ON QUADRUPLES OF LINEARLY CONNECTED PROJECTIONS AND TRANSITIVE SYSTEMS OF SUBSPACES

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Abstract. We study conditions under which the images of irreducible quadruples of linearly connected projections give rise to all transitive systems of subspaces in a finite dimensional Hilbert space.

Introduction

A number of recent papers are devoted to the study of families of projections \( \{ P_i \}_{i=1}^n \), in a complex separable Hilbert space \( \mathcal{H} \), which satisfy the linear relation
\[
\alpha_1 P_1 + \cdots + \alpha_n P_n = \gamma I,
\]
where all \( \alpha_i \) and \( \gamma \) are real non-negative numbers. In particular, the correspondence between such irreducible families and associated systems of \( n \) subspaces in \( \mathcal{H} \), \( S = (\mathcal{H}; \mathcal{H}_1, \ldots, \mathcal{H}_n) \) where \( \mathcal{H}_i = \text{Im}(P_i) \), was noticed and studied in [3, 8].

The system of subspaces \( S \) is transitive (brick) if any operator in \( \mathcal{H} \) which maps any \( \mathcal{H}_i \) into itself is scalar. In this case, we also say that the family \( \{ P_i \}_{i=1}^n \) is transitive. In [3] it was shown that there exists a one-to-one correspondence between transitive quadruples of subspaces in a finite-dimensional Hilbert space and irreducible quadruples of projections, \( P_1, \ldots, P_4 \), such that \( P_1 + P_2 + P_3 + P_4 = \gamma I \) for some \( \gamma \in \mathbb{R} \). For arbitrary \( n \), in the finite-dimensional case the images of an irreducible family of projections \( P_1, \ldots, P_n \) satisfying (1) form a transitive \( n \)-tuple of subspaces (see [8]). In the infinite-dimensional case, the structure of transitive quadruples of subspaces is much more complicated (see, e.g., [2]). Also, it is still unknown if there exist infinite-dimensional transitive triples of subspaces.

In this paper we show directly that all irreducible families of projections that satisfy (1) are transitive in the case where \( n \leq 4 \). The following question arises naturally: given a fixed \( \chi_n = (\alpha_1, \ldots, \alpha_n) \), \( n \leq 4 \), will all transitive systems arise as images of the projections satisfying (1) with an appropriate \( \gamma \)? If \( \chi_n = (1, \ldots, 1) \) then the answer is positive (see [3]). The investigation in the case where \( n < 4 \) is trivial. For the case \( n = 4 \) we use the description of transitive systems in finite dimensional space given in [1] to show that given a fixed \( \chi_4 = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) the irreducible families of projections satisfying (1) generate all transitive finite-dimensional quadruples of subspaces if and only if \( \chi_4 = (1, 1, 1, 1) \).

1. Transitive systems of subspaces

Consider the category \( \text{Sys}(n) \), \( n \in \mathbb{N} \). Each object in this category, \( S \in \text{Sys}_n \), is a system \( S = (\mathcal{H}; \mathcal{H}_1, \ldots, \mathcal{H}_n) \) of subspaces \( \mathcal{H}_i \) in some Hilbert space \( \mathcal{H} \). A morphism \( A \in \text{Mor}(S, \tilde{S}) \) between two systems \( S \in \text{Sys}_n \) and \( \tilde{S} \in \text{Sys}_n \) is a linear bounded operator \( A : \mathcal{H} \rightarrow \tilde{\mathcal{H}} \), such that
\[
A(\mathcal{H}_i) \subset \tilde{\mathcal{H}}_i, \quad \text{for all } i = 1, \ldots, n.
\]

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Definition 1. A system $S \in \text{Sys}_n$ is transitive if the algebra of its endomorphisms is trivial, i.e., $\text{Mor}(S, S) = \mathbb{C}I_n$.

Definition 2. Two systems $S \in \text{Sys}_n$ and $\tilde{S} \in \text{Sys}_n$ are isomorphic if there exists a bijective operator $A \in \text{Mor}(S, \tilde{S})$ such that

$$A(H_i) = \tilde{H}_i, \text{ for all } i = 1, \ldots, n.$$ 

Definition 3. Two systems $S \in \text{Sys}_n$ and $\tilde{S} \in \text{Sys}_n$ are called isomorphic up to a permutation if there exists a permutation $\sigma \in S_n$ such that the systems $S_{\sigma} = (H, H_{\sigma(1)}, \ldots, H_{\sigma(n)})$ and $\tilde{S}$ are isomorphic.

Transitive systems are the simplest objects in the category $\text{Sys}_n$.

Theorem 1 (S. Brenner [1]).

1. For $n = 1$, there exist 2 non-isomorphic transitive systems,

$$S_1^{(1)} = (\mathbb{C}; 0), \quad S_2^{(1)} = (\mathbb{C}; \mathbb{C}).$$

2. For $n = 2$, there exist 4 non-isomorphic transitive systems,

$$S_1^{(2)} = (\mathbb{C}; 0, 0), \quad S_2^{(2)} = (\mathbb{C}; 0, \mathbb{C}), \quad S_3^{(2)} = (\mathbb{C}; 0, 0, \mathbb{C}), \quad S_4^{(2)} = (\mathbb{C}; 0, \mathbb{C}, \mathbb{C}).$$

3. For $n = 3$, there exist 9 non-isomorphic transitive systems, 8 one-dimensional,

$$S_1^{(3)} = (\mathbb{C}; 0, 0, 0), \quad S_2^{(3)} = (\mathbb{C}; 0, 0, \mathbb{C}), \quad S_3^{(3)} = (\mathbb{C}; 0, 0, \mathbb{C}),$$

$$S_4^{(3)} = (\mathbb{C}; 0, \mathbb{C}, 0), \quad S_5^{(3)} = (\mathbb{C}; 0, \mathbb{C}, \mathbb{C}), \quad S_6^{(3)} = (\mathbb{C}; \mathbb{C}, 0, \mathbb{C}), \quad S_7^{(3)} = (\mathbb{C}; \mathbb{C}, \mathbb{C}, 0), \quad S_8^{(3)} = (\mathbb{C}; \mathbb{C}, \mathbb{C}, \mathbb{C}),$$

and 1 two-dimensional,

$$S_9^{(3)} = (\mathbb{C}^2; \mathbb{C}(0, 1), \mathbb{C}(1, 0), \mathbb{C}(1, 1)).$$

For $n = 4$, the description depends in an essential way on an important integer valued invariant $\rho(S)$, called a defect.

Definition 4. For a system $S \in \text{Sys}_n$,

$$\rho(S) = \sum_{i=1}^{n} \dim H_i - 2 \dim H.$$

It turned out that there exist a one-parameter continuous family of transitive systems with defect 0, and four countable series of transitive systems with defect $\rho(S) = \pm 2, \pm 1$, respectively.

Theorem 2 (S. Brenner [1]). Let $B(u, \rho)$ denote the set of systems $S \in \text{Sys}_4$ such that $\dim(H) = u$ and $\rho(S) = \rho$. Then we have the following.

1. For every $u > 2, u \in \mathbb{N}$, there exists a unique system $S \in B(u, \pm 1)$, up to isomorphism and permutation.

2. For every $u = 2k + 1, k \in \mathbb{N}$, there exists a unique system $S \in B(u, \pm 2)$, up to isomorphism and permutation. If the dimension of $H$ is even, then there exist no systems with defect $\rho(S) = \pm 2$.

3. Besides the trivial one-dimensional systems with defect $\rho(S) = 0$, there exists the one-parameter family $B(2, 0)$. If $S_{\lambda} = (\mathbb{C}^2; H_1, H_2, H_3, H_4) \in B(2, 0)$, then

$$H_1 = \mathbb{C}(1, 0), \quad H_2 = \mathbb{C}(0, 1),$$

$$H_3 = \mathbb{C}(1, 1), \quad H_4 = \mathbb{C}(1, \theta), \quad \theta \in \mathbb{C} \setminus \{0, 1\}.$$
There exist no other transitive systems of four subspaces in a finite-dimensional Hilbert space.

2. PROJECTIONS WITH LINEAR RELATION AND COXETER FUNCTORS

Let \( \chi_n = (\alpha_1, \ldots, \alpha_n) \) be a vector in \( \mathbb{R}_+^n \), the components of which are ordered by values. Consider the finitely generated \( * \)-algebra

\[ A_{\chi_n} = C(p_1, \ldots, p_n, q | p_i = p_i^*, [q, p_i] = 0, \alpha_1 p_1 + \cdots + \alpha_n p_n = q). \]

The generator \( q \) belongs to the center of the algebra, therefore, any irreducible \( * \)-representation of this algebra is given by an irreducible collection of projections \( \{P_i\}_{i=1}^n \) that satisfy

\[ \alpha_1 P_1 + \cdots + \alpha_n P_n = \gamma I \]

for some \( \gamma \).

Remark 1. If two vectors \( \tilde{\chi}_n \) and \( \chi_n \) are proportional, then the corresponding algebras \( A_{\tilde{\chi}_n} \) and \( A_{\chi_n} \) are \( * \)-isomorphic, so in what follows we will consider vectors \( \chi_n \) from the projective space \( \mathbb{P}\mathbb{R}_+^n \).

Proposition 1.

1. If \( n < 3 \), then for all vectors \( \chi_n \in \mathbb{P}\mathbb{R}_+^n \) all irreducible representations of the algebras \( A_{\chi_n} \) generate all transitive system of \( n \) subspaces.

2. If \( n = 3 \), then all irreducible representations of the algebras \( A_{\chi_3} \) generate all transitive systems of 3 subspaces iff, for the vector \( \chi_3 = (\alpha_1, \alpha_2, \alpha_3) \), the following holds:

\[ \alpha_3 < \alpha_1 + \alpha_2. \]

Proof. The proof is trivial in the case where \( n < 3 \). Indeed, irreducible \( * \)-representations of the algebra \( A_{\chi_n} \) are one-dimensional and it is easy to see the statement.

For \( n = 3 \), there exist 8 one-dimensional \( * \)-representations of the algebra \( A_{\chi_3} \), but irreducible two-dimensional representations exists iff \( \alpha_3 < \alpha_1 + \alpha_2 \), hence this proves the statement.

In the case of four subspaces the investigation is based on the structure of the set \( \Sigma_{A_4} \), which is the set of those \( \gamma \in \mathbb{R} \) for which there are quadruples of projections that satisfy (2). Such set was completely described in paper [7] using the Coxeter functors technique, developed in [4]. Namely there are two functors \( \Phi^+ \) and \( \Phi^- \) which establish equivalence between the categories of \( * \)-representations of the algebras \( A_{\chi_n} \) with different values of the vector \( \chi_n \) (see [4] for the details). Using these functors it was proved that all irreducible representations of the algebra \( A_{\chi_4} \) are finite dimensional and representations with defect \( \rho(S_n) \neq 0 \) could be obtained starting from one-dimensional representations. But there exists a hyperplane (corresponding to defect \( \rho(S_2) = 0 \) invariant with respect to the action of the Coxeter functors (it is defined by the condition \( \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2\gamma \)). In what follows we conduct an investigation of these two possibilities.

At first we notice that the Coxeter functors preserve the transitivity and the defect.

Proposition 2. Coxeter functors \( \Phi^+ \) and \( \Phi^- \) preserve the defect value of the system in the following sense: if \( \pi \in \text{Rep} A_{\chi_4} \) and \( \pi_+ = \Phi^+(\pi) \) and \( \pi_- = \Phi^-(\pi) \), then

\[ \rho(S_\pi) = \rho(S_{\pi_+}), \quad \rho(S_\pi) = \rho(S_{\pi_-}). \]

Proof. The proof is clear after extending the action of the Coxeter functors to the vectors of generalized dimension of representation \( \pi : A_{\chi_4} \to \mathcal{H} \), i.e., to the vectors

\[ v_\pi = (\dim \mathcal{H}, \dim(\text{Im}(\pi(p_1))), \ldots, \dim(\text{Im}(\pi(p_1)))). \]
In [3] it was proved that the functors $\Phi^+$ and $\Phi^-$ map transitive families of representations of $A_{(1,\ldots,1)}$ into transitive families. The proof can be easily modified for a more general situation of an arbitrary vector $\chi_n$.

**Proposition 3.** The functors $\Phi^+$ and $\Phi^-$ map representations that generate transitive systems into representation that generate transitive systems. That is, if $\pi \in \text{Rep}A_{\chi_n}$ and $S_\pi$ is a transitive system, then the systems $S_{\pi_+}$ and $S_{\pi_-}$ are transitive, where $\pi_+ = \Phi^+(\pi)$, $\pi_- = \Phi^-(\pi)$.

**Corollary 1.** If for a pair $(\chi_4, \gamma)$ there exists an irreducible collection of projections $P_1, P_2, P_3, P_4$ in space $\mathcal{H}$ such that

$$\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = \gamma I,$$

then there exist $\alpha \in \mathbb{R}$ and an irreducible collection of projections $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3, \tilde{P}_4$ in the space $\mathcal{H}$ such that

$$\tilde{P}_1 + \tilde{P}_2 + \tilde{P}_3 + \tilde{P}_4 = \alpha I,$$

and the systems $S = (\mathcal{H}; \text{Im}(P_1), \text{Im}(P_2), \text{Im}(P_3), \text{Im}(P_4))$ and $\tilde{S} = (\mathcal{H}; \text{Im}(\tilde{P}_1), \text{Im}(\tilde{P}_2), \text{Im}(\tilde{P}_3), \text{Im}(\tilde{P}_4))$, are isomorphic in $\text{Sys}_4$.

**Proof.** An arbitrary irreducible quadruple of projections such that $\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = \gamma I$ and $\gamma \neq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)/2$ could be obtained by the functor $\Phi^+$ starting from a one-dimensional quadruple, hence it is a transitive system. On the other hand irreducible collections of projections such that $\tilde{P}_1 + \tilde{P}_2 + \tilde{P}_3 + \tilde{P}_4 = \alpha I$ generate all transitive systems. Hence there exists $\alpha$ such that the statement holds. $\square$

### 3. The Case of Nonzero Defect

**Proposition 4.** 1. If the vector $\chi_4$ satisfies

$$\alpha_1 + \alpha_4 < \alpha_2 + \alpha_3,$$

then all irreducible $*$-representations of the algebra $A_{\chi_4}$ generate all transitive systems of four subspaces with the defect value $\rho(S) = 1$.

2. If the vector $\chi_4$ satisfies

$$\alpha_1 + \alpha_4 > \alpha_2 + \alpha_3,$$

then all irreducible $*$-representations of the algebra $A_{\chi_4}$ generate all transitive systems of four subspaces with the defect value $\rho(S) = -1$.

**Proof.** Let $\chi_4$ satisfy $\alpha_1 + \alpha_4 < \alpha_2 + \alpha_3$. Then (see [7]) the set $\Sigma_{\chi_4}$ includes the infinite series

$$\left\{ \frac{\alpha}{2} - \frac{\alpha_1}{2n} \mid n \in \mathbb{N} \right\}.$$

The corresponding infinite series of $*$-representations are representations of the dimensions $3, 4, \ldots$. Such series is generated by the action of the functor $\Phi^+$ on one dimensional representation $P_1 = 0, P_2 = I, P_3 = I, P_4 = I$ with defect 1. Therefore using Proposition 2 and Theorem 3 we see that such series generate all transitive systems with defect value $\rho(S) = 1$.

The case $\alpha_1 + \alpha_4 < \alpha_2 + \alpha_3$ is similar. $\square$

**Corollary 2.** All irreducible $*$-representations of the algebra $A_{\chi_4}$ generate all transitive systems with defect value $\rho(S) = \pm 1$ if and only if the vector $\chi_4$ satisfies

$$\alpha_1 + \alpha_4 = \alpha_2 + \alpha_3.$$

**Proposition 5.** Irreducible $*$-representations of the algebra $A_{\chi_4}$ generate all transitive systems with defect value $\rho(S) = \pm 2$ if and only if $\chi_4 = (1, 1, 1, 1)$ up to a multiplier.
Proof. All irreducible representations with defect value ±2 are obtained using the functor \( \Phi^+ \) starting from one-dimensional \( P_1 = I, P_2 = I, P_3 = I, P_4 = I \) and \( P_1 = 0, P_2 = 0, P_3 = 0, P_4 = 0 \). But due to the structure of the set \( \Sigma_{\chi_4} \) [7] such series of representations are infinite if and only if \( \chi_4 = (1, 1, 1, 1) \) up to a multiplier. \( \square \)

4. The case of zero defect

Transitive systems of four subspaces with defect value 0 are generated by the quadruples of projections that satisfy

\[
\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = 2,
\]

and with the equation

\[
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 4.
\]

Such irreducible quadruples exist in dimension not greater than 2. It is easy to describe one-dimensional quadruples and to see that for an arbitrary \( \chi_4 = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) such quadruples do not generate all one-dimensional transitive systems with defect value 0.

To investigate two-dimensional case we use explicit formulas for the solutions of (3) (see [5])

\[
P_1 = \frac{1}{2\alpha_1 \lambda} \left( \frac{(\lambda - u_1)(\lambda + u_2)}{\sqrt{-(\lambda^2 - u_1^2)(\lambda^2 - u_2^2)}} \sqrt{-(\lambda^2 - u_1^2)(\lambda^2 - u_2^2)} \right),
\]

\[
P_2 = \frac{1}{2\alpha_2 \lambda} \left( e^{-i\chi} \sqrt{-(\lambda^2 - v_2^2)(\lambda^2 - v_1^2)} \right),
\]

\[
P_3 = \frac{1}{2\alpha_3 \lambda} \left( -e^{-i\chi} \sqrt{-(\lambda^2 - v_2^2)(\lambda^2 - v_1^2)} \right),
\]

\[
P_4 = \frac{1}{2\alpha_4 \lambda} \left( \frac{(\lambda + u_2)(\lambda + u_1)}{\sqrt{-(\lambda^2 - u_2^2)(\lambda^2 - u_1^2)}} \right),
\]

\[
(\alpha_4 - \alpha_1)/2 \leq \lambda \leq \min((\alpha_2 + \alpha_3)/2, (\alpha_1 + \alpha_4)/2), \quad 0 \leq \chi < 2\pi,
\]

where \( u_1 = \frac{1}{2}(\alpha_4 - \alpha_1) \), \( u_2 = \frac{1}{2}(\alpha_4 + \alpha_1) \), \( v_1 = \frac{1}{2}(\alpha_3 - \alpha_2) \), \( v_2 = \frac{1}{2}(\alpha_3 + \alpha_2) \).

The following theorem holds.

**Theorem 3.** All two-dimensional projections \( P_1, P_2, P_3, P_4 \) that satisfy (3) generate all transitive systems of four subspaces with defect value 0 if and only if

\[
\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1
\]

up to a positive multiplier.

Proof. First we prove that the condition is sufficient. Let (4) hold, then the formulas for \( P_1, P_2, P_3, P_4 \) take the following form:

\[
P_1 = \frac{1}{2} \left( \frac{1 + \lambda}{\sqrt{1 - \lambda^2}} \sqrt{\frac{1 - \lambda^2}{1 - \lambda}} \right),
\]

\[
P_2 = \frac{1}{2} \left( \frac{1 - \lambda}{\sqrt{1 - \lambda^2}} e^{i\chi} \sqrt{\frac{1 - \lambda^2}{1 + \lambda}} \right),
\]

\[
P_3 = \frac{1}{2} \left( \frac{-1 - \lambda}{\sqrt{1 - \lambda^2}} e^{-i\chi} \sqrt{\frac{1 - \lambda^2}{1 + \lambda}} \right),
\]

\[
P_4 = \frac{1}{2} \left( \frac{-1 + \lambda}{\sqrt{1 - \lambda^2}} \sqrt{\frac{1 - \lambda^2}{1 - \lambda}} \right),
\]

\[
0 \leq \lambda < 1, \quad \begin{cases} 0 < \chi < \pi, & \lambda = 0, \\ 0 \leq \chi < 2\pi, & 0 < \lambda < 1. \end{cases}
\]

Let \( \Omega \subset \mathbb{C} \) be the set of complex numbers \( z \in \mathbb{C} \) such that

\[
|z| = \frac{1 - \lambda}{1 + \lambda}, \quad \arg z = -\chi.
\]
The set $\Omega$ selects the set of all two-dimensional, unitary non equivalent quadruples of projections that satisfy (3). Topologically this set is homeomorphic to the sphere without three points.

Consider the following complex function (Zhukovski function):

$$\theta(z) = \frac{1}{4} \left( 2 + z + \frac{1}{z} \right).$$

The following proposition proves sufficiency of the statement of the theorem.

**Proposition 6.** The Zhukovski function $\theta(z)$ maps conformally the domain $\Omega$ into the domain $\mathbb{C}\setminus\{0, 1\}$. The system of subspaces that corresponds to the parameter $z \in \Omega$ is isomorphic to transitive quadruples (3) with parameter $\theta = \theta(z)$.

**Proof.** The domain $\Omega$ is the domain of univalence of the function $\theta(z)$. The function $\theta(z)$ maps every circle $|z| \in (0, 1)$ in $\Omega$ to an ellipse with focuses at the points 0 and 1. And the arc $|z| = 1$ maps into the interval $(0, 1)$.

The images of the projections $P_1, P_2, P_3, P_4$ are the following subspaces in $\mathbb{C}^2$:

\[
\begin{align*}
\text{Im}(P_1) &= \mathbb{C}(1, \sqrt{z}) , \quad \text{Im}(P_4) = \mathbb{C}(1, -\sqrt{z}) , \\
\text{Im}(P_2) &= \mathbb{C}(z, \sqrt{z}) , \quad \text{Im}(P_3) = \mathbb{C}(z, -\sqrt{z}) .
\end{align*}
\]

A direct calculation shows that the matrix

\[
M = \begin{pmatrix} 2e^{i\arg z} & -2e^{2i\arg z}\sqrt{|z|} \\ z + 1 & (z + 1)\sqrt{|z|} - 1 \end{pmatrix} \in M^2(\mathbb{C})
\]

establishes an isomorphism between systems of subspaces generated by the images of projections and with transitive quadruples with parameter $\theta = \theta(z)$. \qed

Let us prove the necessary part of the statement. Let $\alpha_1 \neq \alpha_4$, and assume that the projections $P_1, P_2, P_3, P_4$ generate all transitive quadruples with defect value 0. Introduce the following notation:

$$A = \frac{1}{2}(\alpha_4 - \alpha_1), \quad B = \frac{1}{2}(\alpha_4 + \alpha_1), \quad C = \frac{1}{2}(\alpha_3 - \alpha_2), \quad D = \frac{1}{2}(\alpha_3 + \alpha_2),$$

and

\[
\begin{align*}
K_1 &= \frac{(\lambda + A)(B - \lambda)}{(\lambda - A)(B + \lambda)}, \\
K_2 &= \frac{(\lambda - C)(D - \lambda)}{(\lambda + C)(D + \lambda)}, \\
K_3 &= \frac{\lambda + C}{\lambda - C} K_2, \\
K_4 &= \frac{\lambda - A}{\lambda + A} K_1.
\end{align*}
\]

In terms of the latter values the images of the projections could be written as follows:

\[
\begin{align*}
\text{Im}(P_1) &= \mathbb{C}(1, K_1), \quad \text{Im}(P_4) = \mathbb{C}(1, -K_4), \\
\text{Im}(P_2) &= \mathbb{C}(1, e^{-i\chi} K_2), \quad \text{Im}(P_3) = \mathbb{C}(1, -e^{-i\chi} K_3).
\end{align*}
\]

For a fixed $\lambda$ and $\chi$ from the set of possible parameters the system of subspaces (2) is isomorphic to a transitive system with parameter $\theta$ that is defined as follows:

\[
\theta = \frac{1}{(K_1 + K_4)(K_2 + K_3)} (K_1 K_2 + K_3 K_4 + K_1 K_4 e^{i\chi} + K_2 K_3 e^{-i\chi}).
\]

The latter formula is equivalent to the following:

\[
\theta = \frac{1}{4} \left( 2 - 2AC \frac{\lambda^2}{\lambda^2} + \frac{K_1 K_2^{-1} \lambda^2 A (\lambda - C)}{\lambda^2} e^{-i\chi} \right. \\
\left. + \frac{(\lambda^2 - A^2)(\lambda^2 - C^2)}{\lambda^4} \frac{\lambda^2}{K_1 K_2^{-1} \lambda (\lambda - C)} e^{-i\chi} \right). \]
Let \( z \) be a complex number such that
\[
|z| = \frac{K_1 K_2^{-1} (\lambda - A)(\lambda - C)}{\lambda^2}, \quad \arg z = -\chi,
\]
and let \( M \) denote
\[
M = \frac{(\lambda^2 - A^2)(\lambda^2 - C^2)}{\lambda^4},
\]
then formula for \( \theta \) takes the following form:
\[
\theta = \frac{1}{4} \left( 2 - \frac{2AC}{\lambda^2} + z + M \frac{1}{z} \right).
\]

Let us show that this function is not surjective in \( \mathbb{C} \setminus \{0, 1\} \). Indeed, for every fixed \( \lambda \) the set of the corresponding values \( \theta \in \mathbb{C} \) is an ellipse in the complex plane symmetric with respect to the real axis. It is clear that when \( \lambda \) grows, then the axis of the ellipse also grows and the limit point for focuses are the points 0 and 1. Therefore there must exist \( \lambda \) such that one of the half-axis of the ellipse equals zero; in fact this means that \( M = |z|^2 \) and the latter is equivalent to
\[
\frac{B - \lambda}{B + \lambda} = \frac{D + \lambda}{D - \lambda}.
\]
But this means that \( \lambda = 0 \), which is possible if \( \alpha_1 = \alpha_4 \). This contradiction proves the theorem. \( \square \)

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