① $T(x,y,z) = (x-2y+z,\ x+y-z,\ x-5y+3z)$

a) T é linear:

(i) $\underline{T(u+v) = T(u)+T(v)}$: $\quad u=(x,y,z)$, $\quad v=(a,b,c)$

$T(u+v) = T(x+a,\ y+b,\ z+c) =$

$= (x+a-2(y+b)+z+c,\ x+a+y+b-(z+c),\ x+a-5(y+b)+3(z+c)) =$

$= (x-2y+z+a-2b+c,\ x+y-z+a+b-c,\ x-5y+3z+a-5b+3c)$

$= (x-2y+z,\ x+y-z,\ x-5y+3z) + (a-2b+c,\ a+b-c,\ a-5b+3c)$

$= T(u) + T(v)$

(ii) $\underline{T(\lambda u) = \lambda T(u)}$:

$T(\lambda u) = T(\lambda x, \lambda y, \lambda z) = (\lambda x - 2\lambda y + \lambda z,\ \lambda x + \lambda y - \lambda z,\ \lambda x - 5\lambda y + 3\lambda z)$

$= (\lambda(x-2y+z), \lambda(x+y-z), \lambda(x-5y+3z)) = \lambda(x-2y+z, x+y-z, x-5y+3z)$

$= \lambda T(u)$

$\Rightarrow$ T é linear.

(b) $(x,y,z) \in N(T) \Leftrightarrow \begin{cases} x-2y+z=0 & ① \\ x+y-z=0 & ② \\ x-5y+3z=0 & ③ \end{cases}$

① $\Rightarrow \ x = 2y - z$

② $\Rightarrow \ 2y - z + y - z = 0 \Rightarrow 3y - 2z = 0 \Rightarrow y = \frac{2}{3}z$

$\Rightarrow x = \frac{1}{3}z$

③ $\Rightarrow \ \frac{1}{3}z - \frac{10}{3}z + 3z = 0 \Rightarrow 0 = 0$ ✓

$\Rightarrow N(T) = \{(\frac{1}{3}z, \frac{2}{3}z, z) \mid z \in \mathbb{R}\} = [(1/3, 2/3, 1)]$

Base: $B = ((1/3, 2/3, 1))$

$\dim N(T) = 1$

(c) $\mathrm{Im}(T) = \{(x-2y+z,\ x+y-z,\ x-5y+3z) \mid x,y,z \in \mathbb{R}\}$

$= \{x(1,1,1) + y(-2,1,-5) + z(1,-1,3) \mid x,y,z \in \mathbb{R}\}$

$= [\underbrace{(1,1,1)}_{v_1}, \underbrace{(-2,1,-5)}_{v_2}, \underbrace{(1,-1,3)}_{v_3}]$

Vamos verificar se v_1, v_2, v_3 são L.I.

$$x v_1 + y v_2 + z v_3 = 0 \iff \begin{cases} x - 2y + z = 0 \\ x + y - z = 0 \\ x - 5y + 3z = 0 \end{cases}$$

$\Rightarrow \quad x = \frac{1}{3} z, \quad y = \frac{2}{3} z, \quad z = z$ (sistema indeterminado)

Uma solução não trivial: $(z = 3)$

$$v_1 + 2v_2 + 3v_3 = 0$$

$\Rightarrow \mathrm{Im}(T) = [v_1, v_2] = [(1,1,1), (-2,1,-5)]$

$v_1 \nparallel v_2 \Rightarrow$ L.I. $\Rightarrow B = ((1,1,1),(-2,1,-5))$ é base de

$\mathrm{Im}(T)$ e $\dim \mathrm{Im}(T) = 2$.

2) $S_1 = [(1,0,1,-1), (-1,0,-1,2)]$

$S_2 = S_1^\perp$, $S_3 = [(-1,2,-2,1), (2,-1,4,2)]$

a) Sejam $v_1 = (1,0,1,-1)$, $v_2 = (-1,0,-1,2)$ (geradores de S_1)

Note que $u \in S_2 \iff \langle u, v_1 \rangle = 0$ e $\langle u, v_2 \rangle = 0$

De fato: ($\Rightarrow$) é imediato pois $v_1, v_2 \in S_1$

($\Leftarrow$) Suponha $\langle u, v_1 \rangle = 0 = \langle u, v_2 \rangle$. Se $v \in S_1 \Rightarrow v = av_1 + bv_2$

$\Rightarrow \langle u, v \rangle = \langle u, av_1 + bv_2 \rangle = \langle u, av_1 \rangle + \langle u, bv_2 \rangle =$

$= a\langle u, v_1 \rangle + b\langle u, v_2 \rangle = 0 + 0 = 0 \Rightarrow u \in S_2$

Logo, $S_2 = \{(x,y,z,w) \in \mathbb{R}^4 \mid \begin{cases} x+z-w = 0 \\ -x-z+2w = 0 \end{cases} \}$

Resolvendo o sistema: $w = 0$, $x = -z$, y arbitrário, z arbitrário

$\Rightarrow S_2 = \{(-z, y, z, 0) \mid y, z \in \mathbb{R}\} =$

$= [(0,1,0,0), (-1,0,1,0)]$

Base $B_2 = ((0,1,0,0), (-1,0,1,0))$; $\dim S_2 = 2$

(b) $v \in S_3 \iff v = a(-1,2,-2,1) + b(2,-1,4,-2) =$

$= (-a+2b, 2a-b, -2a+4b, a-2b)$

$v \in S_2 \cap S_3 \iff v = (-a+2b, 2a-b, -2a+4b, a-2b)$ e

$v \in S_2$

$\iff \begin{cases} -a+2b - 2a+4b - a+2b = 0 \\ a-2b + 2a - 4b + 2a - 4b = 0 \end{cases} \iff \begin{cases} -4a + 8b = 0 \\ 5a - 10b = 0 \end{cases}$

$\iff a = 2b$

$\iff v = (0, 3b, 0, 0)$ Logo $S_2 \cap S_3 = [(0,1,0,0)]$

$B = ((0,1,0,0))$, $\dim(S_2 \cap S_3) = 1$

(c) $v_1 = (0,1,0,0) \in S_2 \Rightarrow v_1 \in S_2 + S_3$

Vamos encontrar $v_2 \in S_2 + S_3$ t.q. $v_2 \notin [v_1] = S_2 \cap S_3$

Tome $v_2 = (-1,0,1,0) \in S_2 \Rightarrow v_2 \notin S_2 \cap S_3$

v_1, v_2 L.I.

Vamos encontrar $v_3 \in S_3$ ($\Rightarrow v_3 \in S_2 + S_3$) t.q. $v_3 \notin [v_1, v_2]$

Note que se $(x,y,z,w) \in [v_1, v_2] \Rightarrow w = 0$

Logo $v_3 = (-1,2,-2,1) \in S_2 + S_3$ e $v_3 \notin [v_1, v_2]$

Como $\dim(S_2+S_3) = \dim S_2 + \dim S_3 - \dim(S_2 \cap S_3)$

$= 2 + 2 - 1 = 3$

e $v_1, v_2, v_3 \in S_2 + S_3$ são L.I.

$\Rightarrow$ $\boxed{\begin{array}{l} B = (v_1, v_2, v_3) \text{ é base de } S_2+S_3 \\ \dim(S_2+S_3) = 3 \end{array}}$

③ a) <u>Falso:</u>

Contra-exemplo: Em $\mathbb{R}^2$, $S=[(1,0)]$

$v_1=(1,0)$, $v_2=(2,0)$, $v_3=(0,1)$

$v_1, v_2 \in S$, $v_3 \notin S$ e v_1, v_2, v_3 são L.D.

b) <u>Verdadeiro:</u>

$v_1, \dots, v_k$ L.I., $u \notin [v_1, \dots, v_k] \Rightarrow v_1, \dots, v_k, u$ L.I.

<u>Dem:</u> Considere a equação

$$a_1 v_1 + \dots + a_k v_k + bu = 0 \qquad (*)$$

Suponha primeiro que $b \neq 0$

$\Rightarrow u = -\frac{a_1}{b} v_1 - \dots - \frac{a_k}{b} v_k \Rightarrow u \in [v_1, \dots, v_k]$ Absurdo!

Logo $b=0$

e portanto $(*) \Leftrightarrow a_1 v_1 + \dots + a_k v_k = 0 \Leftrightarrow a_1 = \dots = a_k = 0$

pois $v_1, \dots, v_k$ são L.I.

∎

④ $S_1 = [(1,0,1,-1), (-1,1,0,2)]$

$S_2 = [(1,1,2,0), (-1,1,0,2), (-1,2,-1,7)]$

(a) $\underline{S_1 \subset S_2}$: Basta mostrar que $(1,0,1,-1)$, $(-1,1,0,2) \in S_2$

$(1,0,1,-1) \in S_2 \iff \exists\, a,b,c \in \mathbb{R}$ t.q.

$(1,0,1,-1) = a(1,1,2,0) + b(-1,1,0,2) + c(-1,2,-1,7)$

$$\iff \begin{cases} a-b-c = 1 & (1) \\ a+b+2c = 0 & (2) \\ 2a-c = 1 & (3) \\ 2b+7c = -1 & (4) \end{cases}$$

$(1)+(2)+(3) \Rightarrow 4a = 2 \Rightarrow a = 1/2$

$(3) \Rightarrow c = 0$ $\qquad (4) \Rightarrow b = -1/2$

$\Rightarrow \boxed{a = 1/2,\ b = -1/2,\ c = 0} \Rightarrow (1,0,1,-1) \in S_2$

$(-1,1,0,2) \in S_2$ pois $(-1,1,0,2)$ é um dos geradores de S_2

$\Rightarrow S_1 \subset S_2$

(b) $S_2 \not\subset S_1$:
Vamos mostrar que:
$(-1,2,-1,7) \notin S_1$

Note que $S_1 = [(1,0,1,-1), (-1,1,0,2)] =$

$= \{(a-b, b, a, -a-2b) \mid a,b \in \mathbb{R}\}$

Logo, se $(x,y,z,w) \in S_1 \Rightarrow -z-2y = w$

mas $(-1,2,-1,7)$ Não satisfaz esta relação pois

$-(-1) - 2(2) = -3 \neq 7$

$\Rightarrow (-1,2,-1,7) \notin S_1 \Rightarrow S_2 \not\subset S_1$

(c) De (a) e (b) temos que

$\underbrace{(1,0,1,-1)}_{v_1}, \underbrace{(-1,1,0,-2)}_{v_2} \in S_2$, São L.I. pois $v_1 \not\parallel v_2$

e $\underbrace{(-1,2,-1,7)}_{v_3} \notin [v_1, v_2]$

$B = (v_1, v_2, v_3)$ é base de S_2.

5)

(a) $T(f_1) = T(1,1) = (0,3,2) = a(1,1,0) + b(0,1,1) + c(1,0,1)$

$= (a+c, a+b, b+c)$

$$\Rightarrow \begin{cases} a+c = 0 & (1) \\ a+b = 3 & (2) \\ b+c = 2 & (3) \end{cases}$$

$(1) - (3) + (2) \Rightarrow 2a = 1 \Rightarrow a = 1/2 \overset{(1)}{\Rightarrow} c = -1/2 \overset{(3)}{\Rightarrow} b = \frac{5}{2}$

$\Rightarrow T(f_1) = (1/2, 5/2, -1/2)_G$

$T(f_2) = T(1,-1) = (2,1,0) = a g_1 + b g_2 + c g_3$

$$\Rightarrow \begin{cases} a+c = 2 & (1) \\ a+b = 1 & (2) \\ b+c = 0 & (3) \end{cases}$$

$(1) + (2) - (3) \Rightarrow 2a = 3 \Rightarrow a = \frac{3}{2} \Rightarrow b = -\frac{1}{2} \Rightarrow c = \frac{1}{2}$

$\Rightarrow T(f_2) = (3/2, -1/2, 1/2)_G$

$$\Rightarrow [T]_{GF} = \begin{pmatrix} 1/2 & 3/2 \\ 5/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

(b) $T(u)$ Na base G é dado por

$$[T]_{GF} \begin{pmatrix} | \\ u \\ | \end{pmatrix}_F = \begin{pmatrix} | \\ T(u) \\ | \end{pmatrix}_G$$

$$\begin{pmatrix} 1/2 & 3/2 \\ 5/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \boxed{T(u) = (7,3,1)_G}$$