Aula 6 (Seifert-van Kampen)

- Ferramenta principal para calcular $\pi_1(X,x)$
- Descreve $\pi_1(X,x)$ em termos de $\pi_1(U\cap V, x)$, $\pi_1(U,x)$, $\pi_1(V,x)$ $\quad (X = U\cup V)$

I) Um pouco de álgebra

Def: Sejam N, G_1, G_2 grupos e

$$\begin{tikzcd} & G_1 \\ N \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] & \\ & G_2 \end{tikzcd} \quad \text{Homomorfismos}$$

Dizemos que o diagrama

Dados iniciais ; Melhor preenchimento $\qquad \forall H, \psi_1, \psi_2$

$$\begin{tikzcd} & G_1 \arrow[dr, "\varphi_1"] \arrow[drr, bend left, "\psi_1"] & & \\ N \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] & & G \arrow[r, dashed, "\exists!\,\psi"] & H \\ & G_2 \arrow[ur, "\varphi_2"'] \arrow[urr, bend right, "\psi_2"'] & & \end{tikzcd}$$

é um push-out se para todo $\psi_1 : G_1 \longrightarrow H$
$\psi_2 : G_2 \longrightarrow H$

t.q. $\psi_1\phi_1 = \psi_2\phi_2$, $\Rightarrow \exists!\ \psi : G \longrightarrow H$ t.q.

$$\psi\varphi_1 = \psi_1$$
$$\psi\varphi_2 = \psi_2$$

Nomeclatura / Abuso de Linguagem

G é o pushout de
$$\begin{tikzcd} & G_1 \\ N \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] & \\ & G_2 \end{tikzcd}$$

<u>Prop</u>: Se o pushout existe $\Rightarrow$ ele é unico à menos de iso

<u>Dem</u>:

$$\begin{tikzcd} & G_1 \arrow[dr] \arrow[drr] \arrow[drrr, bend left] & & & \\ N \arrow[ur] \arrow[dr] & & G \arrow[r, "\exists!", "\psi"'] & H \arrow[r, dashed, "\exists!", "\theta"'] & G \\ & G_2 \arrow[ur] \arrow[urr] \arrow[urrr, bend right] & & & \end{tikzcd}$$

$\theta \circ \psi : G \to G$ faz comutar $\Longrightarrow$ por unicidade
$\theta \circ \psi = Id_G$

$\blacksquare$

<u>Teorema (Seifert - van Kampen)</u>

Seja $X = U \cup V$ com $U, V, U \cap V$ abertos conexos por caminhos, e $x \in U \cap V$. Então

$$\begin{tikzcd} & \pi_1(U, x) \arrow[dr, "j_{U*}"] & \\ \pi_1(U \cap V, x) \arrow[ur, "i_{U*}"] \arrow[dr, "i_{V*}"'] & & \pi_1(X, x) \\ & \pi_1(V, x) \arrow[ur, "j_{U*}"'] & \end{tikzcd}$$

é um pushout

<u>OBS</u>: Os mapas acima são induzidos por

$$\begin{tikzcd} & U \arrow[dr, "j_U"] & \\ U \cap V \arrow[ur, "i_U"] \arrow[dr, "i_V"'] & & X \\ & V \arrow[ur, "j_V"'] & \end{tikzcd}$$

Casos Particulares:

Caso A: $G_1 = G_2 = \{1\} \implies G = \{1\}$

$$
\begin{tikzcd}
 & \{1\} \arrow[dr] \arrow[drr, bend left, "\varphi_1"] & & \\
N \arrow[ur] \arrow[dr] & & \{1\} \arrow[r, dashed, "\exists!"] & H \\
 & \{1\} \arrow[ur] \arrow[urr, bend right, "\varphi_2"'] & &
\end{tikzcd}
$$

Para todo H, $\exists\ \varphi_1, \varphi_2$. Logo $\forall H$ tem que existir um único $\varphi : G \to H \implies G = \{1\}$

Aplicação: $\pi_1(S^n, p) = \{1\}$, $n \geqslant 2$

$U = S^n \setminus \{q_1\} \cong \mathbb{R}^n$ $\qquad \pi_1(U, p) = \{1\}$

$V = S^n \setminus \{q_2\} \cong \mathbb{R}^n$ $\qquad \pi_1(V, p) = \{1\}$

$U \cap V = S^n \setminus \{q_1, q_2\} \simeq S^{n-1}$

h.e

Caso B: Se $N = G_2 = \{1\} \implies G \cong G_1$

$$
\begin{tikzcd}
 & G_1 \arrow[dr, "\varphi_1"] \arrow[drr, bend left, "\varphi_1"] & & \\
\{1\} \arrow[ur] \arrow[dr] & & G \arrow[r, dashed, "\exists!"] & H \\
 & \{1\} \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}
$$

ou seja Dado qualquer $\varphi_1 : G_1 \to H$, $\exists!\ \varphi : G \to H$ t.q. $\varphi\varphi_1 = \varphi_1$. Basta ver que isso vale para $G = G_1$, $\varphi_1 = Id$

Aplicação: Se M é uma variedade (topológica) $\dim M \geq 3$

$\Rightarrow \pi_1(M_0 - \{q\}, p) \cong \pi_1(M, p)$

Def: Uma variedade topológica de dimensão n é um espaço topológico M Hausdorff, segundo contável (Base enumerável de abertos) e localmente euclideana

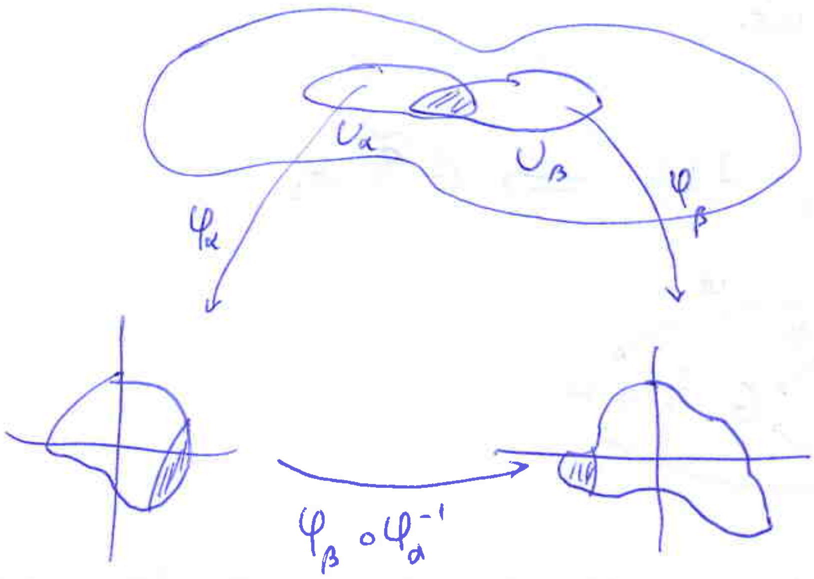
i.e., $\forall x \in M, \exists \underset{x \in}{U} \subset M$ e um homeo $\varphi_U : U \to \varphi(U) \subset \mathbb{R}^n$

OBS: Se $\exists$ $\mathcal{A} = \{U_\alpha, \varphi_\alpha\}$, $M = \bigcup_\alpha U_\alpha$

t.q. $U_\alpha \cap U_\beta \neq \phi \Rightarrow \underbrace{\varphi_\beta \circ \varphi_\alpha^{-1}}_{\in C^k} : \underset{\subset \mathbb{R}^n}{\varphi_\alpha(U_\alpha \cap U_\beta)} \to \underset{\subset \mathbb{R}^n}{\varphi_\beta(U_\alpha \cap U_\beta)}$

$\Rightarrow M^n$ é variedade C^k

- Se $k = \infty \Rightarrow M^n$ é variedade suave



Cuidado: A mesma variedade topológica pode ter mais do que uma estrutura diferenciável. $(S^7, \mathbb{R}^4)$

Dem. da aplicação:

Seja $q \in V \subset M$, t.q. $V \underset{\text{homeo}}{\cong} \mathring{D}^n$.

Assuma que $p \in V$ (Mudar pto base não muda π_1 à menos de $\cong$)

$$\Rightarrow M = V \cup \overbrace{M-\{q\}}^{U}$$

$$V \cap M-\{q\} = V-\{q\} \cong \mathring{D}^n - \{0\} \underset{h.e.}{\simeq} S^{n-1}$$

Se $n \geq 3 \Rightarrow \pi_1(S^{n-1}, *) = \{1\}$

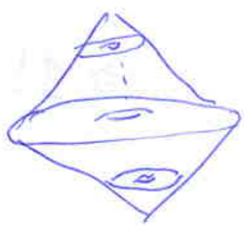
$$\Rightarrow \begin{tikzcd} & \pi_1(V,p) = \{1\} \arrow[dr] & \\ \{1\} = \pi_1(V-\{q\},p) \arrow[ur] \arrow[dr] & & \pi_1(M,p) \\ & \pi_1(M-\{q\},p) \arrow[ur] & \end{tikzcd}$$

Exercício:

1) Seja $U \subset M^n$ aberto. Mostre que U é variedade de $\dim U = \dim M$

2) Seja $\Sigma T^2 =$ suspensão de $T^2 = \dfrac{T^2 \times [-1,1]}{\sim}$

onde $(p,1) \sim (p',1)$, $(p,-1) \sim (p',-1)$ $\forall p, p' \in T^2$



Mostre que ΣT^2 não é variedade topológica (OBS: Se fosse sua dim. seria 3 p q?)

Caso C: $G_2 = \{1\} \Rightarrow G \cong G_1 / i(N)$

• Lembrete de teoria de grupos:

•) $H \subset G$ é normal se $gHg^{-1} = H$

•) $H_1, H_2 \subset G$ normais $\Rightarrow H_1 \cap H_2$ é normal

$\Rightarrow$ (Dado $S \subset G$, $\exists\ \overline{S}$ = menor subgrupo normal que contém S)

•) $\overline{S}$ = fecho normal de $S = \langle \overline{S} \rangle$

•) se $K \subset G$ subgrupo e $f: G \to H$ homo, tal que

$$K \subset \ker f$$

$\Rightarrow \exists!\ \bar{f}: G/\overline{K} \to H$ t.q.

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] & H \\ G/\overline{K} \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

$\left(\bar{f}([g]) = f(g)\right)$

pois $K \subset \overline{K} \subset \ker f$

NORMAL

Voltando:

$$\begin{tikzcd} & G_1 \arrow[dr] \arrow[drr, bend left, "\varphi"] & & \\ N \arrow[ur, "i"] \arrow[dr] & & \frac{G_1}{i(N)} \arrow[r, dashed, "\exists!"] & H \\ & \{1\} \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd}$$

Comuta $\Rightarrow \varphi \circ i = 1$ (trivial)

$\Rightarrow i(N) \subset \ker \varphi$

$\Rightarrow \exists!\ \bar{\varphi}: G_1/i(N) \to H$

$\wr$

$G_1/\overline{i(N)}$

<u>Aplicação</u>: Se $\chi : S^1 \longrightarrow A$ e

$$X = \underbrace{A \sqcup D^2/\sim}_{A \cup_\chi D^2} \qquad a \sim \chi(x), \quad x \in S^1 = \partial D^2$$

$$\Rightarrow \quad i_\# : \pi_1(A,a) \longrightarrow \pi_1(X,a) \quad \text{é sobrejor}$$

$$\& \quad \operatorname{Ker} i_\# = \langle \alpha \rangle \quad \text{onde} \quad \alpha = [\chi \circ \gamma_1]$$

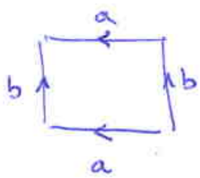
<u>OBS</u>: Dizemos que X foi obtido de A colando uma 2-célula.

<u>Exemplos</u>:

- $S^2 = \dfrac{\{pt\} \sqcup D^2}{\sim} \Rightarrow \pi_1(S^2, p) = \{1\}$

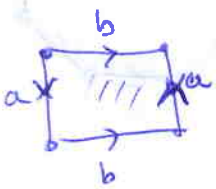
- $T^2 = (S^1 \vee S^1) \cup_\chi D^2, \quad \chi : S^1 \longrightarrow S^1 \vee S^1$

$$\Rightarrow \pi_1(T^2, p) = \frac{\pi_1(S^1 \vee S^1)}{\langle a b a^{-1} b^{-1} \rangle}$$



(Escrevemos $\chi(\gamma_1) = a b a^{-1} b^{-1}$)

- Garrafa de Klein



$$\chi : S^1 \longrightarrow S^1 \vee S^1$$

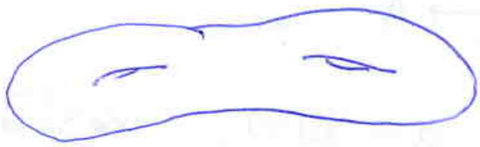
$$\chi(\gamma_1) = a b a b^{-1}$$

$$\Rightarrow \pi_1(K, p) \cong \frac{\pi_1(S^1 \vee S^1, p)}{\langle a b a b^{-1} \rangle}$$

<u>OBS</u>: TODA superfície Aparece colando ~~2-cel~~

$$\chi : S^1 \longrightarrow \underbrace{S^1 \vee \cdots \vee S^1}_{\text{bouquet de círculos}}$$

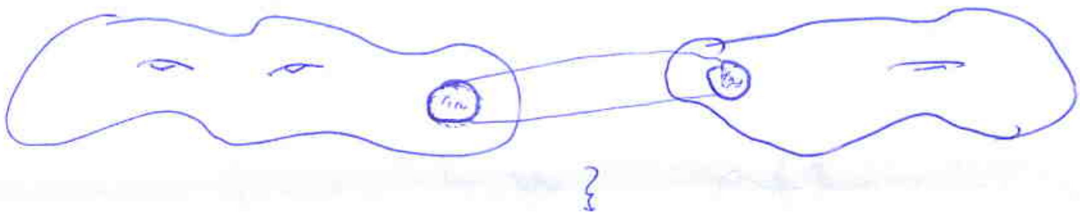
Exemplo:

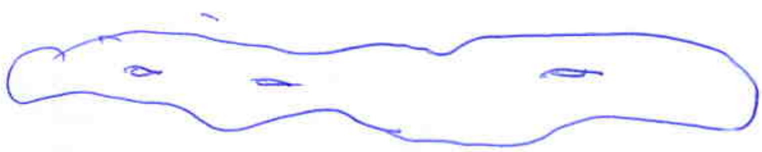
$T_2 = T\#T =$ 

Em geral: $\underbrace{M\#N}_{\text{soma conexa}}$ $(\dim M = \dim N)$ variedades

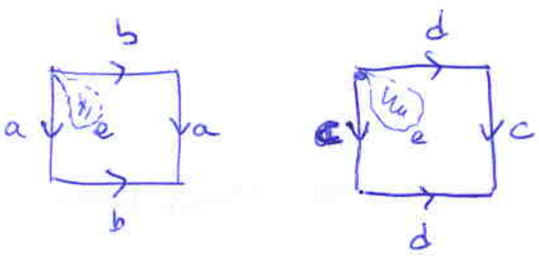
$$M\#N = \frac{M\setminus \mathring{B}_M \sqcup N\setminus \mathring{B}_N}{\sim}$$

onde $\varphi : \partial B_M = S^{n-1} \xrightarrow{\cong} \partial B_N = S^{n-1}$

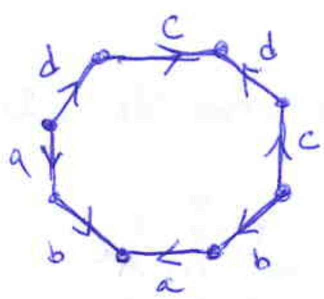




Voltando ao T_2



cola $\longrightarrow$ ao longo de e



$$\Rightarrow T_2 = S^1\vee S^1\vee S^1\vee S^1 \cup_{\chi} D^2$$

$$\boxed{\chi(\gamma_1) = aba^{-1}b^{-1}cdc^{-1}d^{-1}}$$

Dem da aplicação:

$$U = X - \{0\} \xrightarrow{\ 0 \in \mathring{D}^2\ } \underset{h.e.}{\simeq} A \qquad \left(\text{Exercício da lista}\right)$$

$$V = \mathring{D}^2$$

$$U \cap V = \mathring{D}^2 - \{0\} \simeq S^1$$

$$\begin{tikzcd}
 & \pi_1(X - \{0\}, p) \arrow[dr] & \\
\pi_1(S^1, 1) \cong \pi_1(U \cap V, p) \arrow[ur, "i"] \arrow[dr] & & \pi_1(X, p) = \dfrac{\pi_1(X - \{0\}, p)}{\overline{\operatorname{Im} i}} \\
 & \{1\} \arrow[ur] &
\end{tikzcd}$$

Consider: $r_A : X - \{0\} \longrightarrow A$ retrato por deformação $(a = r_A(p))$

$r_{S^1} : D^2 - \{0\} \longrightarrow S^1$ retração

$$\begin{tikzcd}
\pi_1(U \cap V, p) \arrow[r] \arrow[d, "r_{S^1\#}"', "\cong"] & \pi_1(X - \{0\}, p) \arrow[r] \arrow[d, "\cong"', "r_{A\#}"] & \pi_1(X, p) \arrow[d, "\cong"', "\text{mudança de ponto base}"] \\
\pi_1(S^1, 1) \arrow[r, "\chi_\#"'] & \pi_1(A, a) \arrow[r, "i_\#"'] & \pi_1(X, a)
\end{tikzcd}$$