Na Última Aula

Teorema 1

- Existe uma única aplicação multi-linear alternada

$$D : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n\text{-vezes}} \longrightarrow \mathbb{R}$$

tal que

$$D(e_1, \dots, e_n) = 1$$

Def: $\mathrm{Det}\begin{pmatrix} - v_1 - \\ \vdots \\ - v_n - \end{pmatrix} = D(v_1, \dots, v_n)$

Ou seja: $\mathrm{Det} : M_{n\times n} \to \mathbb{R}$ é a única função que é multilinear alternada nas linhas da matriz tal que $\mathrm{Det}(I) = 1$

Para calcular Det, escalona até a forma escalonada reduzida e lembra das operações e seu resultado no determinante

Op 1: $A \xrightarrow{L_i \leftrightarrow L_j} A'$

$\det(A') = -\det A$

Op 2: $A \xrightarrow{L_i \to \lambda L_i} A'$

$\det A' = \lambda \det A$

Op 3: $A \xrightarrow{L_i \to L_i + \lambda L_j} A'$

$\mathrm{Det}\, A = \mathrm{Det}\, A'$

Exemplo:

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ 2 & -1 & 0 & -1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \end{pmatrix} \xrightarrow{L_2 \leftrightarrow L_1} \begin{pmatrix} 2 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow{L_4 \to L_4 - L_1} \begin{pmatrix} 2 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 1 & 3 \end{pmatrix} \xrightarrow{L_1 \to \frac{1}{2} L_1} \begin{pmatrix} 1 & -1/2 & 0 & -1/2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow[L_4 \to L_4 - 2L_2]{L_3 \to L_3 - L_2} \begin{pmatrix} 1 & -1/2 & 0 & -1/2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -1 & 3 \end{pmatrix} \xrightarrow{L_3 \to \frac{1}{2} L_3} \begin{pmatrix} 1 & -1/2 & 0 & -1/2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 3 \end{pmatrix}$$

$$\xrightarrow{L_4 \to L_4 + L_3} \begin{pmatrix} 1 & -1/2 & 0 & -1/2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix} \xrightarrow{L_4 \to \frac{1}{4} L_4} \begin{pmatrix} 1 & -1/2 & 0 & -1/2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{L_3 \to L_3 - L_4} \begin{pmatrix} \quad \end{pmatrix} \xrightarrow{L_2 \to L_2 + L_3} \begin{pmatrix} \quad \end{pmatrix}$$

$$\xrightarrow{L_1 \to L_1 + \frac{1}{2} L_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

$$-\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4} \det A = 1$$

$$\boxed{\det A = -16}$$

- $\operatorname{Det} A = 0 \iff$ Linhas de A L.D. $\iff$

$\iff$ Colunas de A L.D. $\iff$ $\operatorname{Im} T_A \subsetneq \mathbb{R}^n$

$\iff N(T_A) \neq 0 \iff$ ~~A~~ T_A NÃO é injetora

- Da demonstração do Teorema 1

Se $F : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n\text{-vezes}} \longrightarrow \mathbb{R}$ é

multilinear alternada ~~de ...~~

$\Rightarrow$ ~~A~~ $F(v_1, \dots, v_n) = D(v_1, \dots, v_n) \cdot \lambda$

onde $\lambda = F(e_1, \dots, e_n)$

Ou seja, se

$F : M_{n\times n} \longrightarrow \mathbb{R}$ é multlinear alternada nas linhas da matriz

$\Rightarrow F(A) = \lambda \cdot \operatorname{Det}(A)$ onde ~~Det~~ $\lambda = F(I_{n\times n})$

Consequencia:

$$\boxed{Det(A\cdot B) = Det(A)\cdot Det(B)}$$

Dem: Considere

~~Det~~ $F_B : \mathcal{M}_{n\times n} \longrightarrow \mathbb{R}$

$$F_B(A) = Det(A\cdot B)$$

Se $A = \begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix}$

$$\Rightarrow A\cdot B = \begin{pmatrix} - & v_1 B & - \\ - & v_2 B & - \\ & \vdots & \\ - & v_n B & - \end{pmatrix}$$

$\Rightarrow F_B$ é multilinear alternada nas linhas de A

$\Rightarrow F_B(A) = Det(A)\cdot\lambda$ onde $\lambda = F_B(I) = Det(IB) = Det(B)$

$$\Rightarrow \underset{\underset{Det(A\cdot B)}{\parallel}}{F_B(A)} = Det(A)\cdot Det(B)$$

Corolário: Se $A \in \mathcal{M}_{n\times n}$ é invertível

$$\Rightarrow \operatorname{Det}(A^{-1}) = \frac{1}{\operatorname{Det} A}$$

Dem: $\operatorname{Det}(A \cdot A^{-1}) = (\operatorname{Det} A) \cdot \operatorname{Det}(A^{-1})$

$$\operatorname{Det}(A \cdot A^{-1}) = \operatorname{Det}(I) = 1 \Rightarrow \operatorname{Det}(A^{-1}) = \frac{1}{\operatorname{Det} A} \qquad \blacksquare$$

Corolário:

Se $T: \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{F} = (f_1, \dots, f_n)$ base de $\mathbb{R}^n$

$\mathcal{G} = (g_1, \dots, g_n)$

$$\Rightarrow \boxed{\operatorname{Det}[T]_{\mathcal{F}\mathcal{F}} = \operatorname{Det}[T]_{\mathcal{G}\mathcal{G}}}$$

Dem: Seja $\mathcal{M}_{\mathcal{G}\mathcal{F}}$ a matriz de mudança de base de $\mathcal{F}$ pra $\mathcal{G}$

$$\mathcal{M}_{\mathcal{G}\mathcal{F}} = [I]_{\mathcal{G}\mathcal{F}}$$

$$\mathcal{M}_{\mathcal{F}\mathcal{G}} = \mathcal{M}_{\mathcal{G}\mathcal{F}}^{-1}$$

$$[T]_{\mathcal{F}\mathcal{F}} = \mathcal{M}_{\mathcal{F}\mathcal{G}} [T]_{\mathcal{G}\mathcal{G}} \mathcal{M}_{\mathcal{G}\mathcal{F}} = \mathcal{M}_{\mathcal{F}\mathcal{G}} [T]_{\mathcal{G}\mathcal{G}} \mathcal{M}_{\mathcal{F}\mathcal{G}}^{-1}$$

$$\Rightarrow \det[T]_{\mathcal{F}\mathcal{F}} = \frac{\det \mathcal{M}_{\mathcal{F}\mathcal{G}}}{\det \mathcal{M}_{\mathcal{F}\mathcal{G}}} \det[T]_{\mathcal{G}\mathcal{G}} = \det[T]_{\mathcal{G}\mathcal{G}} \qquad \blacksquare$$