## Aula 13:

Lembre que:

- $T: V \to W$ linear
- $E = (e_1, \dots, e_n)$ base de $V$
- $F = (f_1, \dots, f_m)$ base de $W$

Matriz de $T$ com respeito às bases $E$, $F$

$$[T]_{FE} = \begin{pmatrix} | & & | & & | \\ T(e_1)_F & \cdots & T(e_j)_F & \cdots & T(e_n)_F \\ | & & | & & | \end{pmatrix}$$

A $j$-ésima coluna da matriz é formada pelas coordenadas de $T(e_j)$ na base $F$

- $$[T]_{FE} \begin{pmatrix} | \\ v \\ | \end{pmatrix}_E = \begin{pmatrix} | \\ T(v) \\ | \end{pmatrix}_F$$

## Exemplos:

1. $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(x,y,z) = (x+y, x-y+z)$

$E = (e_1, e_2, e_3)$ base canônica de $\mathbb{R}^3$: $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$F = (f_1, f_2)$ base canônica de $\mathbb{R}^2$: $f_1 = (1,0)$, $f_2 = (0,1)$

$$\begin{aligned} T(e_1) &= T(1,0,0) = (1,1) = (1,1)_F \\ T(e_2) &= T(0,1,0) = (1,-1) = (1,-1)_F \\ T(e_3) &= T(0,0,1) = (0,1) = (0,1)_F \end{aligned} \Rightarrow [T]_{FE} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

2) $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad T(x,y,z) = (x+y,\ x-y+z)$

$E = (e_1, e_2, e_3)$, $\quad e_1 = (1,0,1),\ e_2 = (0,1,-1),\ e_3 = (0,1,1)$

$F = (f_1, f_2)$, $\quad f_1 = (1,1),\ f_2 = (1,-1)$

- $T(e_1) = T(1,0,1) = (1,2)$

$T(e_1)$ Na base F: $\quad (1,2) = af_1 + bf_2 \Rightarrow (1,2) = (a+b,\ a-b) \Rightarrow$

$$\Rightarrow \begin{cases} a+b = 1 \\ a-b = 2 \end{cases} \Rightarrow a = \frac{3}{2},\ b = -\frac{1}{2} \Rightarrow \boxed{T(e_1) = \left(\frac{3}{2}, -\frac{1}{2}\right)_F}$$

- $T(e_2) = T(0,1,-1) = (1,-2)$

$T(e_2)$ na base F: $\quad (1,-2) = af_1 + bf_2 \Rightarrow (1,-2) = (a+b,\ a-b) \Rightarrow$

$$\Rightarrow \begin{cases} a+b = 1 \\ a-b = -2 \end{cases} \Rightarrow a = -\frac{1}{2},\ b = \frac{3}{2} \Rightarrow \boxed{T(e_2) = \left(-\frac{1}{2}, \frac{3}{2}\right)_F}$$

- $T(e_3) = T(0,1,1) = (1,0)$

$T(e_3)$ na base F: $\quad (1,0) = af_1 + bf_2 = (a+b,\ a-b) \Rightarrow$

$$\Rightarrow \begin{cases} a+b = 1 \\ a-b = 0 \end{cases} \Rightarrow a = b = \frac{1}{2} \Rightarrow T(e_3) = \left(\frac{1}{2}, \frac{1}{2}\right)_F$$

Logo $\qquad [T]_{FE} = \begin{pmatrix} 3/2 & -1/2 & 1/2 \\ -1/2 & 3/2 & 1/2 \end{pmatrix}$

3) $T: P_2(\mathbb{R}) \longrightarrow \mathcal{M}_{2\times 2}$

$$T(p(x)) = \begin{pmatrix} p'(0) & p(1) - p(0) \\ p(-1) & p''(1) + p'(2) \end{pmatrix}$$

(Mostre que é linear!)

$$E = (\overset{e_1}{1}, \overset{e_2}{x}, \overset{e_3}{x^2})$$

$$F = \left( \underset{f_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{f_2}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}, \underset{f_3}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{f_4}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \right)$$

- $T(e_1) = T(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (1,0,0,0)_F$

- $T(e_2) = T(x) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = a f_1 + b f_2 + c f_3 + d f_4 = \begin{pmatrix} a & b+d \\ b-d & c \end{pmatrix}$

$$\Rightarrow \begin{cases} a = 1 \\ b+d = 1 \\ b-d = -1 \\ c = 1 \end{cases} \Rightarrow \begin{matrix} a=1 \\ b=0 \\ c=1 \\ d=1 \end{matrix} \Rightarrow T(e_2) = (1,0,1,1)_F$$

- $T(e_3) = T(x^2) = \begin{pmatrix} 0 & 1 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} a & b+d \\ b-d & c \end{pmatrix}$

$$\Rightarrow \begin{cases} a = 0 \\ b+d = 1 \\ b-d = 1 \\ c = 6 \end{cases} \Rightarrow \begin{matrix} a=0 \\ b=1 \\ c=6 \\ d=0 \end{matrix} \Rightarrow T(e_3) = (0,1,6,0)_F$$

Logo, $[T]_{FE} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 6 \\ 0 & 1 & 0 \end{pmatrix}$

<u>OBS</u>: $a+bx+cx^2 = (a,b,c)_E$, Logo, para encontrar as coordenadas de $T(a+bx+cx^2)$ na base $F$, temos que calcular

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 6 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+b \\ c \\ b+6c \\ c \end{pmatrix} \Rightarrow T(a+bx+cx^2) = (a+b, c, b+6c, c)_F$$

Por exemplo, $T(1-x^2) = (1,-1,-6,-1)_F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - 6\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & -6 \end{pmatrix}$

- Composta de Transformações Lineares

- Seja $U, V, W$ espaço vetoriais
- Seja $S: V \to W$, $T: U \to V$ lineares

Prop: $S \circ T : U \to W$ é linear

Dem: • $(S\circ T)(u_1+u_2) = S(T(u_1+u_2)) = S(T(u_1)+T(u_2)) = S(T(u_1)) + S(T(u_2))$

$= (S\circ T)(u_1) + (S\circ T)(u_2)$ ✓

• $(S\circ T)(\lambda u) = S(T(\lambda u)) = S(\lambda T(u)) = \lambda S(T(u)) = \lambda (S\circ T)(u)$ ✓

∎

Sejam $E = (e_1, \dots, e_n)$ base de $U$

$F = (f_1, \dots, f_m)$ base de $V$

$G = (g_1, \dots, g_\ell)$ base de $W$

Pergunta: Como descrever $[S\circ T]_{GE}$ em termos de $[S]_{GF}$ e $[T]_{FE}$ ?

Por definição:

$$[S\circ T]_{GE} = \begin{pmatrix} | & & | & & | \\ (S\circ T)(e_1)_G & \cdots & (S\circ T)(e_j)_G & \cdots & (S\circ T)(e_n)_G \\ | & & | & & | \end{pmatrix}$$

Vamos calcular $(S\circ T)(e_j)_G = S(T(e_j))_G$

Sabemos que $\begin{pmatrix} | \\ T(e_j) \\ | \end{pmatrix}_F = [T]_{FE} \begin{pmatrix} | \\ e_j \\ | \end{pmatrix}_E = [T]_{FE} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ j-ésima linha

Suponha que $[T]_{FE} = \begin{pmatrix} t_{11} & \cdots & t_{1j} & \cdots & t_{1n} \\ t_{21} & \cdots & t_{2j} & \cdots & t_{2n} \\ \vdots & & & & \\ t_{m1} & \cdots & t_{mj} & \cdots & t_{mn} \end{pmatrix}$

$$\Rightarrow [T]_{FE}\begin{pmatrix} | \\ e_j \\ | \end{pmatrix}_E = \begin{pmatrix} | \\ T(e_j) \\ | \end{pmatrix}_F = \begin{pmatrix} t_{1j} \\ t_{2j} \\ \vdots \\ t_{mj} \end{pmatrix}$$

Sabemos também que

$$[S]_{GF}\begin{pmatrix} | \\ T(e_j) \\ | \end{pmatrix}_F = \begin{pmatrix} | \\ (S\circ T)(e_j) \\ | \end{pmatrix}_G$$

Logo,

$$[S\circ T]_{GE}\begin{pmatrix} | \\ e_j \\ | \end{pmatrix}_E = \begin{pmatrix} | \\ (S\circ T)(e_j) \\ | \end{pmatrix}_G = \begin{pmatrix} | \\ S(T(e_j)) \\ | \end{pmatrix}_G =$$

$$= [S]_{GF}\begin{pmatrix} | \\ T(e_j) \\ | \end{pmatrix}_F = [S]_{GF}\begin{pmatrix} t_{1j} \\ t_{2j} \\ \vdots \\ t_{mj} \end{pmatrix}$$

Logo, se $[S]_{GF} = \begin{pmatrix} s_{11} & \cdots & s_{1m} \\ \vdots & & \vdots \\ s_{\ell 1} & \cdots & s_{\ell m} \end{pmatrix}$

$$\Rightarrow [S\circ T]_{GE}\begin{pmatrix} | \\ e_j \\ | \end{pmatrix}_E = \begin{pmatrix} | \\ S\circ T(e_j) \\ | \end{pmatrix}_G = \begin{pmatrix} s_{11}t_{1j} + s_{12}t_{2j} + \cdots + s_{1m}t_{mj} \\ s_{21}t_{1j} + s_{22}t_{2j} + \cdots + s_{2m}t_{mj} \\ \vdots \\ s_{\ell 1}t_{1j} + s_{\ell 2}t_{2j} + \cdots + s_{\ell m}t_{mj} \end{pmatrix}$$

Ou seja

Produto de matrizes

$$\boxed{[S\circ T]_{GE} = [S]_{GF}\cdot[T]_{FE}}$$

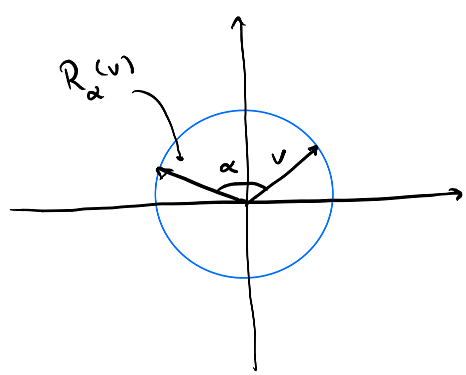
<u>OBS</u>: Se vc nunca viu o produto de matrizes, tome isso como a definição. Ou seja,

$$\begin{pmatrix} s_{11} & \cdots & s_{1m} \\ \vdots & & \vdots \\ s_{l1} & \cdots & s_{lm} \end{pmatrix} \begin{pmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & & \vdots \\ t_{m1} & \cdots & t_{mn} \end{pmatrix} = \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & & \vdots \\ r_{l1} & \cdots & r_{ln} \end{pmatrix}$$

onde

$$\boxed{r_{ij} = \sum_{k=1}^{m} s_{ik} t_{kj}}$$

<u>Aplicação</u>: Seja $\alpha \in \mathbb{R}$ e considere a função $R_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ onde $R_\alpha(v)$ é o vetor obtido de $v$ rotacionando por um ângulo de $\alpha$ no sentido anti-horário




• $R_\alpha$ é linear e $R_\alpha \circ R_\beta = R_{\alpha+\beta}$

Seja $\mathcal{E} = (e_1, e_2)$ a base canônica de $\mathbb{R}^2$

Note que
$$\left.\begin{aligned} R_\alpha(e_1) &= (\cos\alpha, \operatorname{sen}\alpha) \\ R_\alpha(e_2) &= (-\operatorname{sen}\alpha, \cos\alpha) \end{aligned}\right) \text{ verifique! (Exercício)}$$

Segue que $$[R_\alpha]_{\mathcal{E}\mathcal{E}} = \begin{pmatrix} \cos\alpha & \operatorname{sen}\alpha \\ -\operatorname{sen}\alpha & \cos\alpha \end{pmatrix}$$

logo

$$[R_{\alpha+\beta}]_{\mathcal{E}\mathcal{E}} = [R_\alpha \circ R_\beta]_{\mathcal{E}\mathcal{E}} = [R_\alpha]_{\mathcal{E}\mathcal{E}}\,[R_\beta]_{\mathcal{E}\mathcal{E}}$$

ou seja

$$\begin{pmatrix} \cos(\alpha+\beta) & \operatorname{sen}(\alpha+\beta) \\ -\operatorname{sen}(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} = \begin{pmatrix} \cos\alpha & \operatorname{sen}\alpha \\ -\operatorname{sen}\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & \operatorname{sen}\beta \\ -\operatorname{sen}\beta & \cos\beta \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\alpha\cos\beta - \operatorname{sen}\alpha\operatorname{sen}\beta & \cos\alpha\operatorname{sen}\beta + \operatorname{sen}\alpha\cos\beta \\ -\operatorname{sen}\alpha\cos\beta - \cos\alpha\operatorname{sen}\beta & -\operatorname{sen}\alpha\operatorname{sen}\beta + \cos\alpha\cos\beta \end{pmatrix}$$

ou seja:

$$\boxed{\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \operatorname{sen}\alpha\operatorname{sen}\beta \\ \operatorname{sen}(\alpha+\beta) &= \cos\alpha\operatorname{sen}\beta + \operatorname{sen}\alpha\cos\beta \end{aligned}}$$