Aula 11:

Teorema: Sejam $S_1, S_2 \subset V$ subespaços de dimensão finita. Então $\dim(S_1+S_2) = \dim S_1 + \dim S_2 - \dim S_1 \cap S_2$

Dem: Sejam:

$$\dim S_1 \cap S_2 = \ell$$
$$\dim S_1 = m$$
$$\dim S_2 = n$$

Seja $E = (e_1, \dots, e_\ell)$ base de $S_1 \cap S_2$.

Complete $e_1, \dots, e_\ell$ para

- $F = (e_1, \dots, e_\ell, f_{\ell+1}, \dots, f_m)$ Base de S_1
- $G = (e_1, \dots, e_\ell, g_{\ell+1}, \dots, g_n)$ Base de S_2

Afirmação: $B = (e_1, \dots, e_\ell, f_{\ell+1}, \dots, f_m, g_{\ell+1}, \dots, g_n)$ é base de S_1+S_2

De fato:

(i) $S_1+S_2 = [e_1, \dots, e_\ell, f_{\ell+1}, \dots, f_m, g_{\ell+1}, \dots, g_n]$ pois se

$w = v_1+v_2 \in S_1+S_2 \Rightarrow$ $v_1 \in S_1 \Rightarrow v_1 = x_1 e_1 + \dots + x_\ell e_\ell + y_{\ell+1} f_1 + \dots + y_m f_m$

$v_2 \in S_2 \Rightarrow v_2 = a_1 e_1 + \dots + a_\ell e_\ell + z_{\ell+1} g_{\ell+1} + \dots + z_n g_n$

$\Rightarrow v_1+v_2 = (x_1+a_1)e_1 + \dots + (x_\ell+a_\ell)e_\ell + y_{\ell+1} f_{\ell+1} + \dots + y_m f_m + z_{\ell+1} g_{\ell+1} + \dots + z_n g_n$

$\Rightarrow v_1+v_2 \in [e_1, \dots, e_\ell, f_{\ell+1}, \dots, f_m, g_{\ell+1}, \dots, g_n]$.

(ii) $e_1, \dots, e_\ell, f_{\ell+1}, \dots, f_m, g_{\ell+1}, \dots, g_n$ L.I.:

Se $x_1 e_1 + \dots + x_\ell e_\ell + y_{\ell+1} f_{\ell+1} + \dots + y_m f_m + z_{\ell+1} g_{\ell+1} + \dots + z_n g_n = 0$

$\Rightarrow \sum_{i=1}^{n-\ell} z_{\ell+i} g_{\ell+i} \in S_1 \cap S_2 \Rightarrow \sum z_{\ell+i} g_{\ell+i} = \sum a_i e_i \Rightarrow z_{\ell+1} = \dots = z_n = 0$

$\Rightarrow \sum x_i e_i + \sum y_{\ell+j} f_{\ell+j} = 0 \Rightarrow x_1 = x_2 = \dots = x_\ell = y_{\ell+1} = \dots = y_m = 0$

Logo,

$$\dim(S_1+S_2) = \ell + (m-\ell) + (n-\ell) = m+n-\ell = \dim S_1 + \dim S_2 - \dim S_1 \cap S_2$$

$\blacksquare$

<u>Exemplo</u>: $V = \mathbb{R}^4$

$$S_1 = \left\{ (x,y,z,w) \;\middle|\; \begin{cases} x-y+z+w=0 \\ x-z=0 \end{cases} \right\}$$

$$S_2 = [(1,2,1,1),(0,0,0,1)]$$

$\Rightarrow$ $\dim S_1 = 2$ (Exercício!)

$\dim S_2 = 2$

<u>$S_1 \cap S_2$</u>:

$v \in S_2 \Leftrightarrow v = (a, 2a, a, a+b)$, $a, b \in \mathbb{R}$

$v \in S_1 \cap S_2 \Leftrightarrow b = -a$ (substitui nas equações p/ S_1)

$\Rightarrow$ $S_1 \cap S_2 = [(1,2,1,0)]$

$\Rightarrow$ $\dim S_1 \cap S_2 = 1$

$\Rightarrow$ $\dim(S_1+S_2) = 2+2-1 = 3$

$\boxed{\text{Resto da Aula: Plantão de Dúvidas p/ a Prova}}$