Sistema Linear

$$(\circledast)\quad \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = y_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = y_m \end{cases}$$

Soluções: $W = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \circledast \text{ é satisfeito}\}$

- $W = \phi$ Sistema Impossível
- $W = \{X_0\}$ sistema Possível determinado
- W infinito Sistema Possível indeterminado

Operações no Sistema que Não mudam W:

Op 1: $L_i \leftrightarrow L_j$

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = y_1 \\ \vdots \\ a_{i1}x_1 + \cdots + a_{in}x_n = y_i \\ \vdots \\ a_{j1}x_1 + \cdots + a_{jn}x_n = y_j \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = y_m \end{cases} \xrightarrow{L_i \leftrightarrow L_j} \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = y_1 \\ \vdots \\ a_{j1}x_1 + \cdots + a_{jn}x_n = y_j \\ \vdots \\ a_{i1}x_1 + \cdots + a_{in}x_n = y_i \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = y_m \end{cases}$$

Op 2: $L_i \longrightarrow \lambda L_i \qquad \lambda \neq 0$

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = y_1 \\ \vdots \\ a_{i1}x_1 + \cdots + a_{in}x_n = y_i \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = y_m \end{cases} \xrightarrow{L_i \to \lambda L_i} \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = y_1 \\ \vdots \\ \lambda a_{i1}x_1 + \cdots + \lambda a_{in}x_n = \lambda y_i \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = y_m \end{cases}$$

Op 3: $L_i \longrightarrow L_i + \lambda L_j$

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = y_1 \\ \vdots \\ a_{i1}x_1 + \cdots + a_{in}x_n = y_i \\ \vdots \\ a_{j1}x_1 + \cdots + a_{jn}x_n = y_j \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = y_m \end{cases} \xrightarrow{L_i \to L_i + \lambda L_j} \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = y_1 \\ \vdots \\ (a_{i1} + \lambda a_{j1})x_1 + \cdots + (a_{in} + \lambda a_{jn})x_n = y_i + \lambda y_j \\ \vdots \\ a_{j1}x_1 + \cdots + a_{jn}x_n = y_j \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = y_m \end{cases}$$

Ideia: Usar estas operações para deixar o sistema na forma mais simples possível

- Vamos fazer as operasões na MATRIZ ESTENDIDA DO SISTEMA:

$$\tilde{A} = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & y_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & y_m \end{array}\right)$$

Def: Uma matriz está na forma escalonada reduzida se

- 1ª entrada não nula de cada linha é 1 (pivô da linha)
- pivô da linha $i+1$ está a direita do pivô da linha i
- linhas nulas estão em baixo
- Se uma coluna tem um pivô então todas as outras entradas são nulas

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \Bigg| \quad \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & -1 & 1 \end{pmatrix} \times \quad \begin{pmatrix} 2 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \times \quad \Bigg| \quad \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \times \quad \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \times$$

Exemplo

$$\begin{cases} 2x + y + w = 3 \\ x - 2y + z = -1 \\ 3x - y + z + w = 2 \\ -x - 3y + z - w = -4 \end{cases}$$

Matriz estendida do sistema

$$\left(\begin{array}{cccc|c} 2 & 1 & 0 & 1 & 3 \\ 1 & -2 & 1 & 0 & -1 \\ 3 & -1 & 1 & 1 & 2 \\ -1 & -3 & 1 & -1 & -4 \end{array}\right) \xrightarrow{L_1 \to \frac{1}{2} L_1} \left(\begin{array}{cccc|c} 1 & 1/2 & 0 & 1/2 & 3/2 \\ 1 & -2 & 1 & 0 & -1 \\ 3 & -1 & 1 & 1 & 2 \\ -1 & -3 & 1 & -1 & -4 \end{array}\right)$$

$$\xrightarrow{L_2 \to L_2 - L_1} \left(\begin{array}{cccc|c} 1 & 1/2 & 0 & 1/2 & 3/2 \\ 0 & -5/2 & 1 & -1/2 & -5/2 \\ 3 & -1 & 1 & 1 & 2 \\ -1 & -3 & 1 & -1 & -4 \end{array}\right) \xrightarrow{L_3 \to L_3 - 3L_1}$$

$$\left(\begin{array}{cccc|c} 1 & 1/2 & 0 & 1/2 & 3/2 \\ 0 & -5/2 & 1 & -1/2 & -5/2 \\ 0 & -5/2 & 1 & -1/2 & -5/2 \\ -1 & -3 & 1 & -1 & -4 \end{array}\right) \xrightarrow{L_4 \to L_4 + L_1} \left(\begin{array}{cccc|c} 1 & 1/2 & 0 & 1/2 & 3/2 \\ 0 & -5/2 & 1 & -1/2 & -5/2 \\ 0 & -5/2 & 1 & -1/2 & -5/2 \\ 0 & -5/2 & 1 & -1/2 & -5/2 \end{array}\right)$$

$$\xrightarrow[L_4 \to L_4 - L_2]{L_3 \to L_3 - L_2} \left(\begin{array}{cccc|c} 1 & 1/2 & 0 & 1/2 & 3/2 \\ 0 & -5/2 & 1 & -1/2 & -5/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{L_2 \to -\frac{2}{5} L_2}$$

$$\left(\begin{array}{cccc|c} 1 & 1/2 & 0 & 1/2 & 3/2 \\ 0 & 1 & -2/5 & 1/5 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{L_1 \to L_1 - 1/2 L_2} \left(\begin{array}{cccc|c} 1 & 0 & 1/5 & 2/5 & 1 \\ 0 & 1 & -2/5 & 1/5 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} 1 & 0 & 1/5 & 2/5 & 1 & 1 \\ 0 & 1 & -2/5 & 1/5 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(Forma Escalonada Reduzida)

Sistema

$$\begin{cases} x + \frac{1}{5}z + \frac{2}{5}w = 1 \\ y - \frac{2}{5}z + \frac{1}{5}w = 1 \\ 0 = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{array}{l} x = 1 - \frac{1}{5}z - \frac{2}{5}w \\ y = 1 + \frac{2}{5}z - \frac{1}{5}w \\ z, w \text{ arbitrários} \end{array}$$

$$\Rightarrow W = \left\{ \left(1 - \frac{1}{5}z - \frac{2}{5}w, 1 + \frac{2}{5}z - \frac{1}{5}w, z, w\right) \mid z, w \in \mathbb{R} \right\}$$

$$= \left\{ (1,1,0,0) + z\left(-\frac{1}{5}, \frac{2}{5}, 1, 0\right) + w\left(-\frac{2}{5}, -\frac{1}{5}, 0, 1\right) \mid z, w \in \mathbb{R} \right\}$$

$$= \boxed{(1,1,0,0) + [(-1,2,5,0), (-2,-1,0,5)]}$$

<u>Exemplo</u>:

$$\begin{cases} y + 2z = 1 \\ x + z = 1 \\ x - y - z = 1 \end{cases} \qquad \left(\begin{array}{ccc|c} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{array}\right) \xrightarrow{L_2 \leftrightarrow L_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{L_3 \to L_3 - L_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & -2 & 0 \end{array}\right) \xrightarrow{L_3 \to L_3 + L_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \Rightarrow \begin{cases} x + z = 1 \\ y + 2z = 1 \\ 0 = 1 \end{cases}$$

Sistema impossível!