Topologie en Meetkunde 2011

Lecturers: Marius Crainic and Ivan Struchiner

CHAPTER 1

The Classification Problem for Compact Surfaces

1. Introduction

In this Chapter we will introduce and start dealing with the classification problem for compact surfaces. Giving a complete solution to this problem is one of the main goals of the course. It will serve as motivation for many of the concepts that will be introduce.

Our approach to the classification problem will be the following:

(1) We will give a list of compact connected surfaces, all of which will be constructed from a polygonal region in the plane by identifying its edges in pairs.
(2) We will show that any compact connected surface is homeomorphic to one in the list.
(3) We will show that any two surfaces in the list are not homeomorphic to each other.

Parts (1) and (2) will be dealt with in this chapter, while part (3) will be done only after we introduce the fundamental group and learn how to calculate it (via the Seifert - Van Kampen Theorem). To be a bit more precise about part (2) in the plan above, what we will show is that any triangulable compact surface is homeomorphic to one in the list. It turns out that every compact surface is in fact triangulable, and we hope to come back to this at some point in the course.

2. Topological Manifolds

The main objects that will be studied in this chapter are topological surfaces, which are simply 2-dimensional topological manifolds.

Definition 1.1. *An* ***n-dimensional topological Manifold*** *is a topological space* $(X, \mathcal{T})$ *which satisfies the following properties:*

(1) *X is Hausdorff;*
(2) *X admits a countable open cover $\{U_i\}_{i\in\mathbb{N}}$ such that each U_i is homeomorphic to an open set in $\mathbb{R}^n$.*

Each open U_i together with a homeomorphism $\varphi_i : U_i \to V_i \subset \mathbb{R}^n$ will be called a ***coordinate chart*** *of X.*

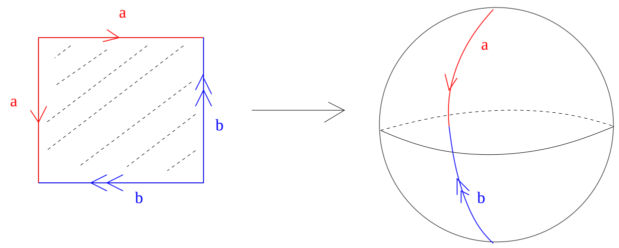
Remark 1.2. *The second condition in the definition above can be restated as: X is second countable and each point of X admits an open neighborhood which is a chart. It the follows that X is locally compact and second countable, and thus metrizable, i.e., the topology of X is induced by a metric.*

Definition 1.3. *A* ***surface*** *is a 2-dimensional topological manifold.*

Example 1.4 (The Sphere $\mathbb{S}^2$). *We define the sphere $\mathbb{S}^2$ to be the quotient space obtained from a square by identifying its border according to the Figure 1. Thus, if we denote the unit interval $[0, 1]$ by I, then*

$$\mathbb{S}^2 = \{(x, y) \in I \times I\}/\sim$$

where we identify $(0, y) \sim (1 - y, 1)$, and $(x, 0) \sim (1, 1 - x)$.



The sphere obtained from a square glueing as indicated in the picture

FIGURE 1.

EXERCISE 1.1. (1) Show that $\mathbb{S}^2$ is homeomorphic to the standard sphere

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1.\}$$

(2) Show that it is a surface.

EXAMPLE 1.5 (The Torus $\mathbb{T}^2$). *We define the torus $\mathbb{T}^2$ to be the quotient space obtained from the unit square by identifying its border according to the Figure 2. Thus,*

$$\mathbb{T}^2 = \{(x, y) \in I \times I\}/\sim$$

where we identify $(0, y) \sim (1, y)$*, and* $(x, 0) \sim (x, 1)$*.*

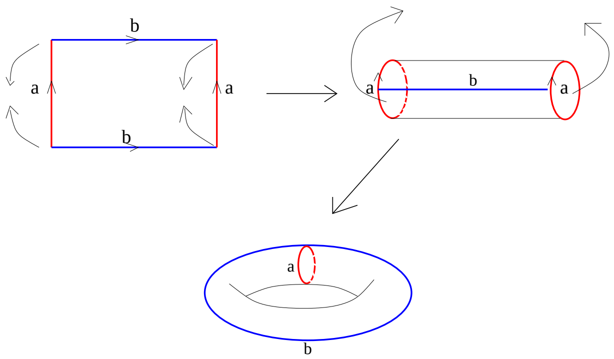


FIGURE 2.

EXERCISE 1.2. Let $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the standard circle.

(1) Show that the torus $\mathbb{T}^2$ is homeomorphic to a $\mathbb{S}^1 \times \mathbb{S}^1$.
(2) Show that it is a surface.

EXAMPLE 1.6 (The Projective Space $\mathbb{P}^2$). *We define the projective space $\mathbb{P}^2$ to be the quotient space obtained from the unit square by identifying its border according to the Figure 3. Thus,*

$$\mathbb{P}^2 = \{(x, y) \in I \times I\}/\sim$$

where we identify $(0, y) \sim (1, y)$*, and* $(x, 0) \sim (1 - x, 1)$*.*

EXERCISE 1.3. Let $D \subset \mathbb{R}^2$ denote the (closed) disk of radius 1.

(1) Show that $\mathbb{P}^2$ is homeomorphic to the quotient space obtained from D by identifying its border $\mathbb{S}^1$ via the antipodal map (Figure 4)

$$A : \mathbb{S}^1 \to \mathbb{S}^1, \quad A(x, y) = (-x, -y).$$

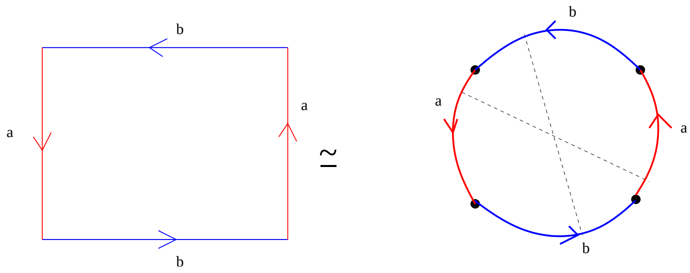


FIGURE 3.

(2) Show that $\mathbb{P}^2$ is homeomorphic to the quotient space obtained from the standard sphere by identifying a point p with its antipodal $-p$.
(3) Show that $\mathbb{P}^2$ is homeomorphic to the space of lines through the origin in $\mathbb{R}^3$.
(4) Show that it is a surface.

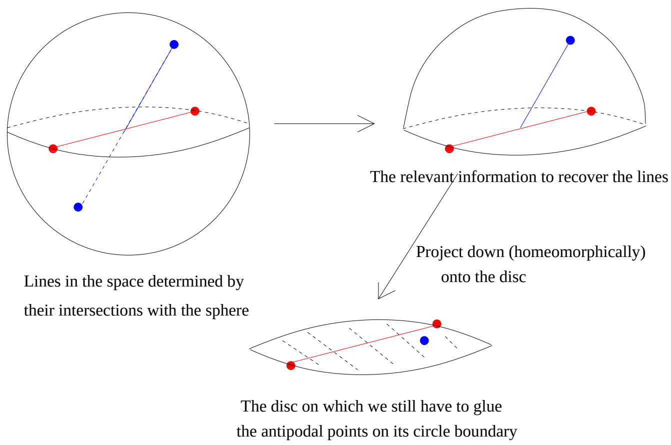


FIGURE 4.

There is a very basic operation which allows us to construct a new manifold out of two given manifolds.

DEFINITION 1.7. *Given two topological manifolds M and N of the same dimension, define their connected sum, denoted $M\#N$ as follows: remove from M and N two "small balls" B_1 and B_2 and glue $M - B_1$ and $N - B_2$ along the sphere $\partial B_1 = \partial B_2$.*

For surfaces, it means that we remove two small disks and we glue the remaininig spaces along the bounday circles (Figure 5). We can describe this operation with more details:

Remove an Open Disk: We remove from M and N an open subset D_1 and D_2 each of which is homeomorphic to an open disk in $\mathbb{R}^2$.

Glue along the Boundary: We fix a homeomorphism $\varphi : \partial D_1 \to \partial D_2$ and we take the quotient space

$$M\#N = (M - D_1)\coprod(N - D_2)/\sim$$

where $x \sim y$ if and only if $x = y$ or $x \in \partial D_1$, $y \in \partial D_2$, and $\varphi(x) = y$.

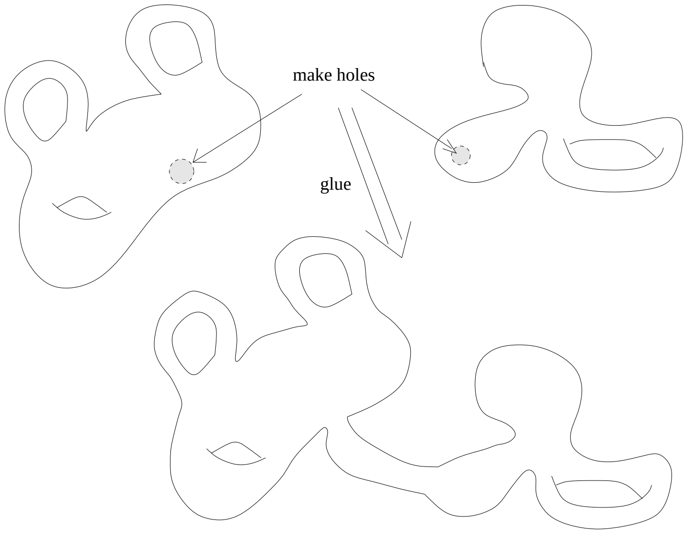


FIGURE 5. Connected Sum

EXAMPLE 1.8. The connected sum of two tori is the double torus $\mathbb{T}_2$. Repeating the operation of connected sum, one obtains all tori with arbitrary number of holes (see Figure 6 for the $g = 2$):

$$\mathbb{T}_g = \underbrace{\mathbb{T}\#\ldots\#\mathbb{T}}_{g \text{ times}}.$$

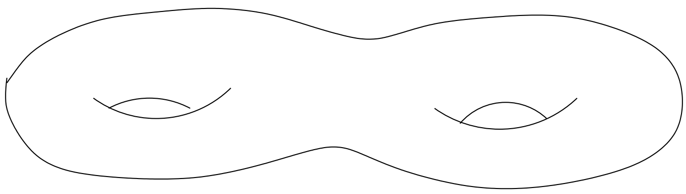


FIGURE 6. Double Torus.

Similarly, one considers the connected sum of h copies of $\mathbb{P}^2$:

$$\mathbb{P}_h = \underbrace{\mathbb{P}^2\#\ldots\#\mathbb{P}^2}_{h \text{ times}}.$$

EXERCISE 1.4. Show that the connected sum $M\#S^2$ of any surface M with the sphere S^2 is homeomorphic to M itself.

We could, in principal, consider more surfaces by considering other examples of connected sums (for example of a torus with a projective space), but as we will soon see, we have already obtained a complete list of all compact connected surfaces:

THEOREM 1.9. *Any compact connected surface is homeomorphic to one of the following:*

(1) *A sphere* $\mathbb{S}^2$,
(2) *A connected sum of Tori (plural of Torus)* $\mathbb{T}_g$, *with* $g \in \mathbb{N}$, *or*
(3) *A connected sum of projective spaces* $\mathbb{P}_h$, *with* $h \in \mathbb{N}$.

3. The Basic Building Blocks: Polygonal Regions

In this section we will show how to construct surfaces out of polygonal regions of the plane, by identifying its edges in pairs. Intuitively, a polygonal region is a subset of the plane which "looks like" in Figure 7. Let us explain how to make this precise.

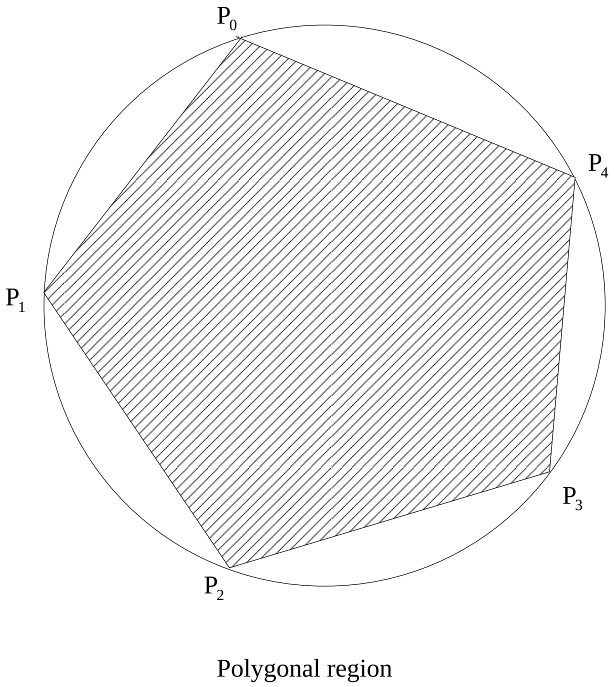


FIGURE 7. Polygonal Region.

- Fix a circle in $\mathbb{R}^2$, pick $n+1$ points on it and order them in counterclockwise direction $\{p_0, \dots, p_n\}$.
- For each $0 < i \leq n$ consider the line passing through p_{i-1} and p_i. It divides $\mathbb{R}^2$ into two half-planes. Let H_i be the half-plane which contains all the other points p_j.
- Let P be the set
$$P = H_1 \cap H_2 \cap \cdots \cap H_n.$$

DEFINITION 1.10. *An n-sided* ***polygonal region*** *of the plane is any subset of $\mathbb{R}^2$ obtained by the "recipe" above.*

Associated to a polygonal region will will use the following notation:

Vertices: The points p_i will be called vertices of P. The set of all vertices of P will be denoted by $V(P)$.

Edges: The line segment joining p_{i-1} and p_i will be denoted by e_i, and will be called an edge of P. The set of all edges of P will be denoted by $E(P)$

Border: The union of all edges of P will be denoted by ∂P and will be called the border of P.

Interior: The complement of ∂P in P will be denoted by $\mathrm{Int}(P)$ and will be called the interior of P.

It will also be important to introduce orientations on the edges of a polygon, and to specify what a "map" between edges is (this is how we will be able the make precise the notion of "glueing one edge to another").

DEFINITION 1.11.

(1) *Let $L \subset \mathbb{R}^2$ be a line segment. An* ***orientation*** *of L is a choice of ordering of its end points. Such an orientation will be represented by an arrow, and we will say that* *L is a line from a to b* *(Figure 8).*
(2) *If L is a line from a to b, and L' is a line segment from c to d, then a* ***positive linear map*** *from L to L' is the homeomorphism $h : L \to L'$ which associates to $x = (1-t)a + tb \in L$ the point $h(x) = (1-t)c + td$.*

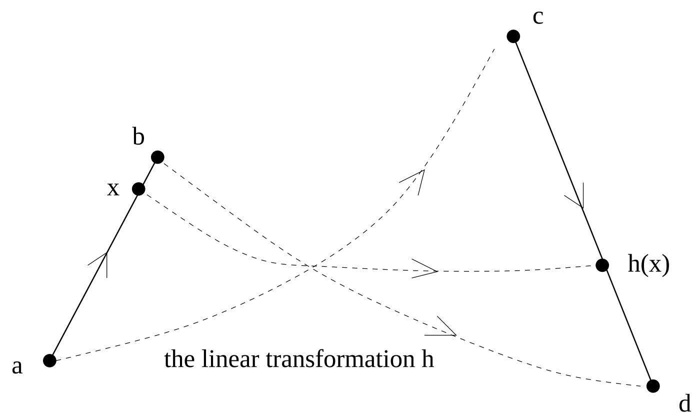


FIGURE 8. Positive Linear Maps.

4. Glueing the Edges of a Polygonal Region

Since we will be considering (disjoint unions of) polygonal regions with several identifications on the borders, we must find a convenient way of keeping track of such "glueing procedures". For this, we will introduce the concept of labels:

DEFINITION 1.12. *A* ***labeling*** *of a polygonal region P is a map $E(P) \to \Lambda$ from the set of edges of P to a set Λ, whose elements will be called* ***labels****.*

Given a polygonal region along with: (1) a labeling of its edges, and (2) an orientation on edge, we consider the space

$$X = P/\sim$$

where

- If $p \in \mathrm{Int}(P)$, then p is equivalent only to itself, i,e,m $p \sim p$;
- If e_i and e_j are edges with the same label, we let $h : e_i \to e_j$ be a positive linear map and we set

$$x \in e_i \sim h(x) \in e_j.$$

In this case we say that X was obtained from P by glueing its edges together according to the orientation and the labeling.

We remark that we also allow X to be obtained from a finite disjoint unit of polygonal regions with identifications on the edges. Thus X may be either connected or disconnected. As an illustration of spaces obtained in this way, consider the following examples:

EXAMPLE 1.13. *The disk can be obtained from a triangle with two labels (a and b) and orientations on the edges as shown in the Figure 9 below.*

EXAMPLE 1.14. *As we have seen in Figure 1 the sphere can be obtained from a square with to labels and orientations on the edges.*

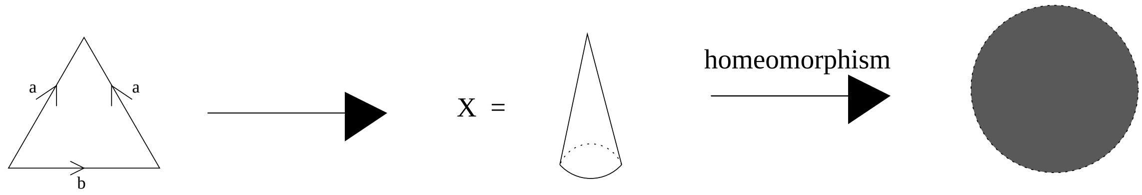


FIGURE 9. The Disk

EXAMPLE 1.15. *In Figure 10 we illustrate the fact that since we allow X to be obtained by glueing the edges of more than one polygonal regions, it follows that X is not necessarily connected.*

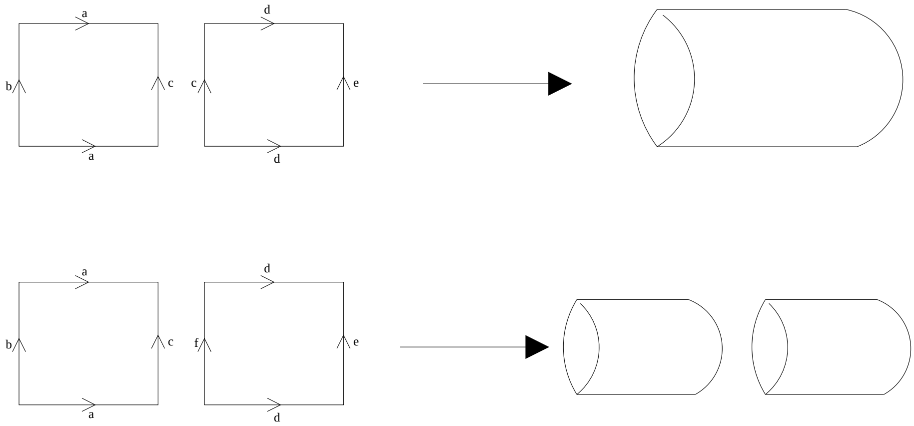


FIGURE 10. X can be connected or disconnected.

Finally, in order to keep track of the orientations of the edges along with the labels, we will now introduce the notion of a labeling scheme. Let e_k is an edge of P with label a_{i_k}. If e_k is oriented from p_{k-1} to p_k, then we put en exponent $+1$ on a_{i_k}. If e_k is oriented from p_k to p_{k-1}, then we put an exponent -1 on a_{i_k}. Then P, its labels, and the orientations on its edges is totally specified up to a homeomorphism which respects the quotient space X by the symbol

$$w = a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \cdots a_{i_n}^{\epsilon_n}, \quad \epsilon_i = \pm 1.$$

DEFINITION 1.16. *The symbol $w = a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \cdots a_{i_n}^{\epsilon_n}$ will be called a **labeling scheme** for P with respect to its labels and orientations.*

In Figure 11 are some examples of how to go back and forth from a (disjoint union of) polygonal regions with labels and orientations to labeling schemes.

5. Operations on Labeling Schemes

It is important to note that we are interested in the quotient space X obtained from a polygonal region by gluing its edges and not on the labeling scheme itself. With this in mind, we will now introduce some operations we can perform on the labeling scheme (or equivalently on the polygonal region) which will leave the resulting quotient space unchanged.

I) Cutting: The operation of cutting is described at the level of labeling schemes as follows. Suppose that $w = a_{i_1}^{\epsilon_1} \cdots a_{i_p}^{\epsilon_p} a_{i_{p+1}}^{\epsilon_{p+1}} \cdots a_{i_n}^{\epsilon_n}$ is a labeling scheme and let b be a

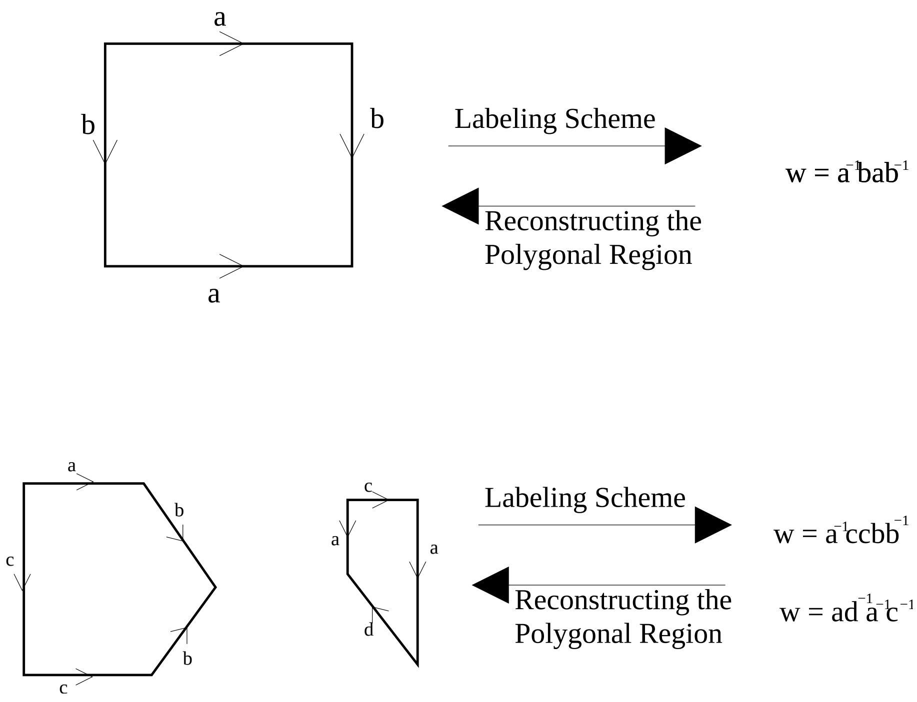


FIGURE 11. Labeling Schemes.

label which does not appear elsewhere in the scheme. Then we may replace w by a pair of labeling schemes

$$w_1 = a_{i_1}^{\epsilon_1} \cdots a_{i_p}^{\epsilon_p} b, \text{ and } w_2 = b^{-1} a_{i_{p+1}}^{\epsilon_{p+1}} \cdots a_{i_n}^{\epsilon_n}.$$

For a geometric interpretation see the Figure 12.

II) Glueing: The reverse operation of cutting is known as glueing. In terms of the labeling scheme it can be described as follows: If

$$w_1 = a_{i_1}^{\epsilon_1} \cdots a_{i_p}^{\epsilon_p} b, \text{ and } b^{-1} a_{i_{p+1}}^{\epsilon_{p+1}} \cdots a_{i_n}^{\epsilon_n}$$

are labeling schemes, and the label b only appears where it is indicated above, then we may replace w_1 and w_2 by the labeling scheme $w = a_{i_1}^{\epsilon_1} \cdots a_{i_p}^{\epsilon_p} a_{i_{p+1}}^{\epsilon_{p+1}} \cdots a_{i_n}^{\epsilon_n}$.

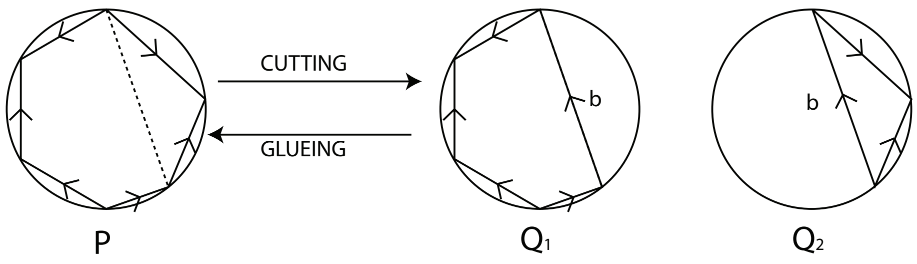


FIGURE 12. Cutting & Glueing

Before we go on with the description of the operations, let us take a small break to write down more formally the result of cutting and glueing:

PROPOSITION 1.17. *Suppose that X is obtained by glueing the edges of n polygonal regions with labeling scheme*

$$w_1 = y_0 y_1, w_2, \ldots, w_n.$$

Let b be a label that does not appear in the scheme. If both y_0 and y_1 have length at least 2, then X can also be obtained by $n+1$ polygonal regions with labeling scheme

$$y_0 b, b^{-1} y_1, w_2, \ldots, w_n.$$

EXERCISE 1.5. The purpose of this exercise is to prove the proposition above. Denote by $P_1, \ldots, P_n$ the original n polygonal regions and by $Q_0, Q_1, P_2, \ldots P_n$ the $n+1$ polygonal regions obtained by cutting P_1. Denote also by $X = (\coprod_{i=1}^n P_i)/\sim$ the space obtained by glueing the edges before cutting, and by $Y = (Q_0 \coprod Q_1 \coprod_{i=2}^n P_i)/\sim$ the space obtained after performing the cutting operation. Consider the obvious map

$$\Phi : Q_0 \coprod Q_1 \coprod_{i=2}^{n} P_i \longrightarrow \coprod_{i=1}^{n} P_i.$$

Show that:

(1) Φ induces a well defined map $\varphi : Y \to X$, i.e., if $q \sim q'$, then $\Phi(q) \sim \Phi(q')$.
(2) φ is continuous (use the definition of the quotient topology).
(3) φ is injective, i.e., if $\Phi(q) \sim \Phi(q')$, then $q \sim q'$.
(4) φ is surjective.
(5) X and Y are both compact and Hausdorff.
(6) φ is a homeomorphism.

With only the operations of cutting and gluing we can now easily understand how to construct the connected sums of tori (and projective spaces) out of polygonal regions with identification on the borders:

EXAMPLE 1.18 (The Double Torus $\mathbb{T}^2$). *Let P be the 8-sided polygonal region with labeling scheme $w = aba^{-1}b^{-1}cdc^{-1}d^{-1}$. In order to see that $X = P/\sim$ is homeomorphic to a double torus, we will apply the cutting and glueing operations described above (see Figure 13). Thus we first cut P into two 5-sided polygonal regions Q_1 and Q_2, with labeling schemes $w_1 = aba^{-1}b^{-1}e$ and $w_2 = e^{-1}cdc^{-1}d^{-1}$ respectively. Now, it is clear that after identifying the vertices of Q_1 correspond to the endpoints of e, we obtain the usual representation of the torus as a quotient of the unit square, but with an open disk removed. The edge e then becomes the border of the open disk. The same is obviously true also for Q_2. Thus, if we now apply the glueing operation, what we obtain is the quotient of two copies of the torus, both with an open disk removed by identifying the border of the disk. This is precisely the construction of the connected sum.*

EXERCISE 1.6. Show that $\mathbb{T}_g$ is obtained from a $4g$-sided polygonal region with labeling scheme

$$w = (a_1 b_1 a_1^{-1} b_1^{-1}) \cdots (a_g b_g a_g^{-1} b_g^{-1}).$$

EXAMPLE 1.19. *A similar argument as the one presented above shows that for $h > 1$, $\mathbb{P}_h$ can be obtained from a $2h$-sided polygonal region with labeling scheme*

$$w = (a_1 a_1) \cdots (a_h a_h).$$

We now continue to describe the rest of the operations that may be performed on the labeling scheme. We suggest that you convince yourself that each of these operations leave the quotient space unchanged.

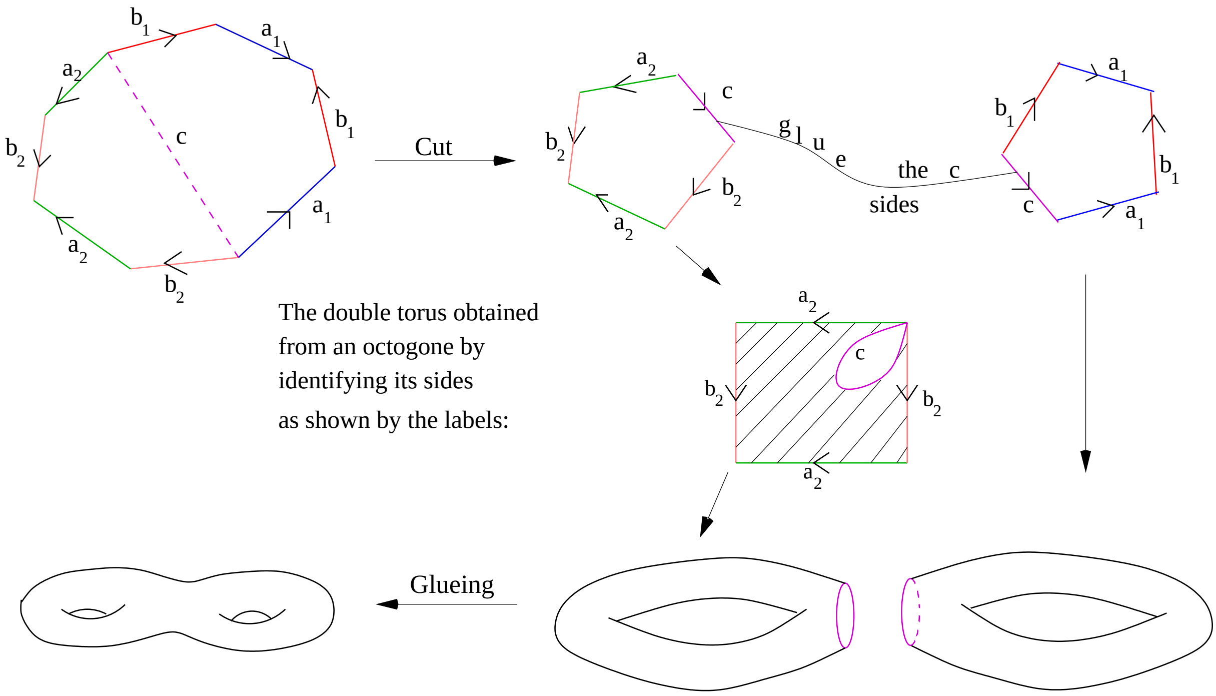


FIGURE 13. Connected Sum of Two Tori.

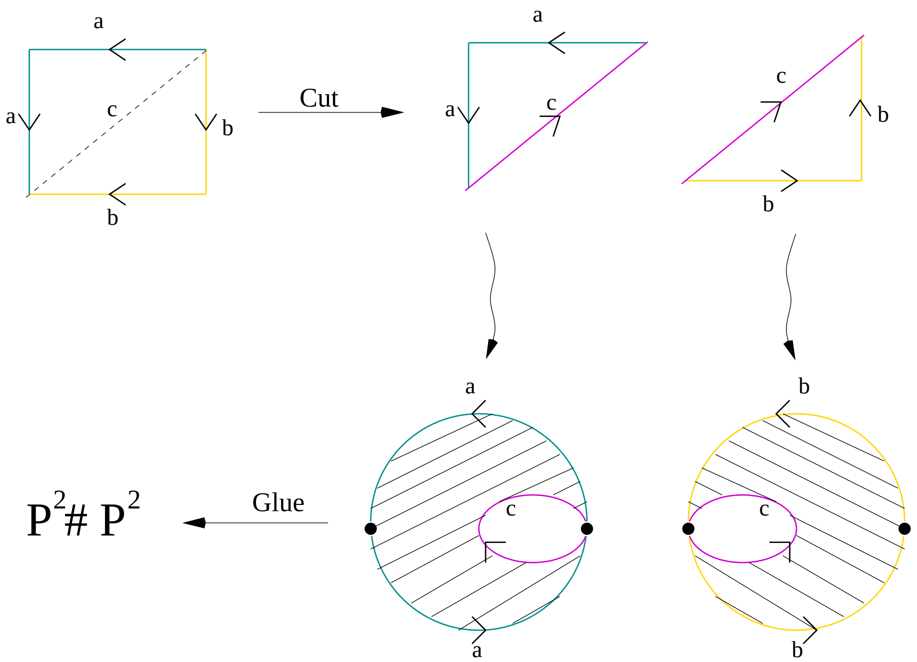


FIGURE 14. Connected Sum of Two Projective Spaces.

DEFINITION 1.20. *Let $w_1, w_2, \ldots, w_n$ be a labeling scheme and let y be a string of labels that appears in the labeling scheme (it may appear in more that one place of the scheme). We will say that y is a* ***removable string*** *if*

(1) *all labels of* y *are distinct, i.e.,* $y = a_{i_1}^{\epsilon_1} \cdots a_{i_k}^{\epsilon_k}$ *with* $a_{i_p} \neq a_{i_q}$ *for all* $i_p \neq i_q$*, and*
(2) *the labels of* y *do not appear elsewhere (outside of* y*) in the labeling scheme, i.e., for all* $1 \leq p \leq k$*, if* a_{i_p} *appears in the labeling scheme, then it belongs to the string* y*.*

III) Unfolding Edges: If y is a removable string of a labeling scheme, then we may replace y by a label that does not appear elsewhere in the scheme. Geometrically, this can be interpreted as replacing a sequence of edges (a "folded line segment"), by a single edge (a line segment) (See Figure 15).

IV) Folding Edges: The reverse operation to unfolding edges is that of folding edges described by: replace all appearances of a single label by a removable string of labels.

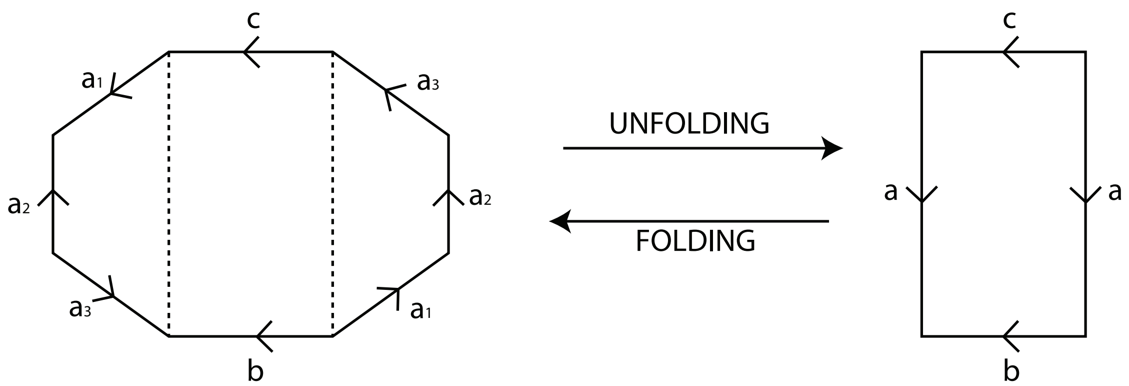


FIGURE 15. Fold/Unfold

V) Reversing Orientations: We may change the the sign of the exponent of all occurrences of a single label in the labeling scheme. In order to understand why the quotient space is left unchanged, recall that we are identifying the points on two oriented edges with the same label by means of a positive linear map. Note that if the orientation on both edges are reversed, the identification remains unchanged.

VI) Cyclic Permutation: It is clear that if instead of writing the labeling scheme of a polygonal region by starting with the label on the edge e_1, we decide to start with the label on a different edge and then continue in the same counterclockwise direction, then the quotient space X is unchanged. We may think of this as performing a rotation on the polygonal region. The effect on the labeling scheme is to take a cyclic permutation of its labels

$$a_{i_1}^{\epsilon_1} \cdots a_{i_n}^{\epsilon_n} \longrightarrow a_{i_2}^{\epsilon_2} \cdots a_{i_n}^{\epsilon_n} a_{i_1}^{\epsilon_1}.$$

VII) Flip: We may replace a labeling scheme by its <u>formal inverse</u>:

$$w = a_{i_1}^{\epsilon_1} \cdots a_{i_n}^{\epsilon_n} \longrightarrow a_{i_n}^{-\epsilon_n} \cdots a_{i_1}^{-\epsilon_1}.$$

Geometrically this corresponds to flipping the polygonal region as in Figure 16 (and then performing a cyclic permutation if necessary) .

REMARK 1.21. The operations of permutation and flipping should be thought of as instances of the same phenomena. If may apply any <u>Euclidean transformation of the plane</u> (i.e., translations, reflections and rotations) to our original polygonal region, the resulting object will be again a polygonal region whose quotient space is homoemorphic to the original one.

Finally, for completeness and also for further reference, we describe two operations that are obtained by composing the operations of cutting/glueing with that of folding/unfolding:

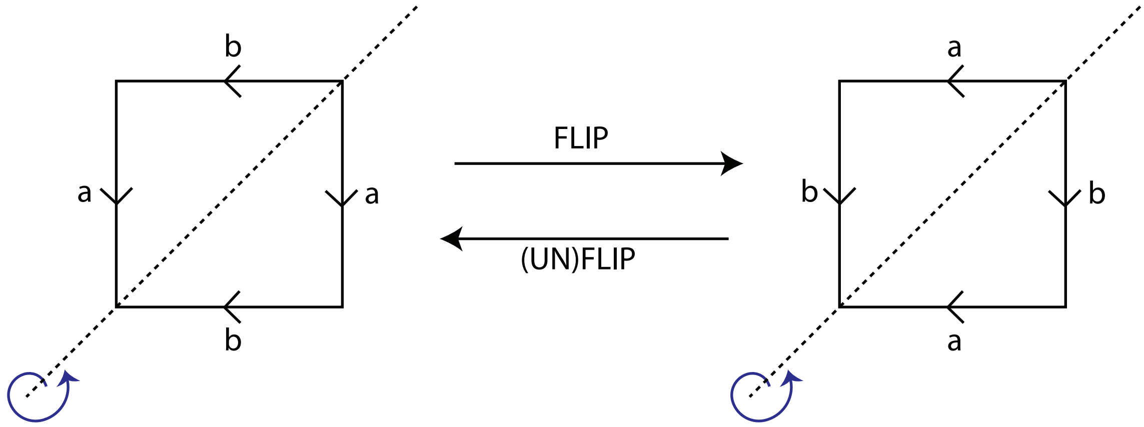


FIGURE 16. Flip/Unflip

VIII) Cancel: We may replace a labeling scheme of the form $y_0aa^{-1}y_1$ by y_0y_1 provided that a does not appear elsewhere in the labeling scheme, and both y_0 and y_1 have length at least 2. Geometrically, this operation is represented by the sequence of diagram in Figure 17.

IX) Uncancel: Under the same conditions as above, we may reverse the operation of canceling by replacing a scheme y_0y_1 by the labeling scheme $y_0aa^{-1}y_1$, as indicated in Figure 17.

It should clear that the operations above leave the quotient space X unchanged. Thus, it is natural to pose the following definition:

DEFINITION 1.22. *Two labeling schemes are* ***equivalent*** *if one can be obtained from the other by applying the operations (I) - (IX) described above.*

EXERCISE 1.7. Show that this defines an equivalence relation on the set of all labeling schemes.

EXAMPLE 1.23. *We have seen that the Klein bottle is the quotient of the unit square by the identification whose labeling scheme is $aba^{-}1b$. Let us prove that the Klein bottle is homeomorphic to the connected sum of two projective spaces:*

$$\begin{aligned}
aba^{-1}b \;&\longrightarrow abc \quad \& \quad c^{-1}a^{-1}b && \textit{(cutting)}\\
&\longrightarrow cab \quad \& \quad b^{-1}ac && \textit{(permuting and flipping)}\\
&\longrightarrow cabb^{-1}ac && \textit{(glueing)}\\
&\longrightarrow caac && \textit{(canceling)}\\
&\longrightarrow aacc && \textit{(permuting)}.
\end{aligned}$$

6. Geometric Surfaces

In this section we will consider surfaces which are obtained from a polygonal region by identifying it edges in pairs. We wil then show that every such surface is homeomorphic to one in the list given in Theorem 1.9.

DEFINITION 1.24. *A compact and connected topological surface X is called a* ***geometric surface*** *if it can be obtained from a polygonal region by glueing its edges in pairs.*

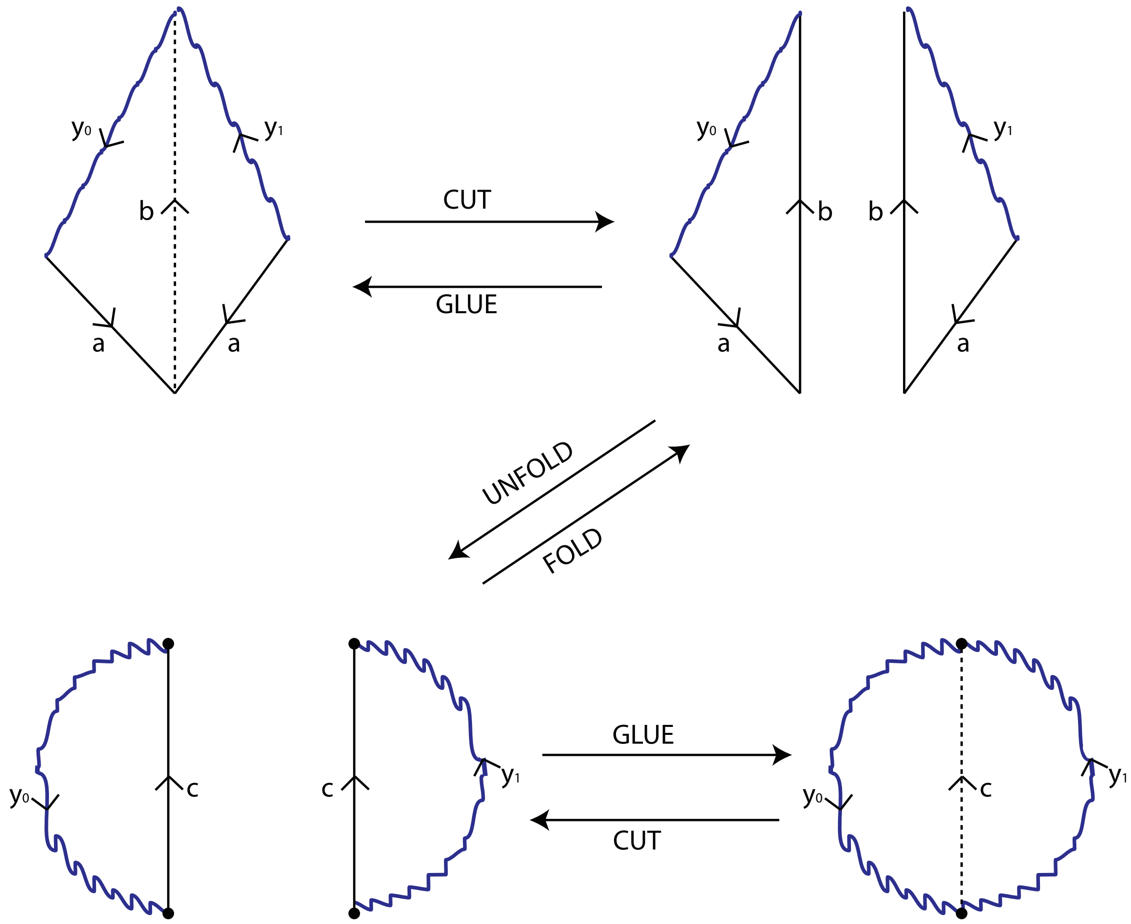


FIGURE 17. Cancel/Uncancel

The remainder of this section will be dedicated to proving the following theorem:

THEOREM 1.25 (Classification of Geometric Surfaces). *Let X be a geometric surface. Then X is homeomorphic to one of the following: $\mathbb{S}^2$, $\mathbb{T}_g$, or $\mathbb{P}_h$ (for some $g, h \in \mathbb{N}$).*

The idea of the proof is to consider labeling schemes which give rise to geometric surfaces (known as <u>proper labeling schemes</u>) and then to show that any proper labeling scheme can be put into a <u>normal form</u> by means of the operations introduced in the last section.

DEFINITION 1.26. *A labeling scheme $w_1, \ldots, w_m$ (for m polygonal regions) is called a* ***proper labeling scheme*** *if each label appears <u>exactly twice</u> in the scheme.*

REMARK 1.27. We note that if we start with a proper labeling scheme, then by applying any of the operations introduced in the preceding section gives rise to another proper labeling scheme.

We can now restate Theorem 1.25 into a more algebraic form:

THEOREM 1.28 (Normal Forms of Proper Labeling Schemes). *Let w be a proper labeling scheme of length greater or equal to 4 (of a single polygonal region). Then w is equivalent to one of the following labeling schemes:*

(1) $aa^{-1}bb^{-1}$,

(2) $abab$,
(3) $(a_1 b_1 a_1^{-1} b_1^{-1}) \cdots (a_g b_g a_g^{-1} b_g^{-1})$, *or*
(4) $(a_1 a_1)(a_2 a_2) \cdots (a_h a_h)$.

Remark 1.29. *Of course, in the list above (1) is a sphere, (2) is a projective space, (3) is a connected sum of tori, and (4) is a connected sum of projective spaces.*

The first step in the proof of Theorem 1.28 is to distinguish between two classes of proper labelings that will then be treated separately:

Definition 1.30. *Let w be a proper labeling scheme for a single polygonal region. If every label of w appears one with an exponent $+1$ and once with exponent -1 we say that w is of* ***torus type****. Otherwise, we say that w is of* ***projective type****.*

We begin by dealing with labeling schemes of projective type:

Proposition 1.31. *Let w be a labeling scheme of projective type. The w is equivalent to a labeling scheme of the following form:*

$$w \sim (a_1 a_1) \cdots (a_k a_k) w_1,$$

where w_1 is a labeling scheme of torus type.

The proof of this proposition will follow from the following lemma:

Lemma 1.32. *If w is a proper labeling scheme of the form $w = [y_0]a[y_1]a[y_2]$, where each $[y_i]$ is a string of labels (which may be empty), the w is equivalent to a labeling scheme of the form*

$$w \sim aa[y_0 y_1^{-1} y_2].$$

Proof. We separate the proof into two cases:
Case 1: $[y_0] = \emptyset$**.** In this case $w = a[y_1]a[y_2]$.

- If $[y_1]$ is empty, then we are done.
- If $[y_2]$ is empty, the we proceed as follows:

$$\begin{aligned} w = a[y_1]a &\longrightarrow a^{-1}[y_1^{-1}]a^{-1} && \text{(flipping)} \\ &\longrightarrow a^{-1}a^{-1}[y_1^{-1}] && \text{(permuting)} \\ &\longrightarrow aa[y_1^{-1}] && \text{(reversing orientation of } a\text{)}. \end{aligned}$$

- If both $[y_1]$ and $[y_2]$ are not empty, the we apply the operations described in Figure 18.

Case 2: $[y_0] \neq \emptyset$**.** Again we exclude the most trivial case first. If both $[y_1]$ and $[y_2]$ are empty, then $w = [y_0]aa$ and a permutation brings w to the desired form. Assume now that either $[y_1]$

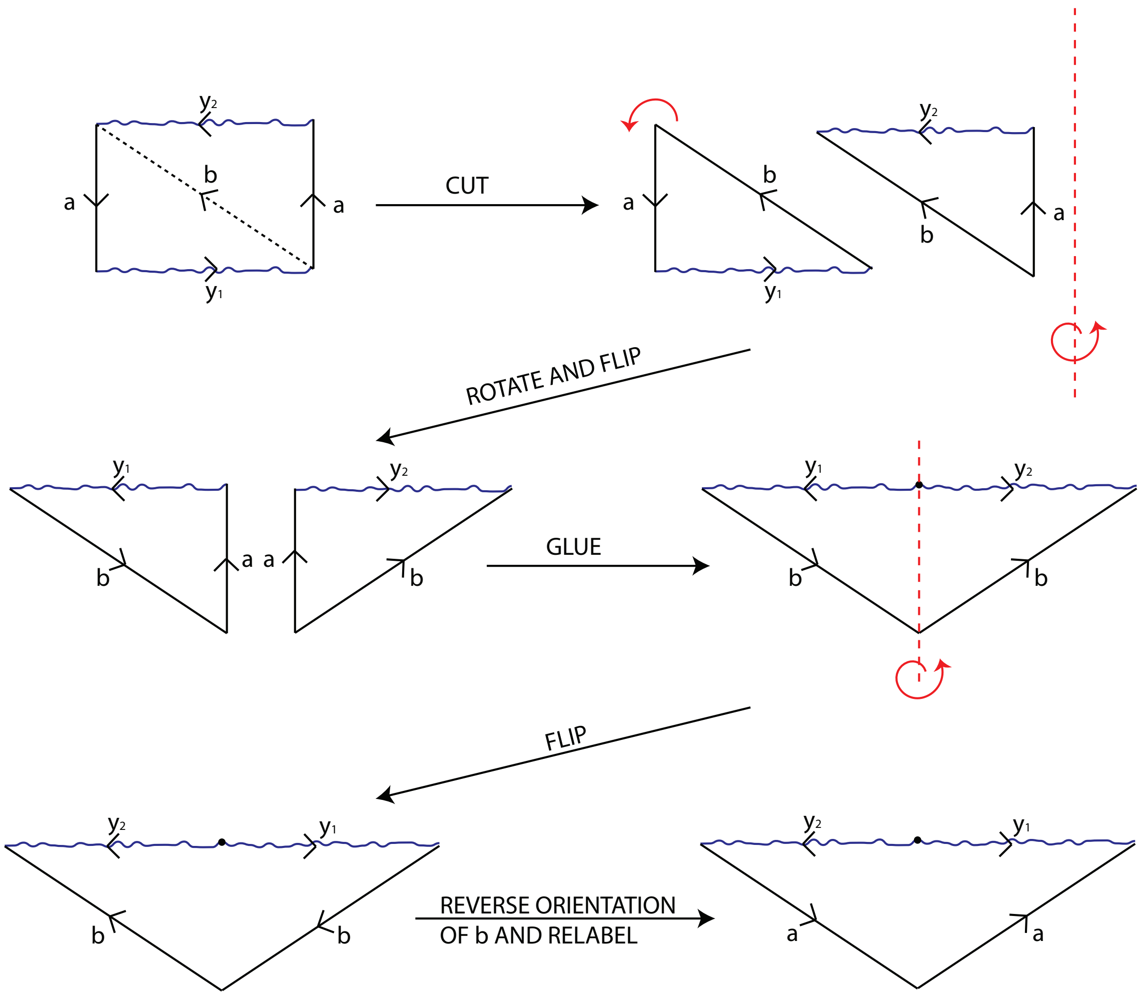


FIGURE 18. Case 1

or $[y_2]$ are non-empty. Then:

$$\begin{aligned}
w = [y_0]a[y_1]a[y_2] &\longrightarrow [y_0]ab \;\&\; b^{-1}[y_1]a[y_2] && \text{(cutting)}\\
&\longrightarrow [y_0^{-1}]b^{-1}a^{-1} \;\&\; a[y_2]b^{-1}[y_1] && \text{(flipping and permuting)}\\
&\longrightarrow [y_0^{-1}]b^{-1}[y_2]b^{-1}[y_1] && \text{(glueing and canceling)}\\
&\longrightarrow b^{-1}[y_2]b^{-1}[y_1y_0^{-1}] && \text{(permuting)}\\
&\longrightarrow b^{-1}b^{-1}[y_2^{-1}y_1y_0^{-1}] && \text{(case 1)}\\
&\longrightarrow [y_0y_1^{-1}y_2]bb && \text{(flipping)}\\
&\longrightarrow aa[y_0y_1^{-1}y_2] && \text{(permuting and relabeling).}
\end{aligned}$$

□

EXERCISE 1.8. Write the algebraic sequence of arguments presented in the proof of case 1, and make diagrams to describe the geometric sequence of arguments presented in case 2 of the proof.

PROOF (OF PROPOSITION 1.31). Let w be a labeling scheme of projective type. Then there is at least one label of w which appears twice with the same sign. Thus,

$$w = [y_0]a[y_1]a[y_2]$$

and by using the lemma, we obtain that w is equivalent to $aa[y_0y_1^{-1}y_2]$. If $[y_0y_1^{-1}y_2]$ is of torus type then we are done. Otherwise, there is a label b is $[y_0y_1^{-1}y_2]$ which appears twice with the same sign, and thus we may assume that w is equivalent to

$$w \sim aa[z_0]b[z_1]b[z_2].$$

We apply the lemma again, this time to the labeling scheme $[aaz_0]b[z_1]b[z_2]$, to obtain that

$$w \sim bbaa[z_0z_1^{-1}z_2].$$

If $[z_0z_1^{-1}z_2]$ is of torus type we are done. Otherwise we continue this process which will end as soon as we have put w into the desired form $w \sim (a_1a_1)\dots(a_ka_k)w_1$ with w_1 a labeling scheme of torus type. □

REMARK 1.33. *We can conclude from Proposition 1.31 that if w is a proper labeling scheme, then either: (1) w is of torus type, or (2) w is of the form $(a_1a_1)\dots(a_ka_k)w_1$ with w_1 a labeling scheme of torus type, or (3) w is of the form $(a_1a_1)\dots(a_ka_k)$ in which case we are done (X is a connected sum of projective spaces).*

We must now examine how to reduce w to a simpler form when w is of the form (1) or (2).

EXERCISE 1.9. Show that if w is a proper labeling scheme of length 4, then w must be equivalent to one of the following labeling schemes:

$$aabb, \quad abab \quad aa^{-1}bb^{-1}, \quad aba^{-1}b^{-1}$$

From now on we assume that w has length greater then 4, and moreover, that it is **irreducible**, i.e., it does not contain any adjacent terms having the same label, but opposite signs (in which case we could perform the operation of canceling to reduce the length of w). In this case we have the following lemma:

LEMMA 1.34. *Suppose that w is a proper labeling scheme of the form $w = w_0w_1$, where w_1 is an irreducible scheme of torus type. Then w is equivalent to a scheme of the form w_0w_2, where w_2 has the same length as w_1, and has the form:*

$$w_2 = aba^{-1}b^{-1}w_3,$$

where w_3 is of torus type or is empty.

PROOF. We will divide the proof of this lemma into several steps.

Step 1: We may assume that w is of the form

$$w = w_0[y_1]a[y_2]b[y_3]a^{-1}[y_4]b^{-1}[y_5],$$

where some of the strings of labels $[y_i]$ may be empty.

To see this we proceed as follows. Let a be the label in w_1 whose occurrences are as close as possible (with the minimal amount of labels in between them). If a appears first with an exponent -1, then we revert the orientation of both appearances of a. Next, let b be any label in between a and a^{-1}. Then, since a and a^{-1} are the closest labels to each other in w_1, it follows that either b^{-1} appears after a^{-1}, in which case we are done, or b^{-1} appears in front of a, in which case we simply exchange the labels of b and a.

Step 2: (1st Surgery) w is equivalent to
$$w \sim w_0 a[y_2]b[y_3]a^{-1}[y_1y_4]b^{-1}[y_5].$$

We may assume that $[y_1] \neq \emptyset$ (or else there is nothing to prove). Then, we perform the following operations:
$$\begin{aligned}
w &= w_0[y_1]a[y_2]b[y_3]a^{-1}[y_4]b^{-1}[y_5] \\
&\longrightarrow [y_2]b[y_3]a^{-1}[y_4]b^{-1}[y_5]w_0c \ \&\ c^{-1}[y_1]a \quad &&\text{(permuting and cutting)} \\
&\longrightarrow [y_4]b^{-1}[y_5]w_0c[y_2]b[y_3]a^{-1} \ \&\ ac^{-1}[y_1] \quad &&\text{(permuting)} \\
&\longrightarrow [y_4]b^{-1}[y_5]w_0c[y_2]b[y_3]c^{-1}[y_1] &&\text{(glueing)} \\
&\longrightarrow w_0a[y_2]b[y_3]a^{-1}[y_1y_4]b^{-1}[y_5]. &&\text{(permuting and relabeling)}
\end{aligned}$$

Step 3: (2nd Surgery) w is equivalent to
$$w \sim w_0 a[y_1y_4y_3]ba^{-1}b^{-1}[y_2y_5].$$

First of all, assume that w_0, y_1, y_4, and y_5 are all empty. Then
$$w \sim a[y_2]b[y_3]a^{-1}b^{-1}$$
and the result follows by permuting and relabeling.

Now assume that at least one of the strings w_0, y_1, y_4, or y_5 is non-empty. Then, we can perform the following sequence of operations:
$$\begin{aligned}
w &\sim w_0a[y_2]b[y_3]a^{-1}[y_1y_4]b^{-1}[y_5] \\
&\longrightarrow a[y_2]b[y_3]a^{-1}c \ \&\ c^{-1}[y_1y_4]b^{-1}[y_5]w_0 \quad &&\text{(permuting and cutting)} \\
&\longrightarrow [y_3]a^{-1}ca[y_2]b \ \&\ b^{-1}[y_5]w_0c^{-1}[y_1y_4] &&\text{(permuting)} \\
&\longrightarrow [y_3]a^{-1}ca[y_2][y_5]w_0c^{-1}[y_1y_4] &&\text{(glueing)} \\
&\longrightarrow w_0c^{-1}[y_1y_4y_3]a^{-1}ca[y_2y_5] &&\text{(permuting)} \\
&\longrightarrow w_0a[y_1y_4y_3]ba^{-1}b^{-1}[y_2y_5]. &&\text{(relabeling)}
\end{aligned}$$

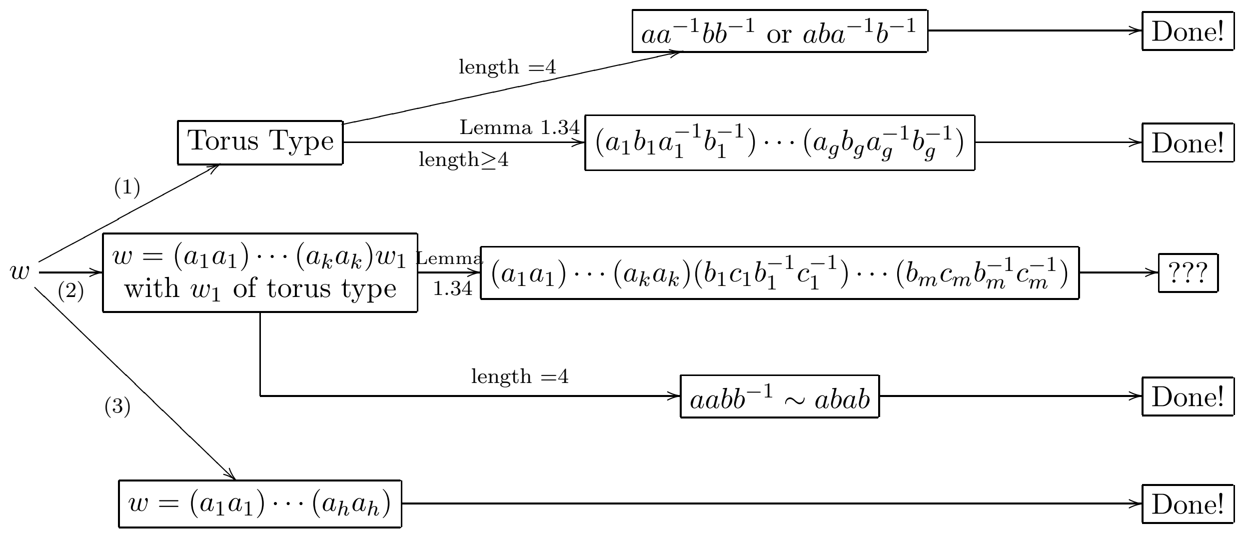
Step 4: (3nd Surgery) w is equivalent to
$$w \sim w_0aba^{-1}b^{-1}[y_1y_4y_3y_2y_5].$$

We perform the following sequence of operations:

$$
\begin{aligned}
w &\sim w_0 a[y_1y_4y_3]ba^{-1}b^{-1}[y_2y_5] \\
&\longrightarrow [y_1y_4y_3]ba^{-1}c \ \&\ c^{-1}b^{-1}[y_2y_5]w_0a && \text{(permuting and cutting)} \\
&\longrightarrow a^{-1}c[y_1y_4y_3]b \ \&\ b^{-1}[y_2y_5]w_0ac^{-1} && \text{(permuting)} \\
&\longrightarrow a^{-1}c[y_1y_4y_3][y_2y_5]w_0ac^{-1} && \text{(glueing)} \\
&\longrightarrow w_0aba^{-1}b^{-1}[y_1y_4y_3y_2y_5]. && \text{(permuting and relabeling)}
\end{aligned}
$$

□

The following graph summarizes the results that we have obtained so far:



Remark 1.35. *The three arrows coming out of w correspond to remark 1.33, while the cases where the length of w is equal to 4 follow from exercise 1.9.*

Thus, in order to conclude the proof of Theorem 1.28 we need to describe what the connected sum of tori and projective spaces correspond to, i.e., to reduce

$$w = (a_1a_1)\cdots(a_ka_k)(b_1c_1b_1^{-1}c_1^{-1})\cdots(b_mc_mb_m^{-1}c_m^{-1})$$

to its normal form. This follows from the following lemma:

Lemma 1.36. *If $w = w_0(aa)(bcb^{-1}c^{-1})w_1$ is a proper scheme, then*

$$w \sim w_0(aabbcc)w_1.$$

Proof. We will make use repeatedly of Lemma 1.32 which states that

$$[y_0]a[y_1]a[y_2] \sim aa[y_0y_1^{-1}y_2].$$

To prove the lemma we consider the following sequence of operations:

$$\begin{aligned}
w &= w_0(aa)(bcb^{-1}c^{-1})w_1 \\
&\longrightarrow (aa)[bc][cb]^{-1}[w_1w_0] && \text{(permuting)} \\
&\longrightarrow [bc]a[cb]a[w_1w_0] && \text{(Lemma 1.32)} \\
&\longrightarrow [b]c[a]c[baw_1w_0] && \text{(regrouping the terms)} \\
&\longrightarrow [cc]b[a^{-1}]b[aw_1w_0] && \text{(Lemma 1.32 and regrouping the terms)} \\
&\longrightarrow w_0(bbccaa)w_1. && \text{(Lemma 1.32 and permuting)}
\end{aligned}$$

The result then follows by relabeling the terms. □

We can thus conclude from the Lemma, by applying it several times if necessary, and then relabeling, that

$$(a_1a_1)\cdots(a_ka_k)(b_1c_1b_1^{-1}c_1^{-1})\cdots(b_mc_mb_m^{-1}c_m^{-1}) \sim (a_1a_1)\cdots(a_{k+2m}a_{k+2m}).$$

This finishes the proof of Theorem 1.28.

Exercise 1.10. Throughout this section we have implicitly described an algorithm to reduce any proper labeling scheme to one in the normal form of Theorem 1.28. Write down this algorithm explicitly.

Exercise 1.11. Use the algorithm you developed in the exercise above to determine which surface corresponds to the following labeling schemes:

(1) $abacb^{-1}c^{-1}$
(2) $abca^{-1}cb$
(3) $abbca^{-1}ddc^{-1}$
(4) $abcda^{-1}c^{-1}b^{-1}d^{-1}$
(5) $abcdabdc$
(6) $abcda^{-1}b^{-1}c^{-1}d^{-1}$

7. Triangulated Surfaces

In this section we will introduce the notion of a triangulation on a compact Hausdorff topological space. We will then show that:

Theorem 1.37. *Any triangulated compact connected surface can be obtained from a single polygonal region by identifying its edges with respect to a proper labeling scheme.*

Theorem 1.9 will then follow from Theorem 1.28 and the following result which we will not prove now (but we intend to come back to it if time permits).

Theorem 1.38. *Every compact connected surface is triangulable (i.e., can be triangulated).*

We begin with the definition of a triangulation:

Definition 1.39. Let X be a compact Hausdorff topological space.

- A **curved triangle** in X is a subspace A together with a homeomorphism $h: T \to A$, where T is a triangular region in the plane. If $v \in T$ is a vertex, then $h(v)$ is called a **vertex** of A. Similarly, if $e \in T$ is an edge, then $h(e)$ is called an **edge** of A.

- A **triangulation** of X is a collection $A_1, \dots, A_n$ of curved triangles of X which cover X,

$$\cup_i A_i = X,$$

and such that if $A_i \cap A_j \neq \emptyset$, then either
  1. $A_i \cap A_j = \{v\}$ is a vertex, or
  2. $A_i \cap A_j = e$ is an edge, and furthermore, the map $h_j^{-1} \circ h_i$ which maps the edge $h_i^{-1}(e)$ of T_i to the edge $h_j^{-1}(e)$ of T_j is a linear homeomorphism.

REMARK 1.40. *By condition (2) in the definition above we mean the that if we pick an orientation on $h_i^{-1}(e)$ then $h_j^{-1} \circ h_i$ induces a choice of orientation on $h_j^{-1}(e)$ for which $h_j^{-1} \cap h_i : h_i^{-1}(e) \to h_j^{-1}(e)$ becomes a positive linear map.*

For an example of a triangulation of the sphere, see Figure 19.

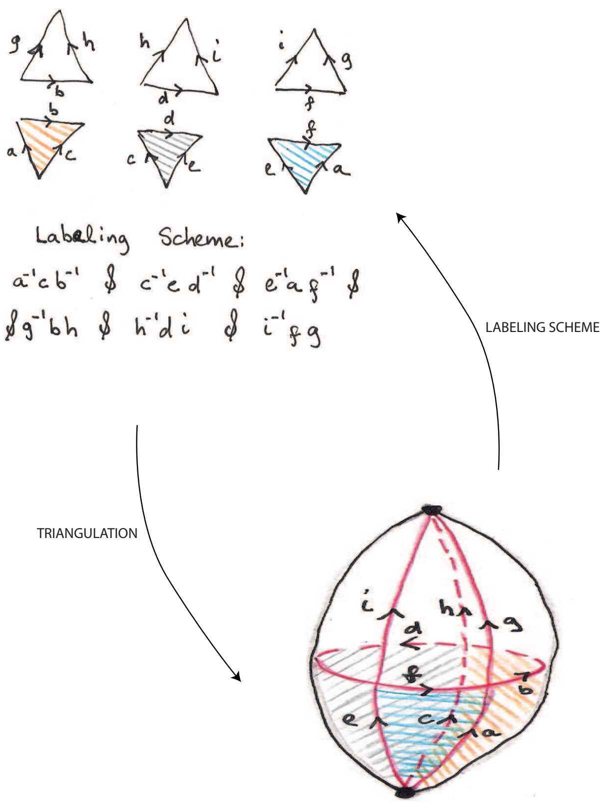


FIGURE 19.

EXERCISE 1.12. For each of the following spaces exhibit an explicit triangulation.
1. A torus
2. a cylinder
3. a cone
4. a projective space
5. a Möbius band,
6. a Klein bottle

A triangulation $\{A_1, \ldots, A_n\}$ of a compact Hausdorff space X induces a labeling scheme, which will be called the **labeling scheme of the triangulation**, as follows:

Polygonal Regions: For each curved triangle A_i, let $h_i : T_i \to A_i$ be the corresponding homeomorphism defined on the triangular region T_i. The polygonal region we will consider is the disjoint union of the triangles T_i's.

Orientation on Edges: Let $e \subset X$ be an edge appearing in the triangulation (i.e., it is the edge of at least one of the curved triangles). Let v and w be the vertices at the endpoints of e. Choose an orientation on e by declaring it to go from v to w. Then if $h_i^{-1}(e)$ is an edge of T_i, we orient it from $h_i^{-1}(v)$ to $h_i^{-1}(w)$.

Labels: Let

$$\Lambda = \{e \subset X : e \text{ is the edge of (at least) one of the curved triangles}\}$$

be the set of edges of the triangulation. Then if $h_i^{-1}(e)$ is an edge of T_i, we associate to it the label $e \in \Lambda$.

Example 1.41. Figure 19 exhibits a triangulation on the sphere, and also the labeling scheme of the triangulation.

Exercise 1.13. For each of the spaces in exercise 1.12, determine the labeling scheme of the triangulation.

Proposition 1.42. *If X is a compact triangulated surface, then the labeling scheme of the triangulation is proper.*

Proof. We need to show that each label appears exactly twice in the labeling scheme. The arguments needed to do this are intuitively clear. However, the easiest way to make them precise is by using the notion of fundamental group. Thus, we will sketch the proof now, but leave the details as an exercise that should be done after the fundamental group is introduced.

The first step is to show that each label appear at least twice in the labeling scheme. Thus, assume that a label appears only once. This means that there is an edge e which is the edge of only one curved triangle. Exercise 1.14 bellow shows that this cannot happen. The intuitive idea is that if $x \in e$, then by removing x we will not "create a hole" in X, but on the other hand, if we remove any point from an open set in $\mathbb{R}^2$, then we do "create a hole"

The next step is to show that there is at most two appearances of each label. Again this will follow by a "removing one point trick". If a label appears more than twice, then there are more than two triangles which intersect in a single edge. Intuitively, this will mean that there is a "multiple corner" which cannot be smoothened into an open subset of $\mathbb{R}^2$ (see figure 20). The precise argument is given in exercise 1.15 bellow.

□

Exercise 1.14 (To be done after the definition of homotopy). Let T be a triangle and $x \in T$ be a point in one of the edges of T and let U be any neighborhood of x. Show that any loop in $U - x$ is homotopic to a constant path. Conclude that x does not have any neighborhood which is homeomorphic to an open set of $\mathbb{R}^2$.

Exercise 1.15 (To be done after the Seifert van Kampen Theorem). Consider the space obtained by glueing together k triangles along a common edge e, with $k > 2$ (Figure 20 shows the case when $k = 3$). Let $x \in e$ be a point in this common edge. Show that any neighborhood U of x contains a possibly smaller neighborhood $V \subset U$ such that $V - x$ is homotopy equivalent to a bouquet of $k - 1$ circles. Conclude by computing the fundamental group of $V - x$ that x does not have any neighborhood which is homeomorphic to an open subset of $\mathbb{R}^2$.

Proposition 1.43. *If X is a compact triangulated surface, then X is homeomorphic to the space obtained from $\coprod T_i$ by glueing its edges according to the labeling scheme of the triangulation.*

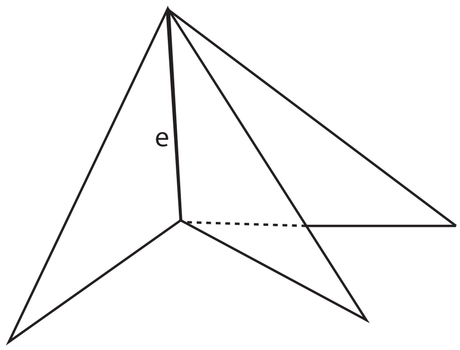


FIGURE 20.

EXERCISE 1.16. Consider the map $h : \coprod T_i \to X$ obtained by putting together all of the maps $h_i : T_i \to A_i \subset X$. Consider the space X' obtained by identifying two points p and q of $\coprod T_i$ if and only if $h(p) = h(q)$. Show that X' is homeomorphic to X.

PROOF. Let us denote Y the quotient space obtained from $\coprod T_i$ by identifying it edges with respect to the labeling scheme of the triangulation. It is an immediate consequence of the exercise above, that h factors through a continuous map $f : Y \to X$, i.e.,

$$\begin{tikzcd} \coprod_i T_i \arrow[d] \arrow[dr, "h"] & \\ Y \arrow[r, "f"'] & X. \end{tikzcd}$$

Moreover, since h is surjective, it follows that f is also surjective. Thus, in order to prove the proposition, it suffices to show that f is injective (because Y is compact and X is Hausdorff).

Let us denote by $[p] \in Y$ the equivalence class – *with respect to the labeling scheme of the triangulation* – of a point p in $\coprod T_i$. Assume that $f([p]) = f([q])$, for some $p \neq q$. Then, by definition, it follows that $h(p) = x = h(q)$. Thus, either x belongs to some edge e of the triangulation on X, in which case it is clear that $[p] = [q]$, or x is a vertex. In this case, in order to show that $[p] = [q]$ (so that f is injective) we must verify that the identification of p with q is "forced" as a consequence of the identification of the edges of the triangles T_i's (see also exercise 1.20).

Suppose that A_i and A_j intersect at a vertex v. What we need to show is that we can find a sequence

$$A_i = A_{i_1}, A_{i_2}, \ldots A_{i_m} = A_j,$$

such that A_{i_k} intersects $A_{i_{k+1}}$ on a common edge which contains v as its endpoint (as illustrated in Figure 21). This is the content of the following exercise.

□

EXERCISE 1.17. Given v, define two curved triangles A_i and A_j with vertex v to be equivalent if we can find a sequence A_{i_k} as above. Use the "remove the one point trick" to show that if there is more that one equivalence class of curved triangles with vertex v, then v does not have any neighborhood in X which is homeomorphic to an open set of $\mathbb{R}^2$, and thus X is not a surface.

We now are ready to finish the proof of Theorem 1.37. What we will show is that we may glue the triangles T_i together in order to obtain the desired polygonal region. In fact, start by

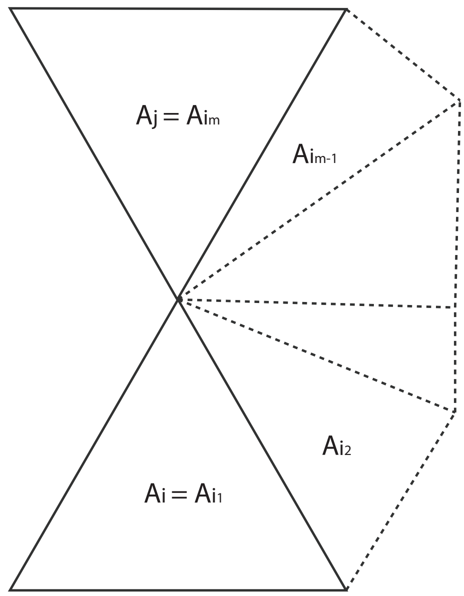


FIGURE 21.

choosing one of the triangles, say T_1. If T_i is another triangle which has a label on one of its edges which is equal to a label of T_1, then (after possibly flipping T_i) we may glue both triangles together. The effect of this is to reduce the original number of triangles by two, at the expense of adding one polygonal region (which in this case has 4 sides) which we denote by P_1. Next, we look at the edges of P_1. If one of the triangles T_j, with $j \neq 1, i$, has a label equal to one of the labels of P_1, then, after flipping T_j if necessary, we may glue it to P_1 obtaining in this way a new polygonal region P_2. We continue this process as long as we have two polygonal regions containing edges that have a common label.

At some point we will reach a situation where either we obtain the polygonal region P_{n+1} that we were looking for, or we obtain more then one disconnected polygonal region in which none of the labels appearing in one of them appear also in the other region. However, it is easy to see that this cannot happen, for in this case the quotient space X will necessarily be disconnected.

EXERCISE 1.18. Determine the space obtained from the following labeling schemes:

(1) abc, dae, bef, cdf.
(2) abc, cba, def, dfe^{-1}.

EXERCISE 1.19. Show that the projective space $\mathbb{P}^2$ can be obtained from two Mobius bands by glueing them along there boundary.

EXERCISE 1.20. Let X be the space obtained from a sphere by identifying its north and south poles (X is not a surface). Find a triangulation on X such that the labeling scheme of the triangulation determines a sphere (i.e., the surface obtained by glueing the edges of the triangles with respect to the labeling scheme of the triangulation is homeomorphic to a sphere). Conclude that two non-homeomorphic compact Hausdorff spaces can have triangulations which induce the same labeling scheme. (We remark that this exercise gives an example of a triangulated space for which the map f from the proof of Proposition 1.43 is not injective.)

CHAPTER 6

Attaching cells

1. Cells

DEFINITION 6.1. *Let X be a Hausdorff topological space. An <u>open n-cell</u> in X is a subspace $e \subset X$ together with a homeomorphism*

$$\overset{\circ}{h}_e : \overset{\circ}{D}{}^n \longrightarrow e \subset X.$$

It is called an <u>n-cell</u> if $\overset{\circ}{h}_e$ extends to a continuous map

$$h_e : D^n \longrightarrow X.$$

We call h_e <u>the defining map</u> of the n-cell, and we also say that e is an n-cell with defining map h_e, or that e is the image of the n-cell. The <u>cell boundary</u> of e is defined by:

$$\partial_{cell}(e) := \overline{e} - e,$$

where $\overline{e}$ is the closure of e in X. The characteristic map χ_e of e is defined as the restriction

$$\chi_e := h_e|_{S^{n-1}} : S^{n-1} \longrightarrow X.$$

REMARK 6.2. Since $\overset{\circ}{D}{}^n$ is dense in D^n (hence each point $x \in D^n$ can be written as the limit of a sequence of points in the interior) and h_e must be continuous (hence preserves the limits), given $\overset{\circ}{h}_e$, the extension h_e will be unique.

In conclusion, an n-cell in the space X is just a subspace $e \subset X$ together woth a continuous map $h_e : D^n \longrightarrow X$ which, when restricted to $\overset{\circ}{D}{}^n$, is a homeomorphism between $\overset{\circ}{D}{}^n$ and e.

Here are some simple examples (more will come later).

EXAMPLE 6.3. Note that, for $n = 0$, a 0-cell in X is the same thing as a point of X.

An interesting 1-cell is $e = S^1 - \{(1,0)\}$ which is a 1-cell in S^1 with defining map

$$h_e(t) = (cos(\pi t), sin(\pi t)).$$

EXAMPLE 6.4. Various cells inside the sphere are shown in Figure 1.

EXAMPLE 6.5. In general, an open n-cell may fail to be an n-cell. This is already clear when $n = 1$ and $X = \mathbb{R}$. A subspace $e \subset \mathbb{R}$ is an open n-cell (with some defining map) if and only if $e = (a, b)$ is an open interval (with a and b- real numbers or plus/minus infinity). Indeed, any such open cell will be a connected subspace of $\mathbb{R}$ hence it must be an interval. From the discussions inthe previous chapter, it must be an open interval.

On the other hand, if $e = (a, b)$ is a bounded interval (hence a, b are finite), then e (together with some defining map) is a 1-cell. However, un-bounded intervals cannot be made into 1-cells. This will also follow from the next proposition (which imply that the closure $\overline{e}$ of e in X must be compact), but let's check it directly here for the open 1-cell $e = (1, \infty)$ together with the defining map:

$$\overset{\circ}{h}_e : (-1, 1) \longrightarrow \mathbb{R}, t \mapsto \frac{2}{t+1}$$

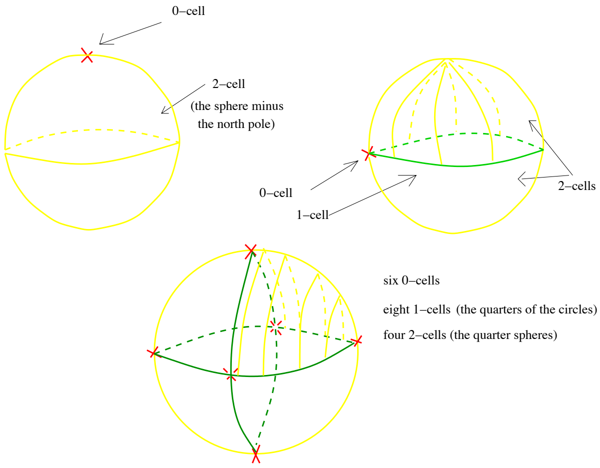


FIGURE 1.

This cannot have a continuous extension to $[-1, 1]$ because $x_n = -1 + \frac{1}{n}$ converges to -1, but $\overset{\circ}{h}_e (x_n) = 2n$ does not have a finite limit.

EXERCISE 6.1. *Let X be the one-point compactification of the space obtained from D^2 by removing two points on its boundary. Describe X in $\mathbb{R}^3$ and show that it is the closure of a 2-cell.*

Here are the main properties of n-cells.

PROPOSITION 6.6. *If $e \subset X$ is an n-cell with defining map $h_e : D^n \longrightarrow X$, then*

$$h_e(D^n) = \overline{e},\ \ h_e(S^{n-1}) = \partial_{cell}(e).$$

Moreover, as a map from D^n into $\overline{e}$, h_e is a topological quotient map.

PROOF. Since h_e is continuous, we have

$$h_e(\overline{B}) \subset \overline{h_e(B)}$$

for all $B \subset D^n$ (prove this!). Choosing $B = \overset{\circ}{D}{}^n$, we obtain $h_e(D^n) \subset \overline{e}$.
On the other hand,

$$e = h_e(\overset{\circ}{D}{}^n) \subset h_e(D^n),$$

and $h_e(D^n)$ is compact (as the image of a compact by a continuous map), hence closed in X (since X is Hausdorff). This implies $\overline{e} \subset h_e(D^n)$. Since the opposite inclusion has been proven, we get $h_e(D^n) = \overline{e}$. We now prove

$$h_e(S^{n-1}) = \overline{e} - e.$$

We first show the inverse inclusion: for $y \in \overline{e} - e$, by the first part, $y = h_e(x)$ for some $x \in D^n$ and, since $e = h_e(\overset{\circ}{D}{}^n)$, x cannot be in $\overset{\circ}{D}{}^n$; hence $x \in S^{n-1}$. We now prove the direct inclusion. So let $y = h_e(x)$ with $x \in S^{n-1}$, and we want to prove that $y \notin e$. Assume the contrary, i.e. $y = h_e(x')$ with $x' \in \overset{\circ}{D}{}^n$. Since x and x' are distinct, we find $U, V \subset D^n$ opens (in D^n) such that

$$x \in U, x' \in V, U \cap V = \emptyset.$$

We may assume that $V \subset \overset{\circ}{D}{}^n$. Since $\overset{\circ}{h}_e$ is a homeomorphism, $h_e(V)$ is open in e, hence also in $\overline{e}$. Since $h_e : D^n \longrightarrow \overline{e}$ is continuous, $h_e^{-1}(h_e(V))$ will be open in D^n; but it contains x, hence we find $\epsilon > 0$ such that

$$D^n \cap B(x,\epsilon) \subset h_e^{-1}(h_e(V))$$

(where the ball is with respect to the usual Euclidean metric). Since $x \in U$ and U is open in D^n, we may choose ϵ so small so that

$$D^n \cap B(x,\epsilon) \subset U.$$

Pick up an element $z \in \overset{\circ}{D}{}^n \cap B(x,\epsilon)$. By the inclusion above, we find $z' \in V$ such that $h_e(z) = h_e(z')$. But then both z and z' are in $\overset{\circ}{D}{}^n$ hence we must have $z = z'$. Hence $z \in V$. But $z \in U$ hence we obtain a contradiction with $U \cap V = \emptyset$. Finally, h_e is a topological quotient map as a continuous surjection from a compact space to a Hausdorff space. □

2. Attaching one n-cell

Definition 6.7. *Let X be a Hausdorff space and $A \subset X$ closed.*

We say that X is obtained from A by attaching an n-cell if there exists an n-cell $e \subset X$ (with some defining map h_e) such that

$$X = A \cup e, A \cap e = \emptyset.$$

Example 6.8. It is clear that the n-ball D^n is obtained from $\partial D^n = S^{n-1}$ by attaching an n-cell (the defining map being the identity map). From example 6.3, we see that S^1 can be obtained from a point (which is a 0-cell!) by attaching a 1-cell. In general, S^n can be obtained from a point by attaching an n-cell (see also example 6.13 below).

To treat examples treated in the previous lectures such as the torus, the Moebius band etc, the following lemma is very useful.

Lemma 6.9. *Let $X = D^n$ or $X = [0,1]^n$, and let Y be a quotient of X obtained by gluing (certain) points on the boundary $\partial(X)$. Assume that Y is Hausdorff,*

Let $\pi : X \longrightarrow Y$ be the quotient map, and let $B = \pi(\partial(X))$ (i.e. the space obtained from $\partial(X)$ by the original gluing). Then Y is obtained from B by attaching an n-cell.

Proof. Since D^n and $X = [0,1]^n$ are homeomorphic by a homeomorphism which preserves their boundary, we may assume that $X = D^n$. We then choose as defining map for the n-cell the quotient map

$$h : D^n \longrightarrow Y,$$

hence the n-cell will be $e := h(\overset{\circ}{D}{}^n)$. Note that $B = \pi(S^{n-1})$. Clearly, $Y = B \cup e$. Next, since no element on the boundary of D^n is equivalent (identified) with an interior element, we have $B \cap e = \emptyset$. Next, since no two interior points of D^n are equivalent, the restriction of h to $\overset{\circ}{D}{}^n$ is injective, hence

$$h|_{\overset{\circ}{D}{}^n} : \overset{\circ}{D}{}^n \longrightarrow e$$

is a continuous bijection. We still have to prove that this map is a homeomorphism, and for this it is enough to show that it sends closed sets to closed sets. So, let F be closed in $\overset{\circ}{D}{}^n$; write $F = \overset{\circ}{D}{}^n \cap K$ with $K \subset D^n$ closed. Then K is compact (as a closed inside a compact), hence $h(K)$ will be compact. Since Y was assumed to be Hausdorff, we deduce that $h(K)$ is closed in Y. So, to show that $h(F)$ is closed in e, it suffices to show that

$$h(\overset{\circ}{D}{}^n \cap K) = e \cap h(K).$$

The direct inclusion is clear. For the converse, note that if y belongs to the right hand side we have $y = h(x)$ and $y = h(x')$ with $x \in \overset{\circ}{D}{}^n$ and $x' \in K$, but since interior points are not being identified with any other points, we must have $x = x'$, hence y belongs to the left hand side. $\square$

Remark 6.10. The case $n = 2$ is particularly important: many spaces X can be obtained from D^2 by identifying certain parts of $\partial D^2 = S^1$, and the identification can be shown on the picture by labeling by letters the parts that are to be identified. In the quotient $B = S^1/\sim = \pi(S^1)$, each letter will appear only once (because we identified all the parts labeled by the same letter). When going one time around the circle, we will meet various labels that will give us a word whose letters are labels. Reading this word in the space B describes the characteristic map.

Example 6.11. Consider the torus as a quotient of $X = [0,1] \times [0,1]$ (see Section 5) of the first chapter, or of D^2. We can apply the previous lemma. The resulting space B is shown in the picture (Figure 2) and it consists of two circles on the torus touching each other in one point (a and b on the picture). This space can be drawn (is homeomorphic to) the space consisting of two circles in the plane touching in one point only (a bouquet of two circles). In conclusion, T^2 can be obtained from a bouquet of two spheres by attaching a 2-cell.

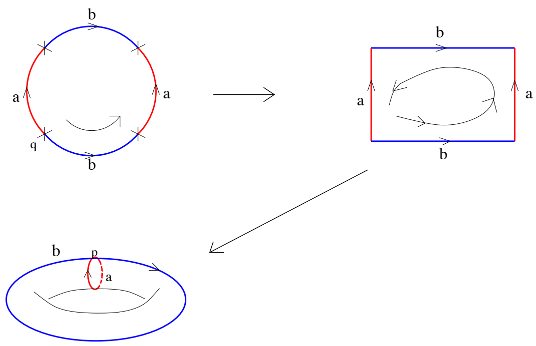


Figure 2.

Note that, according to the conventions from the previous remark, the characteristic map $\chi : S^1 \longrightarrow S^1 \vee S^1$ can be described symbolically as:

$$\chi = bab^{-1}a^{-1}$$

and which is further pictured in Figure 3.

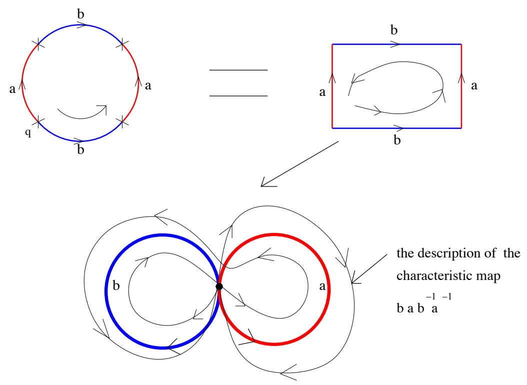


Figure 3.

EXAMPLE 6.12. A similar discussion applies to the Moebius band (section 4 in the first chapetr). The resulting space B, shown in Figure 4) is made of the three segments a, b and c on the Moebius band (hence B is the "boundary" of the Moebius band plus the segment a). The space B alone (as a topological space itself) is homeomorphic to the space consitsting of the circle S^1 together with a segment joining the north and the south pole (see the picture). In conclusion, the Moebius band can be obtained from this space by attaching a 2-cell.

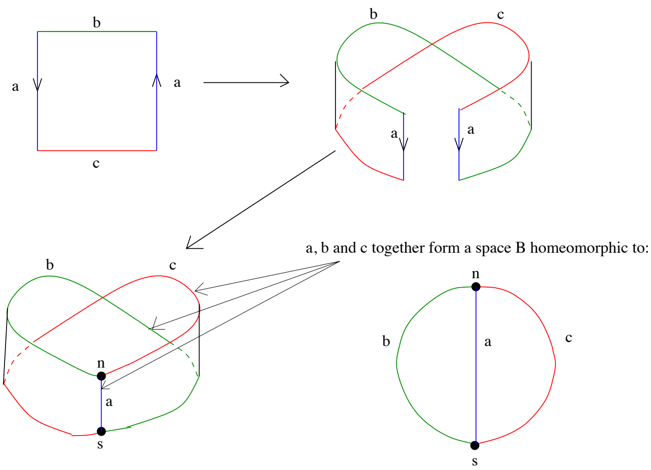


FIGURE 4.

The characteristic map can be described in symbols as:

$$\chi = caba$$

(on the picture, put the directions for b and c so that this formula is correct!). This, as a map defined on S^1, is further pictured in Figure 5.

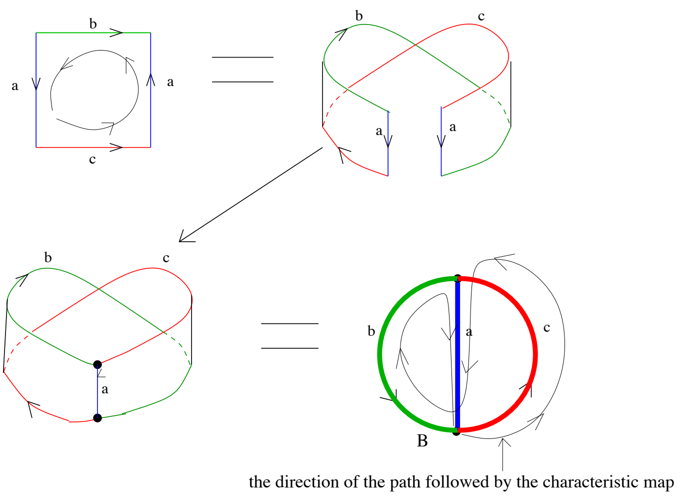


FIGURE 5.

EXERCISE 6.2. *Show that S^2 can be obtained by attaching a 2-cell to the interval $[0,1]$. Describe at least two different ways of realizing this (and explain the characteristic maps).*
(Hint: see Figure 12 and Figure 13 in Chapter 1).

EXERCISE 6.3. *Do the same for $\mathbb{P}^2$ (see Section 7 of the first chapter).*

EXERCISE 6.4. *Do the Klein bottle.*

EXERCISE 6.5. *Show that $\mathbb{P}^2$ can be obtained by attaching a 2-cell to the the Moebius band.*

EXAMPLE 6.13. Consider the sphere S^n. We already know that $S^n = D^n/\partial D^n$, i.e. S^n can be obtained from D^n by gluing all the points of $\partial D^n = S^{n-1}$ together into a single point (See Section 3 and Figure 11 in the first chapter and Example 3.24.iii in the previous chapter). Hence we can apply the previous lemma. The resulting space B clearly consists of one point only, hence S^n can be obtained from one point by adjoining an n-cell (the attaching map is, of course, just the constant map).

EXAMPLE 6.14. Consider the projective space $\mathbb{P}^n$. We recall that $\mathbb{P}^n$ "is equal to" the space $D^n/\sim$ obtained by identifying (gluing) the antipodal points on $\partial D^n = S^{n-1}$ (see e.g. Exercise 1.27 in the first chapter). Hence, we can apply the previous lemma. The space B resulting from the lemma will be $S^{n-1}/\sim$- the space obtained from S^{n-1} by gluing its antipodal points. This is just another description of the projective space and we see that B "is equal to" $\mathbb{P}^{n-1}$. In conclusion, $\mathbb{P}^n$ can be obtained from $\mathbb{P}^{n-1}$ by adjoining an n-cell.

EXERCISE 6.6. *In the previous example, "is equal to" really means that "is homeomorphic to". Via the sequence of "is equal to" that is used in the example, it appears that $\mathbb{P}^{n-1}$ is a subspace of $\mathbb{P}^n$. Write out the "equalities" (i.e. homeomorphisms) that we used to conclude that the way we see $\mathbb{P}^{n-1}$ as a subspace of $\mathbb{P}^n$ is via the canonical inclusion*

$$\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$$

which associates to a line l inside $\mathbb{R}^n$ the line inside $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ given by

$$l \times \{0\} = \{(x, 0) : x \in l\} \subset \mathbb{R}^{n+1}.$$

3. The characteristic map

Note that, using Proposition 6.6, we see that the characteristic map of e can be viewed as a map

$$\chi_e : S^{n-1} \longrightarrow A.$$

The role of the characteristic map is that "it describes the way that A and e interact inside X", or, equivalently, that it describes "the relationship between the inclusion $i : A \hookrightarrow X$ and the defining map $h : D^n \longrightarrow X$. This relationship is $h|_{S^{n-1}} = i \circ \chi_e$ which should be interpreted as a commutative diagram:

$$\begin{tikzcd} S^{n-1} \arrow[r, "\text{inclusion}"] \arrow[d, "\chi_e"'] & D^n \arrow[d, "h_e"] \\ A \arrow[r, "i"] & X \end{tikzcd}$$

But the real reason that the characteristic map χ_e is important comes from the fact that X can be reconstructed from the subspace A and the characteristic map χ_e. The aim of this section is to prove the following:

THEOREM 6.15. *For any Hausdorff space A and any continuous map $\chi : S^{n-1} \longrightarrow A$, there is a space X which is unique up to homeomorphism, with the property that X is obtained from A by adjoining an n-cell with characteristic map equal to χ.*

We first estabilish the following universal property from which we will be able to deduce unqueness by a formal argument.

Proposition 6.16. *(the universal property) Assume that X is obtained from A by adjoining an n-cell e with defining map $h_e : D^n \longrightarrow X$. Let Y be another topological space. Then a map*

$$f : X \longrightarrow Y$$

is continuous if and only if

$$f|_A : A \longrightarrow Y,\ f \circ h_e : D^n \longrightarrow Y$$

are continuous. Moreover, the correspondence

$$f \mapsto (f|_A, f \circ h_e)$$

defines a 1-1 correspondence between

- *continuous maps $f : X \longrightarrow Y$*
- *pairs (f_A, f_e) with $f_A : A \longrightarrow Y$ and $f_e : D^n \longrightarrow Y$ continuous satisfying $f_A \circ \chi_e = f_e|_{S^{n-1}}$.*

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d, "\chi_e"'] & D^n \arrow[d, "h_e"'] \arrow[ddr, "f_e"] & \\
A \arrow[r, "i"] \arrow[drr, "f_A"'] & X \arrow[dr, "f"] & \\
 & & Y
\end{tikzcd}$$

Proof. If f is continuous, it is clear that $f|_A$ and $f \circ h_e$ are continuous. For the converse, assume that $f|_A$ and $f \circ h_e$ are continuous. To show that f is continuous, we will show that, for $B \subset Y$ closed, $f^{-1}(B)$ is closed in X. Note that

$$f^{-1}(B) \cap A = (f|_A)^{-1}(B), f^{-1}(B) \cap \overline{e} = h_e((f \circ h_e)^{-1}(B)).$$

From the first equality and the continuity of $f|_A$, we deduce that $f^{-1}(B) \cap A$ is closed in A, hence (since A is closed in X), also in X. On the other hand, $(f \circ h_e)^{-1}(B)$ is closed in D^n, hence compact. Then $h_e((f \circ h_e)^{-1}(B))$ is a compact inside the Hausdorff X, hence it is closed in X. Since $f^{-1}(B)$ is the union of two closed subspaces of X (namely $f^{-1}(B) \cap A$ and $f^{-1}(B) \cap \overline{e}$), it is itself closed in X. This concludes the proof of the equivalence.

For the second part we remark that, given f_A and f_e, the conditions

$$f_A = f|_A, f_e = f \circ h_e$$

determine f uniquely because $X = A \cup \mathrm{Im}(h_e)$:

$$f(x) = f_A(x) \text{ if } x \in A, f(x) = f_e(v) \text{ if } x = h_e(v) \in \overline{e}.$$

Moreover, under the assumption $f_A \circ \chi_e = f_e|_{S^{n-1}}$, the previous formulas define f un-ambiguously: if x is both in A and of type $h_e(v)$ with $v \in D^n$, then $f_A(x) = f_e(v)$. Indeed, since $h_e(v) \in A$, v must be in S^{n-1}, hence we can write $x = \chi_e(v)$ and we can use the assumption. The continuity of the maps involved follows from the first part. $\square$

The fact that X only depends on A and χ_e is even stronger indicated by the following:

Corollary 6.17. *Let A be a Hausdorff space and let $\chi : S^{n-1} \longrightarrow A$ be a continuous map. For $i \in \{1, 2\}$, assume that X_i is a space which is obtained from A by adjoining an n-cell e_i, and let $\chi_i : S^{n-1} \longrightarrow A$ be the characteristic map of e_i. If $\chi_1 = \chi_2$, then X_1 and X_2 are homeomorphic.*

Proof. Let $\chi = \chi_1 = \chi_2$, and let h_i be the maps defining the n-cell e_i. We apply the last part of the previous proposition to $X = X_1$, $Y = X_2$ and to the pair (i_2, h_2), where $i_2 : A \longrightarrow X_2$ is the inclusion. We find that there is one and only one continuous map $f_{1,2} : X_1 \longrightarrow X_2$ continuous such that

$$f_{1,2} i_1 = i_2, f_{2,1} h_1 = h_2.$$

These conditions mean that: for $a \in A$, $f_{1,2}(a) = a$, while for $y \in h_1(D^n)$, writing $y = h_1(x)$, $f_{1,2}(y) = h_2(x)$. Since $X_1 = A \cup h_{e_1}(D^n)$, these formulas do define $f_{1,2}$ uniquely, but what the universal property is telling us is that $f_{1,2}$ is well defined and it is continuous. Of course, this could have been checked directly (and that is precisely what we have done when proving the universal property), but the hole point of the "universal property philosophy" is that everything can be done by using this property alone (and not the fact that X_i is obtained from A by adjoining and n-cell).

Leaving aside the "philosophical arguments", and choosing one of the two (rather equivalent) ways of defining the map $f_{1,2}$, we end up with our continuous map $f_{1,2} : X_1 \longrightarrow X_2$. Similarly (by exchanging the role of X_1 and X_2), we obtain a map $f_{2,1} : X_1 \longrightarrow X_2$. Our maps fit into the following diagram:

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d, "\chi_e"'] & D^n \arrow[d, "h_1"'] \arrow[ddr, "h_2"] \arrow[dddrr, "h_1"] & & \\
A \arrow[r, "i_1"] \arrow[drr, "i_2"] \arrow[ddrrr, "i_1"'] & X_1 \arrow[dr, "f_{1,2}"] & & \\
& & X_2 \arrow[dr, "f_{2,1}"] & \\
& & & X_1
\end{tikzcd}$$

We claim that $f_{1,2}$ and $f_{2,1}$ are inverse to each other, i.e. $f_{1,2}f_{2,1} = \mathrm{Id}_{X_1}$ and $f_{2,1}f_{1,2} = Id_{X_2}$. If we choose to use the explicit description of $f_{1,2}$ and $f_{2,1}$, this is immediate and concludes the proof. Alternatively (but equivalently!), these relations are, again, a consequence of the universal property. For instance, using the formulas that $f_{1,2}$ and $f_{2,1}$ satisfy (by their definition via the universal property), we deduce that $f := f_{1,2}f_{2,1} : X_1 \longrightarrow X_2$ satisfies:

$$f \circ i_1 = f_{2,1} \circ i_2 = i_1 = \mathrm{Id}_{X_1} \circ i_1,$$

$$f \circ h_{e_1} = f_{2,1} \circ h_{e_2} = h_{e_1} = \mathrm{Id}_{X_1} \circ h_{e_1}.$$

Using now the universal property for $X = X_1$, $Y = X_1$ (the uniqueness part!), we deduce that $f = \mathrm{Id}_{X_1}$. Similarly, $f_{2,1}f_{1,2} = \mathrm{Id}_{X_2}$. This proves that X_1 and X_2 are homeomorphic (... for any two spaces X_1 and X_2 which satisfy the universal property). $\square$

Finally, we show how to reconstruct X from the subspace A and the characteristic map χ_e. So, let's start with a Hausdorff topological space A and a continuous map

$$\chi : S^{n-1} \longrightarrow A.$$

We construct our space in two steps.

- We consider the disjoint union
$$A \coprod D^n.$$
As a set, it is the union of disjoint copies of A and D^n. Any subset of $A \coprod D^n$ is of type $U \coprod V$ with $U \subset A$ and $V \subset D^n$. On $A \coprod D^n$ we consider the topology consisting of those subsets $U \coprod V$ with U open in A and V open in D^n.
- We consider the quotient of $A \coprod D^n$ obtained by identifying each point $x \in S^{n-1}$ with $\chi(x) \in A$; we denote by $A \cup_\chi D^n$ the result. Hence
$$A \cup_\chi D^n = A \coprod D^n / \sim,$$
where $\sim$ is the smallest equivalence relation in $A \coprod D^n$ with the property that
$$x \sim \chi(x), \quad \forall\, x \in S^{n-1}.$$

Remark 6.18. Note that the construction of X is inspired by the universal property; actually, X is constructed by "brute force", forcing the universal property to hold. To explain this, we remind that maps out of X (i.e. defined on X) should correspond to certain pairs of maps, one out of A and one out of D^n. The first step (taking the disjoint union of A and D^n) takes care of this property. However, the inclusion of A into the disjoint union, and the inclusion of D^n into the disjoint union (a map out of A together with a map out of D^n!) do not satisfy the formula appearing in the universal property (for

$x \in S^{n-1}$, x itself and $\chi(x) \in A$ are distinct elements in the disjoint union!). The second step in our construction forces the desired formula (by identifying x with $\chi(x)$).

Some more explanations now.

- First about A. In the first step, A is a subspace of $A \coprod D^n$. Passing to the quotient $A \cup_\chi D^n$, note that the equivalence class of an element a coming from A can be safely denoted by the same symbol a (two elements coming from A are equivalent only if they are equal!). In this way we view A as a *subset* of $A \cup_\chi D^n$.
- Now D^n. As before, D^n is a subspace of $A \coprod D^n$. Passing to the quotient, we obtain a map which associates to x the equivalence class of x, and which we denote by
$$h : D^n \longrightarrow A \cup_\chi D^n.$$
In other words, h is the restriction of the quotient map $\pi : A \coprod D^n \longrightarrow A \cup_\chi D^n$ to D^n. The map h will be the defining map for our n-cell.
- In particular, we define the n-cell
$$e := h(\overset{\circ}{D}{}^n) \subset A \cup_\chi D^n.$$

Exercise 6.7. Show that A is a subspace of $A \cup_\chi D^n$, i.e. the topology induced on A (the restricted topology) coincides with the original topology of A.

Exercise 6.8. Show that the restriction of h to $\overset{\circ}{D}{}^n$ defines a homeomorphism between $\overset{\circ}{D}{}^n$ and e.

Exercise 6.9. We now know that $A \cup_\chi D^n$ is obtained from A by adjoining the n-cell e. Check that the characteristic map of e coincides with χ.

Exercise 6.10. Haven't we forgotten something? Prove that $A \cup_\chi D^n$ is Hausdorff.

Putting together everything, we deduce Theorem 6.15.

4. Adjoining more cells; cell complexes

Definition 6.19. *Let X be a Hausdorff space and let $A \subset X$ be closed. We say that X is obtained from A by adjoining n-cells if we are given n-cells $e_i \subset X$ (with i running in an index set I) such that*
$$X = A \cup (\cup_{i\in I} e_i)$$
and the following conditions are satisfied:

1. *$A \cap e_i = \emptyset$ for all i, $\partial_{cell}(e) \subset A$ and $e_i \cap e_j = \emptyset$ for all $i \neq j$.*
2. *$F \subset X$ is closed if and only if $F \cap A$ is closed, and $F \cap \overline{e_i}$-closed for all $i \in I$.*

Remark 6.20. It follows that

- e_i is open in X (for this one uses the last condition for $F = X - e_i$).
- since e_i's are open and $e_i \cap e_j = \emptyset$ for $i \neq j$, we deduce that $\overline{e}_i \cap e_j = \emptyset$ for all $i \neq j$ (indeed, $X - e_j$ will be a closed set containing e_i, hence also its closure).

Hence the characteristic maps χ_i of e_i will be continuous maps
$$\chi_i : S^{n-1} \longrightarrow A.$$
As in the case of one n-cell, the characteristic maps together with A determine X uniquely (the arguments we have presented before extend to an arbitrary number of cells without much trouble).

Definition 6.21. *Let X be a Hausdorff space. A* <u>cell decomposition</u> *of X is a sequence of closed subspaces*
$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset \dots$$
with:

- $\cup_n X_n = X$.
- *For each $n \geq 0$ integer, X_n is obtained from X_{n-1} by attaching n-cells (the n-cells, together with the defining maps, are part of the structure!).*
- *A subset $F \subset X$ is closed if and only if $F \cap X_n$ is closed for all $n \geq 0$.*

The space X together with a cell decomposition is called a cellular space (note that the cells themselves, together with their defining maps, are viewed as part of the structure). The subspace X^n is called n-skeleton of X.

Given a cell complex X which is compact, define the Euler number of the cell complex X as

$$\chi(X) = \#(0 - cells) - \#(1 - cells) + \#(2 - cells) - \ldots.$$

REMARK 6.22. Hence, for each n, X_n is obtained from X_{n-1} by adjoining a family of n-cells:

$$\{e_i^n : i \in I_n\}$$

(I_n- a set indexing the n-cells). Note that

$$X_0 = \{e_i^0 : i \in I_0\},$$

is a closed set of points in X,

$$X = \cup_{n, i \in I_n} e_i^n,$$

and each e_i^n is open in X_n, but not in X.

The following exercise explains the compactness condition we added when defining the Euler number of a cell complex.

EXERCISE 6.11. *Let X be a cell complex. Show that the total number of cells is finite if and only if X is compact.*

EXAMPLE 6.23. Given a topological space X, it may admit many different cell decompositions. For instance, for $X = S^1$, we know that S^1 can be obtained from a point by adjoining a 1-cell, i.e. it has a cell decomposition with one 0-cell and one 1-cell. However, it can also has a cell decomposition with two 0-cells and two 1-cells, or one with three 0-cells and three 1-cells, etc (see Figure 6).

We would also like to mention here that, although the definition of $\chi(X)$ uses a cell-decomposition of X, it does not depend on the cell-decomposition one uses, but only X as a topological space X. This is a non-trivial result of algebraic topology which will not be proven in this course.

EXAMPLE 6.24. Here is a cell-decomposition of $\mathbb{R}$ (figure 7) the zero cells are the integers, while the one cells are

$$e_n = (n, n+1),$$

one for each integer n, with defining map

$$h_n : D^1 = [-1, 1] \longrightarrow \mathbb{R}, t \mapsto \frac{t + 2n + 1}{2}.$$

EXERCISE 6.12. *Deduce Euler's formula: for any polyhedra, $V - E + F = 2$ where V = number of vertices, E = number of edges, F = number of faces.*

EXERCISE 6.13. *Describe a cell-decomposition of the plane $\mathbb{R}^2$.*

EXERCISE 6.14. *Describe a cell decomposition and compute the Euler number of the space X consisting of two circles joined by a line (Figure 8). Do the same for the space drawn in Figure 9.*

And also for the space consisting of the sides and the diagonal of a square (the left hand side of Figure 10).

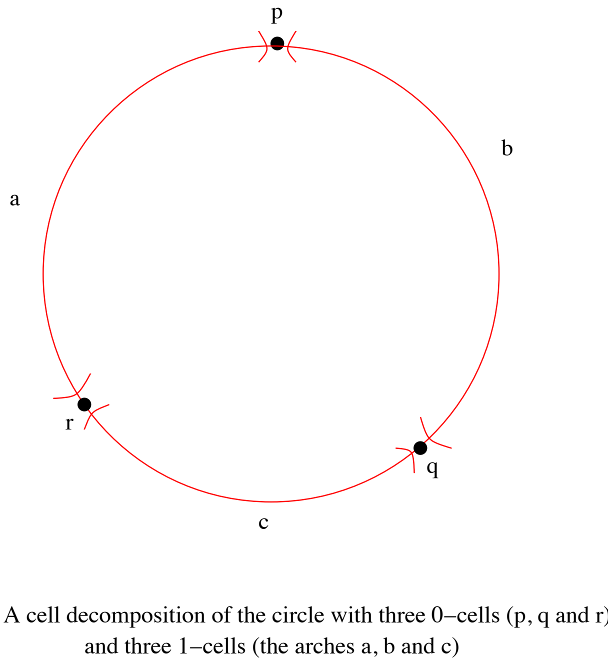


FIGURE 6.

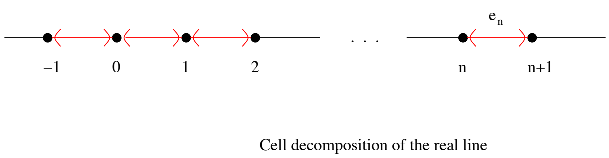


FIGURE 7.

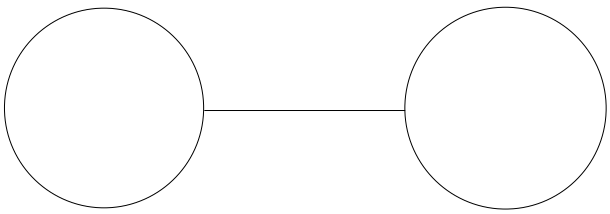


FIGURE 8.

EXERCISE 6.15. *Describe a cell decomposition of the cylinder, $C = S^1 \times [0, 1]$.*

EXAMPLE 6.25. Consider B to be a bouquet of two circles (two unit circles touching each other in one point). It is obtained from a point (the common point) by attaching two 1-cells (the connected components of the space that remains after removing the point).

On the other hand, we know that by (suitably) attaching a 2-cell to X one ends up with the torus T^2 (see Example 6.11). Hence T^2 has a cell decomposition with one 0-cell (the point), two 1-cells, and a 2-cell (see Figure 11). Symbolically, we write

$$T^2 = c_0 \cup c_1 \cup c_1 \cup c_2.$$

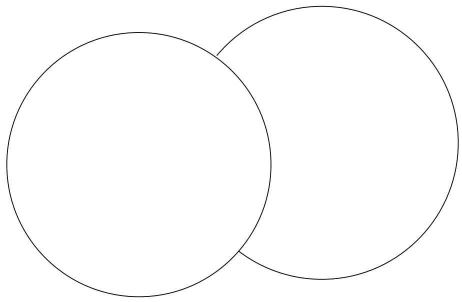


FIGURE 9.

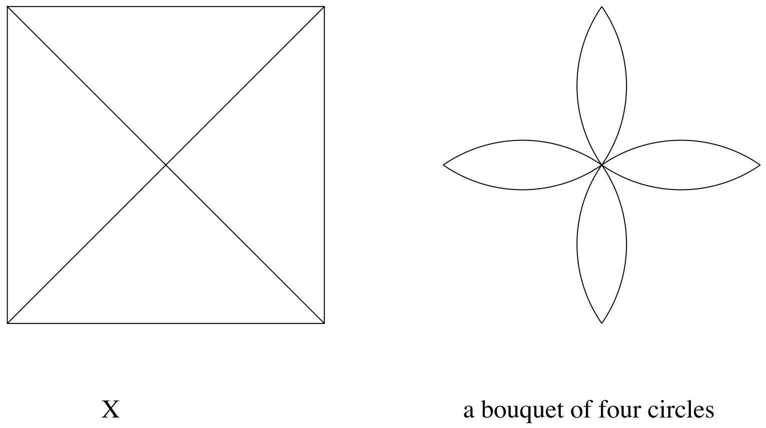


FIGURE 10.

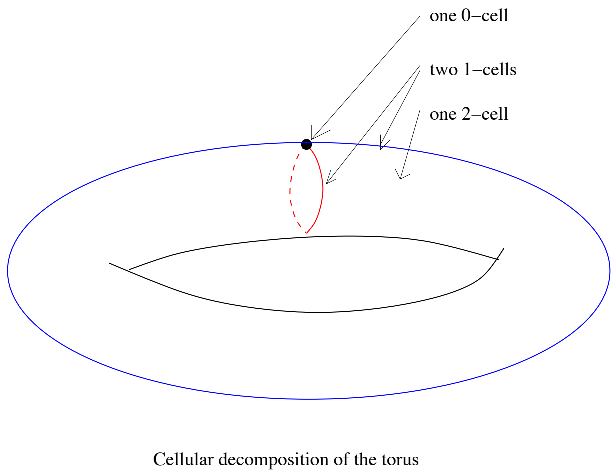
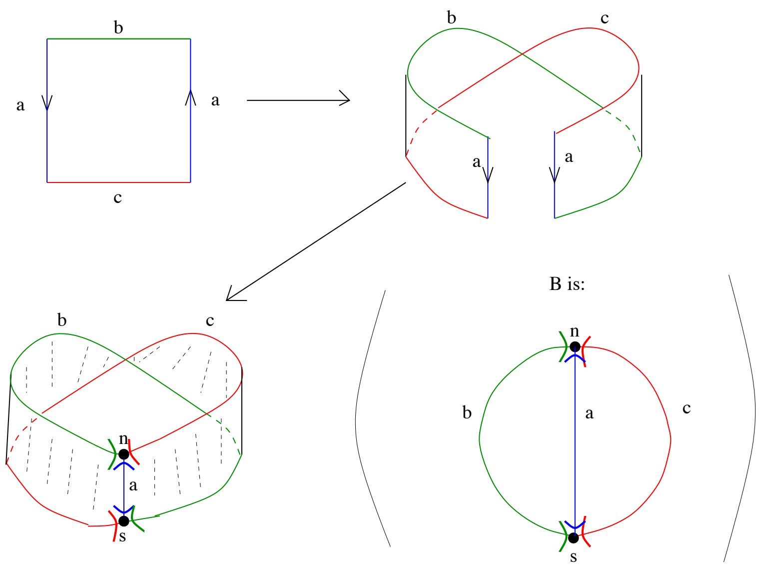


FIGURE 11.

EXAMPLE 6.26. Consider the space B which, after adjoining a 2-cell, gives the Moebius band M (see Example 6.12). This is obtained the south and the north pole by adjoining three 1-cells, as shown in Figure 12: the 0-cells are denoted n and s there, while the 1-cells are a, b and c. Hence the Moebius band has a cell decomposition with two 0-cells, three 1-cells and one 2-cell (see the picture):

$$M = c_0 \cup c_0 \cup c_1 \cup c_1 \cup c_1 \cup c_2.$$



Moebius band: two 0–cells (n and s), three 1–cells (a, b and c), one 2–cell.

FIGURE 12.

EXERCISE 6.16. *Do the Klein bottle?*

EXERCISE 6.17. *Show that T^2 and S^2 are not homeomorphic. Do the same for S^2 and S^3.*

EXERCISE 6.18. *For $\mathbb{P}^2$:*

(i) *describe a cell decomposition of type*

$$c_0 \cup c_0 \cup c_1 \cup c_1 \cup c_1 \cup c_2 \cup c_2.$$

(ii) *can you find one of type*

$$c_0 \cup c_0 \cup c_0 \cup c_1 \cup c_1 \cup c_1 \cup c_1 \cup c_2 \cup c_2 \cup c_2?$$

(Hint for (i): Moebius).

EXERCISE 6.19. *Find a cell decomposition of the annulus*

$$A(R, r) = \{(x, y) \in \mathbb{R}^2 : r^2 \leq x^2 + y^2 \leq R^2\}.$$

Then generalize this to arbitrary dimensions.

EXERCISE 6.20. *Show that the annulus $A(R, r)$ is not homeomorphic to the closed disk D^2.*

EXERCISE 6.21. *Consider the spaces from Figure 13.*

(a) *For each of these spaces, describe a cell decomposition (indicate it on the picture).*
(b) *Which of these spaces are homeomorphic among them? Explain your answer.*

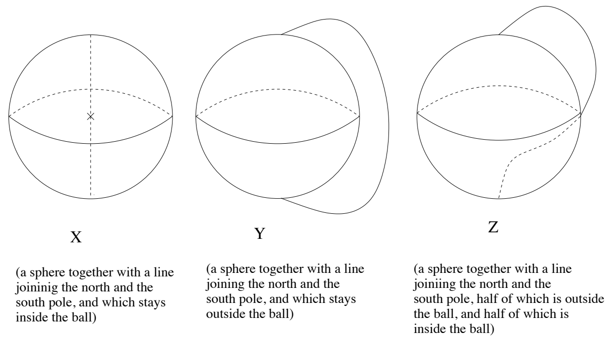


FIGURE 13.

EXAMPLE 6.27. From example 6.13 we know that

$$S^n = c_0 \cup c_n$$

(i.e. the n-sphere has a cell decomposition with one 0-cell and one n-cell). This can be pictured as shown in the first picture in Figure 14. However, there is yet another interesting cell decomposition of S^n. We already know that the two (open) semi-spheres S^n_+ and S^n_- are n-cells, hence S^n can be obtained from S^{n-1} by adjoining two n-cells (see Figure 14).

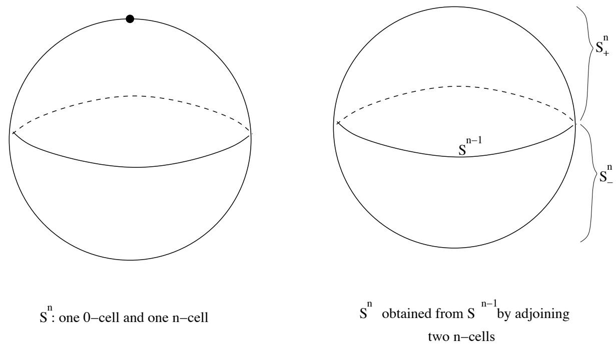


FIGURE 14.

One could now use the previous cell decomposition applied to S^{n-1} to deduce that

$$S^n = c_0 \cup c_{n-1} \cup c_n \cup c_n.$$

Alternatively, one could iterate our argument and deduce that S^n can be obtained from S^0 (two points!) by adjoining two 1-cells, and then two 2-cells, etc, and at the end two n-cells:

$$S^n = c_0 \cup c_0 \cup c_1 \cup c_1 \cup \ldots \cup c_n \cup c_n.$$

EXAMPLE 6.28. The way we expressed S^n as obtained from S^{n-1} by adjoining two n-cells (previous example) is interesting when looking at the projective space $\mathbb{P}^n$. Realizing $\mathbb{P}^n$ as $S^n/\mathbb{Z}_2$, note that the two cells will be identified in the quotient, and will define a single n-cell in $\mathbb{P}^n$. Since S^{n-1} goes in the quotient to a copy of $\mathbb{P}^{n-1}$, we deduce that $\mathbb{P}^n$ can be obtained from $\mathbb{P}^{n-1}$ by adjoining an n-cell. One can check that this coincides with the decomposition that we have discussed in Example 6.14- this is a matter of unraveling the definitions (exercise).

EXAMPLE 6.29. Since $\mathbb{P}^n$ can be obtained from $\mathbb{P}^{n-1}$ by adjoining and n-cell, continuing inductively, we find a cell decomposition of $\mathbb{P}^n$ of type

$$\mathbb{P}^n = c_0 \cup c_1 \cup \ldots \cup c_n.$$

EXERCISE 6.22. *Consider the square* $X = [0,1] \times [0,1]$ *and let* A *be the subset consisting of its corners:*

$$A = \{(0,0), (0,1), (1,0), (1,1)\}.$$

Consider the quotient space:

$$Y = X/A$$

(with the induced quotient topology).

(i) *Make a picture of* Y.
(ii) *PROVE that, indeed,* Y *can be embedded in* $\mathbb{R}^3$.
(iii) *Describe a cell-decomposition of* Y.

EXERCISE 6.23. *Draw a picture, describe a cell decomposition and compute the Euler number of the space* X *which is the one point compactification of the space consisting of the* xOy *plane together with the upper half plane of the unit sphere (Figure 15).*

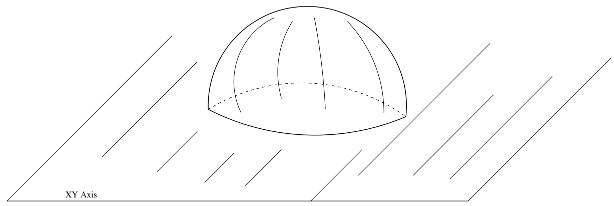


FIGURE 15.

EXERCISE 6.24. *Draw a picture, describe a cell decomposition and compute the Euler number of the space* X *which is the one point compactification of a plane with a handle (i.e. the* XOY *plane together with a segment whose end points belong to the plane, but which does not intersect the plane in any other point). (Figure 16).*

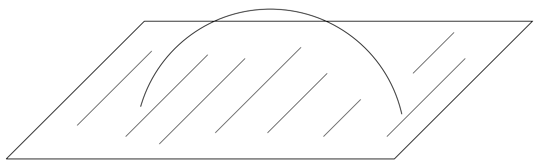


FIGURE 16.

EXERCISE 6.25. *Draw a picture, describe a cell decompositionand compute the Euler number of the space* X *which is the one point compactification of the space consisting of the* xOy *plane together with the upper half of the* Z *axis (Figure 17).*

EXERCISE 6.26. *Consider an octogon* X *in the plane and the equivalence relation* $\sim$ *on* X *which identifies the sides of* ∂X *as shown in the Figure 18.*

(1) *Show that* $\partial X/\sim$ *is homeomorohic to a bouquet* $S^1 \vee S^1 \vee S^1 \vee S^1$ *of four circles.*
(2) *Show that* $X/\sim$ *us homeomorphic to the double torus* T_2. *(Figure 19).*
(3) *Describe a cell decomposition of the double torus* T_2.
(4) *Compute the Euler number of the double torus.*

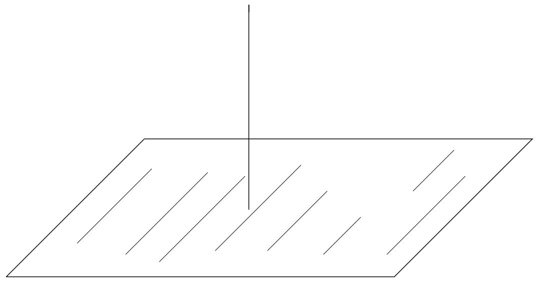


FIGURE 17.

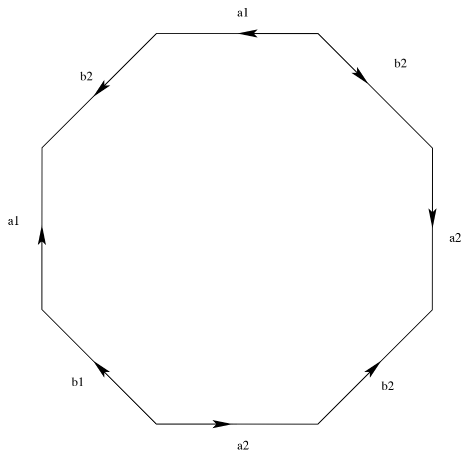


FIGURE 18.

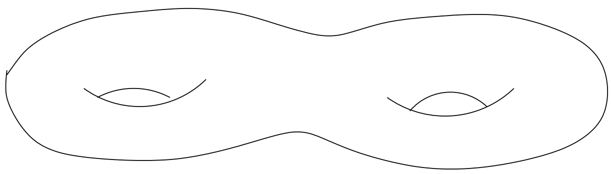


FIGURE 19.

EXERCISE 6.27. *Consider the unit circle*

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

Let $n \geq 1$ *integer and define*

$$f_n : S^1 \to S^1, f_n(z) = z^n.$$

Let X_n *be the space obtained from* S^1 *by attaching a two cell with characteristic map* f_n. *Describe* X_n *as a quotient of* D^2. *Do you recognize* X_2*?*

CHAPTER 7

The fundamental group

1. Homotopies and homotopy equivalences

DEFINITION 7.1. *Let X and Y be topological spaces. We denote by* <u>$\mathrm{Cont}(X,Y)$</u> *the set of all continuous maps from X to Y. Given $f_0, f_1 \in \mathrm{Cont}(X,Y)$, a* <u>*homotopy*</u> *between f_0 and f_1 is a continuous map*

$$H : X \times [0,1] \longrightarrow Y$$

with the property that

$$H(x,0) = f_0(x),\ H(x,1) = f_1(x),$$

for all $x \in X$. We say that f_0 and f_1 are <u>*homotopic*</u>*, and we write*

$$f_0 \sim f_1,$$

if there exists a homotopy H between f_0 and f_1.

REMARK 7.2. A homotopy between f_0 and f_1 should be viewed as a "continuous deformation" of f_0 into f_1: the homotopy H defines for each $t \in [0,1]$ a continuous map $H_t \in \mathrm{Cont}(X,Y)$,

$$H_t : X \longrightarrow Y,\ x \mapsto H(t,x),$$

and the family $\{H_t\}$ goes from $H_0 = f_0$ to $H_1 = f_1$.

PROPOSITION 7.3. *Given topological spaces X, Y, Z,*

(i) *The homotopy relation $\sim$ is an equivalence relation on* $\mathrm{Cont}(X,Y)$.
(ii) *If $f_0, f_1 \in \mathrm{Cont}(Y,Z)$ are homotopic, and $g_0, g_1 \in \mathrm{Cont}(X,Y)$ are homotopic, then f_0g_0 and f_1g_1 are homotopic.*

PROOF. We first check (i). For any $f : X \longrightarrow Y$ we have $f \sim f$ via the homotopy $h(x,t) = f(x)$. If $f \sim g$ via the homotopy H, then $g \sim f$ via the homotopy H^- defined by $H^-(x,t) = H(x, 1-t)$. Assume now that $f \sim g$ via a homotopy H and $g \sim h$ via a homotopy H'. Then we define H'' by

$$H''(x,t) = \begin{cases} H(x,2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ H'(x,2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases} .$$

Clearly, it satisfies $H''_0 = f$, $H''_1 = h$ (why is it continuous?). □

EXAMPLE 7.4. If $Y = \mathbb{R}^n$, any two continuous maps $f_0, f_1 : X \longrightarrow \mathbb{R}^n$ are homotopic. Indeed,

$$H(x,t) = (1-t)f_0(x) + tf_1(x)$$

is a homotopy between f_0 and f_1. The same happens if $\mathbb{R}^n$ is replaced by any $C \subset \mathbb{R}^n$ which is convex (i.e. $tx + (1-t)y \in C$ for all $x, y \in C$ and all $t \in [0,1]$).

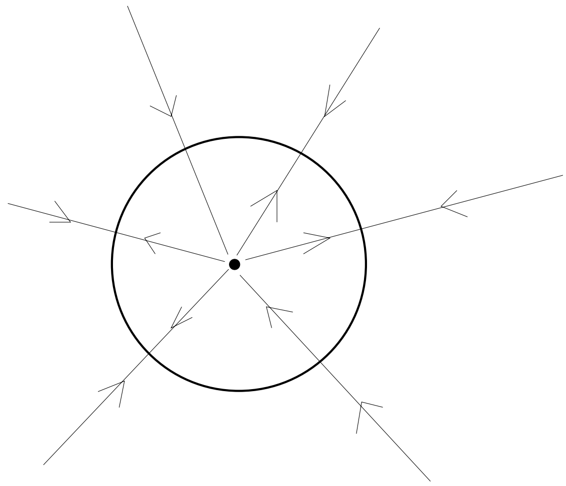
EXAMPLE 7.5. The map

$$f : \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}^2 - \{0\}, x \mapsto \frac{x}{||x||}$$

is homotopic to the identity map. For this we consider

$$H(x,t) = (1-t)\frac{x}{||x||} + tx$$

(which takes values in $\mathbb{R}^2 - \{0\}$!). See Figure 1.



$\mathbb{R}^2$ minus a point is homotopic equivalent to the circle

FIGURE 1.

DEFINITION 7.6. *We say that a continuous function $f : X \longrightarrow Y$ is a* <u>*homotopy equivalence*</u> *if there exists a continuous function $g : Y \longrightarrow X$ such that*

$$g \circ f \sim Id_X, \ f \circ g \sim Id_Y.$$

The map g will be called a <u>*homotopy inverse*</u> *of f. Given two topological spaces X and Y, we say that they are* <u>*homotopic equivalent*</u> *if there exists a homotopy equivalence $f : X \longrightarrow Y$.*

We say that a space X is <u>*contractible*</u> *if it is homotopic equivalent to the one-point space (a topological space with only one point).*

EXAMPLE 7.7. Example 7.4 implies that $\mathbb{R}^n$ is contractible. Indeed, taking an arbitrary $X = \{x_0\}$ with $x_0 \in \mathbb{R}^n$ arbitrary, $f : \mathbb{R}^n \longrightarrow X$ the constant map and $g : X \longrightarrow \mathbb{R}^n$ the inclusion, $f \circ g$ is the identity, while $g \circ f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is homotopic to (any other map hence also to) the identity. The same applies to any convex subset $C \subset \mathbb{R}^n$.

EXERCISE 7.1. *Give an example of a subspace $C \subset \mathbb{R}^n$ which is not convex but is contractible.*

Regarding contractibility, one has the following characterization.

EXERCISE 7.2. *For a topological space C, show that the following are equivalent:*

(i) *C is contractible.*
(ii) *The identity map $Id_C : C \longrightarrow C$ is homotopic to a constant map.*
(iii) *For any other topological space X, any two continuous maps $f, g : X \longrightarrow C$ are homotopic.*

EXAMPLE 7.8. Similar to the previous example, $\mathbb{R}^2 - \{0\}$ is homotopic equivalent to S^1. To see this, we consider the inclusion $i : S^1 \longrightarrow \mathbb{R}^2$ and the map

$$r : \mathbb{R}^2 - \{0\} \longrightarrow S^1, r(x) = \frac{x}{||x||}.$$

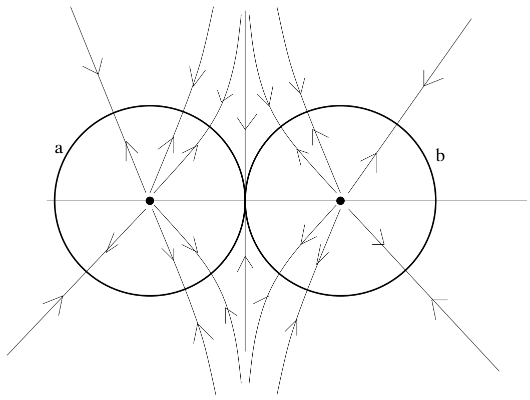
Clearly, $r \circ i = \mathrm{Id}_{S^1}$. We still have to check that

$$i \circ r : \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}^2 - \{0\},$$

is homotopic to the identity map, but this is precisely the map from Example 7.5.

Note that the same argument shows that $D^2 - \{0\}$ is homotopic equivalent to S^1. Similarly, $\mathbb{R}^n - \{0\}$ and to $D^n - \{0\}$ are homotopic equivalent to S^{n-1} for any $n \geq 1$ integer, with the homotopy equivalences given by the inclusion of S^{n-1}.

EXAMPLE 7.9. Similarly, the space obtained by removing two points from $\mathbb{R}^2$ is homotopic equivalent to a bouquet of two circles, as indicated in Figure 2.



$\mathbb{R}^2$ minus two points is homotopic equivalent to a bouquet of two circles

FIGURE 2.

EXERCISE 7.3. *Show that the cylinder $C = S^1 \times [0, 1]$ is homotopic equivalent to a circle. Similarly, the Moebius band is homotopic equivalent to a circle. Try to write down the explicit formulas.*

EXERCISE 7.4. *More generally, show that for any space X, the associated cylinder $Cyl(X)$ is homotopic equivalent to X.*

EXERCISE 7.5. *Show that, for any space X, the cone of X, $C(X)$, is contractible.*

EXERCISE 7.6. *Show that the spaces in Figure 3 are homotopic equivalent.*

EXERCISE 7.7. *Go back to Exercise 6.14. Show that the spaces from Figure 8 and Figure 9 are both homotopic equivalent to a bouquet of two circles. Then show that the two spaces drawn in Figure 10 are homtopic equivalent.*

EXERCISE 7.8. *Consider the cylinder $C = S^1 \times [0, 1]$. Show that*

(1) *For any $p = (x, t) \in C$ with $t \notin \{0, 1\}$, $C - \{p\}$ is homotopic equivalent to $S^1 \vee S^1$.*
(2) *For any $p = (x, t) \in C$ with $t \in \{0, 1\}$, $C - \{p\}$ is homotopic equivalent to S^1.*

EXERCISE 7.9. *Show that, after removing two points from the sphere, the resulting space is homotopic equivalent to a circle.*

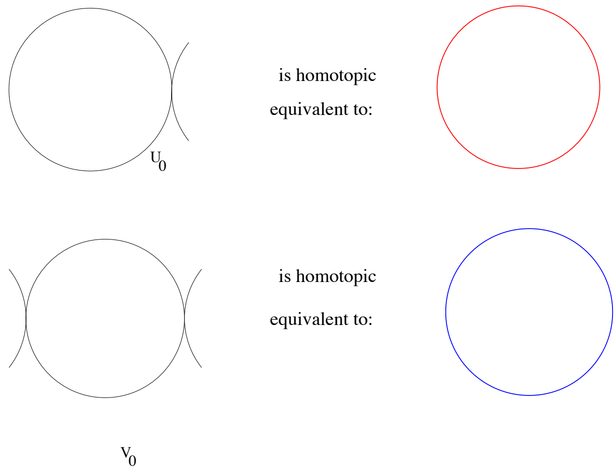


FIGURE 3.

EXAMPLE 7.10. Similar to Example 7.8, the space obtained from the square by removing one point from its interior is homotopic equivalent to its boundary, hence (also) to S^1. See Figure 4. This also follows directly from Example 7.8 because the square is homeomorphic to the disk. More generally, for any convex subspace $X \subset \mathbb{R}^n$ and any interior point $x_0 \in \overset{\circ}{X}$, one can prove

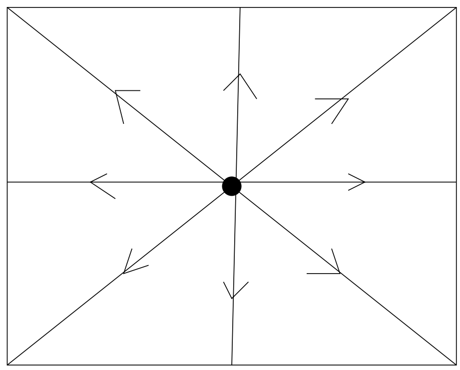


FIGURE 4.

that $X - \{x_0\}$ is homotopic equivalent to S^{n-1}.

EXERCISE 7.10. *Show that the space after removing three points from the plane is homotopic equivalent to a bouquet of three circles.*

EXAMPLE 7.11. The torus minus a point is homotopic equivalent to a bouquet of two circles. To see this, recall first that the torus can be obtained from the square by gluing its opposite sides. Hence the torus minus a point can be obtained from the same procedure, but starting with X which a square minus a point in its interior. We have remarked that this last space is homotopic equivalent to a circle. Onthe other hand, this homotopy equivalence preserves the boundary of the square, hence it induces a homotopy equivalence between the space X/R (X is the square minus a point, and R is the equivalence relation encoding the gluing) and S^1/R- the space obtained from S^1 by performing the induced gluings. The result is $S^1 \vee S^1$. See Figure 5.

EXERCISE 7.11. *What if we remove a small disk from the torus?*

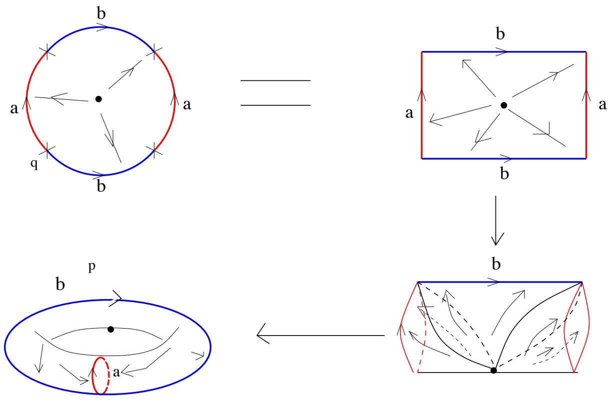


FIGURE 5.

EXERCISE 7.12. *Prove that the space obtained by removing a point from* $\mathbb{P}^2$ *is homotopic equivalent to a circle.*

EXERCISE 7.13. *What happens for the Moebius band?*

More generally, one has the following.

EXERCISE 7.14. *Assume that* X *is obtained from* A *by adjoining an* n*-cell. Prove that, for any point* $x \in X - A$*, the inclusion*

$$i : A \longrightarrow X - \{x\}$$

is a homotopy equivalence. Write down the complete <u>*proof*</u> *carefully.*

EXERCISE 7.15. *Go back to Exercise 6.21. Which one of the spaces there do you think is homtopic equivalent to the space* W *from Figure 6?*

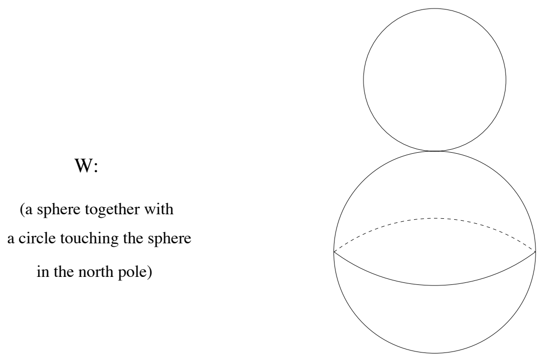


FIGURE 6.

REMARK 7.12 ((The homotopy category)). Some of the constructions here can be nicely interpreted using the language of "category theory". Precisely, a category $\mathcal{C}$ consists of

- certain objects of $\mathcal{C}$, which we will denote by capital letters (e.g. X, A, etc).
- for any two objects X and Y, a set of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ which are called morphisms (or arrows) from X to Y, and which we will denote by small letters. Given an arrow $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, we also write $f : X \longrightarrow Y$.

- for any three objects X, Y and Z, a map
$$\mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z),$$
which is called the composition of arrows, and which is denoted by $(a,b) \mapsto a \circ b$. The composition is required to satisfy:
$$(a \circ b) \circ c = a \circ (b \circ c),$$
(whenever the composition is defined).
- for any object X, there is a specified arrow $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$, called the identity arrow of X. It is required that, for any $a \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$,
$$a \circ \mathrm{Id}_X = \mathrm{Id}_Y \circ a = a.$$

Each category $\mathcal{C}$ has its own notion of isomorphisms: we say that a morphism $a \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an isomorphism if there exists $b \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ such that
$$b \circ a = \mathrm{Id}_X,\ a \circ b = \mathrm{Id}_Y.$$

Standard examples are:

(i) The category of sets has as objects the sets, the morphisms in this category between two sets X and Y are the usual functions $f : X \longrightarrow Y$, and the composition is the usual composition of functions. The isomorphisms of this category correspond to bijective functions.

(ii) The category of groups has as objects all groups, the morphisms in this category between two groups G and H are the usual group homomorphisms $f : G \longrightarrow H$, and the composition is the usual composition of maps. The isomorphisms in this category correspond to the usual group isomorphisms.

(iii) The category of topological spaces has as objects the topological spaces, the morphisms between two space X and Y are the continuous maps $f : X \longrightarrow Y$ and, again, the composition is the usual composition of maps. The isomorphisms of this category are the homeomorphisms.

Back to the previous proposition, for any two spaces X and Y we consider the set of homotopy classes of maps from X to Y:
$$[X,Y] := \mathrm{Cont}(X,Y)/\sim .$$
For $f : X \longrightarrow Y$ continuous, we denote by $[f] \in [X,Y]$ the equivalence class of f. The proposition ensures that there is a well-defined map
$$[Y,Z] \times [X,Y] \longrightarrow [X,Z], ([f],[g]) \mapsto [f] \circ [g] := [f \circ g]$$
What we have really done was to define a new category, called <u>the homotopy category of spaces</u>: it has as objects all topological spaces, while the set of morphisms from X to Y is $[X,Y]$ with the composition that we have just described. In this language, two spaces are homotopic equivalent if and only if they are isomorphic in the homotopy category,

2. Path homotopies

Recall that a path in a topological space X is a continuous map
$$\gamma : [0,1] \longrightarrow X.$$
The point $x = \gamma(0)$ is called the initial (or start) point of γ, while the point $y = \gamma(1)$ is called the final (or end) point of γ. We also say that γ is a path (in X) from x to y. We denote by
$$P(X,x,y)$$
the set of all paths in X from x to y.

Definition 7.13. *Given a space X, $x, y \in X$, we say that two paths*
$$\gamma, \gamma' \in P(X,x,y)$$
are path homotopic, and we write
$$\gamma \sim_p \gamma'$$

if there exists a homotopy H between γ and γ' such that each $H_t(-) = H(-,t) : [0,1] \longrightarrow X$ is a path from x to y.

We say that a path $\gamma \in P(X,x,x)$ is null-homotopic if it is path homotopic to the constant path $c_x \in P(X,x,x)$ ($c_x(t) = t$ for all t).

REMARK 7.14. A path homotopy H should be viewed a "continuous deformation" of γ into γ' through paths from x to y. Explicitly,

$$H : [0,1] \times [0,1] \longrightarrow X$$

must satisfy

$$H(s,0) = \gamma(s), H(s,1) = \gamma'(s),$$
$$H(0,t) = x, H(1,t) = y.$$

for all $s,t \in [0,1]$.

PROPOSITION 7.15. *Let X be a space, $x,y \in X$. The path homotopy relation $\sim_p$ is an equivalence relation on $P(X,x,y)$.*

PROOF. One just remarks that all the homotopies in the proof of Proposition 7.3 are path-homotopies. □

How can one "compose" paths?

DEFINITION 7.16. *Let X be a topological space, $x,y,z \in X$. For $\gamma \in P(X,x,y)$ and $\gamma' \in P(X,y,z)$, we define the concatenation of γ and γ' as the new path $\gamma * \gamma' \in P(x,z)$ defined by*

$$(\gamma * \gamma')(s) = \begin{cases} \gamma(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \gamma'(2s-1) & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases} .$$

Note that, by the pasting lemma (Lemma **??**), $\gamma * \gamma'$ is continuous.

Note that the concatenation of paths does not behave "like a composition"- for instance, it is not associative. Things become much nicer is we pass to homotopy classes of paths.

DEFINITION 7.17. *Given a space X and $x,y \in X$, we define the set of homotopy classes of paths from x to y as*

$$\Pi(X,x,y) := P(X,x,y)/\sim_p .$$

For $\gamma \in P(X,x,y)$, the induced equivalence class is denote by $[\gamma] \in \Pi(X,x,y)$.

PROPOSITION 7.18. *Let X be a topological space. Then, for any $x,y,z \in X$, the map*

$$\Pi(X,x,y) \times \Pi(X,y,z) \longrightarrow \Pi(X,x,z),$$
$$([\gamma],[\gamma']) \mapsto [\gamma] * [\gamma'] := [\gamma * \gamma']$$

is well defined. Moreover

1. *For $x \in X$, denote by $c_x \in P(X,x,x)$ the constant path, and $1_x = [c_x] \in \Pi(X,x,x)$. Then for any $a \in \Pi(X,x,y)$,*
$$1_x * a = a * 1_y = a.$$
2. *Given $a = [\gamma] \in \Pi(X,x,y)$, define $a^{-1} \in \Pi(X,y,x)$ as follows: write $a = [\gamma]$ with $\gamma \in P(X,x,y)$ and put $a^{-1} = [\gamma^-]$ where $\gamma^- \in P(X,y,x)$ is given by $\gamma^-(t) = \gamma(1-t)$. Then a^{-1} is well-defined and*
$$a * a^{-1} = 1_x, \ a^{-1} * a = 1_y.$$
3. *The new operation $*$ is associative:*
$$(a * b) * c = a * (b * c),$$
for all $a \in \Pi(X,x,y)$, $b \in \Pi(X,y,z)$, $c \in \Pi(X,z,u)$ (with $x,y,z,u \in X$).

PROOF. We will use the following construction: for $a, b \in \mathbb{R}$, we consider the affine function $l_{a,b}$ which sends a to 0 and b to 1:

$$l_{a,b} : [a, b] \longrightarrow [0, 1], \ l_{a,b}(s) = \frac{l - a}{b - a}.$$

With the help of this, if we have a path $\gamma : [0, 1] \longrightarrow X$, and we want to reparametrize it to obtain a path defined on $[a, b]$ which has the same image as γ, we will consider

$$\gamma \circ l_{a,b} : [a, b] \longrightarrow X, \gamma_{a,b}(s) = \gamma(\frac{l - a}{b - a}).$$

For the first part of the theorem, write $a = [\gamma]$, and we have to show that $c_x * \gamma \sim_p \gamma$, $\gamma * c_y \sim_p \gamma$. We prove the first one (the second one being similar). To produce paths h_t with $h_0 = \gamma$, $h_1 = c_x * \gamma$, look first at $c_x * \gamma$:

$$(c_x * \gamma)(s) = \begin{cases} x & \text{if } 0 \leq s \leq \frac{1}{2} \\ \gamma(2s - 1) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}.$$

We will construct H_t by considering the points $p_t \in [0, 1]$ and defining $H_t|_{[0,p_t]}$ to be constant equal to x, and $H_t|_{[p_t,1]}$ to be γ reparametrized (i.e. γ_{1,p_t} described above). We need to choose p_t so that $p_0 = 0$ (so that $H_0 = \gamma$) and $p_1 = \frac{1}{2}$ (so that $H_1 = c_x * \gamma$). The simples choice is the linear one: $p_t = \frac{t}{2}$, which produces the homotopy

$$H(s, t) = H_t(s) = \begin{cases} x & \text{if } 0 \leq s \leq \frac{t}{2} \\ \gamma(\frac{2s-t}{2-t}) & \text{if } \frac{t}{2} \leq s \leq 1 \end{cases}.$$

Next, for $\gamma \in P(X, x, y)$, we have to show that $\gamma * \gamma^{-} \sim_p c_x$ and $\gamma^{-} * \gamma \sim_p c_y$. We prove the first one (the second one being similar). The intuition is the following: $(\gamma * \gamma^{-})(s)$ covers $\gamma([0, 1])$ when s covers the first half of $[0, 1]$, and then comes back covering the same path on the second half of the interval. To construct the homotopy H_t, we consider the path which, in the first half of the interval covers $\gamma([0, t])$ and then comes back. With this intuition, we define

$$H(s, t) = \begin{cases} \gamma(2st) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \gamma(2(1 - s)t) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

which realizes the desired path homotopy.

For the last part of the theorem, assume that $a = [\gamma]$, $b = [\gamma']$, $c = [\gamma'']$, and consider the paths $H_0 = (\gamma * \gamma') * \gamma''$, $H_1 = \gamma * (\gamma' * \gamma'')$. Hence

$$H_0(s) = \begin{cases} \gamma(4s) & \text{if } 0 \leq s \leq \frac{1}{4} \\ \gamma'(4s - 1) & \text{if } \frac{1}{4} \leq s \leq \frac{1}{2} \\ \gamma''(2s - 1) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases},$$

$$H_1(s) = \begin{cases} \gamma(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \gamma'(4s - 2) & \text{if } \frac{1}{2} \leq s \leq \frac{3}{4} \\ \gamma''(4s - 3) & \text{if } \frac{3}{4} \leq s \leq 1 \end{cases}.$$

Remark that both H_0 and H_1 are of the following type. One divides the interval $[0, 1]$ into three intervals, by using two numbers p and q $(0 < p < q < 1)$. On the first interval, i.e. on $[0, p]$, one consider γ reparametrized by $l_{0,p}$, (which, when s goes from 0 to p, will cover the whole path γ). Next, on $[p, q]$, one considers γ' reparametrized by $l_{p,q}$ and, on $[q, 1]$, γ'' reparametrized by $l_{q,1}$. Putting together these three pieces, we find a curve constructed out of the curves γ, γ' and γ'', and of the numbers p and q. With these, H_0 is obtained for

$$p = p_0 = \frac{1}{4}, q = q_0 = \frac{1}{2},$$

while H_1 is obtained for

$$p = p_1 = \frac{1}{2},\ q = q_1 = \frac{3}{4}.$$

We are looking for a homotopy consisting of paths H_t (between H_0 and H_1). For this we consider (see Figure 7)

$$p_t = \frac{t+1}{4},\ q_t = \frac{t+2}{4}$$

(the affine maps which have the described values $p_0, p_1,\ q_0, q_1$), and we consider the path H_t given by the procedure we have just described, applied to $p = p_t$, $q = q_t$. We find the desired homotopy

$$H(s,t) = H_t(s) = \begin{cases} \gamma(\frac{4s}{t+1}) & \text{if } 0 \le s \le \frac{t+1}{4} \\ \gamma'(4s - t - 1) & \text{if } \frac{t+1}{4} \le s \le \frac{t+2}{4} \\ \gamma''(\frac{4s-t-2}{2-t}) & \text{if } \frac{t+2}{4} \le s \le 1 \end{cases}.$$

□

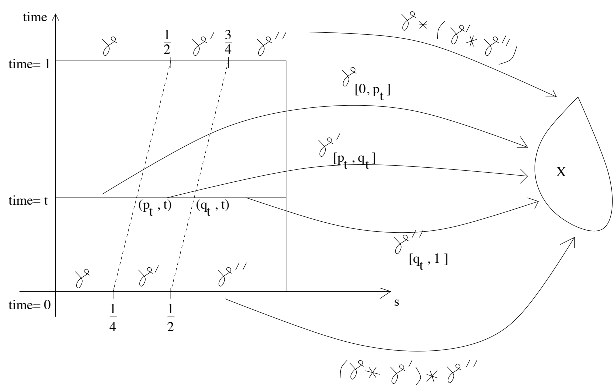


FIGURE 7.

EXERCISE 7.16. Let $\gamma : [0,1] \longrightarrow X$ be a path from x to y, and $0 = a_0 < a_1 < \ldots < a_p < a_{p+1} = 1$. For each $0 \le i \le p$ integer define

$$\gamma_i : [0,1] \longrightarrow X, \gamma_i(s) = \gamma((1-s)a_i + s a_{i+1}).$$

Show that

$$[\gamma] = [\gamma_p] * \ldots * [\gamma_1] * [\gamma_0].$$

REMARK 7.19 ((The homotopy groupoid)). The previous proposition too, should be viewed from the point of view of category theory. It says that, given the space X, there is an associated category $\Pi(X)$ with

- The objects of $\Pi(X)$ are the points of X.
- The set of morphisms from $x \in X$ to $y \in Y$ is $\Pi(X, x, y)$.
- The composition of morphisms comes from the concatenation of paths.

The last part of the proposition says that this category has one more property: any morphism is an isomorphism. Categories with this property are called groupoids. For this reason, $\Pi(X)$ is called <u>the fundamental groupoid of X</u>.

3. The fundamental group

DEFINITION 7.20. *The fundamental group of X with base point x, (X-a topological space, $x \in X$), is defined as*

$$\pi(X, x) := \Pi(X, x, x),$$

with the group structure is the one induced by the concatenation of paths.

We first discuss the functoriality of the fundamental group (what happens if we change the space?).

THEOREM 7.21. *If $f : X \longrightarrow Y$ is a continuous map and $x \in X$, then the map*

$$f_* : \pi(X, x) \longrightarrow \pi(Y, f(x)), [\gamma] \mapsto [f \circ \gamma]$$

is well defined and is a morphism of groups. If $g : Y \longrightarrow Z$ is another continuous map, then

$$g_* \circ f_* = (g \circ f)_*.$$

In particular, if $f : X \longrightarrow Y$ is a homeomorphism, then $\pi(X, x)$ is isomorphic to $\pi(Y, f(x))$, for all $x \in X$.

PROOF. The map $\gamma \mapsto f \circ \gamma$ (from paths in X to paths in Y) is compatible with path homotopies: if H is a path homotopy between γ and γ', then $f \circ f$ is a path homotopy between $f \circ \gamma$ and $f \circ \gamma'$. This implies that f_* is well defined. Next, using the definition of the concatenation of paths, we see that

$$f \circ (\gamma * \gamma') = (f \circ \gamma) * (f \circ \gamma')$$

or all $\gamma, \gamma' \in P(X, x, x)$, which implies that f_* is a group homomorphism. Given another function g, we have:

$$(g_* \circ f_*)([\gamma]) = g_*([f \circ \gamma]) = [g \circ f \circ \gamma] = (g \circ f)_*([\gamma],$$

proving that $(g_* \circ f_*) = (g \circ f)_*$. Finally, if f is a homeomorphism, denoting by $g : Y \longrightarrow X$ its inverse and by $g_* : \pi(Y, y) \longrightarrow \pi(X, g(y))$ the induced map, where $y = f(x)$, we have $g_* \circ f_* = (g \circ f)_* = (Id_X)_* = Id$, and similarly $f_* \circ g_* = Id$, proving that f_* is an isomorphism. $\square$

Next, what happens if we change the base point?

THEOREM 7.22. *Given a space X, and α a path in X from x to y, the map*

$$\widehat{\alpha} : \pi(X, x) \longrightarrow \pi(X, y),$$
$$\widehat{\alpha}([\gamma]) = [\alpha]^{-1} * [\gamma] * [\alpha]$$

is well defined and is an isomorphism of groups.

In particular, if X is path connected, then for any two points $x, y \in X$ the groups $\pi(X, x)$ and $\pi(X, y)$ are isomorphic.

PROOF. Let $a = [\alpha] \in \Pi(X, x, y)$. We have to show that

$$\widehat{a} : \pi(X, x) \longrightarrow \pi(X, y), \phi(u) = a^{-1} * \gamma * u$$

is a group isomorphism. First of all, note that if $b \in \Pi(Y, Z)$, then

$$\widehat{b} \circ \widehat{a} = \widehat{a * b}.$$

Indeed, for all $u \in \Pi(X, x, x)$,

$$\begin{aligned} (\widehat{b} \circ \widehat{a})(u) &= b^{-1} * (a^{-1} * u * a) * b \\ &= (b^{-1} * a^{-1}) * u * (a * b) \\ &= (a * b)^{-1} * u * (a * b) = \widehat{a * b}(u). \end{aligned}$$

In particular, this implies that $\widehat{a}$ is bijective with inverse $\widehat{b}$ with $b = a^{-1}$. Next, $\widehat{a}$ is a group homomorphism:

$$\begin{aligned}\widehat{a}(u) * \widehat{a}(v) &= (a^{-1} * u * a) * (a^{-1} * v * a) \\ &= a^{-1} * u * (a * a^{-1}) * v * a \\ &= a^{-1} * u * 1_y * v * a \\ &= a^{-1} * (u * v) * a = \widehat{a}(u * v).\end{aligned} \tag{3.1}$$

When X is path connected, then for any two points $x, y \in X$ we can find a path γ from x to y, and then $\hat{\gamma}$ will provide an isomorphism between $\pi(X, x)$ and $\pi(X, y)$. $\square$

Finally, we show that two spaces which are homotopic equivalent (Definition 7.1) have isomorphic fundamental groups.

THEOREM 7.23. *If $f : X \longrightarrow Y$ is a homotopy equivalence then, for any $x_0 \in X$, the induced map*

$$f_* : \pi(X, x_0) \longrightarrow \pi(Y, f(x_0))$$

is an isomorphism of groups.

PROOF. Let $g : Y \longrightarrow X$ be a homotopy inverse of f (see Definition 7.1), $x_0 \in X$, and the maps

$$f_* : \pi(X, x_0) \longrightarrow \pi(Y, f(x_0)), \;\; g_* : \pi(Y, f(x_0)) \longrightarrow \pi(X, g(f(x_0)))$$

the maps induced in the fundamental groups. It suffices to show that $g_* f_*$ and $f_* g_*$ are isomorphism. Indeed, the injectivity of $g_* f_*$ implies that f_* is injective, while the surjectivity of $f_* g_*$ implies that f_* is surjective (show this!).

Due to the symmetry, it suffices to show that $g_* f_* = (gf)_*$ is bijective. Let $h = gf$. We are in the following situation: we have a continuous function $h : X \longrightarrow X$ which is homotopic to the identity map, $x_0 \in X$, and we want to prove that

$$h_* : \pi(X, x_0) \longrightarrow \pi(X, h(x_0))$$

is an isomorphism. Since h is homotopic to Id_X, we find

$$H : X \times [0, 1] \longrightarrow X$$

such that $H(x, 0) = x$, $H(x, 1) = h(x)$ for all $x \in X$. Consider

$$\alpha : [0, 1] \longrightarrow X, \alpha(s) = H(x_0, s).$$

This is a path from x_0 to $h(x_0)$, hence the previous theorem (Theorem 7.22) provides us with a group isomorphism

$$\widehat{\alpha} : \pi(X, x_0) \longrightarrow \pi(X, h(x_0)), \;\; [\gamma] \mapsto [\alpha]^{-1} * [\gamma] * [\alpha].$$

We will show that $h_* = \widehat{\alpha}$. Given $[\gamma] \in \pi(X, x_0)$, we have to check that

$$[h \circ \gamma] = [\alpha]^{-1} * [\gamma] * [\alpha].$$

Using the properties of the concatenation (see Proposition 7.18), we have to check that

$$[\alpha] * [h \circ \gamma] = [\gamma] * [\alpha],$$

i.e. that the paths $\gamma * \alpha$ and $\alpha * (h \circ \gamma)$ are path homotopic. For this, we define

$$G : [0, 1] \times [0, 1] \longrightarrow X,$$

$$G(s, t) = \begin{cases} H(\alpha(2st), 2s(1 - t)) & \text{if } 0 \leq s \leq \frac{1}{2} \\ H(\alpha(1 - 2(1 - s)(1 - t)), 1 - 2(1 - s)t) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}.$$

One checks directly that G is a path homotopy between $\gamma * \alpha$ and $\alpha * (h \circ \gamma)$ (check it!). $\square$

Exercise 7.17. Explain (on a picture) the formula for the homotopy G given in the previous proof.

Example 7.24.

- If X is a contractible space, the $\pi(X, x)$ is trivial (i.e. consisting only of the identity element 1_x) for all $x \in X$. In particular,
$$\pi(\mathbb{R}^n, x) = 0.$$
- Consider the inclusion $i : S^{n-1} \longrightarrow \mathbb{R}^n - \{0\}$. Since this is a homotopy equivalence (Example 7.8), we deduce that we have an isomorphism
$$i_* : \pi_1(S^{n-1}, x) \xrightarrow{\sim} \pi_1(\mathbb{R}^n - \{0\}, x),$$
for all $x \in S^{n-1}$. We will see that the left hand side equals to $\mathbb{Z}$ if $n = 1$, and is trivial otherwise.
- Consider the Moebius band M. Inside it, one finds a copy of the circle S^1, and, as in the previous example, the inclusion $S^1 \longrightarrow M$ is a homotopy equivalence (Exercise 7.3). We deduce that the homotopy group of M is isomorphic to the homotopy group of S^1.

Definition 7.25. *We say that a space X is* <u>*simply connected*</u> *if $\pi(X, x)$ is trivial for any $x \in X$. We say that X is* <u>*1-connected*</u> *if it is path connected and simply connected.*

Exercise 7.18. *We are not quite ready to prove that the sphere S^2 (and S^n for all $n \geq 2$) is simply connected. However, using the stereographic projection, try to explain this result.*

Exercise 7.19. Show that, for any two topological spaces X and Y, and $x \in X$, $y \in Y$, one has an isomorphism of groups:
$$\pi(X \times Y, (x, y)) \cong \pi(X, x) \times \pi(Y, y).$$

4. Covering spaces and the homotopy group of S^1

So far, we know that the homotopy group of the spaces $\mathbb{R}^n$ (or any contractible space) is trivial. In this section we will compute the fundamental group of the circle; more precisely, we will show that it isomorphic to the cyclic group in one generator, i.e. to $(\mathbb{Z}, +)$, and we will produce an explicit isomorphism
$$\deg : \pi(S^1, 1) \longrightarrow \mathbb{Z}.$$
The idea behind this map is quite simple: it associates to the (equivalence class) of a path γ (starting and ending at $1 \in S^1$) its rotation number, i.e. the total number of rotation of S^1, counted in a the counterclockwise direction (so that a path that goes once around the circle, but on the clockwise direction, has degree -1). There are some standard paths: for each n there is the path γ_n which rotates n times around the circle (and has degree n). In general, any path γ in S^1 satrting end nding at $1 \in S^1$ is path homotopic to the path γ_n for $n = \deg(\gamma)$. Intuitively this can be explained as follows. Interpret it as a rope which goes around a rigid S^1. It may go for some time in one direction, then turn back for a while, then change direction again, etc. See Figure 8 for a path of degree two. Hold now the piece of rope by its ends, and start pulling it, as long as it is possible. Note that the pulling process does not change the degree. We can keep on pulling untill all the "turnings" of the rope will be smoothened out, and we will end up with γ_n. The process of pulling defines the homotopy between γ and γ_n.

The remaining part of this section will make these idea more precise. In particular, we will introduce some tools (covering spaces) which may be used in various other situations. The basic idea is to relate S^1 to $\mathbb{R}$, by the map
$$p : \mathbb{R} \longrightarrow S^1, p(t) = (cos(2\pi t), sin(2\pi t)).$$

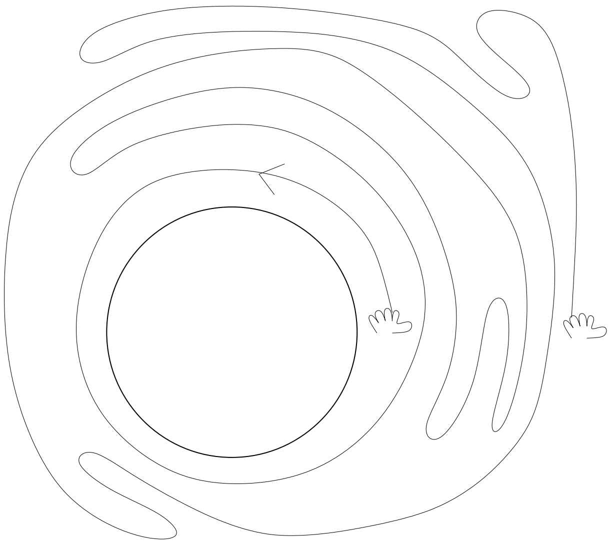


FIGURE 8.

We will use the base point $1 \in S^1$. Note that $p^{-1} = \mathbb{Z}$. It is useful to picture this map as follows: spiral $\mathbb{R}$ above the circle (on a cylinder), as shown in Figure 9, and then p is just the projection into the circle. In particular, we see that the pre-image p^{-1} of small opens $U \subset S^1$ will consist of a disjoint family of copies of U. It is precisely this property that allows us to compute $\pi(S^1, 1)$. What we will actually do is to look at more general maps which have this property (which

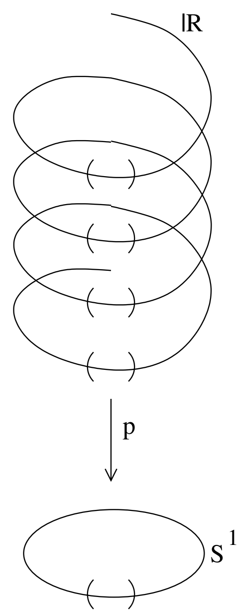


FIGURE 9.

will be called covering maps), and extract the relevant information about fundamental groups. Returning to our example, we will be able to conclude the computation of $\pi(S^1, 1)$.

DEFINITION 7.26. *Let $p : E \longrightarrow B$ be a continuous map. Given $U \subset B$ an open subset, a partition of $p^{-1}(U)$ into slices is a family $\{V_i\}_{i \in I}$ of opens in E such that*

- *$\{V_i\}$ is a partition of $p^{-1}(U)$;*
- *$p|_{V_i} : V_i \longrightarrow U$ is a homeomorphism for each $i \in I$.*

A covering map is a continuous surjective map $p : E \longrightarrow B$ with the property that each point $b \in B$ has an open neighborhood U such that $p^{-1}(U)$ admits a partition into slices.

We also say that p is a covering of B, or that E is a covering space of B with covering projection p.

EXERCISE 7.20. Show that $p : \mathbb{R} \longrightarrow S^1$ is a covering map.

EXERCISE 7.21. *View the unit circle as*

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

Show that for any integer $n \geq 1$, the map

$$f_n : S^1 \to S^1, f_n(z) = z^n.$$

is a covering map.

PROPOSITION 7.27. *Let $p : E \longrightarrow B$ be a covering map, and let $b_0 \in B$. Consider*

- *a path $\gamma : [0,1] \longrightarrow B$ starting at b_0.*
- *a lift e_0 of b_0, i.e. $e_0 \in p^{-1}(b_0)$.*

Then there exists an unique path $\tilde{\gamma} : [0,1] \longrightarrow E$ such that

- *$\tilde{\gamma}$ starts at e_0;*
- *$\tilde{\gamma}$ is a lift of γ, i.e. $p \circ \tilde{\gamma} = \gamma$.*

PROOF. First we prove the uniqueness. Assume that $\tilde{\gamma}$ and $\tilde{\tilde{\gamma}}$ are two lifts of γ starting at the same point e_0. We consider

$$S = \{s \in [0,1] : \tilde{\gamma}|_{[0,s]} = \tilde{\tilde{\gamma}}\tilde{\tilde{\gamma}}\},$$

and we prove that S is both closed and open in $[0,1]$this will imply that $S = [0,1]$, proving the uniqueness.

We first show that S is open. Let $s_0 \in S$. Choose an open neighborhood U of $\gamma(s_0)$ such that $p^{-1}(U) = \cup V_i$ is a partition into slices. Choose i such that $\tilde{\gamma}(s_0) \in V_i$. Since $\tilde{\gamma}$ and $\tilde{\tilde{\gamma}}$ are continuous, we find a neighborhood $D = (s_0 - \epsilon, s_0 + \epsilon) \cap [0,1]$ of s_0 in $[0,1]$ such that

$$\tilde{\gamma}(D) \subset V_i, \ \tilde{\tilde{\gamma}}(D) \subset V_i.$$

Since $p|_{V_i} : V_i \longrightarrow U$ is a homeomorphism and $p \circ \tilde{\gamma} = p \circ \tilde{\tilde{\gamma}}$, we deduce that $\tilde{\gamma}|_D = \tilde{\tilde{\gamma}}|_D$, which implies that $D \subset S$.

We now show that S is closed. Let $s_0 \in \overline{S}$, and we show that $s_0 \in S$. For any $s < s_0$, $(s, s_0) \cap S \neq \emptyset$, which implies that $\tilde{\gamma}(s) = \tilde{\tilde{\gamma}}(s)$ (for all $s < s_0$). We still have to show that this equality also holds for $s = s_0$. Assume it does not, and let $e = \tilde{\gamma}(s_0)$, $e' = \tilde{\tilde{\gamma}}(s_0)$. Let U be as above, with $p^{-1}(U) = \cup V_i$, and choose i and j such that $e \in V_i$, $e' \in V_j$. Note that, due to the assumption $e \neq e'$ and the fact that $p|_{V_i}$ is a homeomorphism, we must have $i \neq j$, hence $V_i \cap V_j = \emptyset$. On the other hand, due to continuity, we find a neighborhood D of s_0 such that

$$\tilde{\gamma}(D) \subset V_i, \ \tilde{\tilde{\gamma}}(D) \subset V_j.$$

But this contradicts the fact that $\tilde{\gamma}(s) = \tilde{\tilde{\gamma}}(s)$ for all $s < s_0$. This concludes the proof of the fact that S is closed.

Before we prove the existence part, let us point out the following consequence.

Corollary 7.28. *Let $p : E \longrightarrow B$ be a covering map, and let $f : X \longrightarrow B$ be a continuous function defined on a space X. A lift of f is a continuous map $\tilde{f} : X \longrightarrow E$ such that $p\tilde{f} = f$.*

If X is path connected, then any two lifts of f which coincide in the same point of X must coincide everywhere.

Proof. Let $x_0 \in X$ such that $\tilde{f}(x_0) = \tilde{\tilde{f}}(x_0)$. For $x \in X$ arbitrary, choose a path α from x_0 to x. Then $\tilde{f} \circ \alpha$ and $\tilde{\tilde{f}} \circ \alpha$ are two lifts of $f \circ \alpha$ which coincide at the initial point. By the uniqueness proven above, they coincide everywhere; in particular, the end points (i.e. $\tilde{f}(x)$ and $\tilde{\tilde{f}}(x)$) must coincide. □

We now return to the proof of the proposition (existence part). Let $S' \subset [0, 1]$ be the set of those s with the property that $\gamma|_{[0,s]}$ admits a lift starting at e_0. One can proceed as above and prove that S' is open and closed in $[0, 1]$. We give here a slightly different argument. Remark first that if $s \in S'$, then all $t < s$ are in S'. This implies that $S' = [0, s_0]$ or $S' = [0, s_0)$ for some $s_0 \in [0, 1]$. Also, we can find a lift $\tilde{\gamma} : [0, s_0) \longrightarrow E$ of $\gamma|_{[0,s_0)}$. To see this, we may assume $s_0 \neq 0$, and then $s_0 - \frac{1}{n} \in S'$ for n large enough. Choose $\tilde{\gamma}_n : D_n \longrightarrow E$ lifts of $\gamma|_{D_n}$ starting at e_0 defined on $D_n = [0, s_0 - \frac{1}{n}]$. The uniqueness proven above implies that $\tilde{\gamma}_n$ and $\tilde{\gamma}_m$ coincide on $D_n \cap D_m$ for all n and m, hence we can define $\tilde{\gamma}$ on $D_0 = [0, s_0)$ so that $\tilde{\gamma}|_{D_n} = \tilde{\gamma}_n$ for all n, and this will be the desired lift.

Hence $S' = [0, s_0]$ or $S' = [0, s_0)$ for some $s_0 \in [0, 1]$, and γ has a lift $\tilde{\gamma}$ on $[0, s_0)$, starting at e. Let U be an open neighborhood of $\gamma(s_0)$ such that $p^{-1}(U) = \cup V_i$ is a partition into slices. Choose ϵ such that $D = (s_0 - \epsilon, s_0 + \epsilon) \cap [0, 1]$ is sent by γ inside U. Consider also $D' = (s_0 - \epsilon, s_0)$. Since continuous maps send connected spaces to connected spaces, $\tilde{\gamma}(D')$ must be connected; but it lies in the disjoint union of the V_i's, hence we find i such that

$$\tilde{\gamma}(D') \subset V_i.$$

Since $p|_{V_i}$ is a homeomorphism, we must have

$$\tilde{\gamma}(s) = (p|_{V_i})^{-1}(\gamma(s))$$

for all $s \in D'$. But we can take this formula as the definition of $\tilde{\gamma}$ also for $s \in D$, and we obtain a lift $\tilde{\gamma}$ defined on D. Hence

$$D = (s_0 - \epsilon, s_0 + \epsilon) \cap [0, 1] \subset S'.$$

Recalling that $S' = [0, s_0]$ or $S' = [0, s_0)$, we see that the only possibility is when $S = [0, 1]$. □

Remark 7.29. In particular, we obtain a map

$$\phi_{e_0} : P(B, b_0, b_0) \longrightarrow p^{-1}(b_0) \tag{4.1}$$

which associates to a path γ the end-point $\tilde{\gamma}(1)$ of its lift evaluated at the end point.

Example 7.30. Apply the previous lemma to the covering map $p : \mathbb{R} \longrightarrow S^1$ and $e_0 = 0$. For any path in S^1 starting and ending at $1 \in S^1$ we find a path $\tilde{\gamma}$ in $\mathbb{R}$ starting at 0 such that

$$\gamma(t) = (cos(2\pi\tilde{\gamma}(t)), sin(2\pi\tilde{\gamma}(t))).$$

Note that $\tilde{\gamma}(1) \in p^{-1}(0) = \mathbb{Z}$.

Definition 7.31. *Given a path γ in S^1 starting and ending in $1 \in S^1$, we define the degree of γ by*

$$deg(\gamma) := \tilde{\gamma}(1) \in \mathbb{Z}.$$

For instance, for each $n \in \mathbb{Z}$, the path which goes around the circle n times:

$$\gamma_n(t) = (cos(2\pi nti), sin(2\pi nti)),$$

has as lift

$$\tilde{\gamma}_n(t) = nt,$$

hence

$$\deg(\gamma_n) = n.$$

Our next aim is to show that the degree (or, more generally, the map (4.1)) only depends on path homotopy classes of paths.

LEMMA 7.32. *Let $p : E \longrightarrow B$ be a covering map, $b_0 \in B$, $e_0 \in E$ a lift of b_0. Let $\gamma, \gamma' \in P(B, b_0, b_0)$, and consider their lifts $\tilde{\gamma}$, $\tilde{\gamma}'$ which start at e_0. If γ and γ' are path homotopic, then $\tilde{\gamma}$ and $\tilde{\gamma}'$ have the same end-point and they are path homotopic.*

PROOF. Let $H : [0,1] \times [0,1]$ be a path homotopy between γ and γ'. Let $\gamma_t(s) = H(s,t)$ (so that $\gamma_0 = \gamma$, $\gamma_1 = \gamma'$). For each t we consider the lift $\tilde{\gamma}_t$ of γ_t starting at e_0 (which, by the previous proposition, exists and is unique). Put $\tilde{H}(s,t) = \tilde{\gamma}_t(s)$. Locally, H is the inverse of the restriction of p to a slice, composed with H, from which one deduces that H is continuous (fill in the details!). On the other hand, $t \mapsto \tilde{H}(1,t)$ is a lift of the constant path $H(1,t) = b_0$, hence will be constant (use the uniqueness part of previous proposition). In conclusion, $\tilde{H}$ is a path homotopy between $\tilde{\gamma}_0 = \tilde{\gamma}$ and $\tilde{\gamma}_1 = \tilde{\gamma}'$. □

REMARK 7.33. Continuing the previous remark, the map (4.1) will define a map

$$\phi_{e_0} : \pi(B, b_0) \longrightarrow p^{-1}(b_0).$$

In particular, for the covering map $p : \mathbb{R} \longrightarrow S^1$ we find that the notion of degree induces a map

$$\deg : \pi(S^1, 1) \longrightarrow \mathbb{Z}.$$

THEOREM 7.34. *$deg : \pi(S^1, 1) \longrightarrow \mathbb{Z}$ is an isomorphism of groups.*

PROOF. We have already seen that $\deg(\gamma_n) = n$, where $\gamma_n(t) = p(nt)$ for $t \in [0,1]$. Hence deg is surjective. To show it is injective, assume that

$$\deg(\gamma) = \deg(\gamma').$$

This means that, choosing lifts $\tilde{\gamma}$ and $\tilde{\gamma}'$ as above (starting at 0), $\tilde{\gamma}(1) = \tilde{\gamma}'$. But we know that $\mathbb{R}$ is simply connected- any two paths which start and end at the same point are path homotopic. Hence $\tilde{\gamma}$ and $\tilde{\gamma}'$ are path homotopic, which implies that γ and γ' are path homotopic (in explicit formulas, $\tilde{H}(s,t) = (1-t)\tilde{\gamma}(s) + t\tilde{\gamma}'(s)$ is a path homotopy between $\tilde{\gamma}$ and $\tilde{\gamma}'$, while $H(s,t) = p(H(s,t))$ is a path homotopy between γ and γ'.

Hence deg is bijective. To see it is a group homomorphism it suffices to check that $\deg(\gamma_n * \gamma_n) = \deg(\gamma_n) + \deg(\gamma_m)$ for all $n, m \in \mathbb{Z}$. By the definition of concatenation, we have

$$(\gamma_n * \gamma_m)(t) = \begin{cases} p(2nt) & \text{if } 0 \leq t \leq \frac{1}{2} \\ p(m(2t-1)) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases},$$

which has as lift

$$t \mapsto \begin{cases} 2nt & \text{if } 0 \leq t \leq \frac{1}{2} \\ m(2t-1) + n & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases},$$

hence

$$\deg(\gamma_n * \gamma_m) = m(2 \cdot 1 - 1) + n = m + n = \deg(\gamma_n) + \deg(\gamma_m).$$

□

EXERCISE 7.22. *Consider again* $f_n : S^1 \longrightarrow S^1$, $f_n(z) = z^n$ *and consider the induced map*

$$(f_n)_* : \pi(S^1, 1) \to \pi(S^1, 1).$$

After identifying $\pi(S^1, 1)$ *with* $(\mathbb{Z}, +)$, *show that* $(f_n)_*$ *is identified with*

$$mult_n : \mathbb{Z} \to \mathbb{Z}, \quad r \mapsto nr.$$

EXERCISE 7.23. *If* $p : E \longrightarrow B$ *is a covering map defined on a simply connected space* E, $b_0 \in B$, $e_0 \in p^{-1}(b_0)$, *then*

$$\phi_{e_0} : \pi(B, b_0) \longrightarrow p^{-1}(b_0)$$

is a bijection.

EXERCISE 7.24. *Let* $f : [0, 1] \longrightarrow S^1$ *be a continuous map with* $f(0) = f(1) = 1$. *If* $deg(f) = n$, *prove that the equation*

$$f(x) = 1$$

has at least $n + 1$ *solutions.*

COROLLARY 7.35. *There are no continuous retractions of* D^2 *into* S^1 *(i.e. continuous maps* $r : D^2 \longrightarrow S^1$ *such that* $r(x) = x$ *for all* $x \in S^1$*).*

PROOF. Assume such a retraction $r : D^2 \longrightarrow S^1$ exists. Let $i : S^1 \longrightarrow D^2$ be the inclusion. Consider the maps induced in the fundamental groups

$$\pi(S^1, 1) \xrightarrow{i_*} \pi(D^2, 1) \xrightarrow{r_*} \pi(S^1, 1).$$

Since $r \circ i = \mathrm{Id}_{S^1}$, we have $r_* \circ i_* = \mathrm{Id} : \pi(S^1, 1) \longrightarrow \pi(S^1, 1)$. But $\pi(D^2, 1) = 0$ (trivial) because D^2 is contractible, hence $\mathrm{Id} = r_* i_*$ will be constant. But this can only happen if $\pi(S^1, 1)$ was trivial- which is not. Hence r does not exist. □

COROLLARY 7.36. *(Brouwer fixed point theorem) Any continuous function* $f : D^2 \longrightarrow D^2$ *has at least one fixed point.*

PROOF. Assume that $f : D^2 \longrightarrow D^2$ has no fixed point. For each $x \in D^2$, since $f(x) \neq x$, we can talk about the line through x and $f(x)$. This line intersects S^1 in two points, and let $r(x)$ be the point with the property that x lies between $r(x)$ and $f(x)$. Then $r : D^2 \longrightarrow S^1$ will be a retraction of of D^2 into S^1- contradiction. □

Related to the last corollary, here is one exercise.

EXERCISE 7.25. *In general, given a topological space* X *and a subset* $A \subset X$, *a retraction of* X *into* A *is a continuous map* $r : X \longrightarrow A$ *with the property that* $r(a) = a$ *for all* $a \in A$.

Prove that if r *is a retraction of* X *into* A, *then the map induced in the fundamental groups:*

$$r_* : \pi(X, a) \longrightarrow \pi(A, a)$$

is surjective for all $a \in A$. *What about the map induced by the inclusion of* A *into* X*?*

EXERCISE 7.26. *Prove that the Moebius band cannot be retracted into its boundary.*

ADDENDUM TO CHAPTER 7

1. Action of the Fundamental Group

Throughout these notes, $p : E \to B$ will denote a covering map, and E and B are path connected topological spaces.

We have seen that for any path $\gamma : [0, 1] \to B$ and $e \in p^{-1}(\gamma(0))$ there exists a unique lift $\tilde{\gamma}_e : [0, 1] \to E$ of γ starting at e (Proposition 7.27). Moreover, it was shown that if γ_1 and γ_2 are path homotopic, then their lifts (starting at the same point) have the same end points and are also path homotopic (Lemma 7.32). What we will now explain is that from both of these properties we obtain a (right) action of $\pi(B, b)$ on the fiber $p^{-1}(b)$.

Definition 1.1. Let X be a topological space and G a topological group (i.e., a topological space endowed with a group structure such that the multiplication $m : G \times G \to G$ and the inversion $\iota : G \to G$ are continuous). An (left) **action of G on X** is a continuous map

$$\Psi : G \times X \to X, \quad (g, x) \mapsto \Psi(g, x) = g \cdot x$$

which satisfies:

- $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$, and
- $1 \cdot x = x$ for all $x \in X$.

Similarly, a *right* action of G on X is a continuous map $X \times G \to X$ which satisfies:

- $x \cdot (gh) = (x \cdot g) \cdot h$ for all $g, h \in G$ and $x \in X$, and
- $x \cdot 1 = x$ for all $x \in X$.

For an action of G on X we define for each x the **orbit of G through x** to be the set

$$\mathcal{O}_x = \{g \cdot x : g \in G\} \subset X,$$

and the **isotropy of G at x** to be

$$G_x = \{g \in G : g \cdot x = x\} \subset G.$$

Exercise 1. *Show that G_x is a subgroup of G.*

Exercise 2. *Show that for any $x \in X$ there is a bijection between $\mathcal{O}_x$ and G/G_x, where G/G_x denotes the quotient of G by the equivalence relation $g \sim h$ if and only if $gh^{-1} \in G_x$.*

We can now explain the action of $\pi(B, b)$ on $p^{-1}(b)$. It is defined as follows: for each pair $(e, [\gamma]) \in p^{-1}(b) \times \pi(B, b)$ we take $e \cdot [\gamma] = \tilde{\gamma}_e(1)$ where $\tilde{\gamma}_e$ is the unique lift of γ which starts at e.

Exercise 3. *Let E and B be path connected topological spaces and $p : E \to B$ a covering map:*

(1) *Show that the map $(e, [\gamma]) \mapsto \tilde{\gamma}_e(1)$ is well defined and determines a right action of $\pi(B, b)$ on $p^{-1}(b)$.*
(2) *Show that the action is transitive, i.e., for every $e \in p^{-1}(b)$ we have that $\mathcal{O}_e = p^{-1}(b)$.*
(3) *Show that the isotropy of $\pi(B, b)$ at e is isomorphic to $\pi(E, e)$.*
(4) *Show that there is a bijection between $p^{-1}(b)$ and $\pi(B, b)/p_*\pi(E, e)$.*
(5) *Conclude that if E is simply connected then there is a bijection between $p^{-1}(b)$ and $\pi(B, b)$.*

Exercise 4. *Consider the map $f : \mathbb{S}^n \to \mathbb{P}^n$ which associates to each x in the sphere $\mathbb{S}^n$ the line in $\mathbb{R}^n$ which passes through x and the origin.*

(1) *Show that f is a covering map.*
(2) *Assuming that $\mathbb{S}^n$ is simply connected, for $n \geq 2$, compute the fundamental group of $\mathbb{P}^n$.*

2. Properly Discontinuous Actions

Definition 2.1. A (continuous) action of G on X is said to be **properly discontinuous** if for every $x \in X$ there exists a neighborhood U_x of x in X such that $g \cdot U_x \cap U_x \neq \emptyset$ implies that $g = 1$.

The importance of properly discontinuous actions for us is given by the following proposition:

Proposition 2.2. *If E is a path connected and simply connected topological space, and G acts properly discontinuously on E, then the quotient map $p : E \to E/G$ is a covering map, and moreover, $\pi(E/G, x) \cong G$, for any $x \in E/G$.*

Exercise 5. *Prove the proposition above by following these steps:*

(1) *For $e \in E$, let U_e be a neighborhood such that $g \cdot U_e \cap U_e \neq \emptyset$ implies that $g = 1$. Show that $g \cdot U_e$ is a neighborhood of $g \cdot e$ which satisfies the same property.*
(2) *Show that $V_{[e]} = p(U_e)$ is an open neighborhood of $[e]$ in E/G.*
(3) *Show that $V_{[e]}$ is evenly covered (i.e., that $p^{-1}(V_{[e]})$ admits a partition into slices). Conclude that p is a covering map.*
(4) *Fix $e \in E$ and show that the map $G \to \pi(B, b)$ which associates to any $g \in G$ the homotopy class of $p \circ \tilde{\gamma}$ (where $\tilde{\gamma}$ is any path joining e to $g \cdot e$), is an isomorphism of groups.*

Exercise 6. *Show that the action of $\mathbb{Z}$ on $\mathbb{R}$ given by $(n, x) \mapsto n + x$ is properly discontinuous. Conclude that $\pi(\mathbb{S}^1, p) = \mathbb{Z}$.*

Exercise 7. *Show that the action of $\mathbb{Z}$ on $\mathbb{R}^2$ given by $(n, (x, y)) \mapsto (n + x, y)$ is properly discontinuous. Conclude that $\pi(\text{Cylinder}, p) = \mathbb{Z}$.*

Exercise 8. *Show that the action of* $\mathbb{Z}^2$ *on* $\mathbb{R}^2$ *given by* $((n,m),(x,y)) \mapsto (n+x, m+y)$ *is properly discontinuous. Conclude that* $\pi(\mathbb{T}^2, p) = \mathbb{Z}^2$*.*

Exercise 9. *Show that* $\mathbb{Z}_2 = \{1, -1\}$ *acts properly discontinuously on the sphere* $\mathbb{S}^n$*. What is the quotient space?*

Exercise 10. *Show that a covering of a simply connected space is a homeomorphism.*

Exercise 11. *Can* $\mathbb{S}^2$ *be obtained from a properly discontinuous action of a group on* $\mathbb{R}^2$*? What about* $\mathbb{P}^2$*?*

Exercise 12. *Let* K *be the Klein bottle.*

1. *Show that there is a covering map* $\mathbb{T}^2 \to K$*.*
2. *Show that* K *can be obtained from* $\mathbb{R}^2$ *as the quotient by a properly discontinuous action.*
3. *Compute the fundamental group of* K*.*

Exercise 13. *Show that the quotient of a topological manifold by a properly discontinuous action of a group is also a topological manifold.*

Exercise 14. *Show that if* $p : E \to B$ *is a covering, and* B *is a topological manifold of dimension* n*, then* E *is a topological manifold of dimension* n*.*

CHAPTER 8

The Seifert- van Kampen theorem

1. The statement and first explanations

The Seifert- van Kampen theorem allows us to compute the fundamental group of a space by breaking it into pieces.

THEOREM 8.1. *(Seifert- van Kampen theorem) Let X be a topological space and assume that $X = U \cup V$ with $U, V \subset X$ opens such that $U \cap V$ is path connected and let $x_0 \in U \cap V$. We consider the a commutative diagram*

$$\begin{tikzcd}
 & \pi(U,x_0) \arrow[dr, "j_1"] & \\
\pi(U\cap V,x_0) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & \pi(X,x_0) \\
 & \pi(V,x_0) \arrow[ur, "j_2"'] &
\end{tikzcd}$$

where all the maps are induced by the inclusions. Then, for any group H and any group homomorphisms $\phi_1 : \pi(U, x_0) \longrightarrow H$, $\phi_2 : \pi(V, x_0) \longrightarrow H$ such that

$$\phi_1 i_1 = \phi_2 i_2,$$

there exists and is unique a group homomorphism $\phi : \pi(X, x_0) \longrightarrow H$ such that $\phi_1 = \phi j_1$, $\phi_2 = \phi j_2$.

$$\begin{tikzcd}
 & \pi(U,x_0) \arrow[dr, "j_1"] \arrow[drr, "\phi_1"] & & \\
\pi(U\cap V,x_0) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & \pi(X,x_0) \arrow[r, "\phi"] & H \\
 & \pi(V,x_0) \arrow[ur, "j_2"'] \arrow[urr, "\phi_2"'] & &
\end{tikzcd}$$

1.1. Explanation. Although it may not be clear from the statement, the theorem describes the way that $\pi(X, x_0)$ is determined by $\pi(U, x_0)$, $\pi(V, x_0)$ and $\pi(U \cap V, x_0)$. To understand this, we will look at the property described in the theorem in a slightly more general setting.

Starting data: Start with (N, G_1, G_2, i_1, i_2) where N, G_1 and G_2 are groups and $i_1 : H \longrightarrow G_1$, $i_2 : N \longrightarrow G_2$ are group homomorphisms:

(1.1)
$$\begin{tikzcd} & G_1 \\ N \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \\ & G_2 \end{tikzcd}$$

Consider: triples (G, j_1, j_2) consisting of a group G and group homomorphisms $j_1 : G_1 \longrightarrow G$, $j_2 : G_2 \longrightarrow G$ such that $j_1 i_1 = j_2 i_2$, i.e. "commutative fillings" of the starting data (1.1):

$$\begin{tikzcd} & G_1 \arrow[dr, "j_1"] & \\ N \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & G \\ & G_2 \arrow[ur, "j_2"'] & \end{tikzcd}$$

Definition 8.2. *We say that a triple (G, j_1, j_2) as above satisfies the universal property (with respect to the starting data (1.1)) if for any other such triple (H, ϕ_1, ϕ_2), there exists and is unique a group homomorphism $\phi : G \longrightarrow G'$ such that $\phi_1 = \phi j_1$, $\phi_2 = \phi j_2$:*

$$\begin{tikzcd} & G_1 \arrow[dr, "j_1"] \arrow[drr, "\phi_1"] & & \\ N \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & G \arrow[r, "\phi"] & H \\ & G_2 \arrow[ur, "j_2"'] \arrow[urr, "\phi_2"'] & & \end{tikzcd}$$

We now show that the universal property determines G uniquely up to isomorphism.

Lemma 8.3. *If (G, j_1, j_2) and (G', j'_1, j'_2) are two triples which satisfy the universal property with respect to the starting data (1.1), then there exists (and is unique) an isomorphism of groups*

$$\phi : G \xrightarrow{\sim} G'$$

such that $j'_1 = \phi j_1$, $j'_2 = \phi j_2$

Proof. Due to the universal property of G, there exists a unique homomorphism $\phi : G \longrightarrow G'$ satisfying the equations in the lemma. We have to prove that it is an isomorphism. Interchanging G and G' (and applying the universal property for G'), we find a unique homomorphism $\phi : G' \longrightarrow G$ satisfying $j_1 = \phi j'_1$, $j_2 = \phi j'_2$. We will prove that $\phi'\phi = \mathrm{Id}_G$. Due to the universal property of G (the uniqueness part!), it suffices to check that

$$(\phi'\phi)j_1 = j_1, (\phi'\phi)j_2 = j_2,$$

which is immediate from the identities defining ϕ and ϕ'. Hence $\phi'\phi = \mathrm{Id}_G$ and, similarly, $\phi\phi' = \mathrm{Id}_{G'}$, proving that ϕ is an isomorphism. □

Definition 8.4. *The amalgamated free product associated to the starting data (1.1) is a triple (G, j_1, j_2) satisfying the universal property. We also say that G is the free product of G_1 and G_2 over N (but keep in mind the maps involved!).*

REMARK 8.5. Hence, what the Seifert- van Kampen tells us is that $\pi(X, x_0)$ is uniquely determined by $\pi(U, x_0)$, $\pi(V, x_0)$ and $\pi(U \cap V, x_0)$, as the amalgamated free product associated to the starting data

$$\begin{tikzcd} & \pi(U, x_0) \\ \pi(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \\ & \pi(V, x_0) \end{tikzcd}$$

Back to the general discussion, we know from the previous lemma that the amalgamated free product is unique up to isomorphism. One can also prove that it exists (for any starting data (1.1)), but the explicit *general* construction is not allways useful. Instead, it willbe more interesting to look at particular cases first (Cases A-D below). But first, let us reformulate the Seifert- van Kampen theorem using this new language.

THEOREM 8.6. *(Seifert- van Kampen theorem reformulated) Let X be a topological space and assume that $X = U \cup V$ with $U, V \subset X$ opens such that $U \cap V$ is path connected and let $x_0 \in U \cap V$. We consider the a commutative diagram*

$$\begin{tikzcd} & \pi(U, x_0) \arrow[dr, "j_1"] & \\ \pi(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & \pi(X, x_0) \\ & \pi(V, x_0) \arrow[ur, "j_2"'] & \end{tikzcd}$$

where all the maps are induced by the inclusions. Then $\pi(X, x_0)$, together with the maps j_1 and j_2, is the amalgamated free product associated to $\pi(U, x_0)$, $\pi(V, x_0)$ and $\pi(U \cap V, x_0)$ (with maps i_1, i_2).

2. The case $G_1 = G_2 = \{1\}$

Hence assume that G_1 and G_2 are trivial. Then all the maps involved (i_1, i_2, j_1 etc) are the constant maps and all the equations are automatically satisfied. Hence we are looking for a group G with the property that for any other group H, there exists a unique group homomorphism $\phi : G \longrightarrow H$. Of course, $G = \{1\}$ does the job.

COROLLARY 8.7. *(Corollary A) Let X, U, V, x_0 be as in the Seifert-van Kampen theorem. If $\pi(U, x_0)$ and $\pi(V, x_0)$ are trivial, then so is $\pi(X, x_0)$.*

In particular, we deduce

COROLLARY 8.8. *$\pi(S^n, p) = \{1\}$ for all $n \geq 2$ and all $p \in S^n$.*

PROOF. Choose $U = S^n - \{p_N\}$ and $V = S^n - \{p_S\}$, where p_N is the north pole, p_S is the south pole. We know that U and V are homeomorphic to $\mathbb{R}^n$ (by the stereographic projection), hence they have trivial fundamental groups. Also, since $n \geq 2$, $U \cap V$ is path connected, hence Corollary A applies. $\square$

EXERCISE 8.1. Use the covering projection $S^n \longrightarrow \mathbb{P}^n$ to deduce that $\pi(\mathbb{P}^n, p)$ is isomorphic to $\mathbb{Z}_2$ for all $n \geq 2$ and $p \in \mathbb{P}^n$.

3. The case $N = G_2 = \{1\}$

Look now at the case when N and G_2 are trivial. Also in this case there is an obvious solution to the universal property: namely $G = G_1$, $j_1 = \mathrm{Id}$ (while j_2 can only be one thing: the trivial map).

COROLLARY 8.9. *(Corollary B) Let X, U, V, x_0 be as in the Seifert-van Kampen theorem. If $\pi(V, x_0)$ and $\pi(U \cap V, x_0)$ are trivial, then the map induced by inclusion*

$$i_1 : \pi(U, x_0) \longrightarrow \pi(X, x_0)$$

is an isomorphism of groups.

Using this, we will prove the following result whose importance comes from the fact that it shows that cells of dimension higher then three do not affect the fundamental group.

PROPOSITION 8.10. *Let X be space which is obtained from a path connected subspace $A \subset X$ by adjoining an n-cell e with $n \geq 3$. Then, for any $a \in A$, the inclusion $i : A \longrightarrow X$ induces an isomorphism of groups:*

$$i_* : \pi(A, a) \longrightarrow \pi(X, a).$$

Before we prove the proposition let us mention the following consequence which follows after applying the proposition repeatedly to eliminate cells of dimension ≥ 3.

COROLLARY 8.11. *If X is a compact, path connected space then, for any cell decomposition of X, the inclusion of the 2-skeleton X_2 into X, $i : X_2 \longrightarrow X$, induces an isomorphism of groups (for any base point $x \in X_2$):*

$$i_* : \pi(X_2, x) \xrightarrow{\sim} \pi(X, x).$$

PROOF. (of the Proposition 8.10) Let $h : D^n \longrightarrow X$ be the defining map for e, so that the restriction to the open ball is a homeomorphism onto e, and the restriction to $S^{n-1} = \partial D^n$ is the characteristic map of e, $\chi : S^{n-1} \longrightarrow A$. We consider

$$U = X - \{h(0)\}, V = e.$$

Remark that

1. V is simply connected, because it is homeomorphic to $\overset{\circ}{D}{}^n$ (via h).
2. $U \cap V$ is simply connected. Indeed, $U \cap V = h(\overset{\circ}{D}{}^n - \{0\})$ is homeomorphic to $\overset{\circ}{D}{}^n - \{0\}$ (via h) which, in turn, is homotopic equivalent to S^{n-1}. Since S^{n-1} has trivial fundamental group for $n - 1 \geq 2$, the same will be true for $U \cap V$.
3. U is homotopic equivalent to A. More precisely, the inclusion
$$i : A \longrightarrow U$$
is a homotopy equivalence (Exercise 7.14). In particular, the map induced in the fundamental groups,
$$k_* : \pi(A, a) \longrightarrow \pi(U, a)$$
is an isomorphism for all $a \in A$.

Due to the first two remarks, we can apply Corollary B, hence

$$j_1^x : \pi(U, x) \longrightarrow \pi(X, x)$$

is an isomorphism for all $x \in U \cap V = e - \{h(0)\}$. We use the super-script "x" to indicate that we use x as a base point. We claim that the same holds also for base points $a \in A$ (i.e. for j_1^a). So, let $a \in A$. Choose $x \in U \cap V$. There exists a path α in U starting at a and ending at x. Indeed, since $\overline{e} - \{h(0)\} = h(D^n - \{0\})$, this space is path connected, hence we can join x with any point $y \in \partial e = \chi(S^{n-1}) \subset A$ by a path inside $\overline{e} - \{h(0)\} \subset U$. On the other hand, since

A is path connected, y can be connected with a by a path inside $A \subset U$. Using the path α to change the base point (from a to x), we have a commutative diagram:

$$\begin{array}{ccc} \pi(U,a) & \xrightarrow{j_1^a} & \pi(X,a) \\ \widehat{\alpha}\downarrow & & \widehat{\alpha}\downarrow \\ \pi(U,x) & \xrightarrow{j_1^x} & \pi(X,x) \end{array}$$

(the commutativity follows immediately from the definition of j_1 and $\widehat{\alpha}$. Since both vertical maps $\widehat{\alpha}$, as well as j_1^a, are isomorphisms, from $\widehat{\alpha} j_1^a = j_1^x \widehat{\alpha}$ it follows that

$$j_1^a = \widehat{\alpha}^{-1} j_1^x \widehat{\alpha}$$

is an isomorphism. Composing with the map k_* (see the third remark above), the statement follows. $\square$

4. The case $G_2 = \{1\}$

Let us now look at a slightly more general case, when only G_2 is assumed to be trivial (this is what we need if we want to have an analogue of Proposition 8.10 in the case of two-cells). Assume first that N is a subgroup of G_1 and

$$i_1 : N \hookrightarrow G_1$$

is the inclusion map. In this case j_2 must be trivial and the equation $j_1 i_1 = j_2 i_2$ becomes $j_1 i_1 = 1$, i.e.

$$j_1(n) = 1, \quad \forall\, n \in N.$$

Recall that for any group homomorphism j_1, its kernel is defined by

$$\mathrm{Ker}(j_1) := \{g \in g_1 : j_1(g) = 1\}.$$

Hence, what we are looking at are pairs (G, j_1), with $j_1 : G_1 \longrightarrow G$ a group homomorphism with the property that $N \subset \mathrm{Ker}(j_1)$. The universal property reads: for any other such pair (H, ϕ_1), there exists a unique group homomorphism $\phi : G \longrightarrow H$ such that $\phi_1 = \phi j_1$. The existence of an universal pair (G, j_1) brings us to some elementary constructions on groups which we now recall:

- For a subgroup N of G_1 and $g \in G_1$, the N-coset defined by g is
$$gN = \{gn : n \in N\} \subset G_1.$$
These subsets (when g varies) form a partition of G_1. The quotient of G_1 by N is the set
$$G_1/N := \{gN : g \in G_1\}.$$
Equivalently, the action of N on G_1 defines an equivalence relation on G_1:
$$g_1 \sim g_2 \Longleftrightarrow \exists \;\; n \in N : \;\; g_2 = g_1 n,$$
the orbit through $g \in G_1$ is precisely the coset gN, hence
$$G_1/N = G_1/\sim .$$
- A subgroup N of G_1 is called a normal subgroup if
$$gng^{-1} \in N, \quad \forall \;\; g \in G_1, n \in N.$$
The importance of this notion is that, in this case, G_1/N can be made into a group with the multiplication defined by:
$$(gN) \cdot (g'N) = (gg')N$$

(if N was not normal, this multiplication would not have been well defined). Denoting by j_1 the quotient map

$$j_1 : G_1 \longrightarrow G_1/N, \; g \mapsto gN,$$

one knows from group theory (and it is an easy exercise- see also below) that $(G_1/N, j_1)$ has precisely the universal property that we are looking for.

- Let's now go back to the case where N is a subgroup of G_1 which is not necessarily normal. Since arbitrary intersections of normal subgroups of G_1 is a normal subgroup, we can talk about the smallest normal subgroup of G_1 containing N (which is the intersection of all normal subgroups of G_1 containing N), which we will denote by $\overline{N}$. Since the kernel $\mathrm{Ker}(j_1)$ of any group homomorphism $j_1 : G_1 \longrightarrow G$ is a normal group, we have:

$$j_1 \circ i_1 = 1 \Longleftrightarrow N \subset \mathrm{Ker}(j_1) \Longleftrightarrow \overline{N} \subset \mathrm{Ker}(j_1).$$

Let us now consider the general case when $i_1 : N \longrightarrow G_1$ is a group homomorphism (not necessarily an inclusion). In this case $i_1(N)$ is a subgroup of G_1 and we can apply the discussion above to $i_1(N)$.

Lemma 8.12. *Consider the starting data (1.1) with $G_2 = \{1\}$. Let $i_1(N) \subset G_1$ be the image of i_1 (a subgroup of G_1), let $\overline{i_1(N)}$ be the smallest normal subgroup of G_1 containing $i_1(N)$ and consider the quotient group and the associated quotient map*

$$\pi : G_1 \longrightarrow G_1/\overline{i_1(N)}.$$

Then $(G_1/\overline{i_1(N)}, \pi, 1)$ has the universal property with respect to the starting data (1.1) with $G_2 = 1$.

Another triple $(G, j_1, 1)$ has the universal property with respect to this data if an only if

$$j_1 : G_1 \longrightarrow G$$

is surjective with kernel equal to $\overline{i_1(N)}$. In particular j_1 will induce a group isomorphism

$$G \cong G_1/\overline{i_1(N)}.$$

Proof. The universal property with respect to the starting data (1.1) is equivalent to the universal property with respect to:

$$\begin{tikzcd} & G_1 \\ i_1(N) \arrow[ur, "\mathrm{incl}"] \arrow[dr, "i_2"'] & \\ & \{1\} \end{tikzcd}$$

(where "incl" is the inclusion). Hence we may assume that N is a subgroup of G_1 and $i_1 : N \longrightarrow G_1$ is the inclusion.

Let (H, ϕ_1) be an arbitrary pair with $\phi_1 i_1 = 1$. We have remarked that this implies

$$\overline{N} \subset \mathrm{Ker}(\phi_1).$$

We have to show that there exists an unique homomorphism $\phi : G_1/\overline{N} \longrightarrow H$ such that $\phi_1 = \phi\pi$. Explicitly, this equation translates into:

$$\phi(g\overline{N}) = \phi_1(g)$$

for all $g \in G_1$. This equation determines ϕ uniquely: it proves that ϕ will be unique, and provides us with the formula defining ϕ. We still have to check that this ϕ is well defined, i.e. we still have to show that, if $g'\overline{N} = g''\overline{N}$ then $\phi_1(g') = \phi_1(g'')$. Taking $n = g'-1g''$, we have $n \in \overline{N}$ and we have to show that $\phi_1(n) = 1$, and this follows from the inclusion we mentioned at the beginning of the proof. The last part of the lemma follows from the uniqueness insured by Lemma 8.3. $\square$

COROLLARY 8.13. *(Corollary C) Let X, U, V, x_0 be as in the Seifert-van Kampen theorem. If $\pi(V, x_0)$ is trivial, then the map*

$$j_1 : \pi(U, x_0) \longrightarrow \pi(X, x_0)$$

is surjective and it induces an isomorphism of groups

$$\pi(U, x_0)/\overline{Im(i_1)} \xrightarrow{\sim} \pi(X, x_0),$$

where $\overline{Im(i_1)}$ is the smallest normal subgroup of $\pi(U, x_0)$ containing the image of $i_1 : \pi(U \cap V, x_0) \longrightarrow \pi(U, x_0)$.

We can now prove the following version of Proposition 8.10 in the case of two-cells.

PROPOSITION 8.14. *Assume that X is obtained from A by adjoining a two-cell e with characteristic map $\chi_e : S^1 \longrightarrow A$. Consider $a = \chi_e(1)$, and denote by $\alpha \in \pi(A, a)$ the class of the path $t \mapsto \chi_e(e^{2\pi ti})$. Then the map induced by inclusion $i : A \longrightarrow X$,*

$$i_* : \pi(A, a) \longrightarrow \pi(X, x)$$

is surjective and has as kernel the smallest normal subgroup of $\pi(A, a)$ containing α, denoted by $\langle \alpha \rangle$. In particular, there is an isomorphism of groups:

$$\pi(X, a) \cong \pi(A, a)/\langle \alpha \rangle.$$

PROOF. We proceed like in the proof of Proposition 8.10. First of all, we use the same notations, and the Remarks 1 and 3 apply also to this case. In particular we can apply Corollary C to deduce that, for any $x \in U \cap V = e - \{h(0)\}$, we have a sequence of maps:

$$\pi(U \cap V, x) \xrightarrow{i_1^x} \pi(U, x) \xrightarrow{j_1^x} \pi(X, x)$$

with j_1^x- surjective with the kernel equal to the smallest normal subgroup containing the image of i_1^x. Take $x = h(v_0)$, where $v_0 \in \overset{\circ}{D}{}^2 - \{0\}$. Since $U \cap V$ is homeomorphic to $\overset{\circ}{D}{}^2 - \{0\}$ which is homotopic equivalent $D^2 - \{0\}$ (the last two spaces are both homotopic equivalent to the circle), we find that the image of i_1^x coincides with the image of the map induced by $h : D^2 - \{0\} \longrightarrow U$,

$$h_* : \pi(D^2 - \{0\}, v_0) \longrightarrow \pi(U, x).$$

We now proceed again as in the proof of Proposition 8.10. Take a path α_0 in $D^2 - \{0\}$ going from 1 to v_0, and consider the path $\alpha = h \circ \alpha_0$- a path in U going from $a == h(1)$ to $x = h(v_0)$. We then have a commutative diagram

$$\begin{tikzcd}
\pi(D^2 - \{0\}, 1) \arrow[r, "h_*"] \arrow[d, "\widehat{\alpha_0}"'] & \pi(U, a) \arrow[r, "j_1^a"] \arrow[d, "\widehat{\alpha}"'] & \pi(X, a) \arrow[d, "\widehat{\alpha}"'] \\
\pi(D^2 - \{0\}, v_0) \arrow[r, "h_*"] & \pi(U, x) \arrow[r, "j_1^x"] & \pi(X, x)
\end{tikzcd}$$

where all the vertical maps are isomorphisms. From the similar property of the bottom line, we deduce that the upper line has the property that j_1^a is surjective with the kernel equal to the smallest normal subgroup containing the image of h_*. But, since $S^1 \hookrightarrow D^2 - \{0\}$ is a homotopy equivalence, $\pi(D^2 - \{0\}, 1)$ is the free group in one generator $[\gamma_1]$, where $\gamma_1(t) = e^{2\pi t i}$. Hence the image of

$$h_* : \pi(D^2 - \{0\}, 1) \longrightarrow \pi(U, a)$$

coincides with the group generated by

$$h_*([\gamma_1]) = [h \circ \gamma_1] \in \pi(U, a).$$

Finally, by Remark 3,

$$k_* : \pi(A, a) \longrightarrow \pi(U, a)$$

is an isomorphism, and, from the definition of α in the statement, we have

$$k_*(\alpha) = [h \circ \gamma_1],$$

hence i_* is surjective and has as kernel the subgroup generated by α. □

Example 8.15. Consider the torus $T = X/\sim$, where X is the square and $\sim$ is the equivalence relation which identifies the opposite sides of ∂X (see Section 5). Recall (see Example 6.11) that T is obtained from a bouquet of two circles by attaching a 2-cell (see also the picture which comes with that example). The bouquet of two circles arises as $\partial X/\sim$, and we view it as a subspace of the torus as shown in the picture (Figure 1), where a labels the first circle $S^1 \vee \{p\} \subset T$ and b labels the second circle $\{p\} \vee S^1 \subset T$. The attaching map comes from the homeomorphism $\tilde{\chi} : S^1 \longrightarrow \partial X$ which transforms the circle to the boundary of the square (e.g. by taking 4 points on the circle and pulling them apart straightening the arcs between the points). The attaching map itself is $\chi = \pi \circ \tilde{\chi}$ where $\pi_0 : \partial X \longrightarrow \partial X/\sim = S^1 \vee S^1$ is the quotient map. In other words, when $x \in S^1$ goes one time around the circle in the counterclockwise direction starting at q, during the first quarter of the circle $\chi(x)$ goes around the first circle $S^1 \vee \{p\} \subset T$ (in the direction of a), during the next quarter $\chi(x)$ covers the second circle $\{p\} \vee S^1 \subset T$ (in the direction of b), during the third quarter $\chi(x)$ goes again around the first circle $S^1 \vee \{p\} \subset T$ but in the direction opposite to a, and in the last quarter it goes again around the second circle $\{p\} \vee S^1 \subset T$ (in the direction opposite to b). Symbolically, we write $\chi(S^1) = aba^{-1}b^{-1}$.

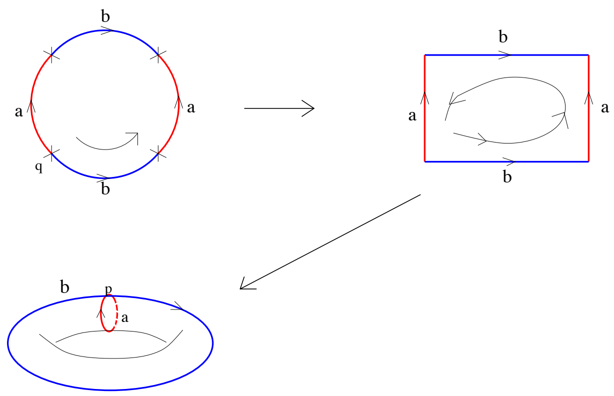


Figure 1.

We denote by the same letters a and b the elements

$$a, b \in \pi(T, p)$$

represented by the paths in T which go once around the two circles $S^1 \vee \{p\}$ and $\{p\} \vee S^1$, in the direction indicated in the picture. We see that the path defined by χ in $\pi(T, p)$ is precisely

$$\alpha = aba^{-1}b^{-1},$$

hence

$$\pi(T, p) = \pi(S^1 \vee S^1)/\langle aba^{-1}b^{-1}\rangle.$$

We will complete this computation in Example 8.20 below.

5. The case $N = \{1\}$

Assume now that $N = \{1\}$. The way to recognize the groups which have the universal property is described in the following lemma.

LEMMA 8.16. *Consider the starting data (1.1) with $N = \{1\}$. Then a triple (G, j_1, j_2) has the universal property if and only if, for any $g \in G$, there exists and are unique elements*

$$a_1, \ldots, a_n \in G_1, \; b_1, \ldots, b_n \in G_2$$

with $a_i \neq 1$ for all $i \geq 2$ and $b_j \neq 1$ for all $j \leq n-1$, such that

$$g = j_1(a_1)j_2(b_1)\ldots j_n(a_n)j_n(b_n). \tag{5.1}$$

Before we prove this lemma, we show how to construct a group with this property. We start by considering "words in G_1 and G_2", i.e. sequences

$$w = (g_1)(g_2)\ldots(g_n), \tag{5.2}$$

where each g_i is either in G_1 or in G_2. To avoid confusion and/or too complicated notations, we assume that G_1 and G_2 are disjoint (otherwise, if $a \in G_1 \cap G_2$, we would have to make the distinction between the word which contains a as an element of G_1, and the word which contains a as an element of G_2). One can realize this by taking a copy of G_2 which is disjoint from G_1.

We also allow the empty word

$$w_\emptyset = (\;).$$

For any two words w and w', one can consider a new word, ww', which is made from w and w' put next to each other (juxtaposition).

A word (5.2) is called reduced if, for each i

- g_i is neither the unit of G_1, nor the unit of G_2.
- g_i and g_{i+1} are not in the same group.

Starting with an arbitrary word w, one can always produce a reduced word w_{red} by applying repeatedly the following operations

- If g_i is the unit of G_1 or of G_2, delete it (and the parenthesis around it) from the w.
- If g_i and g_{i+1} belong to the same group, then replace w by

$$(g_1)\ldots(g_{i-1})(g_ig_{i+1})(g_{i+2})\ldots(g_n).$$

We now define

$$G_1 * G_2 := \{w : w \text{ is a reduced word in } G_1 \text{ and } G_2\},$$

which is a group with:

- the multiplication of two words given by

$$w * w' := (ww')_{\text{red}}.$$

- the unit element $w_\emptyset = (\;)$ (the empty word).

There are two group homomorphisms,

$$k_1 : G_1 \longrightarrow G, k_2 : G_2 \longrightarrow G$$

sending an element $g \in G_1$ or $g \in G_2$ to the word of length one (g).

We can now improve Lemma 8.16 as follows:

LEMMA 8.17. *Consider the starting data (1.1) with* $N = \{1\}$. *Then* $(G_1 * G_2, k_1, k_2)$ *has the universal property.*

For any other triple (G, j_1, j_2) *which has the universal property with respect to this data, one has a group isomorphism*

$$\phi : G_1 * G_2 \longrightarrow G,$$

uniquely determined by the condition that it sends $g_1 \in G_1$ *to* $j_1(g_1) \in G$ *and* $g_2 \in G_2$ *to* $j_2(g_2) \in G$.

PROOF. (of Lemma 8.16 and of Lemma 8.17). We first show that if (G, j_1, j_2) has the property described in Lemma 8.16, then it satisfies the universal property. This will apply in particular to $(G_1 * G_2, k_1, k_2)$ (which clearly has that property,). So, starting with (H, ϕ_1, ϕ_2), we have to find $\phi : G \longrightarrow H$ such that $\phi_1 = \phi j_1$, $\phi_2 = \phi j_2$. This means that

$$\phi(j_1(g_1)) = \phi_1(g_1), \phi(j_2(g_2)) = \phi_2(g_2)$$

for all $g_1 \in G_1$, $g_2 \in G_2$. For $g \in G$ arbitrary, consider its unique decomposition (5.1), we must have

$$\phi(g) = \phi_1(a_1)\phi_2(b_1)\dots\phi_1(a_n)\phi_2(b_n).$$

This formula determines ϕ: it proves the uniqueness of ϕ, and also provides us with the defining formula. One still has to show that ϕ is a group homomorphism, which is left as an exercise.

The last part of the Lemma 8.17 follows from the uniqueness insured by Lemma 8.3. This also implies the remaining part of Lemma 8.16, namely that if (G, j_1, j_2) has the universal property, then any g has a unique decomposition as in the lemma. Indeed, using the isomorphism $\phi : G_1 * G_2 \longrightarrow G$, it suffices to prove the similar property for $(G_1 * G_2, k_1, k_2)$, which is clear from the construction of the free product. □

COROLLARY 8.18. *(Corollary D) Let* X, U, V, x_0 *be as in the Seifert-van Kampen theorem. If* $\pi(U \cap V, x_0)$ *is trivial, then* $\pi(X, x_0)$ *is isomorphic to the free product of* $\pi(U, x_0)$ *and* $\pi(V, x_0)$. *More precisely, there is a unique group homomorphism*

$$\phi : \pi(U, x_0) * \pi(V, x_0) \longrightarrow \pi(X, x_0)$$

which sends $g_1 \in \pi(U, x_0)$ *to* $j_1(g_1)$ *and* $g_2 \in \pi(V, x_0)$ *to* $j_2(g_2)$, *and* ϕ *is an isomorphism.*

REMARK 8.19. (generators and relations): To be able to write down some of the groups we obtain, it is useful to represent groups by generators and relations.

First of all, starting with $G_1 = F(a_1)$-the free group (= the infinite cyclic group) with generator a_1:

$$F(a) = \{(a_1)^n : n \in \mathbb{Z}\},$$

and G_2-the free group with generator a_2, the free product $G_1 * G_2$ is the free group with two generators, denoted $F(a, b)$:

$$F(a_1, a_2) = F(a_1) * F(a_2).$$

The free group in three generators is defined similarly

$$F(a_1, a_2, a_3) = F(a_1, a_2) * F(a_3).$$

Inductively, one defines the free group generated by finite number of generators. More generally, for any set S, we can define the free group generated by S, denoted by

$$F(S).$$

This means that we have generators a_s, one for each $s \in S$, and each element of $g \in F(S)$ different from the identity element can be uniquely written as a product

$$g = a_{s_1}^{n_1} \dots a_{s_p}^{n_p},$$

with $p \in \mathbb{Z}$ positive integer, $n_1, \ldots, n_p \in \mathbb{Z}$ non-zero, and $s_1, \ldots, s_p \in S$ with $s_i \neq s_{i+1}$ for all i.

Given a subset $R \subset F(S)$, we denote by

$$\langle R \rangle \subset F(S)$$

the smallest normal subgroup of $F(S)$ containing R (the intersection of all normal subgroups of $F(S)$ containing R), and we will consider the quotient:

$$F(S)/\langle R \rangle.$$

By considering this quotient, what we actually do is to force the elements of R to be trivial. To write a group as such a quotient $F(S)/\langle R \rangle$ is known as describing the group by generators (the elements of S) and relations (the elements of R).

A group can be represented by generators and relations in many ways. For instance, $(\mathbb{Z}^2, +)$ can be written as

$$F(a, b)/\langle aba^{-1}b^{-1} \rangle, \ F(a, b, c)/\langle aba^{-1}c, bc \rangle.$$

On the other hand, any group G can be realized in this way, i.e. there exist S and R such that G is isomorphic to $F(S)/\langle R \rangle$ (the problem is that there are many such choices). For instance, one could take $S = G$, $R = \{a_{gg'}a_{g'}^{-1}a_g^{-1} : g, g' \in G\} \cup \{a_1\}$.

Example 8.20. Let us consider a bouquet of two circles $X = S^1 \vee S^1$, and let $p \in X$ be the common point of the two circles. See Figure 2. Let U_0 be a small neighborhood of p in the

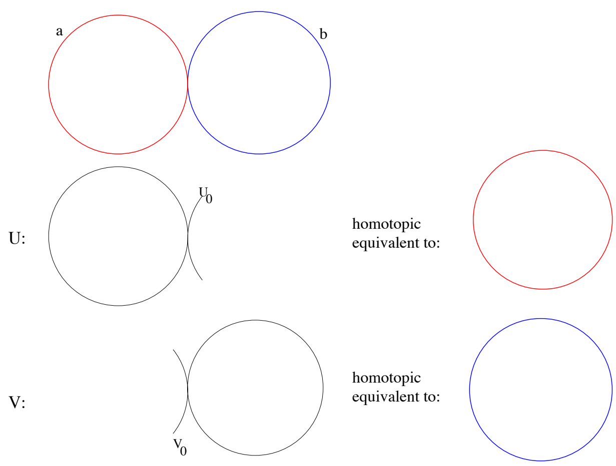


Figure 2.

second circle, let V_0 be a small neighborhood of p in the first circle, and let

$$U = S^1 \vee U_0, \ V = V_0 \vee S^1.$$

The intersection $U \cap V$ is contractible to p, hence we can apply the previous result. Note also that the inclusion

$$f_1 : S^1 = S^1 \vee \{p\} \hookrightarrow U$$

is a homotopy equivalence. Passing to fundamental groups, since $\pi(S^1)$ is free in one generator, we find that $\pi(U, p) = \langle a \rangle$ is the free group in one generator a, where a is the homotopy class of the path which starts and ends at p going around the first circle once. Similarly, $\pi(V, p) = \langle b \rangle$ is the free group in one generator b, where b is the homotopy class of the path which starts and ends at p going around the second circle once. We deduce from the previous corollary that $\pi(S^1 \vee S^1, p)$

$$\pi(S^1 \vee S^1, p) = F(a, b)$$

is the free group in two generators a and b shown in the picture.

We can now complete the computation of the fundamental group of the torus started in Example 8.15. We find

$$\pi(T^2, p) = F(a,b)/\langle aba^{-1}b^{-1}\rangle$$

which is the commutative group in two generators, hence isomorphic to $\mathbb{Z}^2$.

EXAMPLE 8.21. Consider $Y = S^1 \vee S^1 \vee S^1$ a bouquet of three circles. We can use the previous example in two ways: both the result obtained there, as well as the idea for the computation. First of all, take $X = S^1 \vee S^1 \vee \{p\}$ (p is the common point of the circles), $U = S^1 \vee S^1 \vee U_0$, $V = W_0 \vee V_0 \vee S^1$, with U_0, V_0, W_0-small neighborhoods of p. The intersection $U \cap V$ is contractible. The inclusion $X \hookrightarrow U$ is a homotopy equivalence, hence $\pi(U,p) = \langle a_1, a_2\rangle$ is the free group in two generators: a_1-represented by the path which goes once around the first circle, and a_2-represented by the path which goes once around the second circle. Similarly, $\pi(V,p) = \langle a_3\rangle$ is the free group in one generator with a_3-represented by the path which goes once around the third circle. Applying the previous proposition, we find $\pi(Y,p) = \langle a_1, a_2, a_3\rangle$. Applying repeatedly this argument, we find the fundamental group of any finite bouquet of circles:

$$\pi(\underbrace{S^1 \vee \ldots \vee S^1}_{n-\text{times}}, p) = F(a_1, \ldots, a_n),$$

the free group in n generators.

6. The general case

To explain the general case (i.e. when no restriction are made on the starting data (1.1)), we will represent our groups using generators and relations (explained in Remark 8.19). The Seifert- van Kampen theorem takes the following form.

COROLLARY 8.22. *Let X, U, V, x_0 be as in the Seifert-van Kampen theorem. Assume that*

$$\pi(U, x_0) = F(S_1)/\langle R_1\rangle,$$
$$\pi(V, x_0) = F(S_2)/\langle R_2\rangle$$
$$\pi(U \cap V, x_0) = F(S)/\langle R\rangle,$$

where S_1 and S_2 are chosen to be disjoint. For each $s \in S$, choose $f_s \in F(R_1)$ such that

$$i_1(a_s) = f_s\langle R_1\rangle \in F(S_1)/\langle R_1\rangle,$$

and similarly choose $g_s \in F(R_2)$ such that $i_2(a_s) = g_s\langle R_2$. Consider

$$R' = \{f_s g_s^{-1} : s \in S\} \subset F(S_1 \cup S_2).$$

Then

$$\pi(X, x_0) = F(S_1 \cup S_2)/\langle R_1 \cup R_2 \cup R'\rangle.$$

7. Some more exercises

Here are some more exercises about the fundamental group.

EXERCISE 8.2. Prove that

$$X = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \neq 0\}$$

is not homeomorphic to $\mathbb{R}^3$. Also, compute the fundamental group of X.

EXERCISE 8.3. Compute the fundamental group of the spaces appearing in Figures 8 and 9 of Section 4, Chapter 6.

EXERCISE 8.4. Compute the fundamental group of the spaces appearing in Figures 6 of Chapter 7, Section 1.

EXERCISE 8.5. Compute the fundamental group of the spaces appearing in Figure 10 of Section 4, Chapter 6. Do the same for the spaces from Figure 13 of the same section.

EXERCISE 8.6. Compute the fundamental group of all the spaces appearing in Exercises 6.22, 4.21, 6.24, 6.25.

EXERCISE 8.7. Compute the fundamental group obtained by removing one point from the torus.

EXERCISE 8.8. Compute the fundamental group obtained by removing two points from the sphere. Similarly for the torus instead of the sphere.

EXERCISE 8.9. Go back to exercise 6.26, and compute the fundamental group of the double torus.

EXERCISE 8.10. Compute the fundamental group of the Klein bottle.

EXERCISE 8.11. Find a topological space whose fundamental group is isomorphic to G in each of the following cases:

(i) $G = \mathbb{Z}_n$.
(ii) $G = \mathbb{Z}_n \times \mathbb{Z}_m$.
(iii) $G = \mathbb{Z}_n * \mathbb{Z}_m$.

where n and m are arbitrary positive integers.
(Hint: look at exercise 6.27).

EXERCISE 8.12. Let x_0 be a point in the torus p. We know that any homeomorphism $f : T \longrightarrow T$ such that $f(x_0) = x_0$ induces a group isomorphism $f_* : \pi(T, x_0) \longrightarrow \pi(T, x_0)$. Show the converse: any group isomorphism arises in this way (for some f).

EXERCISE 8.13. Find a map $f : T \longrightarrow S^1$ which is not null-homotopic (i.e. is not homotopic to the constant map). Similarly for the Moebius band instead of the torus.

EXERCISE 8.14. Show that any map $f : S^n \longrightarrow S^1$ is null-homotopic, when $n \geq 2$.

CHAPTER 9

How to compute fundamental groups: a summary and examples

1. Summary and some example

So: how does one compute fundamental groups? Assume one wants to compute the fundamental group of a space X. Recall that, if X is path connected, one may choose any (convenient) point as the base point (cf. Theorem 7.22). Here are some possible steps.

1.1. Step 0: Try to see if your space is homotopic equivalent to a space X' which is simpler or whose fundamental group you already know. Then you can replace X by X' (cf. Theorem 7.23).

1.2. Step 1: Try to find a cell decomposition of X. Then you can throw away the n-cells with $n \geq 3$ (cf. Proposition 8.10 or Corollary 8.11).

Cell decompositions can be found by looking at the picture. Another useful way to find a cell decomposition is to realize your space as a quotient space obtained from the square or the disc, by identifying certain points on the boundary: then one can use (maybe repeatedly) Lemma 6.9 which we now recall (in slightly different notations): Let X be a Hausdorff space which is obtained from D^n (or any other space homeomorphic to D^n) by identifying certain points on $\partial D^n = S^{n-1}$ and let $A = \partial D^n / \sim$. We denote by

$$\chi : S^{n-1} = \partial D^n \longrightarrow \partial D^n / \sim = A$$

the quotient map. Then X is obtained from A by attaching an n-cell whose characteristic map is χ.

The case $n = 2$ is particularly important: many spaces X can be obtained from D^2 by identifying certain parts of $\partial D^2 = S^1$, and the identification can be shown on the picture by labeling by letters the parts that are to be identified. In the quotient $A = \partial D^2 / \sim$, each letter will appear only once (because we identified all the parts labeled by the same letter). When going once around the circle, we will meet various labels that will give us a word whose letters are labels. Reading this word in the space A describes the characteristic map $\chi : S^1 \longrightarrow A$. We have already seen this in the case of the torus (see Figure 1): in that case A was a bouquet of two circles A and B (two circles touching each other in one point), and the characteristic map $\chi : S^1 \longrightarrow A$ gave us the word $aba^{-1}b^{-1}$, which we can read in the picture of A to describe the characteristic map itself.

1.3. Step 2: You have reduced the problem to the computation of the fundamental group of the 2-skeleton X_2 (what remains after throwing away the n-cells with $n \geq 3$). Try to apply Step 0 to X_2. Otherwise, use Proposition 8.14 to get rid of the 2-cells. Applying the proposition repeatedly for each 2-cell, you will find that the fundamental group you are interested in is isomorphic to the quotient of $\pi(X_1)$ by the smallest normal subgroup generated by paths induced by the characteristic maps of the 2-cells.

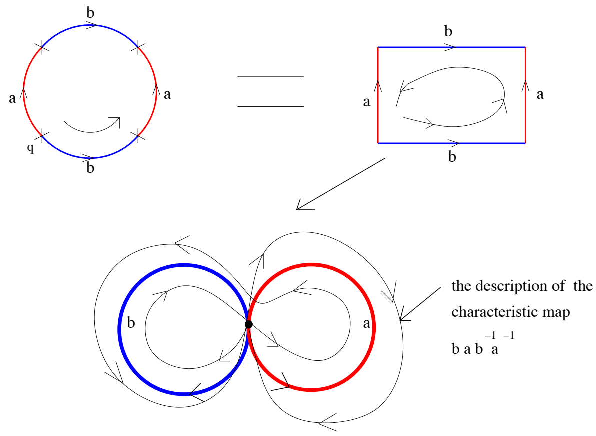


FIGURE 1.

1.4. Step 3: Hence you reduced the computation to the computation of the fundamental group of X_1. Try to apply Step 0 again. Otherwise, choose U and V inside X_1 so that you can apply Corollary D.

One should be aware that these steps should not be followed blindly- also think about simplifying them! For instance, the use of covering maps (e.g. of Exercise 7.23), of the fundamental group of the product (Exercise 7.19), or the trick of taking out points (Exercise 7.14) should not be underestimated.

Here are a few spaces whose fundamental group has been computed:

- The fundamental group of spaces like $\mathbb{R}^n$, D^n, $\overset{\circ}{D}{}^n$, or any other convex space in $\mathbb{R}^n$ is zero. In particular, since $S^n - \{p\}$ (a sphere minus a point) is homeomorphic to $\mathbb{R}^n$, its fundamental group is zero as well.
- The fundamental group of S^n is zero for all $n \geq 2$.
- The fundamental group of S^1 is the free group in one generator (hence isomorphic to $\mathbb{Z}$), where the generator is induced by the path $t \mapsto e^{2\pi t i}$. In particular, since spaces like
$$\mathbb{R}^2 - \{0\}, D^2 - \{0\}, \overset{\circ}{D}{}^2 - \{0\}$$
are all homotopic equivalent to S^1, the fundamental group of all such spaces is isomorphic to $(\mathbb{Z}, +)$.
- The fundamental group of a bouquet of n circles is the free group in n generators, each generator being defined by the path which goes once around one of the n circles.
- Various other spaces are homotopy equivalent to bouquets of circles. For instance -as we have seen- the torus T from which we remove one point is homotopic equivalent to a bouquet of two circles, hence its fundamental group is isomorphic to the free group in two generators.
- The fundamental group of the torus is isomorphic to $\mathbb{Z}^2$. This has been explained in this chapter using the Seifert van Kampen theorem.

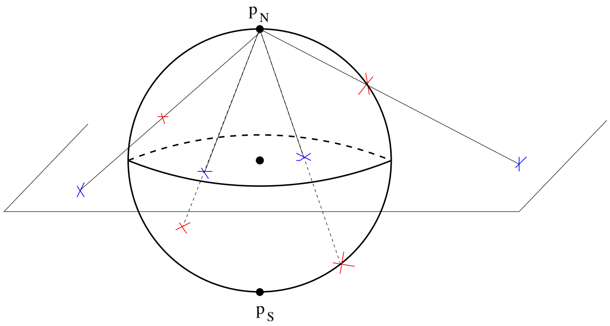
EXAMPLE 9.1. The fundamental group of $S^2 \times S^2$ can be computed using Exercise 7.19, and the result is the trivial group.

EXAMPLE 9.2. We have seen the computation of the fundamental group of the torus based on the Seifert van Kampen theorem. The advantage of this proof is that the same idea applies to many other examples. However, since the torus is homeomorphic to the product of two circles, it suffices to use Exercise 7.19 and immediately get the result.

EXAMPLE 9.3. Consider X obtained from the sphere by removing two points:

$$X = S^2 - \{p_N, p_S\}.$$

Since S^2 minus a point is homeomorphic to $\mathbb{R}^2$, X is homeomorphic to $\mathbb{R}^2 - \{0\}$ (by stereographic projection, see Figure 2), hence it is homotopic equivalent to the circle $S^1 \subset X$ (homotopy that can be seen directly). Hence the fundamental group of X (let's say with base point $(1, 0, 0)$) is isomorphic to $(\mathbb{Z}, +)$, with the generator induced by the path that goes once around the middle circle.



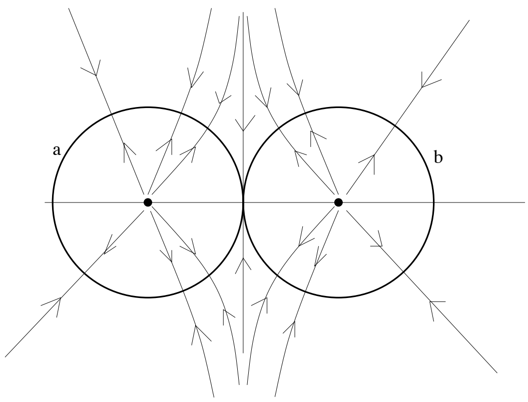
By stereographic projection (sending the red points to the blue ones), the spehere minus two points is homeomorphic to the plane minus the origin, hence it is homotopic equivalent to the midle circle

FIGURE 2.

EXAMPLE 9.4. Consider the Euclidean plane from which we remove two points. For instance, take

$$X = \mathbb{R}^2 - \{(-1, 0), (0, 1)\}.$$

Similar to the fact that $\mathbb{R}^2$ minus a point is homotopic equivalent to the circle, we have already seen that X is homotopic equivalent to a bouquet of two circles (Figure 3). We conclude that $\pi(X, 0)$ is a free group in two generators a and b shown in the picture.



$\mathbb{R}^2$ minus two points is homotopic equivalent to a bouquet of two circles

FIGURE 3.

2. The Moebius band

Let us look at the Moebius band M. We have already remarked that M is homotopic equivalent to S^1 (with S^1 sitting inside M as the middle circle, hence the homotopy group of M is isomorphic to $(\mathbb{Z}, +)$.

Let us compute this group differently, using a cell-decomposition of M. Since M can be described as a quotient of the square, as mentioned above, we can use Lemma 6.9 to find a cell decomposition of M. But that is precisely what we have done in the second part of the Example 6.12. The conclusion is that M is obtained from the space B shown in Figure 4 by adjoining a 2-cell, with characteristic map described in the picture. Hence $\pi(M)$ is isomorphic to $\pi(B)/\langle\alpha\rangle$, where α is the path which the characteristic map follows ("$cab^{-1}a$"). On the other hand, by

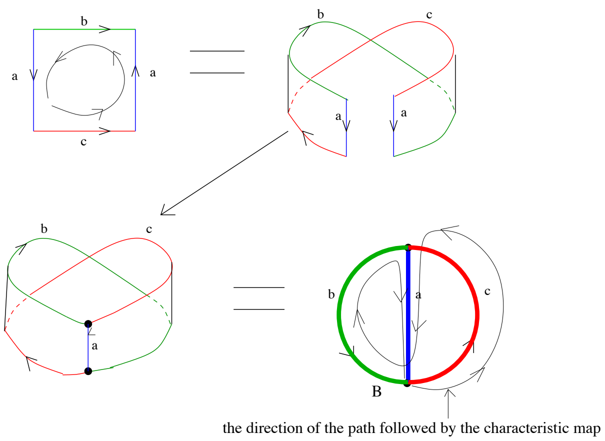


FIGURE 4.

collapsing "a" in the picture, we see that B is homotopic equivalent to a bouquet of two circles ("b" and "c"), and α is sent to cb^{-1}. Hence the group we are interested in is isomorphic to

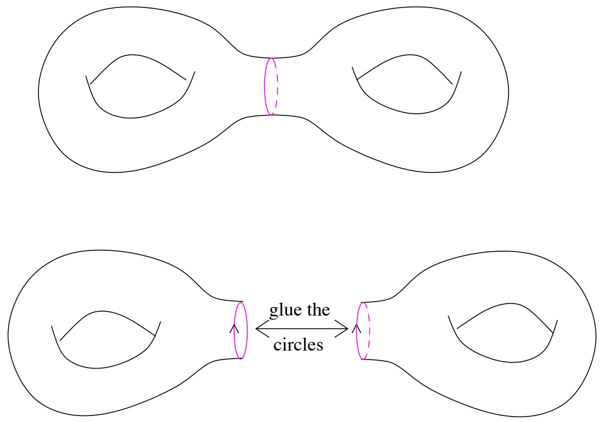
$$F(c, b)/\langle cb^{-1}\rangle$$

the quotient obtained from the free group in two generators c and b by imposing $cb^{-1} = 1$, i.e. the free group in one generator.

3. The double torus

Consider the double torus T_2 (Figure 5). It can be obtained as a quotient of D^2 in the following way (indicated already in the last two exercises of Section 5). First cut the double torus by a circle in the middle. We obtain two "cut tori", where a cut torus is obtained from a torus from which we remove a small open ball. Going backward, i.e. identifying the two cut torus along their boundary circles as shown in the picture, gives us back T_2. This is the description of T_2 as the "connected sum $T\#T$" (to be explained below).

So, let's first look at the cut torus T_c. Since the torus is obtained from the square by identifying its opposite points, a cut torus can be obtained from a square from which we remove a small ball e as shown in the picture (Figure 6), followed by the identification of its opposite sides. In turn, $[0, 1] \times [0, 1] - e$ can be cut at the origin $(0, 0)$ to produce a pentagon. Going back from the pentagon to the cut-tours amounts to making the identifications described by the labeling in the picture (the c-side is not identified with any other segment!). This describes the cut-torus as obtained from te pentagon by identifying its sides according to the labeling.



The double torus obtained from two cut–tori by glueing the boundary circles

FIGURE 5.

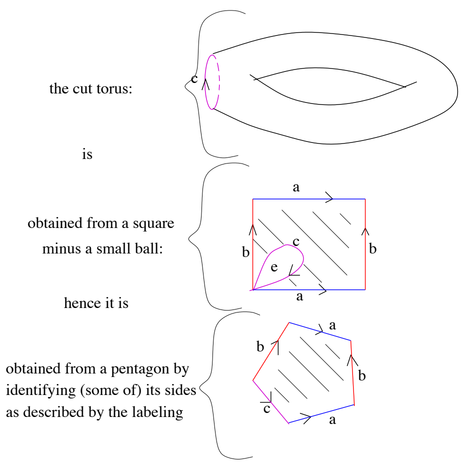


FIGURE 6.

Back to the double torus, we apply the previous construction to one of the cut torus, and we obtained a pentagon whose sides are labeled by a_1, b_1, c. We do the same for the other cut torus, and we obtain another pentagon whose sides are labelled by a_2, b_2, c. See Figure 22. The label c is the same because it represents the initial cut in the double torus; so, when glueing back, we also have to glue the two copies of c. However, one can start with gluing the two copies of c first, and leave the other identifications for later. This produces an octogone with the sides labelled as in the picture (Figure 7), and this gives a description of T_2 as obtained from an octagon by identifying its sides according to the labeling. To compute the fundamental group, we continue as before (e.g. as in the case of the torus, of the Moebius band, or of $\mathbb{P}^2$). We find that T_2 is obtained by adjoining to a bouquet of four circles (labeled a_1, b_1, a_2, b_2):

$$S^1 \vee S^1 \vee S^1 \vee S^1$$

a 2-cell with characteristic map $\chi : S^1 \longrightarrow S^1 \vee S^1 \vee S^1 \vee S^1$ described by

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}.$$

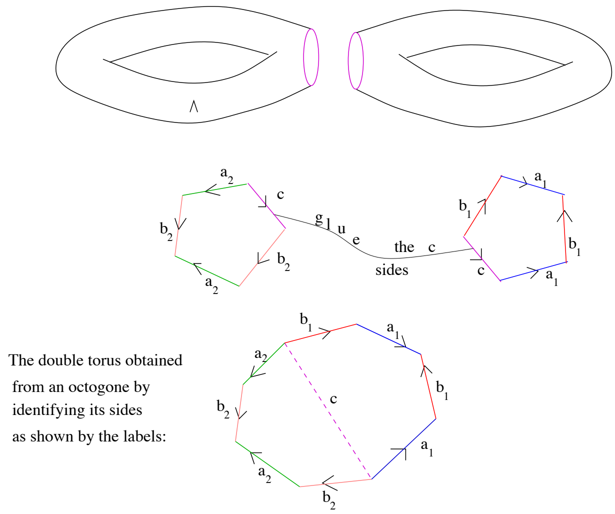


Figure 7.

Denoting by a_i, b_i the resulting elements in the fundamental group of $S^1 \vee S^1 \vee S^1 \vee S^1$, we find

$$\pi(T_2) \sim F(a_1, b_1, a_2, b_2)/\langle a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \rangle.$$

More generally, consider

$$T_g = \underbrace{T\# \ldots \# T}_{g \text{ times}},$$

the "connected sum" of g copies of the torus, which can be visualized as the torus with g wholes. Repeating the arguments above, one obtains a description of T_g as obtained starting from a $4g$-sided polygon with the labeling of its sides:

$$c_g = a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1},$$

and making the identifications dictated by the labeling. The fundamental group of T_g will be the quotient

$$\pi(T_g) \sim F(a_1, b_1, \ldots, a_g, b_g)/\langle c_g \rangle.$$

4. The projective space

Let us look at the projective space $\mathbb{P}^n$. As in the case of the torus and the Moebius band, it is not really necessary to use the Seifert-van Kampen theorem (however, it is again a good illustration of the theorem).

First of all, recall that $\mathbb{P}^n$ admits a cell decomposition with 2-skeleton equal to $\mathbb{P}^2$ -see Example 6.29- hence we reduce the computation to the case $n = 2$ (cf. Proposition 8.10 or Corollary 8.11). But we have seen (see Section 7) that $\mathbb{P}^2$ can be obtained from the square $[0,1] \times [0,1]$ by identifying its opposite sides as shown in the picture, hence, as above, we can use Lemma 6.9. We can actually use the picture of the Moebius band because $\mathbb{P}^2$ can be obtained by the identifications made to obtain the Moebius band, plus one more: one also identifies b^{-1} and c in the picture (the notation "b^{-1}" refers to the fact that we change the orientation). Hence, denoting by $\sim$ the equivalence relation on B which identifies b^{-1} and c, $\mathbb{P}^2$ is obtained from $A = B/\sim$ by attaching a 2-cell. Hence A is obtained from B by folding it along the middle vertical line, and the result is clearly (homeomorphic to) a circle. See Figure 8. Hence $\pi(A) = F(u)$ (with generator $u = ca$ on the picture). Also, the path followed by the characteristic map (which was "$cab^{-1}a$" in B) becomes "$caca$", i.e. it goes twice around the circle. Hence

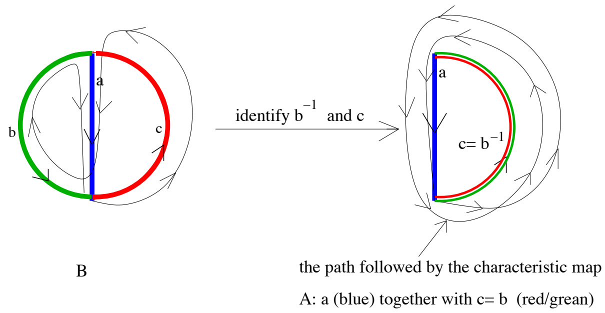


FIGURE 8.

the characteristic map induces in the fundamental group the element $u^2 \in \pi(A)$. Hence the fundamental group of $\mathbb{P}^2$ is $F(u)/\langle u^2 \rangle$, or, in additive notation, $\mathbb{Z}/2\mathbb{Z}$. Hence, for each $n \geq 2$,

$$\pi(\mathbb{P}^n) \cong \mathbb{Z}_2.$$

5. Surfaces

DEFINITION 9.5. *An n-dimensional topological manifold is a topological spaces M with the following properties:*

- *M is Hausdorff and second countable.*
- *it is locally homeomorphic to $\mathbb{R}^n$ i.e. for each $x \in M$, there exists a neighborhood U of x in M which is homeomorphic to $\mathbb{R}^n$.*

A (topological) surface is a 2-dimensional (topological) manifold.

There is a very basic operation which allows us to construct a new manifold out of two given one.

DEFINITION 9.6. *Given two topological manifolds M and N of the same dimension, define their connected sum, denoted $M \# N$ as follows: remove from M and N two "small balls" B_1 and B_2 and glue $M - B_1$ and $N - B_2$ along the sphere $\partial B_1 = \partial B_2$.*

For surfaces, it means that we remove two small disks and we glue the remaininig spaces along the bounday circles.

EXAMPLE 9.7. The connected sum $M \# S^2$ of any surface M with the sphere S^2 is homeomorphic to M itself.

EXAMPLE 9.8. The connected sum of two tori is the double torus. Repeating the operation of connected sum, one obtains all tori with arbitrary number of wholes:

$$T_g = \underbrace{T \# \ldots \# T}_{g \text{ times}}.$$

Similarly, one considers the connected sum of g copies of $\mathbb{P}^2$:

$$U_h = \underbrace{\mathbb{P}^2 \# \ldots \# \mathbb{P}^2}_{h \text{ times}}.$$

EXAMPLE 9.9. Example 1.23 tells us that, after removing a small disk from $\mathbb{P}^2$, one obtaines a Moebius band. On the other hand, Exercise 1.21 tells us that the Klein bottle can be obtained by gluing two copies of the Moebius band along the boundary circle. In other words, the Klein bottle can be constructed out of $\mathbb{P}^2$, as the connected sum $\mathbb{P}^2 \# \mathbb{P}^2$.

And here is one of the most beautiful theorems of topology:

THEOREM 9.10. *Any compact surface is homeomorphich to one, and only one, of the surfaces T_g with $g \geq 0$, or U_h with $h \geq 1$ (we define $T_0 = S^2$).*

What we are able to prove here is that any two of these spaces are not homeomorphic. The proof is based on the computation of the fundamental group. We have already done it for T_g. The computation for U_h is completely analogous. Starting from the fact that $U_1 = \mathbb{P}^2$ can be obtained from a square with its sides label led in the order

$$aabb = a^2b^2,$$

we find that U_h is obtained by starting from a $2h$-sided polygon with the labeling of its sides:

$$d_h = (a_1)^2 \dots (a_h)^2,$$

and making the identifications dictated by the labeling. Finally,

$$\pi(U_h) = F(a_1, \dots, a_h)/\langle (a_1)^2 \dots (a_h)^2 \rangle.$$

Since these fundamental groups are rather wild, in order to compare them, we will use one trick coming from group theory: their abelianization. Recall that, for a group G, one defines the abelianization of G as:

$$G_{\text{ab}} = G/\overline{\{ghg^{-1}h^{-1} : g, h \in G\}},$$

the quotient of G by the smallest normal subgroup containing all the elements of type $ghg^{-1}h^{-1}$ (in this way we force in the quotients all commutation relations). Two isomorphic groups have isomorphic abelianizations. Hence, to prove that the fundamental groups of the surfaces T_g and U_h are not isomorphic, it suffices to show that their abelianizations are not isomorphic. First of all, for T_g, we use the computation from Section 3: the abelianization is obtained by imposing the new relations which say that any two of the generators of $F(a_1, \dots, a_h)$ commute, and we obtain the abelian group in $2g$ generators

$$\pi(T_g)_{\text{ab}} \sim (\mathbb{Z}^{2g}, +).$$

In particular, T_g and $T_{g'}$ cannot be homeomorphic if $g \neq g'$. Similarly, for U_h, we will have $F(a_1, \dots, a_h)/\langle (a_1)^2 \dots (a_h)^2 \rangle$ to which we have to impose the new relations $a_ia_j = a_ja_i$ for all i and j. We find that $\pi(U_h)$ is isomorphic to the quotient of $\mathbb{Z}^h$ by the subgroup generated by

$$2a_1 + 2a_2 + \dots 2a_h$$

(note that we passed from the multiplicative notation for the group composition, to the additive notation- which is the one used fro $\mathbb{Z}^r$) where $a_i = (0, \dots, 0, 1, 0, \dots, 0)$ is the i-the generator (1 is on the i-th place). Hence we are talking about the subgroup

$$N_h = \{(2n, \dots, 2n) : n \in \mathbb{Z}\} \subset \mathbb{Z}^h.$$

It is not difficult to see that

$$\mathbb{Z}^h/N_h \mapsto \mathbb{Z}^{h-1} \times \mathbb{Z}_2, (n_1, \dots, n_h) \mapsto (n_1, \dots, n_{h-1}, n_1 + \widehat{\dots +}\, n_{h-1})$$

is an isomorphism, hence

$$\pi(U_h)_{\text{ab}} \sim \mathbb{Z}^{h-1} \times \mathbb{Z}_2.$$

This implies that U_h and $U_{h'}$ are not homeomorphic if $h \neq h'$ and that T_g is not homeomorphic to U_h for all g and h.

Contents