LISTA 19 DE MAT 0111

Prof. Jean Cerqueira Berni*

"Eu ouço, eu esqueço. Eu vejo, eu lembro. Eu faço, eu aprendo."

(1) Calcular as seguintes integrais indefinidas:

(a) $\displaystyle\int \frac{dx}{(1+\cos(x))^2}$

(b) $\displaystyle\int \frac{dx}{5-3\cdot\cos(x)}$

(c) $\displaystyle\int \frac{dx}{4-5\cdot\sin(x)}$

(d) $\displaystyle\int \frac{\sin(x)}{1+\sin(x)}dx$

Solução:

(a) Neste caso, temos a função racional:

$$q(z) = \frac{1}{(1+z)^2}$$

calculada para $z = \cos(x)$, de modo que convém efetuar a mudança de variável $u = \tan\left(\frac{x}{2}\right)$.
Neste caso, temos:

$$\cos(x) = \frac{1-u^2}{1+u^2} \text{ e } dx = \frac{2\cdot du}{1+u^2}$$

Assim,

$$\int \frac{dx}{(1+\cos(x))^2} = \int \frac{\frac{2\cdot du}{1+u^2}}{\left(1+\frac{1-u^2}{1+u^2}\right)^2} = \int \frac{\frac{2\cdot du}{1+u^2}}{\left(\frac{2}{1+u^2}\right)^2} =$$

$$= \int \frac{\frac{2\cdot du}{1+u^2}}{\frac{4}{(1+u^2)^2}} = \frac{1}{2}\cdot\int(1+u^2)du = \frac{1}{2}\left(u+\frac{u^3}{3}\right)+C =$$

*jeancb@ime.usp.br

$$= \frac{1}{2} \cdot \tan\left(\frac{x}{2}\right) + \frac{1}{6} \cdot \tan^3\left(\frac{x}{2}\right) + C$$

(b) Neste caso temos uma função racional de $\cos(x)$, de modo que convém efetuar a mudança de variável $u = \tan\left(\frac{x}{2}\right)$, e por conseguinte obter $dx = \frac{2}{1+u^2}du$. Segue que:

$$\int \frac{dx}{5 - 3 \cdot \cos(x)} = \int \frac{\frac{2 \cdot du}{1+u^2}}{5 - 3 \cdot \left(\frac{1-u^2}{1+u^2}\right)} = \int \frac{\frac{2}{1+u^2}}{\frac{8u^2+2}{1+u^2}} du =$$

$$= \int \frac{2}{8u^2 + 2} du = \int \frac{du}{4u^2 + 1} = \frac{1}{2} \int \frac{2du}{(2u)^2 + 1} = \frac{1}{2} \cdot \arctan(2u) + C =$$

$$= \frac{1}{2} \cdot \arctan\left(2 \cdot \tan\left(\frac{x}{2}\right)\right) + C$$

(c) Neste caso temos uma função racional de $\sin(x)$, de modo que convém efetuar a mudança de variável $u = \tan\left(\frac{x}{2}\right)$, e por conseguinte obter $dx = \frac{2}{1+u^2}du$. Segue que:

$$\int \frac{dx}{4 - 5 \cdot \sin(x)} = \int \frac{\frac{2 \cdot du}{1+u^2}}{4 - 5 \cdot \left(\frac{2u}{1+u^2}\right)} = \int \frac{\frac{2}{1+u^2}}{\frac{4u^2-10u+4}{1+u^2}} du =$$

$$= \int \frac{1}{2u^2 - 5u + 2} du$$

Decompondo em frações parciais:

$$\frac{1}{2u^2 - 5u + 2} = -\frac{2}{3 \cdot (2u - 1)} + \frac{1}{3 \cdot (u - 2)}$$

De modo que:

$$\int \frac{1}{2u^2 - 5u + 2} du = -\int \frac{2}{3 \cdot (2u - 1)} du + \int \frac{1}{3 \cdot (u - 2)} du =$$

$$= -\frac{1}{3} \ln|2u - 1| + \frac{1}{3} \ln|u - 2| + C$$

Voltando à variável x, tem-se:

$$= -\frac{1}{3}\ln\left|2\cdot\tan\left(\frac{x}{2}\right)-1\right| + \frac{1}{3}\ln\left|\tan\left(\frac{x}{2}\right)-2\right| + C$$

$$\therefore \int \frac{dx}{4-5\cdot\sin(x)} = \ln\left|\frac{\tan\left(\frac{x}{2}\right)-2}{2\cdot\tan\left(\frac{x}{2}\right)-1}\right|^{\frac{1}{3}} + C$$

(d) Neste caso temos uma função racional de $\sin(x)$, de modo que convém efetuar a mudança de variável $u = \tan\left(\frac{x}{2}\right)$, e por conseguinte obter $dx = \frac{2}{1+u^2}du$. Segue que:

$$\int \frac{\sin(x)}{1+\sin(x)}dx = \int \frac{\frac{2u}{1+u^2}\frac{2\cdot du}{1+u^2}}{1+\left(\frac{2u}{1+u^2}\right)} = \int \frac{4u}{(1+u^2)\cdot(u^2+2u+1)}du =$$

$$= \int \frac{4u}{(1+u^2)\cdot(u+1)^2}du$$

Decompondo em frações parciais:

$$\frac{4u}{(1+u^2)\cdot(u+1)^2} = \frac{2}{u^2+1} - \frac{2}{(u+1)^2}$$

De modo que:

$$\int \frac{4u}{(1+u^2)\cdot(u+1)^2}du = \int \frac{2}{u^2+1}du - \int \frac{2}{(u+1)^2}du =$$

$$= 2\arctan(u) + \frac{2}{u+1} + C$$

Voltando à variável x, tem-se:

$$x + \frac{2}{\tan\left(\frac{x}{2}\right)+1} + C$$

$$\therefore \int \frac{\sin(x)}{1+\sin(x)} = x + \frac{2}{\tan\left(\frac{x}{2}\right)+1} + C$$

(2) Resolver as seguintes integrais:

(a) $\int \sin^3(x) \cdot \cos^2(x) dx$

(b) $\int \cos^3(x) dx$

(c) $\int \sin^4(x) \cdot \cos^2(x) dx$

(d) $\int \cos^7(x) dx$

Solução:

(a) Temos:

$$\int \sin^3(x) \cdot \cos^2(x) dx = \int \sin^2(x) \cdot \cos^2(x) \cdot \sin(x) dx =$$

$$= \int (1 - \cos^2(x)) \cdot \cos^2(x) \cdot \sin(x) dx \overset{\substack{y=\cos(x) \\ dy=-\sin(x)dx \\ \uparrow}}{=} - \int (1 - y^2) \cdot y^2 dy =$$

$$= -\frac{y^3}{3} + \frac{y^5}{5} + C = -\frac{\cos^3(x)}{3} + \frac{\cos^5(x)}{5} + C$$

(b) Temos:

$$\int \cos^3(x) dx = \int \cos^2(x) \cdot \cos(x) dx = \int (1 - \sin^2(x)) \cdot \cos(x) dx \overset{\substack{y=\sin(x) \\ dy=\cos(x)dx \\ \uparrow}}{=}$$

$$= \int (1 - y^2) dy = y - \frac{y^3}{3} + C$$

$$\therefore \int \cos^3(x) dx = \sin(x) - \frac{\sin^3(x)}{3} + C$$

(c) Temos:

$$\int \sin^4(x) \cdot \cos^2(x) dx = \int \left(\frac{1 - \cos(2x)}{2} \right)^2 \cdot \frac{1 + \cos(2x)}{2} dx =$$

$$= \int \frac{1 - 2\cdot\cos(2x) + \cos^2(2x)}{4} \cdot \frac{1+\cos(2x)}{2} dx =$$

$$= \frac{1}{8} \cdot \int (1 - \cos(2x) - \cos^2(2x) + \cos^3(2x)) dx =$$

$$= \frac{1}{8} \cdot \int \left[\frac{1}{2} - \cos(2x) - \frac{1}{2}\cos(4x) + \cos^3(2x)\right] dx =$$

$$= \frac{1}{8}\left[\int \frac{1}{2} dx - \int \cos(2x)dx - \frac{1}{2}\int \cos(4x)dx + \int \cos^3(2x)dx\right] =$$

$$= \frac{1}{8} \cdot \left[\frac{1}{2}x - \frac{1}{2}\sin(2x) - \frac{1}{8}\sin(4x) + \frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x)\right] =$$

$$= \frac{1}{16}x - \frac{1}{16}\sin(4x) - \frac{1}{48}\sin^3(2x) + C$$

(d) Temos:

$$\int \cos^7(x)dx = \int (1 - \sin^2(x))^3 \cdot \cos(x)dx \overset{\substack{y=\sin(x) \\ dy=\cos(x)dx \\ \uparrow}}{=}$$

$$= \int (1-y^2)^3 dy = \int (1 - 3y^2 + 3y^4 - y^6)dy = y - \frac{y^3}{3} + \frac{3y^5}{5} - \frac{y^7}{7} + C$$

Retornando à variável x, obtemos:

$$\int \cos^7(x)dx = \sin(x) - \sin^3(x) + \frac{3\cdot\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C$$

(3) Calcular as seguintes integrais:

(a) $\int \sqrt{a^2 - x^2}dx$

(b) $\int \frac{dx}{x \cdot \sqrt{x^2 - 1}}$

(c) $\int \frac{x^3}{\sqrt{3 - x^2}}dx$

(d) $\int \frac{dx}{\sqrt{4 + x^2}}$

(e) $\int \frac{x^3}{\sqrt{9 - x^2}}dx$

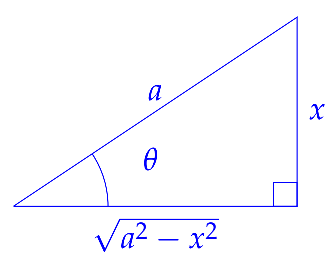
(f) $\int \frac{dx}{\sqrt{25x^2 - 4}}$

Solução:

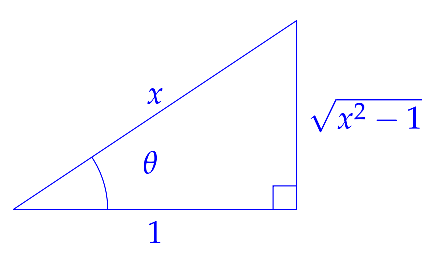
(a) Neste caso, convém fazermos $x = a \cdot \sin(\theta)$, de modo que $dx = a \cdot \cos(\theta)d\theta$ e $\sqrt{a^2 - x^2} = a \cdot \cos(\theta)$. Assim,

$$\int \sqrt{a^2 - x^2}dx = \int a \cdot \cos(\theta) \cdot a \cdot \cos(\theta)d\theta = \int a^2 \cdot \cos^2(\theta)d\theta = a^2 \cdot \left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right) =$$

$$= \frac{a^2}{2} \cdot \arcsin\left(\frac{x}{a}\right) + \frac{x}{2} \cdot \sqrt{a^2 - x^2}$$



(b) Neste caso, fazemos $x = \sec(\theta)$, de modo que $dx = \sec(\theta) \cdot \tan(\theta)d\theta$ e $\sqrt{x^2 - 1} = \tan(\theta)$

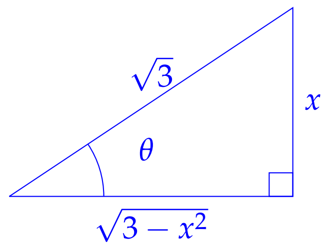


Segue que:

$$\int \frac{dx}{x \cdot \sqrt{x^2 - 1}} = \int \frac{\sec(\theta) \cdot \tan(\theta)}{\sec(\theta) \cdot \tan(\theta)}d\theta = \int d\theta = \theta + C =$$

$$= \operatorname{arcsec}(x) + C$$

(c) Neste caso, convém tomarmos $x = \sqrt{3} \cdot \cos(\theta)$, de modo que $dx = -\sqrt{3} \cdot \sin(\theta) d\theta$ e $\sqrt{3 - x^2} = \sqrt{3} \cdot \sin(\theta)$



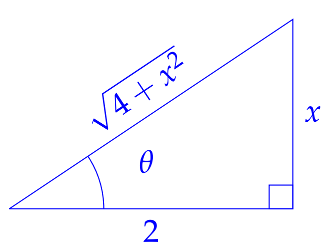
Tem-se, então:

$$\int \frac{x^3}{\sqrt{3 - x^2}} dx = \int \frac{3^{\frac{3}{2}} \cdot \cos^3(\theta)}{\sqrt{3 - 3 \cdot \cos^2(\theta)}} \cdot \sqrt{3} \cdot (-\sin(\theta)) d\theta =$$

$$= -3^{\frac{3}{2}} \cdot \int \cos^3(\theta) d\theta = -3^{\frac{3}{2}} \cdot \sin(\theta) + 3^{\frac{3}{2}} \cdot \frac{\sin^3(\theta)}{3} + C =$$

$$= -3^{\frac{3}{2}} \cdot \left[\frac{\sqrt{3 - x^2}}{\sqrt{3}} - \frac{\left(\frac{\sqrt{3-x^2}}{\sqrt{3}}\right)^3}{3} \right] + C$$

(d) Neste caso, fazemos $x = 2 \cdot \tan(\theta)$, de modo que $dx = 2 \cdot \sec^2(\theta) \cdot d\theta$



Podemos escrever:

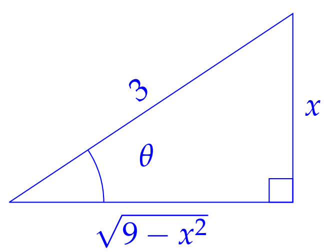
$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\cdot\sec^2(\theta)}{\sqrt{4\cdot\sec^2(\theta)}}d\theta = \int \frac{\sec^2(\theta)}{|\sec(\theta)|}d\theta =$$

$$= \int \sec(\theta)d\theta = \ln|\sec(\theta)+\tan(\theta)| + C =$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C = \ln\left|\frac{\sqrt{4+x^2}+x}{2}\right| + C =$$

$$= \ln|\sqrt{4+x^2}+x| - \ln(2) + C = \ln|\sqrt{4+x^2}+x| + C'.$$

(e) Neste caso, fazemos $x = 3\cdot\sin(\theta)$, de modo que $dx = 3\cdot\cos(\theta)\cdot d\theta$



Podemos escrever:

$$\int \frac{x^3}{\sqrt{9-x^2}}dx = \int \frac{27\cdot\sin^3(\theta)\cdot 3\cdot\cos(\theta)}{|3\cdot\cos(\theta)|}d\theta = \int 27\cdot\sin^3(\theta)d\theta =$$

$$= 27\cdot\int \sin^3(\theta)d\theta = -27\cdot\cos(\theta) + 9\cdot\cos^3(\theta) + C =$$

$$= -27\cdot\frac{\sqrt{9-x^2}}{3} + 9\cdot\left(\frac{\sqrt{9-x^2}}{3}\right)^3 + C =$$

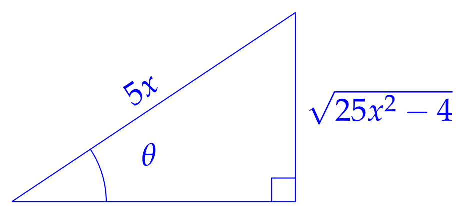
$$= -9\cdot\sqrt{9-x^2} + \frac{(9-x^2)^{3/2}}{3} + C$$

$$\therefore \int \frac{x^3}{\sqrt{9-x^2}}dx = -9\cdot\sqrt{9-x^2} + \frac{(9-x^2)^{3/2}}{3} + C$$

(e) Primeiramente reescrevemos o radical como:

$$\sqrt{25x^2-4} = \sqrt{25\cdot\left(x^2-\frac{4}{25}\right)} =$$

$$= 5\cdot\sqrt{x^2-\left(\frac{2}{5}\right)^2}$$



Neste caso, fazemos $x = \frac{2}{5}\cdot\sec(\theta)$, de modo que $dx = \frac{2}{5}\cdot\sec(\theta)\cdot\tan(\theta)d\theta$
Temos:

$$x^2-\left(\frac{2}{5}\right)^2 = \frac{4}{25}\cdot\sec^2(\theta) - \frac{4}{25} = \frac{4}{25}\cdot(\sec^2(\theta)-1) = \frac{4}{25}\cdot\tan^2(\theta)$$

$$\therefore \sqrt{x^2-\left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan(\theta)|$$

Podemos escrever:

$$\int \frac{dx}{\sqrt{25x^2-4}}dx = \int \frac{dx}{5\sqrt{x^2-(4/25)}}d\theta = \int \frac{(2/5)\cdot\sec(\theta)\cdot\tan(\theta)}{5\cdot(2/5)\cdot\tan(\theta)}d\theta =$$

$$= \frac{1}{5}\cdot\int \sec(\theta)d\theta = \frac{1}{5}\cdot\ln|\sec(\theta)+\tan(\theta)| + C =$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C$$

$$\therefore \int \frac{dx}{\sqrt{25x^2 - 4}} dx = \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C$$