Vimos:

$$\mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m) = \{T : \mathbb{R}^n \to \mathbb{R}^m \mid T \text{ é linear}\} \xleftrightarrow{1-1} \mathcal{M}_{m\times n}$$

$$T \longmapsto A_T = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}$$

$$T_A \longleftarrow A$$

$$T_A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

<u>Prop</u>: Se $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \xrightarrow{S} \mathbb{R}^\ell$ são lineares

$\Rightarrow$ $S \circ T$ é linear.

<u>Dem</u>: • $(S\circ T)(u+v) = S(T(u+v)) = S(T(u)+T(v)) = S(T(u)) + S(T(v))$

$= (S\circ T)(u) + (S\circ T)(v)$

• $(S\circ T)(\lambda u) = S(T(\lambda u)) = S(\lambda T(u)) = \lambda S(T(u)) =$

$= \lambda (S\circ T)(u)$ ∎

<u>Lembre que</u>:

Se $A \in \mathcal{M}_{\ell\times m}$, $B \in \mathcal{M}_{m\times n}$ $\Rightarrow$ $A\cdot B \in \mathcal{M}_{\ell\times n}$

é a matriz definida por

$$(A\cdot B)_{i,j} = \sum_{k=1}^{m} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$$

Por exemplo:

$$\underbrace{\begin{pmatrix} -1 & 2 \\ 0 & 1 \\ 3 & -2 \end{pmatrix}}_{\mathcal{M}_{3\times 2}} \underbrace{\begin{pmatrix} 0 & 2 & 5 & -1 \\ -1 & 3 & 1 & 2 \end{pmatrix}}_{\mathcal{M}_{2\times 4}} = \begin{pmatrix} -2 & 4 & -3 & 5 \\ -1 & 3 & 1 & 2 \\ 2 & 0 & 13 & -7 \end{pmatrix}$$

$$\mathcal{M}_{3\times 4}$$

__Prop:__ Seja $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \xrightarrow{S} \mathbb{R}^\ell$ aplicações lineares e denote por $A_T$, $A_S$ as matrizes de $T$, $S$.

$$\Rightarrow \quad A_{S\circ T} = A_S \cdot A_T$$

__Dem:__ Seja $A_S = \begin{pmatrix} s_{11} & \cdots & s_{1m} \\ \vdots & & \vdots \\ s_{\ell 1} & \cdots & s_{\ell m} \end{pmatrix}$, $A_T = \begin{pmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & & \vdots \\ t_{m1} & \cdots & t_{mn} \end{pmatrix}$

A j-ésima coluna de $A_{S\circ T}$ é $S\circ T(e_j) =$

$$= S(T(e_j)) = S(t_{1j}, t_{2j}, \dots, t_{mj}) = S(t_{1j} f_1 + t_{2j} f_2 + \dots + t_{mj} f_m)$$

$$= \sum_{k=1}^{m} t_{kj}\, S(f_k) = \sum_{i=1}^{\ell} \sum_k s_{ik} t_{kj}\, g_i = \left(\sum_k s_{1k} t_{kj}, \sum_k s_{2k} t_{kj}, \dots, \sum_k s_{\ell k} t_{kj}\right)$$

$$S(f_k) = \sum s_{ik}\, g_i$$

$$\Rightarrow \quad (A_{S\circ T})_{ij} = \sum_k s_{ik} t_{kj} \qquad \blacksquare$$

**Exemplo:**

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$(x,y) \longmapsto (2x-y,\ x+3y,\ x)$$

$$S: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$(x,y,z) \longmapsto (-y,\ x+z)$$

$$\Rightarrow \quad S \circ T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(S \circ T)(x,y) = S(T(x,y)) = S(2x-y,\ x+3y,\ x) =$$
$$= (-x-3y,\ 3x-y)$$

$$\Rightarrow \boxed{A_{S\circ T} = \begin{pmatrix} -1 & -3 \\ 3 & -1 \end{pmatrix}}$$

$$A_S = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A_T = \begin{pmatrix} 2 & -1 \\ 1 & 3 \\ 1 & 0 \end{pmatrix}$$

$$A_S \cdot A_T = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ 3 & -1 \end{pmatrix} = A_{S\circ T}$$