# Aula 22

## Homologia Celular

1

- $X = \bigcup_{n \in \mathbb{N}} X_n$ , $\phi \subset X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$

complexo CW

Considere, para cada $n$, a sequencia do par

$$0 \to C_k(X_n) \xrightarrow{i} C_k(X_{n+1}) \xrightarrow{j} C_k(X_{n+1}, X_n) \to 0$$

que induz a sequencia longa exata

$$\cdots \to H_{k+1}(X_n) \xrightarrow{i_*} H_{k+1}(X_{n+1}) \xrightarrow{p_{k+1}} H_{k+1}(X_{n+1}, X_n) \xrightarrow{\partial_{k+1}} H_k(X_n) \to \cdots$$

Podemos juntar essas sequencias para formar

$$\begin{tikzcd}
 & & H_n(X_n) \arrow[dr, "p_n"] & & & & {} \\
\cdots \arrow[r] & H_{n+1}(X_{n+1}, X_n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X_n, X_{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r] \arrow[ur] & \cdots \\
H_{n+1}(X_{n+1}) \arrow[ur] & & & & H_{n-1}(X_{n-1}) \arrow[ur, "p_{n-1}"'] & &
\end{tikzcd}$$

onde $\boxed{d_n = p_{n-1} \circ \partial_n}$

**Lema**: $d_n \circ d_{n+1} = 0$ $\quad (d^2 = 0)$

seq. longa do par.

**Dem**: $(p_{n-1} \circ \partial_n) \circ (p_n \circ \partial_{n+1}) = p_{n-1} \circ \overbrace{(\partial_n \circ p_n)}^{0} \circ \partial_{n+1} = 0$

<u>Definição</u>: A <u>homologia Celular</u> do complexo CW $X$ é a homologia do complexo de cadeias

$$C_k^{cw}(X) = H_k(X_k, X_{k-1})$$

$$d_k : H_k(X_k, X_{k-1}) \longrightarrow H_{k-1}(X_{k-1}, X_{k-2})$$

ou seja

$$H_n^{cw}(X) = \frac{\ker d_n}{\operatorname{Im} d_{n+1}}$$

<u>Pergunta</u>: . O que esta homologia mede?

<u>Teorema</u>: Seja $X$ um complexo CW.

$\Rightarrow$ $H_n^{cw}(X) \cong H_n(X)$ (homologia singular)

<u>Lema</u>: Seja $X$ complexo CW. Então

(a) $H_k(X_n, X_{n-1}) = \begin{cases} 0 & \text{se } k \neq n \\ \mathbb{Z}\langle e_\alpha^n \rangle_{\alpha \in \Lambda} & \text{se } k = n \end{cases}$ (grupo abeliano livre gerado pelas $n$-células)

(b) $H_k(X_n) = 0$ se $k > n$

(c) $i : X_n \hookrightarrow X$ induz isomorfismo $H_k(X_n) \xrightarrow[\cong]{i_*} H_k(X)$ $\forall k < n$

<u>Dem</u>: (a) $(X_n, X_{n-1})$ é um bom par. Para ver isto, seja $p_\alpha \in \mathring{e}_\alpha^n$ e tome $\mathcal{U} = X_n - \bigcup_\alpha p_\alpha$ (vizinhança de $X_{n-1}$). Usando o retrato por deformação $D^n - \{p\} \rightleftarrows \partial D^n$ ($p \in \mathring{D}^n$) obtemos o retrato por deformação $\mathcal{U} \xrightarrow{r} X_{n-1}$ (Exercício! preencha os detalhes)

Logo, $H_k(X_n, X_{n-1}) \cong \tilde{H}_k(X_n/X_{n-1})$

mas $$X_n/X_{n-1} \cong \coprod_\alpha D^n_\alpha \Big/ \coprod \partial D^n_\alpha \cong \bigvee_\alpha S^n_\alpha$$

$$\Rightarrow H_k(X_n/X_{n-1}) \cong \begin{cases} 0 & \text{se } k \neq n \\ \underbrace{\bigoplus_\alpha \mathbb{Z}} & \text{se } k = n \end{cases}$$

um gerador para cada $S^n_\alpha$ (ou para cada n-célula)

OBS: Pensando em $\Delta^n \cong D^n$, $h_\alpha : D^n_\alpha \to X_{n-1}$ (aplicação definidora de $e^n_\alpha$)

$\Rightarrow h_\alpha \in C_n(X_n, X_{n-1})$ & $\partial h_\alpha \in X_{n-1} \Rightarrow [h_\alpha] \in H_n(X_n, X_{n-1})$.

a identificação do bom par é dada por

$$H_n(X, A) \xrightarrow{q_*} H_n(X/A, A/A)$$

mas $q \circ h_\alpha : D^n_\alpha \to X_n \to X_n/X_{n-1}$, $\operatorname{Im}(q \circ h_\alpha) = S^n_\alpha$

$\Rightarrow H_n(X_{n-1}, X_n) = \mathbb{Z}\langle [h_\alpha] \rangle = \mathbb{Z}([e^n_\alpha])$ (OBS: Daqui segue que $\chi_{cell}(X)$ só depende da homologia de X)

notação

b) Pela sequencia longa do par $(X_n, X_{n-1})$ temos

$$\cdots \to \underbrace{H_{k+1}(X_n, X_{n-1})}_{\substack{= \\ 0 \\ k>n}} \to H_k(X_{n-1}) \to H_k(X_n) \to \underbrace{H_k(X_n, X_{n-1})}_{\substack{= \\ 0 \\ k>n}} \to \cdots$$

$$\Rightarrow H_k(X_n) \cong H_k(X_{n-1}) \cong \cdots \cong H_k(X_0) = 0 \quad \forall k > n$$

(c) Assum primeiro que $\exists m$ tal que $X_m = X_{m+1} = \cdots = X$ ⊛

$$\Rightarrow \underbrace{H_{k+1}(X_{n+1}, X_n)}_{\substack{= \\ 0 \\ k<n}} \to H_k(X_n) \xrightarrow{i_*} H_k(X_{n+1}) \to \underbrace{H_k(X_{n+1}, X_n)}_{\substack{= \\ 0 \\ k<n}} \to \cdots$$

$$\Rightarrow H_k(X_n) \xrightarrow[\cong]{i_*} H_k(X_{n+1}) \xrightarrow[\cong]{i_*} \cdots \xrightarrow[\cong]{i_*} H_k(X_m) = H_k(X)$$

- Agora usamos um argumento de compacidade para provar o caso geral.

Considere $i_*: H_k(X_n) \longrightarrow H_k(X)$. $(k<n)$

- $i_*$ é sobrejetor:

se $c \in C_k(X)$, $\partial c = 0$ $\Rightarrow$ $c = \underbrace{\sum n_i \sigma_i}_{\text{soma finita}}$ & $\bigcup_i \operatorname{Im} \sigma_i$ é compacto $\Rightarrow \subseteq X_\ell$ para algum $\ell \geq n$

Logo, $c \in C_k(X_\ell)$, $\partial c = 0$ $\Rightarrow$ $[c] \in H_k(X_\ell) \cong H_k(X_n)$

pelo argumento no caso ⊛

$\Rightarrow [c] \in \operatorname{Im} i_*$

(Mais preciso: $[c]_X = i_*[c]_{X_\ell} = i_*[c]_{X_n}$)

- $i_*$ é injetor:

se $i_*[c] = 0 \in H_k(X) \Rightarrow i_* c = \partial b$ mas "$\operatorname{Im} b$" é compacto

$\Rightarrow i_*[c] = [\partial b] \in H_k(X_\ell)$ para algum $\ell \geq n$

$\Rightarrow [c] = 0 \in H_k(X_n)$. □

**Teorema:** $H_n(X) \cong H_n^{CW}(X)$

**Dem:** Temos

$$\cdots \longrightarrow H_{n+1}(X_{n+1}, X_n) \xrightarrow{d_{n+1}} H_n(X_n, X_{n-1}) \xrightarrow{d_n} H_{n-1}(X_{n-1}, X_{n-2}) \to \cdots$$

e $H_n^{CW}(X) = \dfrac{\ker d_n}{\operatorname{Im} d_{n+1}}$

$\bullet$ $\underline{\operatorname{Im} d_{n+1}}$ :

$$\begin{tikzcd}
 & & H_n(X_{n-1})=0 \arrow[d] & & \\
 & & H_n(X_n) \arrow[dr, "p_n"] & & \\
\cdots \arrow[r] & H_{n+1}(X_{n+1},X_n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X_n,X_{n-1}) \arrow[r] & \cdots
\end{tikzcd}$$

5

$\Rightarrow p_n$ é injetor

$\Rightarrow \operatorname{Im} d_{n+1} = p_n(\operatorname{Im}\partial_{n+1}) \cong \operatorname{Im}\partial_{n+1}$

$\bullet$ $\underline{\operatorname{Ker} d_n}$ :

$$\begin{tikzcd}
 & 0 \arrow[d] & & & & \\
 & H_n(X_n) \arrow[dr, "p_n"] & & & & \\
\cdots \arrow[r] & & H_n(X_n,X_{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X_{n-1},X_{n-2}) \arrow[r] & \cdots \\
 & & & H_{n-1}(X_{n-1}) \arrow[ur, "p_{n-1}"'] & & \\
 & & & H_{n-1}(X_{n-2})=0 \arrow[u] & &
\end{tikzcd}$$

$p_n$ injetor

$\Rightarrow p_{n-1}$ é injetor $\Rightarrow \operatorname{Ker} d_n \cong \operatorname{Ker}\partial_n \cong \operatorname{Im} p_n \cong H_n(X_n)$

Logo

$$\frac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}} \cong \frac{H_n(X_n)}{\operatorname{Im}\partial_{n+1}}$$

Mas da sequencia longa do par $(X_{n+1},X_n)$ temos

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X_{n+1},X_n) \arrow[r, "\partial_{n+1}"] & H_n(X_n) \arrow[r, "i_n"] & H_n(X_{n+1}) \arrow[r] \arrow[d, equal, "\text{(lema)}"] & H_n(X_{n+1},X_n) \arrow[r] \arrow[d, equal, "\text{(lema)}"] & \cdots \\
 & & & H_n(X) & 0 &
\end{tikzcd}$$

$\Rightarrow i_n$ é sobrejetor

$$\Rightarrow \frac{H_{n+1}(X_n)}{\operatorname{Im}\partial_{n+1}} \cong H_n(X_{n+1}) \cong H_n(X) \qquad \square$$

Exemplo: $\mathbb{C}P^n$ tem uma estrutura celular com uma $2k$-célula $\forall\ 0 \le k \le n$ (Exercício!)

$$\Rightarrow \mathbb{C}P^n = e^0 \cup e^2 \cup e^4 \cup \cdots \cup e^{2n}$$

$\Rightarrow$ Complexo de Cadeias Celular:

$$0 \to \underset{2n}{\mathbb{Z}} \to 0 \to \mathbb{Z} \to \cdots \to \underset{2}{\mathbb{Z}} \to 0 \to \underset{0}{\mathbb{Z}} \to 0$$

$$\Rightarrow H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & \text{se } 0 \le k \le 2n \text{ é par} \\ 0 & \text{caso contrário} \end{cases}$$

Pergunta: Como calcular $d_n : C_n^{CW}(X) \to C_{n-1}^{CW}(X)$ ?

- Lembre que $C_n^{CW}(X) = H_n(X_n, X_{n-1})$ é gerado pelas $n$-células $\{e_\alpha^n\}$

$\Rightarrow d_n(e_\alpha^n) = \sum \lambda_{\alpha\beta} e_\beta^{n-1}$, e precisamos determinar os coeficientes $\lambda_{\alpha\beta} \in \mathbb{Z}$.

Teorema: $\lambda_{\alpha\beta} = \deg(f_{\alpha\beta})$ onde

$$\begin{tikzcd} S_\alpha^{n-1} \arrow[r, "\chi_{e_\alpha^n}"] \arrow[rrr, bend right, "f_{\alpha\beta}"'] & X^{n-1} \arrow[r, "q_\beta"] & X^{n-1}/X^{n-1} - e_\beta^{n-1} \arrow[r, phantom, "\cong"] & S_\beta^{n-1} \end{tikzcd}$$

Dem: Temos $H_n(X_n, X_{n-1}) \cong H_n(\amalg D^n_\alpha, \amalg \partial D^n_\alpha)$

ou seja, estamos identificando

$$1 = [D^n_\alpha] \in H_n(D^n_\alpha, \partial D^n_\alpha) \xmapsto{h_{e^\alpha_n *}} e^n_\alpha \in H_n(X_n, X_{n-1})$$

Logo,

$$\begin{tikzcd}
1 \arrow[dd, mapsto] & H_n(D^n_\alpha, \partial D^n_\alpha) \arrow[r, "\underset{\text{(seq. do par)}}{\simeq}"] \arrow[dd, "h_{e^n_\alpha *}"'] & H_{n-1}(\underbrace{S^{n-1}_\alpha}_{\partial D^n_\alpha}) \arrow[r, "f_{\alpha\beta}"] \arrow[d, "\chi_{e^n_\alpha *}"] & H_{n-1}(S^{n-1}_\beta) & \lambda_{\alpha\beta} \\
 & 1 \arrow[r, mapsto] & 1 \arrow[r, mapsto] & \deg(f_{\alpha\beta}) & \\
e^n_\alpha & H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[rr, bend right, "d_n"'] & H_{n-1}(X_{n-1}) \arrow[r, "p_{n-1}"] & H_{n-1}(X_{n-1}, X_{n-2}) \arrow[uu, "q_{\beta *}"'] & \\
 & e^n_\alpha \arrow[rr, mapsto] & & \Sigma \lambda_{\alpha\beta} e^{n-1}_\beta \arrow[uuur, mapsto, bend right] &
\end{tikzcd}$$

onde $q_\beta : X_{n-1} \longrightarrow X_{n-1} / X_{n-1} - e^{n-1}_\beta \cong S^{n-1}_\beta$

Ou seja:

Por um lado $q_{\beta *} d_n (e^n_\alpha) = \lambda_{\alpha\beta}$

mas

$$d_n(e^n_\alpha) = d_n h_{e^n_\alpha *}(1) = p_{n-1} \circ \partial_n \circ h_{e^n_\alpha *}(1) = p_{n-1} \circ \chi_{e^n_\alpha *}(1) =$$

$$\Rightarrow q_{\beta *}(e^n_\alpha) = q_{\beta *} p_{n-1} \chi_{e^n_\alpha *}(1) = \deg(f_{\alpha\beta}) \qquad \square$$

## Exemplo: Homologia de $\mathbb{R}P^n$

Decomposição Celular:

$$\mathbb{R}P^0 \subset \mathbb{R}P^1 \subset \cdots\cdots \subset \mathbb{R}P^{n-1} \subset \mathbb{R}P^n$$

$$\left(\text{ou } \mathbb{R}P^n = e^0 \cup e^1 \cup \cdots\cdots \cup e^n\right)$$

Complexo de Cadeias Celular:

$$0 \longrightarrow \underset{n}{\mathbb{Z}} \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} \mathbb{Z} \xrightarrow{d_{n-2}} \cdots\cdots \longrightarrow \underset{1}{\mathbb{Z}} \xrightarrow{d_1} \underset{0}{\mathbb{Z}} \longrightarrow 0$$

Temos que calcular $d_k$

$$X_k = \mathbb{R}P^k, \quad X_{k-1} = \mathbb{R}P^{k-1}$$

$$\begin{tikzcd}
\chi_{e^k} : S^{k-1} \arrow[r] \arrow[dr, "f_k"'] & \mathbb{R}P^{k-1} \arrow[d, "q_{k-1}"] \\
& S^{k-1} = \mathbb{R}P^{k-1} \Big/ \underbrace{\mathbb{R}P^{k-1} - e^{k-1}}_{\mathbb{R}P^{k-2}}
\end{tikzcd}$$

recobrimento duplo

temos que calcular $\deg f_k$.
Fazemos isso através do grau local. Escolhe $\tilde{y} \in e^{k-1} \subset \mathbb{R}P^{k-1}$ e seja $y = q_{k-1}(\tilde{y})$. Então, como $q_{k-1}$ é um homeo em uma vizinhança de $\tilde{y}$, temos que $f_k^{-1}(y) = \{y, -y\}$ & $f_k = Id$ em uma viz. de $y$ & $f_k = A$ (antípoda) em um viz. de $-y$

Segue que $\deg(f_k) = 1 + (-1)^k = \begin{cases} 0 & \text{se } k \text{ é ímpar} \\ 2 & \text{se } k \text{ é par} \end{cases}$

• $\underline{H_k(\mathbb{R}P^{2m})}$:

$$0 \to \underset{2m}{\mathbb{Z}} \xrightarrow{\cdot 2} \underset{2m-1}{\mathbb{Z}} \xrightarrow{0} \underset{2m-2}{\mathbb{Z}} \xrightarrow{\cdot 2} \underset{2m-3}{\mathbb{Z}} \xrightarrow{0} \cdots \xrightarrow{0} \underset{2}{\mathbb{Z}} \xrightarrow{\cdot 2} \underset{1}{\mathbb{Z}} \xrightarrow{0} \underset{0}{\mathbb{Z}} \to 0$$

$$\Rightarrow H_k(\mathbb{R}P^{2m}) \cong \begin{cases} 0 & \text{se } k \text{ é par} \\ \mathbb{Z}_2 & \text{se } k \text{ é ímpar } \& \ 0 \leq k \leq 2m \end{cases}$$

• $H_k(\mathbb{R}P^{2m+1}) = \begin{cases} 0 & \text{se } k \text{ é par} \\ \mathbb{Z}_2 & \text{se } k \text{ é ímpar } \& \ 0 \leq k \leq 2m+1 \end{cases}$

$\square$