \beta + \sigma \epsilon_{ij}, $$

for $i = 1, \ldots, k$, $j = 1, \ldots, n_i$, where $T_{ij}$ represents the time between the $(j - 1)$ -th and the $j$-th failure of the $i$-th sample unit, $b_i$ are independent and identically distributed (unobserved) random effects with null means and common variance $\sigma^2_b$, $x_{ij}$ is a $(p \times 1)$ vector of covariates with the first component equal to 1, $\beta$ is a $(p \times 1)$ vector whose elements are fixed (but unknown) parameters, $\sigma$ is a (unknown) scale parameter and $\epsilon_{ij}$ are independent and identically distributed unobserved random errors with known mean and common variance $\sigma^2$. Furthermore, we assume that $\text{Cov}(b_i, \epsilon_{ij}) = 0$. When $\sigma^2_b = 0$, this model reduces to the usual AFT model [see Lawless (2003) and Bolfarine and Valença (2005), for example].
Because of censoring, the response variable $\ln T_{ij}$ is not observed for all sampling units. In fact, we observe

$$Y_{ij} = \delta_{ij} \ln T_{ij} + (1 - \delta_{ij}) \ln C_{ij},$$

where $C_{ij}$ is the $j$-th censoring time for the $i$-th sample unit and $\delta_{ij} = I(T_{ij} \leq C_{ij})$ is an indicator of failures.

Consider initially a sample without censoring, i.e., suppose that $\delta_{ij} = 1$ for $i = 1, \ldots, k$, $j = 1, \ldots, n_i$ in (2) and let $n = \sum_{i=1}^{k} n_i$.

Then the AFT model with random effects (1) can be represented as a linear mixed model, namely

$$Y = X\beta + Zb + e,$$

where $Y = (Y_1^T, \ldots, Y_k^T)^T$, with $Y_i = (Y_{i1}, \ldots, Y_{in_i})^T$, $i = 1, \ldots, k$, $X = (X_1^T, \ldots, X_k^T)^T$, $X_i = (x_{i1}, x_{i2}, \ldots, x_{in_i})^T$, $Z = \oplus_{i=1}^{k} 1_{n_i}$, $b = (b_1, \ldots, b_k)^T$ is a $(k \times 1)$ vector of random effects such that $E(b) = 0$, $Var(b) = \sigma_b^2 I_k$ and $e = \sigma [e - E(e)]$ is an $(n \times 1)$ vector of random uncorrelated errors and $e$ is a $(n \times 1)$ vector having the random errors $\epsilon_{ij}$ as components. Thus, $E(e) = 0$, $Var(e) = \sigma_e^2 I_n$, with $\sigma_e^2 = \sigma^2 Var(\epsilon_{ij})$. When $\epsilon_{ij}$ follows a standard extreme value distribution, $\sigma_e^2 = \sigma^2 \pi^2 / 6$. These definitions imply that

$$E(Y) = X\beta$$

$$Var(Y) = V = \sigma_b^2 ZZ^T + \sigma_e^2 I_n$$

$$Cov(b, Y^T) = C = \sigma_b^2 Z.$$

Conditionally on the knowledge of $V$ and $C$, the best linear unbiased estimator (BLUE) of $\beta$ and the best linear unbiased predictor (BLUP) of $b$ are respectively:

$$\hat{\beta} = (X^T V^{-1} X)^{-1} X^T V^{-1} Y$$

$$\tilde{b} = CV^{-1}(Y - X\hat{\beta}).$$

In practice, as the variance components $\sigma_b^2$ and $\sigma_e^2$ (and therefore, $V$ and $C$) are unknown, it is reasonable to use empirical estimates and predictors (EBLUE and EBLUP, respectively), obtained by replacing $V$ and $C$ with suitable estimators. Details can be found in Robinson (1991) and Jiang and Verbeke (1998). Estimates of the variance com-
ponents may be obtained via the nonparametric MINQUE and I-MINQUE methods [see Rao (1971a), Rao (1971b) and Searle et al. (1992), for example].

Under model (3) we may consider three types of residuals, namely:

- **Marginal residuals**: $\hat{\xi} = Y - X\hat{\beta}$, that predict the marginal errors, $\xi = Y - E(Y) = Y - X\beta = Zb + e$

- **Conditional residuals**: $\hat{e} = Y - X\hat{\beta} - Z\tilde{b}$, that predict the conditional errors, $e = Y - E(Y|b) = Y - X\beta - Zb$

- **Random effects residuals**: $Z\tilde{b}$, that predict the random effects, $Zb = E(Y|b) = (Y - X\beta - Zb)$.

The marginal residuals may be used to evaluate the linearity of fixed effects, to detect outliers as well as to check the adequacy of the within-units covariance structure. Lesaffre and Verbeke (1998) comment that when the within-unit covariance structure is adequate, $V_i = \|I_{n_i} - \widehat{R}_i\widehat{R}_i^\top\|^2$, $i = 1, \ldots, k$, where $\widehat{R}_i = \widehat{V}_i^{-1/2}\hat{\xi}_i$ with $\widehat{V}_i = V_i(\hat{\theta})$ being the $i$-th diagonal block of $V_i$, should be close to zero. Units with large values of $V_i$ are those for which the proposed covariance structure might not be adequate. Given that the true variance of $\hat{\xi}_i$ is $V(\hat{\xi}_i) = [V_i - X_i(X_i^\top V_i^{-1}X_i)^{-1}X_i^\top]$ and not $V_i$, Singer et al. (2015) consider replacing $\widehat{R}_i$ in $V_i$ with $\hat{\xi}_i = [\widehat{V}(\hat{\xi}_i)]^{-1/2}\hat{\xi}_i$ where $\widehat{V}(\hat{\xi}_i)$ corresponds to the diagonal block of $\widehat{V} - X(X^\top\widehat{V}^{-1}X)^{-1}X^\top$ associated to the $i$-unit. Furthermore, to avoid giving much weight to units with many observations, these authors consider taking $V_i^* = \sqrt{V_i}/n_i$ as a standardized measure of adequacy of the within-unit covariance structure. Unit index-plots of $V_i^*$ may help to identify units for which the covariance structure should be modified.

To evaluate the linearity of the fixed effects in model (3), Singer et al. (2015) consider plotting the elements of the standardized marginal residuals $\hat{\xi}_{ij}^* = \hat{\xi}_{ij}/[\text{diag}_j(\widehat{V}(\hat{\xi}_i))]^{1/2}$, where $\text{diag}_j(\widehat{V}(\hat{\xi}_i))$ is the $j$-th element of the main diagonal of $\widehat{V}(\hat{\xi}_i)$, versus the values of each explanatory variable as well as versus the fitted values. They also recommend plotting $\hat{\xi}_{ij}$ versus the observation index as a tool to detect outlying observations.

Given that $V(\hat{e}) = \sigma_e^4[V^{-1} - V^{-1}X(X^\top V^{-1}X)^{-1}X^\top V^{-1}]$, Nobre and Singer (2007) observe that the conditional residuals may have different variances. They suggest plots of standardized conditional residuals, $\hat{e}_{ij}^* = \hat{e}_{ij}/[\text{diag}_{ij}(\widehat{V}(\hat{e}))]^{1/2}$, with $\text{diag}_{ij}(\widehat{V}(\hat{e}))$ denoting...
the main diagonal element of $\hat{V}(\hat{e})$ corresponding to the $j$-th observation of the $i$-th unit versus fitted values to check for homoskedasticity of the conditional errors or versus unit index to check for outlying observations.

The use of conditional residuals is not recommended for assessment of the assumption on the distribution of errors, because this type of residuals involves confounding between $b$ and $e$. In other words, the distribution of $\hat{e}$ depends on both the distributions of $e$ and $b$. Alternatively, residuals with minimal confounding can be used [see Hilden-Minton (1995) and Singer et al. (2015)].

When there is no confounding and the random effects follow a $q$-dimensional gaussian distribution, $M_i = \tilde{b}_i^\top \{\hat{V}^\top [\tilde{b}_i - b_i]\}^{-1} b_i$ (the Mahalanobis’s distance between $\tilde{b}_i$ and $E(b_i) = 0$) should have an approximate chi-squared distribution with $q$ degrees of freedom. Therefore, a $\chi^2_q$ QQ plot for $M_i$ may be used to verify whether the random effects follow a (q-variate) gaussian distribution. Unit index-plots of $M_i$ may also be employed to detect outliers.

More details on residual analysis for mixed models may be obtained in Singer et al. (2015).

3 Taking censored observations into account

The response variables (2) underestimate the true times between failures when $\delta_{ij} = 0$ so that an appropriate use of the standard linear mixed model (3) requires some form of imputation. In this context, we propose two procedures to replace the censored values $C_{ij}$ in (2) by estimates $\hat{T}_{ij}$ of the true (unobserved) failure time $T_{ij}$, i.e., we consider the response variable

$$Y_{ij}^* = \delta_{ij} \ln T_{ij} + (1 - \delta_{ij}) \ln \hat{T}_{ij},$$

for $i = 1, \ldots, k$ and $j = 1, \ldots, n_i$. 

3.1 Extension of Ageel’s method (EAM)

Ageel (2002) proposes parametric and non-parametric methods to impute the censored observations in a survival model with independent random and right censored observations. Carvalho et al. (2014) consider an adaptation of Ageel’s non-parametric approach to deal with correlated data. Here, we adapt the parametric approach to incorporate covariates based on (1) assuming that conditionally to random effects $b_i$, $T_{ij}$ follows a two parameter Weibull distribution and replacing each censored value with an estimate of $E(T_{ij}|T_{ij} > c_{ij}, b_i, x_{ij})$.

The density and survival functions of the conditional distribution of $T_{ij}$ given $b_i$ are, respectively:

$$f(t_{ij}|b_i, \phi, x_{ij}) = \frac{\gamma}{\exp(b_i + \beta^\top x_{ij})} \left(\frac{t_{ij}}{\exp(b_i + \beta^\top x_{ij})}\right)^{\gamma - 1} \exp\left[-\left(\frac{t_{ij}}{\exp(b_i + \beta^\top x_{ij})}\right)^\gamma\right] I_{(0, \infty)}(t_{ij})$$

and

$$S(t_{ij}|b_i, \phi, x_{ij}) = \exp\left[-\left(\frac{t_{ij}}{\exp(b_i + \beta^\top x_{ij})}\right)^\gamma\right] I_{(0, \infty)}(t_{ij}),$$

where $\phi = (\beta^\top, \sigma^2, \sigma_b^2)^\top$ is the vector of parameters, $\gamma = 1/\sigma > 0$ denotes the shape parameter and $\exp(b_i + \beta^\top x_{ij}) > 0$ represents the scale parameter. The density function and the expected value for the conditional distribution of $T_{ij}$ (given $T_{ij} > c_{ij}$ and the random effects vector $b_i$) are, respectively

$$f(t_{ij}| t_{ij} > c_{ij}; b_i, \phi, x_{ij}) = \frac{f(t_{ij}|b_i, x_{ij})}{S(c_{ij}|b_i, x_{ij})} = \frac{\gamma/[\exp(b_i + \beta^\top x_{ij})] \left(t_{ij}/[\exp(b_i + \beta^\top x_{ij})]\right)^{\gamma - 1}}{\exp[-(c_{ij}/\exp(b_i + \beta^\top x_{ij}))^\gamma]} \times \frac{\exp[-(t_{ij}/[\exp(b_i + \beta^\top x_{ij})])^\gamma]}{\exp[-(c_{ij}/\exp(b_i + \beta^\top x_{ij}))^\gamma]}$$

and

$$E(T_{ij}|T_{ij} > c_{ij}; b_i, \phi, x_{ij}) = \frac{\exp(b_i + \beta^\top x_{ij}) \Gamma(1 + 1/\gamma, (c_{ij}/\exp(b_i + \beta^\top x_{ij}))^\gamma)}{\exp[-(c_{ij}/\exp(b_i + \beta^\top x_{ij}))^\gamma]}, \quad (7)$$

where $c_{ij}$ represents the time of occurrence of the $j$-th censoring for the $i$-th sample unit and $\Gamma(\eta, \tau)$ denotes the incomplete gamma function with parameters $\eta = 1 + 1/\gamma$ and $\tau = (c_{ij}/\exp(b_i + \beta^\top x_i))^\gamma$ [Olver et al. (2010)].
3.2 Inverse transform method (ITM)

A second imputation approach consists in replacing the censored observations in (2) by a value generated from the conditional distribution of \( T_{ij} \) given the random effect \( b_i \) and \( T_{ij} > c_{ij} \). We generate random observations \( \hat{T}_{ij} \) from the cumulative distribution function \( F(t_{ij}|T_{ij} > c_{ij}, b_i, x_{ij}) \) using the method of inverse transform [see, for example, Stewart (2009)] and plug in the generated values in (6).

The distribution function of \( T_{ij} \) given the vector of random effects \( b_i \), the vector of covariates \( x_{ij} \) and \( T_{ij} > c_{ij} \) is

\[
F(t_{ij}|t_{ij} > c_{ij}; b_i, \phi, x_{ij}) = 1 - \frac{S(t_{ij}|b_i, \phi, x_{ij})}{S(c_{ij}|b_i, \phi, x_{ij})}.
\] (8)

We obtain \( t_{ij} \) through the inverse of the distribution function given in (8), and thus, \( \hat{T}_{ij} \) in (6) can be generated from

\[
\exp(b_i + \beta^\top x_{ij}) \left[ -\log(1 - u_{ij}) + \left( \frac{c_{ij}}{\exp(b_i + \beta^\top x_{ij})} \right)^{\gamma} \right]^{1/\gamma},
\] (9)

where \( u_{ij} \) is a value generated from a uniform distribution on \((0,1)\), \( i = 1, \ldots, k, j = 1, \ldots, n_i \).

3.3 Procedures for imputation

To obtain estimates of (7) and (9) we require preliminary estimates of \( \phi \) and of the predictors of \( b_i \) which may be obtained from the penalized log-likelihood [see Therneau et al. (2003)]

\[
l_{pen}(\phi|b) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \log f(t_{ij}|b_i, \phi, x_{ij}) \delta_{ij} S(t_{ij}|b_i, \phi, x_{ij})^{1-\delta_{ij}} - h(b; \sigma^2_b),
\] (10)

where \( f(t_{ij}|b_i, \phi, x_{ij}) \) and \( S(t_{ij}|b_i, \phi, x_{ij}) \) are, respectively, the density and survival functions of the conditional distribution of \( T_{ij} \) given \( b_i \) and \( h(b; \sigma^2_b) \) is the penalty function. When the distribution of the random effects is normal, Therneau et al. (2003) suggest the following penalty function

\[
h(b; \sigma^2_b) = \frac{1}{2\sigma^2_b} \sum_{i=1}^{k} b_i^2.
\] (11)
Maximizing (10) we obtain the maximum penalized likelihood estimator $\widehat{\phi}^*$ for $\phi$ and predictors $\widehat{b}_i^*$ for the random effects $b_i$, $i = 1, \ldots, k$. In this context we may use the function `survreg` in the library `survival` available in the free software package R (R Development Core Team, 2015). Details can be found in Santos and Valença (2012).

The procedure for imputing the censored data and obtaining the residuals may be summarized as follows.

i) Assume that $b_i \sim N(0, \sigma_b^2)$ in (1),

ii) Obtain $\widehat{\phi}^*$ and $\widehat{b}_i^*$ based on the penalized likelihood (10),

iii) Select the significant covariates using likelihood ratio tests.

iv) Compute $Y_{ij}^*$ in (6), considering for each censored value, an estimate $\widehat{T}_{ij}$ from either (7) or (9) with $\phi$ and $b_i$ replaced by $\widehat{\phi}^*$ and $\widehat{b}_i^*$ and $u_{ij}$ generated from a standard uniform distribution on (0,1).

v) Repeat iv) for all censored values and obtain the vector $Y^* = (Y_1^T, \ldots, Y_k^T)^T$ where $Y_i^* = (Y_{i1}^*, \ldots, Y_{in_i}^*)^T$, $i = 1, \ldots, k$.

After replacing the censored observations with the imputed values, we may treat the vector $Y^*$ as the true vector of responses in (3). The estimates of the parameters (EBLUE) and predictors of the random effects (EBLUP) do not require a specification for the distribution of the random effects.

4 Simulation

A brief simulation study was conducted to compare the two proposed procedures, (EAM) and (ITM), for imputation of the censored observations.

We considered model (1) with a single covariate $x_{ij}$ drawn from the standard normal distribution. In order to mimic the structure of the data motivating this study, we considered samples with $k = 200$ units, with sizes $n_i$ for $i = 1, \ldots, k$, generated from a Poisson distribution with mean 6, where units with size zero were disregarded. The true parameter values were taken as $\beta_0 = 7.5$, $\beta_1 = 0.85$ and $\sigma = 1.15$. The random effects $b_i$ were
generated as independent and identically distributed (i.i.d.) normal random variables with zero mean and variance $\sigma_b^2 = 0.35$ and the errors $\epsilon_{ij}$ were generated as i.i.d. standard extreme value random variables. This yields failure times $t_{ij}$ that, conditionally on the random effects $b_i$, follow a Weibull distribution. Besides uncensored samples, we considered samples with 30% and 50% censored observations. Censoring times $c_{ij}$ were generated as independent uniform random variables on $(0, u)$, where $u$ varies according to the specified censoring proportions. Censored values were replaced by using both imputation methods described in Section 3 and model (3) with response variable (6) was fitted to the data.

To evaluate the performance of the two imputation procedures, we considered 1,000 data sets for each of the three simulation scenarios (uncensored, 30% and 50% censoring). To show how the diagnostic tools may be used to check the model assumptions, we randomly selected one of the data sets generated in each simulation scenario.

All simulations and residuals analysis were performed using R.

4.1 Results

The Monte Carlo mean, bias and standard error (SE) of the estimates for each parameter $(\beta_0, \beta_1)$ where respectively computed as

$$\text{Mean}(\hat{\beta}_i) = \frac{\sum_{l=1}^{1,000} \hat{\beta}_{il}}{1,000},$$

$$\text{Bias}(\hat{\beta}_i) = \text{Mean}(\hat{\beta}_i) - \beta_i,$$

and

$$\text{SE} = \sqrt{\frac{\sum_{l=1}^{1,000} (\hat{\beta}_{il} - \text{Mean}(\hat{\beta}_i))^2}{1,000}},$$

where $\hat{\beta}_{il}$ denotes the estimate of $\beta_i$ obtained from the $l$-th simulated sample, $i = 0, 1, l = 1, \ldots, 1000$. The results are displayed in Table I.

Estimates obtained from imputed samples have smaller bias and standard errors than those obtained from the original (censored) samples for both percentages of censoring with the performance of imputation via EAM being slightly better than that obtained with ITM.

To illustrate the use of the diagnostic tools, we selected one sample from each censoring scenario, fitted the proposed AFT model using the original (censored) simulated data as
Table 1: Bias and standard errors (SE) of the parameter estimates ($\beta_0 = 7.5$ and $\beta_1 = 0.85$).

<table>
<thead>
<tr>
<th>Censoring</th>
<th>Parameter</th>
<th>No imputation</th>
<th>EAM</th>
<th>ITM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Bias</td>
<td>SE</td>
<td>Bias</td>
</tr>
<tr>
<td>30%</td>
<td>$\beta_0$</td>
<td>-0.397</td>
<td>0.018</td>
<td>-0.092</td>
</tr>
<tr>
<td>30%</td>
<td>$\beta_1$</td>
<td>-0.256</td>
<td>0.019</td>
<td>-0.028</td>
</tr>
<tr>
<td>50%</td>
<td>$\beta_0$</td>
<td>-0.800</td>
<td>0.021</td>
<td>-0.140</td>
</tr>
<tr>
<td>50%</td>
<td>$\beta_1$</td>
<td>-0.425</td>
<td>0.020</td>
<td>-0.033</td>
</tr>
</tbody>
</table>

As well as the imputed counterparts and obtained the corresponding residual plots. Estimates of the model parameters and corresponding standard errors for the selected samples are presented in Table 2.

Table 2: Parameter estimates and standard errors (SE) for the selected simulated samples.

<table>
<thead>
<tr>
<th>Censoring</th>
<th>Parameter</th>
<th>No imputation</th>
<th>EAM</th>
<th>ITM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Estimates</td>
<td>SE</td>
<td>Estimates</td>
</tr>
<tr>
<td>0%</td>
<td>$\beta_0$</td>
<td>7.628</td>
<td>0.068</td>
<td>-</td>
</tr>
<tr>
<td>0%</td>
<td>$\beta_1$</td>
<td>0.816</td>
<td>0.048</td>
<td>-</td>
</tr>
<tr>
<td>30%</td>
<td>$\beta_0$</td>
<td>7.207</td>
<td>0.056</td>
<td>7.648</td>
</tr>
<tr>
<td>30%</td>
<td>$\beta_1$</td>
<td>0.534</td>
<td>0.043</td>
<td>0.837</td>
</tr>
<tr>
<td>50%</td>
<td>$\beta_0$</td>
<td>6.819</td>
<td>0.047</td>
<td>7.674</td>
</tr>
<tr>
<td>50%</td>
<td>$\beta_1$</td>
<td>0.420</td>
<td>0.039</td>
<td>0.841</td>
</tr>
</tbody>
</table>

In both censoring scenarios, the bias of the intercept is slightly smaller under the ITM procedure than that obtained under the EAM method; the situation is reversed when the slope is considered. Standard errors obtained under both imputation methods are comparable. Estimates of both parameters are more biased when no imputation is considered.

The corresponding residual plots are shown in Figures 1-5.

Although the EAM method is less biased, for both censoring percentages, the standardized marginal residuals obtained under ITM imputation show a better resemblance with those produced from the uncensored data (Figure 1). Furthermore, no evidence against the homoskedasticity assumption stems from these plots. A similar conclusion holds for the standardized conditional residuals as depicted in Figure 3. From Figure 2 we may...
Figure 1: Standardized marginal residuals ($\hat{\xi}_i^*$) vs fitted values.
Figure 2: Modified Lesaffre-Verbeke index ($V_i^*$) unit index plots.
Figure 3: Standardized conditional residuals ($\hat{e}_k^*$) vs fitted values.

(a) uncensored

(b) $p = 30\%$, No imputation

(c) $p = 50\%$, No imputation

(d) $p = 30\%$, EAM

(e) $p = 30\%$, ITM

(f) $p = 50\%$, EAM

(g) $p = 50\%$, ITM
Figure 4: QQ plot for standardized conditional residual ($\hat{e}_k^*$).
Figure 5: QQ for Mahalanobis distance ($M_i$).
conclude that the adopted covariance structure may be accepted except for a small num-
er of observations and displays a similar pattern for the uncensored data and the imputed
data obtained under both imputation methods and censoring percentages. The QQ-plots in
Figures 4 and 5 also do not present evidence against the assumptions adopted for either the
random effects (normal distribution) or for the errors (extreme value distribution) except,
perhaps, for the random effects associated to the data imputed via the EAM procedure
when the censoring percentage is 50%. Note, however, that the adopted model does not
impose a specific distribution for the random effects.
5 Analysis of the oil-well data.

5.1 Model Specification

To illustrate the proposed methodology we consider 2374 observations from 616 oil wells involving 1811 failure times and 563 censored times. The failures were characterized by the total interruption of the operation of the well due to malfunction of one or more components of the sub-surface equipment. The covariates were, Oil production in m$^3$/day (PRO), Elevation method: [pumpjack (PJ = 1) or progressive cavity (PJ = 0)], Age at failure in years (AGE), Location of operating unit: [Region B (RB = 1 and RC = RD = 0), C (RC = 1 and RB = RD = 0), D (RD = 1 and RB = RC = 0) and A (RB = RC = RD = 0)] and Depth of the oil pump in meters (DEP).

Following the suggestions of Carvalho et al. (2014) we adopted the model following

$$
\ln T_{ij} = b_i + \beta_{pro} PRO_{ij} + \beta_{pj} PJ_{ij} + \beta_{age} AGE_{ij} + \beta_{rb} RB + \beta_{rc} RC + \beta_{rd} RD + \\
\beta_{dep} DEP_i + \beta_{prorb} PRO_{ij} * RB + \beta_{prorc} PRO_{ij} * RC + \\
+ \beta_{prosr} PRO_{ij} * RD + \beta_{depsrb} DEP_i * RB + \\
+ \beta_{depsrc} DEP_i * RC + \beta_{depsrd} DEP_i * RD + \sigma_{ij},
$$

(12)

assuming that, conditionally on the random effects $b_i$, the $T_{ij}$, $i = 1, \ldots, 616$ and $j = 1, \ldots, n_i$ follow Weibull distributions. The $b_i$ are i.i.d. random effects with null means and variance $\sigma_b^2$. The form of their distribution is not specified.

Censored observations were imputed via both the ITM and the EAM procedures and the EBLUE of the parameters were computed using (5). The variance components were estimated by the MINQUE method. Estimates of the fixed parameters and of the variance components are displayed in Table 3. Both approaches produce comparable estimates of the model parameters.

5.2 Residual analysis of the fitted model

The residual plots corresponding to the fit of model (12) are shown in Figures 6-12. The results suggest that:
Table 3: Estimates and standard errors (SE) for model (12) parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>SE</th>
<th>Estimate</th>
<th>SE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_0$</td>
<td>6.577</td>
<td>0.302</td>
<td>6.479</td>
<td>0.302</td>
</tr>
<tr>
<td>$\beta_{pro}$</td>
<td>-0.041</td>
<td>0.012</td>
<td>-0.041</td>
<td>0.012</td>
</tr>
<tr>
<td>$\beta_{pj}$</td>
<td>0.623</td>
<td>0.152</td>
<td>0.646</td>
<td>0.152</td>
</tr>
<tr>
<td>$\beta_{age}$</td>
<td>0.088</td>
<td>0.008</td>
<td>0.092</td>
<td>0.008</td>
</tr>
<tr>
<td>$\beta_{rb}$</td>
<td>1.334</td>
<td>0.371</td>
<td>1.455</td>
<td>0.371</td>
</tr>
<tr>
<td>$\beta_{rc}$</td>
<td>1.125</td>
<td>0.277</td>
<td>1.177</td>
<td>0.277</td>
</tr>
<tr>
<td>$\beta_{rd}$</td>
<td>1.804</td>
<td>0.394</td>
<td>1.950</td>
<td>0.394</td>
</tr>
<tr>
<td>$\beta_{dep}$</td>
<td>0.002</td>
<td>&lt;0.001</td>
<td>0.002</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>$\beta_{pro*rb}$</td>
<td>0.042</td>
<td>0.019</td>
<td>0.041</td>
<td>0.019</td>
</tr>
<tr>
<td>$\beta_{pro*rc}$</td>
<td>0.004</td>
<td>0.019</td>
<td>0.003</td>
<td>0.019</td>
</tr>
<tr>
<td>$\beta_{pro*rd}$</td>
<td>-0.026</td>
<td>0.024</td>
<td>-0.027</td>
<td>0.024</td>
</tr>
<tr>
<td>$\beta_{dep*rb}$</td>
<td>-0.002</td>
<td>0.001</td>
<td>-0.002</td>
<td>0.001</td>
</tr>
<tr>
<td>$\beta_{dep*rc}$</td>
<td>-0.003</td>
<td>0.001</td>
<td>-0.003</td>
<td>0.001</td>
</tr>
<tr>
<td>$\beta_{dep*rd}$</td>
<td>-0.003</td>
<td>0.001</td>
<td>-0.003</td>
<td>0.001</td>
</tr>
<tr>
<td>$\sigma_e$</td>
<td>1.451</td>
<td>-</td>
<td>1.440</td>
<td>-</td>
</tr>
<tr>
<td>$\sigma_b^2$</td>
<td>0.896</td>
<td>-</td>
<td>0.908</td>
<td>-</td>
</tr>
</tbody>
</table>

- The hypothesis of linearity of the fixed effects appears satisfactory (Figure 6) and the distribution of the marginal residuals are slightly negatively skewed as expected (Figure 7).
- The structure of the within-units covariance matrix proposal may not be considered adequate only for a small number (13%) of some sample units (Figure 8).
- There is no evidence of outliers (Figure 9).
- There is evidence of violation of the homoskedasticity assumption for the conditional errors (Figure 10).
- The standard extreme value distribution assumption seems acceptable for the conditional errors (Figure 11).
- There is some evidence against normality of the random effects (Figure 12); this however, is not a model requirement.
Figure 6: Oil well data - Standardized marginal residuals ($\hat{\xi}_i^*$) vs fitted values.

Figure 7: Oil well data - Histogram for standardized marginal residuals ($\hat{\xi}_i^*$).
Figure 8: Oil well data - Modified Lesaffre-Verbeke index (\(V_i^\ast\)) unit index plots.

Figure 9: Oil well data - Standardized conditional residuals (\(\hat{e}_i^\ast\)) unit index plots.
Figure 10: Oil well data - Standardized conditional residuals ($\hat{e}_{k}^{*}$) vs fitted values.

Figure 11: Oil well data - QQ plot for standardized conditional residual ($\hat{e}_{k}^{*}$).
Figure 12: Oil wells data - QQ for Mahalanobis distance ($M_i$).
6 Conclusion

There are few proposals for residual analysis in the context of correlated survival data. In particular, we are not aware of any techniques to evaluate validity of the assumptions usually considered in AFT random effects models for the analysis of correlated censored lifetime data. For such purposes, we consider the use of residual plots originally developed for mixed models to assess the assumptions of AFT models with random effects used to analyze correlated censored lifetime data. Because censoring can compromise the analysis of residuals, we propose and compare two imputation procedures (one based on Ageel (2002) and the other on the method of inverse transform) by means of a simulation study. Although the latter produces slightly more biased estimates, the proposed diagnostic tools suggest that the associated residuals are closer to those obtained with the uncensored data. We applied these techniques to evaluate data related to the time between failures of oil wells and observed a similar behaviour of the imputed data.

SUPPLEMENTARY MATERIAL

R code and datasets: R codes containing code used for imputation of the censored data, for parameter estimation and to residual analysis as well as the simulated data can be downloaded from [http://www.ime.usp.br/~jmsinger/oilwell.zip](http://www.ime.usp.br/~jmsinger/oilwell.zip) as a .zip file.

References


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