Methods of Functional Analysis and Topology
Vol. 13 (2007), no. 1, pp. 43–49

ON QUADRUPLES OF LINEARLY CONNECTED PROJECTIONS AND TRANSITIVE SYSTEMS OF SUBSPACES

YULIA MOSKALEVA, VASYL OSTROVSKYI, AND KOSTYANTYN YUSENKO

Abstract. We study conditions under which the images of irreducible quadruples of linearly connected projections give rise to all transitive systems of subspaces in a finite dimensional Hilbert space.

Introduction

A number of recent papers are devoted to the study of families of projections $\{P_i\}_{i=1}^n$, in a complex separable Hilbert space $\mathcal{H}$, which satisfy the linear relation

$$(1) \qquad \alpha_1 P_1 + \cdots + \alpha_n P_n = \gamma I,$$

where all α_i and γ are real non-negative numbers. In particular, the correspondence between such irreducible families and associated systems of n subspaces in $\mathcal{H}$, $S = (\mathcal{H}; \mathcal{H}_1, \dots, \mathcal{H}_n)$ where $\mathcal{H}_i = \mathrm{Im}(P_i)$, was noticed and studied in [3, 8].

The system of subspaces S is transitive (brick) if any operator in $\mathcal{H}$ which maps any $\mathcal{H}_i$ into itself is scalar. In this case, we also say that the family $\{P_i\}_{i=1}^n$ is transitive. In [3] it was shown that there exists a one-to-one correspondence between transitive quadruples of subspaces in a finite-dimensional Hilbert space and irreducible quadruples of projections, $P_1,\dots,P_4$, such that $P_1+P_2+P_3+P_4 = \gamma I$ for some $\gamma \in \mathbb{R}$. For arbitrary n, in the finite-dimensional case the images of an irreducible family of projections $P_1,\dots,P_n$ satisfying (1) form a transitive n-tuple of subspaces (see [8]). In the infinite-dimensional case, the structure of transitive quadruples of subspaces is much more complicated (see, e.g., [2]). Also, it is still unknown if there exist infinite-dimensional transitive triples of subspaces.

In this paper we show directly that all irreducible families of projections that satisfy (1) are transitive in the case where $n \le 4$. The following question arises naturally: given a fixed $\chi_n = (\alpha_1, \dots, \alpha_n)$, $n \le 4$, will all transitive systems arise as images of the projections satisfying (1) with an appropriate γ? If $\chi_n = (1, \dots, 1)$ then the answer is positive (see [3]). The investigation in the case where $n < 4$ is trivial. For the case $n = 4$ we use the description of transitive systems in finite dimensional space given in [1] to show that given a fixed $\chi_4 = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ the irreducible families of projections satisfying (1) generate all transitive finite-dimensional quadruples of subspaces if and only if $\chi_4 = (1, 1, 1, 1)$.

1. Transitive systems of subspaces

Consider the category $\mathrm{Sys}(n)$, $n \in \mathbb{N}$. Each object in this category, $S \in \mathrm{Sys}_n$, is a system $S = (\mathcal{H}; \mathcal{H}_1, \dots, \mathcal{H}_n)$ of subspaces $\mathcal{H}_i$ in some Hilbert space $\mathcal{H}$. A morphism $A \in \mathrm{Mor}(S, \tilde{S})$ between two systems $S \in \mathrm{Sys}_n$ and $\tilde{S} \in \mathrm{Sys}_n$ is a linear bounded operator $A : \mathcal{H} \to \tilde{\mathcal{H}}$, such that

$$A(\mathcal{H}_i) \subset \tilde{\mathcal{H}}_i, \quad \text{for all } i = 1, \dots, n.$$

2000 *Mathematics Subject Classification.* Primary 47A62, 17B10, 16G20.
Key words and phrases. Operator algebras, additive spectral problem, transitive systems of subspaces, ∗-representations, Coxeter functors.

Definition 1. *A system $S \in \mathrm{Sys}_n$ is transitive if the algebra of its endomorphisms is trivial, i.e.,* $\mathrm{Mor}(S,S) = \mathbb{C}I_{\mathcal{H}}$.

Definition 2. *Two systems $S \in \mathrm{Sys}_n$ and $\tilde{S} \in \mathrm{Sys}_n$ are isomorphic if there exists a bijective operator $A \in \mathrm{Mor}(S,\tilde{S})$ such that*

$$A(\mathcal{H}_i) = \tilde{\mathcal{H}}_i, \quad \text{for all } i = 1, \dots, n.$$

Definition 3. *Two systems $S \in \mathrm{Sys}_n$ and $\tilde{S} \in \mathrm{Sys}_n$ are called isomorphic up to a permutation if there exists a permutation $\sigma \in S_n$ such that the systems $S_\sigma = (\mathcal{H}, \mathcal{H}_{\sigma(1)}, \dots, \mathcal{H}_{\sigma(n)})$ and $\tilde{S}$ are isomorphic.*

Transitive systems are the simplest objects in the category Sys_n.

Theorem 1 (S. Brenner [1])**.**

(1) *For $n = 1$, there exist 2 non-isomorphic transitive systems,*

$$S_1^{(1)} = (\mathbb{C}; 0), \quad S_2^{(1)} = (\mathbb{C}; \mathbb{C}).$$

(2) *For $n = 2$, there exist 4 non-isomorphic transitive systems,*

$$S_1^{(2)} = (\mathbb{C}; 0, 0), \quad S_1^{(2)} = (\mathbb{C}; \mathbb{C}, 0), \quad S_3^{(2)} = (\mathbb{C}; 0, \mathbb{C}), \quad S_4^{(2)} = (\mathbb{C}; \mathbb{C}, \mathbb{C}).$$

(3) *For $n = 3$, there exist 9 non-isomorphic transitive systems, 8 one-dimensional,*

$$\begin{aligned} &S_1^{(3)} = (\mathbb{C}; 0, 0, 0), && S_2^{(3)} = (\mathbb{C}; \mathbb{C}, 0, 0), && S_3^{(3)} = (\mathbb{C}; 0, \mathbb{C}, 0), \\ &S_4^{(3)} = (\mathbb{C}; 0, 0, \mathbb{C}), && S_5^{(3)} = (\mathbb{C}; \mathbb{C}, \mathbb{C}, 0), && S_6^{(3)} = (\mathbb{C}; \mathbb{C}, 0, \mathbb{C}), \\ &S_7^{(3)} = (\mathbb{C}; 0, \mathbb{C}, \mathbb{C}), && S_8^{(3)} = (\mathbb{C}; \mathbb{C}, \mathbb{C}, \mathbb{C}), \end{aligned}$$

and 1 two-dimensional,

$$S_9^3 = (\mathbb{C}^2; \mathbb{C}(0,1), \mathbb{C}(1,0), \mathbb{C}(1,1)).$$

For $n = 4$, the description depends in an essential way on an important integer valued invariant $\rho(S)$, called a *defect*.

Definition 4. *For a system $S \in \mathrm{Sys}_n$,*

$$\rho(S) = \sum_{i=1}^{n} \dim \mathcal{H}_i - 2 \dim \mathcal{H}.$$

It turned out that there exist a one-parameter continuous family of transitive systems with defect 0, and four countable series of transitive systems with defect $\rho(S) = \pm 2, \pm 1$, respectively.

Theorem 2 (S. Brenner [1])**.** *Let $B(u,\rho)$ denote the set of systems $S \in \mathrm{Sys}_4$ such that* $\dim(\mathcal{H}) = u$ *and $\rho(S) = \rho$. Then we have the following.*

(1) *For every $u > 2, u \in \mathbb{N}$, there exists a unique system $S \in B(u, \pm 1)$, up to isomorphism and permutation.*
(2) *For every $u = 2k + 1, k \in \mathbb{N}$, there exists a unique system $S \in B(u, \pm 2)$, up to isomorphism and permutation. If the dimension of $\mathcal{H}$ is even, then there exist no systems with defect $\rho(S) = \pm 2$.*
(3) *Besides the trivial one-dimensional systems with defect $\rho(S) = 0$, there exists the one-parameter family $B(2,0)$. If $S_\lambda = (\mathbb{C}^2; \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4) \in B(2,0)$, then*

$$\begin{aligned} &\mathcal{H}_1 = \mathbb{C}(1,0), && \mathcal{H}_2 = \mathbb{C}(0,1), \\ &\mathcal{H}_3 = \mathbb{C}(1,1), && \mathcal{H}_4 = \mathbb{C}(1,\theta), \quad \theta \in \mathbb{C} \setminus \{0,1\}. \end{aligned}$$

There exist no other transitive systems of four subspaces in a finite-dimensional Hilbert space.

2. Projections with linear relation and Coxeter functors

Let $\chi_n = (\alpha_1, \dots, \alpha_n)$ be a vector in $\mathbb{R}_+^n$ the components of which are ordered by values. Consider the finitely generated $*$-algebra

$$\mathcal{A}_{\chi_n} = \mathbb{C}\langle p_1, \dots, p_n, q | p_i = p_i^* = p_i^2,\ [q, p_i] = 0,\ \alpha_1 p_1 + \dots + \alpha_n p_n = q\rangle.$$

The generator q belongs to the center of the algebra, therefore, any irreducible $*$-representation of this algebra is given by an irreducible collections of projections $\{P_i\}_{i=1}^n$ that satisfy

$$\alpha_1 P_1 + \dots + \alpha_n P_n = \gamma I \tag{2}$$

for some γ.

Remark 1. If two vectors $\tilde{\chi}_n$ and χ_n are proportional, then the corresponding algebras $\mathcal{A}_{\chi_n}$ and $\mathcal{A}_{\tilde{\chi}_n}$ are $*$-isomorphic, so in what follows we will consider vectors χ_n from the projective space $\mathbb{PR}_+^n$.

Proposition 1.

(1) *If $n < 3$, then for all vectors $\chi_n \in \mathbb{PR}_+^n$ all irreducible representations of the algebras $\mathcal{A}_{\chi_n}$ generate all transitive system of n subspaces.*

(2) *If $n = 3$, then all irreducible representations of the algebras $\mathcal{A}_{\chi_3}$ generate all transitive systems of 3 subspaces iff, for the vector $\chi_3 = (\alpha_1, \alpha_2, \alpha_3)$, the following holds:*

$$\alpha_3 < \alpha_1 + \alpha_2.$$

Proof. The proof is trivial in the case where $n < 3$. Indeed, irreducible $*$-representations of the algebra $\mathcal{A}_{\chi_n}$ are one-dimensional and it is easy to see the statement.

For $n = 3$, there exist 8 one-dimensional $*$-representations of the algebra $\mathcal{A}_{\chi_3}$, but irreducible two-dimensional representations exists iff $\alpha_3 < \alpha_1 + \alpha_2$, hence this proves the statement. □

In the case of four subspaces the investigation is based on the structure of the set Σ_{χ_4}, which is the set of those $\gamma \in \mathbb{R}$ for which there are quadruples of projections that satisfy (2). Such set was completely described in paper [7] using the Coxeter functors technique, developed in [4]. Namely there are two functors Φ^+ and Φ^- which establish equivalence between the categories of $*$-representations of the algebras $\mathcal{A}_{\chi_n}$ with different values of the vector χ_n (see [4] for the details). Using these functors it was proved that all irreducible representations of the algebra $\mathcal{A}_{\chi_4}$ are finite dimensional and representations with defect $\rho(S_\pi) \neq 0$ could be obtained starting from one-dimensional representations. But there exists a hyperplane (corresponding to defect $\rho(S_\pi) = 0$) invariant with respect to the action of the Coxeter functors (it is defined by the condition $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = 2\gamma$). In what follows we conduct an investigation of these two possibilities.

At first we notice that the Coxeter functors preserve the transitivity and the defect.

Proposition 2. *Coxeter functors Φ^+ and Φ^- preserve the defect value of the system in the following sense: if $\pi \in \operatorname{Rep} \mathcal{A}_{\chi_4}$ and $\pi_+ = \Phi^+(\pi)$ and $\pi_- = \Phi^-(\pi)$, then*

$$\rho(S_\pi) = \rho(S_{\pi_+}), \quad \rho(S_\pi) = \rho(S_{\pi_-}).$$

Proof. The proof is clear after extending the action of the Coxeter functors to the vectors of generalized dimension of representation $\pi : \mathcal{A}_{\chi_4} \to \mathcal{H}$, i.e., to the vectors

$$v_\pi = (\dim \mathcal{H}, \dim(\operatorname{Im}(\pi(p_1))), \dots, \dim(\operatorname{Im}(\pi(p_1)))).$$

□

In [3] it was proved that the functors Φ^+ and Φ^- map transitive families of representations of $\mathcal{A}_{(1,\dots,1)}$ into transitive families. The proof can be easily modified for a more general situation of an arbitrary vector χ_n.

Proposition 3. *The functors Φ^+ and Φ^- map representations that generate transitive systems into representation that generate transitive systems. That is, if $\pi \in Rep\mathcal{A}_{\chi_n}$ and S_π is a transitive system, then the systems S_{π_+} and S_{π_-} are transitive, where $\pi_+ = \Phi^+(\pi)$, $\pi_- = \Phi^-(\pi)$.*

Corollary 1. *If for a pair (χ_4, γ) there exists an irreducible collection of projections P_1, P_2, P_3, P_4 in space $\mathcal{H}$ such that*

$$\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = \gamma I,$$

then there exist $\alpha \in \mathbb{R}$ and an irreducible collection of projections $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3, \tilde{P}_4$ in the space $\tilde{\mathcal{H}}$ such that

$$\tilde{P}_1 + \tilde{P}_2 + \tilde{P}_3 + \tilde{P}_4 = \alpha I,$$

and the systems $S = (\mathcal{H}; \mathrm{Im}(P_1), \mathrm{Im}(P_2), \mathrm{Im}(P_3), \mathrm{Im}(P_4))$ and $\tilde{S} = (\mathcal{H}; \mathrm{Im}(\tilde{P}_1), \mathrm{Im}(\tilde{P}_2), \mathrm{Im}(\tilde{P}_3), \mathrm{Im}(\tilde{P}_4))$, are isomorphic in Sys_4.

Proof. An arbitrary irreducible quadruple of projections such that $\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = \gamma I$ and $\gamma \neq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)/2$ could be obtained by the functor Φ^+ starting from a one-dimensional quadruple, hence it is a transitive system. On the other hand irreducible collections of projections such that $\tilde{P}_1 + \tilde{P}_2 + \tilde{P}_3 + \tilde{P}_4 = \alpha I$ generate all transitive systems. Hence there exists α such that the statement holds. □

3. The case of nonzero defect

Proposition 4. *1. If the vector χ_4 satisfies*

$$\alpha_1 + \alpha_4 < \alpha_2 + \alpha_3,$$

then all irreducible $*$-representations of the algebra $\mathcal{A}_{\chi_4}$ generate all transitive systems of four subspaces with the defect value $\rho(S) = 1$.

2. If the vector χ_4 satisfies

$$\alpha_1 + \alpha_4 > \alpha_2 + \alpha_3,$$

then all irreducible $*$-representations of the algebra $\mathcal{A}_{\chi_4}$ generate all transitive systems of four subspaces with the defect value $\rho(S) = -1$.

Proof. Let χ_4 satisfy $\alpha_1 + \alpha_4 < \alpha_2 + \alpha_3$. Then (see [7]) the set Σ_{χ_4} includes the infinite series

$$\left\{ \frac{\alpha}{2} - \frac{\alpha_1}{2n} \mid n \in \mathbb{N} \right\}.$$

The corresponding infinite series of $*$-representations are representations of the dimensions $3, 4, \dots$. Such series is generated by the action of the functor Φ^+ on one dimensional representation $P_1 = 0, P_2 = I, P_3 = I, P_4 = I$ with defect 1. Therefore using Proposition 2 and Theorem 3 we see that such series generate all transitive systems with defect value $\rho(S) = 1$.

The case $\alpha_1 + \alpha_4 < \alpha_2 + \alpha_3$ is similar. □

Corollary 2. *All irreducible $*$-representations of the algebra $\mathcal{A}_{\chi_4}$ generate all transitive systems with defect value $\rho(S) = \pm 1$ if and only if the vector χ_4 satisfies*

$$\alpha_1 + \alpha_4 = \alpha_2 + \alpha_3.$$

Proposition 5. *Irreducible $*$-representations of the algebra $\mathcal{A}_{\chi_4}$ generate all transitive systems with defect value $\rho(S) = \pm 2$ if and only if $\chi_4 = (1, 1, 1, 1)$ up to a multiplier.*

Proof. All irreducible representations with defect value ± 2 are obtained using the functor Φ^+ starting from one-dimensional $P_1 = I, P_2 = I, P_3 = I, P_4 = I$ and $P_1 = 0, P_2 = 0, P_3 = 0, P_4 = 0$. But due to the structure of the set Σ_{χ_4} [7] such series of representations are infinite if and only if $\chi_4 = (1,1,1,1)$ up to a multiplier. □

4. The case of zero defect

Transitive systems of four subspaces with defect value 0 are generated by the quadruples of projections that satisfy

$$\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = 2, \tag{3}$$

and with the equation

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 4.$$

Such irreducible quadruples exist in dimension not grater than 2. It is easy to describe one-dimensional quadruples and to see that for an arbitrary $\chi_4 = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ such quadruples do not generate all one-dimensional transitive systems with defect value 0.

To investigate two-dimensional case we use explicit formulas for the solutions of (3) (see [5])

$$P_1 = \frac{1}{2\alpha_1\lambda}\begin{pmatrix} (\lambda - u_1)(\lambda + u_2) & \sqrt{-(\lambda^2 - u_1^2)(\lambda^2 - u_2^2)} \\ \sqrt{-(\lambda^2 - u_1^2)(\lambda^2 - u_2^2)} & -(\lambda + u_1)(\lambda - u_2) \end{pmatrix},$$
$$P_2 = \frac{1}{2\alpha_2\lambda}\begin{pmatrix} -(\lambda - v_2)(\lambda + v_1) & e^{i\chi}\sqrt{-(\lambda^2 - v_2^2)(\lambda^2 - v_1^2)} \\ e^{-i\chi}\sqrt{-(\lambda^2 - v_2^2)(\lambda^2 - v_1^2)} & (\lambda + v_2)(\lambda - v_1) \end{pmatrix},$$
$$P_3 = \frac{1}{2\alpha_3\lambda}\begin{pmatrix} -(\lambda - v_2)(\lambda - v_1) & -e^{i\chi}\sqrt{-(\lambda^2 - v_2^2)(\lambda^2 - v_1^2)} \\ -e^{-i\chi}\sqrt{-(\lambda^2 - v_2^2)(\lambda^2 - v_1^2)} & (\lambda + v_2)(\lambda + v_1) \end{pmatrix},$$
$$P_4 = \frac{1}{2\alpha_4\lambda}\begin{pmatrix} (\lambda + u_2)(\lambda + u_1) & -\sqrt{-(\lambda^2 - u_1^2)(\lambda^2 - u_2^2)} \\ -\sqrt{-(\lambda^2 - u_1^2)(\lambda^2 - u_2^2)} & -(\lambda - u_2)(\lambda - u_1) \end{pmatrix},$$
$$(\alpha_4 - \alpha_1)/2 \le \lambda \le \min((\alpha_2 + \alpha_3)/2, (\alpha_1 + \alpha_4)/2)), \quad 0 \le \chi < 2\pi,$$

where $u_1 = \frac{1}{2}(\alpha_4 - \alpha_1)$, $u_2 = \frac{1}{2}(\alpha_4 + \alpha_1)$, $v_1 = \frac{1}{2}(\alpha_3 - \alpha_2)$, $v_2 = \frac{1}{2}(\alpha_3 + \alpha_2)$.

The following theorem holds.

Theorem 3. *All two-dimensional projections P_1, P_2, P_3, P_4 that satisfy* (3) *generate all transitive systems of four subspaces with defect value 0 if and only if*

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1 \tag{4}$$

up to a positive multiplier.

Proof. First we prove that the condition is sufficient. Let (4) hold, then the formulas for P_1, P_2, P_3, P_4 take the following form:

$$P_1 = \frac{1}{2}\begin{pmatrix} 1+\lambda & \sqrt{1-\lambda^2} \\ \sqrt{1-\lambda^2} & 1-\lambda \end{pmatrix}, \qquad P_2 = \frac{1}{2}\begin{pmatrix} 1-\lambda & e^{i\chi}\sqrt{1-\lambda^2} \\ e^{-i\chi}\sqrt{1-\lambda^2} & 1+\lambda \end{pmatrix},$$
$$P_3 = \frac{1}{2}\begin{pmatrix} 1-\lambda & -e^{i\chi}\sqrt{1-\lambda^2} \\ -e^{-i\chi}\sqrt{1-\lambda^2} & 1+\lambda \end{pmatrix}, \qquad P_4 = \frac{1}{2}\begin{pmatrix} 1+\lambda & -\sqrt{1-\lambda^2} \\ -\sqrt{1-\lambda^2} & 1-\lambda \end{pmatrix},$$

$$0 \le \lambda < 1, \quad \begin{cases} 0 < \chi < \pi, & \lambda = 0, \\ 0 \le \chi < 2\pi, & 0 < \lambda < 1. \end{cases}$$

Let $\Omega \subset \mathbb{C}$ be the set of complex numbers $z \in \mathbb{C}$ such that

$$|z| = \frac{1-\lambda}{1+\lambda}, \quad \arg z = -\chi.$$

The set Ω selects the set of all two-dimensional, unitary non equivalent quadruples of projections that satisfy (3). Topologically this set is homeomorphic to the sphere without three points.

Consider the following complex function (Zhukovski function):

$$\theta(z) = \frac{1}{4}\left(2 + z + \frac{1}{z}\right). \tag{5}$$

The following proposition proves sufficiency of the statement of the theorem.

Proposition 6. *The Zhukovski function $\theta(z)$ maps conformally the domain Ω into the domain $\mathbb{C}\setminus\{0,1\}$. The system of subspaces that corresponds to the parameter $z \in \Omega$ is isomorphic to transitive quadruples* (3) *with parameter $\theta = \theta(z)$.*

Proof. The domain Ω is the domain of univalence of the function $\theta(z)$. The function $\theta(z)$ maps every circle $|z| \in (0,1)$ in Ω to an ellipse with focuses at the points 0 and 1. And the arc $|z| = 1$ maps into the interval $(0,1)$.

The images of the projections P_1, P_2, P_3, P_4 are the following subspaces in $\mathbb{C}^2$:

$$\operatorname{Im}(P_1) = \mathbb{C}(1, \sqrt{|z|}), \quad \operatorname{Im}(P_4) = \mathbb{C}(1, -\sqrt{|z|}),$$
$$\operatorname{Im}(P_2) = \mathbb{C}(z, \sqrt{|z|}), \quad \operatorname{Im}(P_3) = \mathbb{C}(z, -\sqrt{|z|}).$$

A direct calculation shows that the matrix

$$M = \begin{pmatrix} 2e^{i\arg z} & -2e^{2i\arg z}\sqrt{|z|} \\ z+1 & (z+1)\sqrt{|z|}^{\,-1} \end{pmatrix} \in \mathcal{M}^2(\mathbb{C})$$

establishes an isomorphism between systems of subspaces generated by the images of projections and with transitive quadruples with parameter $\theta = \theta(z)$. □

Let us prove the necessary part of the statement. Let $\alpha_1 \neq \alpha_4$, and assume that the projections P_1, P_2, P_3, P_4 generate all transitive quadruples with defect value 0. Introduce the following notation:

$$A = \frac{1}{2}(\alpha_4 - \alpha_1), \quad B = \frac{1}{2}(\alpha_4 + \alpha_1), \quad C = \frac{1}{2}(\alpha_3 - \alpha_2), \quad D = \frac{1}{2}(\alpha_3 + \alpha_2),$$

and

$$K_1 = \sqrt{\frac{(\lambda + A)(B - \lambda)}{(\lambda - A)(B + \lambda)}}, \qquad K_2 = \sqrt{\frac{(\lambda - C)(D - \lambda)}{(\lambda + C)(D + \lambda)}},$$
$$K_3 = \frac{\lambda + C}{\lambda - C} K_2, \qquad K_4 = \frac{\lambda - A}{\lambda + A} K_1.$$

In terms of the latter values the images of the projections could be written as follows:

$$\operatorname{Im}(P_1) = \mathbb{C}(1, K_1), \qquad \operatorname{Im}(P_4) = \mathbb{C}(1, -K_4),$$
$$\operatorname{Im}(P_2) = \mathbb{C}(1, e^{-i\chi}K_2), \qquad \operatorname{Im}(P_3) = \mathbb{C}(1, -e^{-i\chi}K_3).$$

For a fixed λ and χ from the set of possible parameters the system of subspaces (2) is isomorphic to a transitive system with parameter θ that is defined as follows:

$$\theta = \frac{1}{(K_1 + K_4)(K_2 + K_3)} \left(K_1K_2 + K_3K_4 + K_1K_4e^{i\chi} + K_2K_3e^{-i\chi}\right).$$

The latter formula is equivalent to the following:

$$\theta = \frac{1}{4}\Bigg(2 - \frac{2AC}{\lambda^2} + \frac{K_1K_2^{-1}(\lambda - A)(\lambda - C)}{\lambda^2} e^{-i\chi}$$
$$+ \frac{(\lambda^2 - A^2)(\lambda^2 - C^2)}{\lambda^4} \frac{\lambda^2}{K_1K_2^{-1}(\lambda - A)(\lambda - C)} e^{-i\chi}\Bigg).$$

Let z be a complex number such that

$$|z| = \frac{K_1 K_2^{-1}(\lambda - A)(\lambda - C)}{\lambda^2}, \quad \arg z = -\chi,$$

and let M denote

$$M = \frac{(\lambda^2 - A^2)(\lambda^2 - C^2)}{\lambda^4},$$

then formula for θ takes the following form:

$$\theta = \frac{1}{4}\left(2 - \frac{2AC}{\lambda^2} + z + M\frac{1}{z}\right).$$

Let us show that this function is not surjective in $\mathbb{C} \setminus \{0, 1\}$. Indeed, for every fixed λ the set of the corresponding values $\theta \in \mathbb{C}$ is an ellipse in the complex plane symmetric with respect to the real axis. It is clear that when λ grows, then the axis of the ellipse also grows and the limit point for focuses are the points 0 and 1. Therefore there must exist λ such that one of the half-axis of the ellipse equals zero; in fact this means that $M = |z|^2$ and the latter is equivalent to

$$\frac{B - \lambda}{B + \lambda} = \frac{D + \lambda}{D - \lambda}.$$

But this means that $\lambda = 0$, which is possible if $\alpha_1 = \alpha_4$. This contradiction proves the theorem. □

References

1. S. Brenner, *Endomorphism algebras of vector spaces with distinguished sets of subspaces*, J. Algebra **6** (1967), 100–114.
2. M. Enomoto and Ya. Watatani, *Relative position of four subspaces in a Hilbert space*, arXive: 2004.
3. Yu. P. Moskaleva and Yu. S. Samoilenko, *Systems of n subspaces and representations of $*$-algebras generated by projections*, Methods Funct. Anal. Topology **12** (2006), no. 1, 57–73.
4. S. A. Kruglyak, V. I. Rabanovich, Yu. S. Samoilenko, *On sums of projections*, Funct. Anal. Appl. **36** (2002), no. 3, 182–195.
5. S. A. Kruglyak, L. A. Nazarova, A. V. Roiter, *On regular locally scalar representations of the graph D_4 in Hilbert spaces*, arXiv math.RT/0610931.
6. V. Ostrovskyi, Yu. Samoilenko, *Introduction to the Theory of Representations of Finitely Presented $*$-Algebras.* I. *Representations by Bounded Operators*, Reviews in Mathematics and Mathematical Physics, 11, pt. 1. Harwood Academic Publishers, Amsterdam, 1999.
7. K. A. Yusenko, *On quadruples of projections connected with linear equation*, Ukrain. Mat. Zh. **58** (2006), no. 9, 1289–1295. (Ukrainian)
8. S. A. Kruglyak, L. A. Nazarova, A. V. Roiter, *On the indecomposability in the category of representations of a quiver and in its subcategory of orthoscalar representations*, arXiv math.RT/0611385.

Taurida National University, 4 Vernads'ky, Simferopol, 95007, Ukraine
E-mail address: YulMosk@mail.ru

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine
E-mail address: vo@imath.kiev.ua

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine
E-mail address: kay@imath.kiev.ua

Received 05/01/2007