Exercises for coalgebras, pseudocompact algebras and their representations

Kostiantyn Iusenko[a] and John William MacQuarrie[b]

[a]Instituto de Matemática e Estatística, Univ. de São Paulo, São Paulo, SP, Brazil
[b]Universidade Federal de Minas Gerais, Belo Horizonte, MG, Brazil

January 2023

Lecture 1

Exercise 1.1. Prove Schur's Lemma.

Exercise 1.2. Prove that any submodule of a semisimple module has complement.

Exercise 1.3. Show that module is semisimple iff it is a sum of simple modules. Conclude that each submodule of a semisimples module is semisimples.

Exercise 1.4. Suppose that module M is finitely-generated and semisimple. Show that M is a direct sum of finitely many simples.

Exercise 1.5. Let S be a simple module. Show that $\mathrm{End}_A(S^n) \cong M_n(\mathrm{End}_A(S))$.

Exercise 1.6. Show that $M_n(D)^{op} \cong M_n(D^{op})$

Exercise 1.7. Given a finite-dimensional algebra A show that $J^n(A) = 0$ for some n.

Exercise 1.8. Suppose that A is f.d. algebra with $J(A) = 0$, show that there is a finite set X so that

$$J(A) = \bigcap_{i \in X} M_i$$

(*Hint:* Consider all possible finite intersection:

$$I_\Phi := \bigcap_{M \in \Phi} M.$$

The collection of all such ideals I_Φ has a minimal element I_X. Prove that this element has a desired property).

Exercise 1.9. Let $A = \mathbb{U}_2(K) = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$. Show that

$$J(A) = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix},$$

and describe radical of algebra $\mathbb{U}_n(k)$.

Exercise 1.10. Suppose that char$k = 0$. Show that the group algebra $k[G]$ of finite group G is semi-simple.

Exercise 1.11. Let $f : A \to B$ be a surjective homomorphism of finite-dimensional algebras show that $f(J(A)) = J(B)$. (*Hint:* Treat B as an A-module via f. Then use the property that $\mathrm{rad}_A(M) = J(A)M$ for each A-module).

Exercise 1.12. Show that the algebra kQ is generated by $p_i, i \in Q_0$ and $a_h, h \in Q_1$ with the following relations:

1) $p_i^2 = p_i$, $p_i p_j = 0$ if $i \neq j$;

2) $a_h p_{s(h)} = a_h$ $a_h p_j = 0$ if $j \neq s(h)$;

2) $p_{t(h)} a_h = a_h$ $p_i a_h = 0$ if $i \neq t(h)$.

Exercise 1.13. Supposing that Q is acyclic, show that the radical of the algebra kQ is generated by all the arrows in Q.

Exercise 1.14. Using Exercise 1.13, show that $kQ/J(kQ) \simeq \prod_{i \in Q_0} k$.

Lecture 2

Exercise 2.1. Let C be a coalgebra and let $c \in C$ such that $\Delta(c) = c \otimes c$. Prove that if $\varepsilon(c) \neq 0$, then $\varepsilon(c) = 1$

Exercise 2.2. Let C be a k-vector space with basis $\{s, c\}$. Prove that C is a coalgebra with comultiplication Δ and counit ε defined by:

$$\begin{aligned} \Delta(s) &= s \otimes c + c \otimes s, \qquad & \varepsilon(s) &= 0, \\ \Delta(c) &= c \otimes c - s \otimes s, \qquad & \varepsilon(s) &= 1. \end{aligned}$$

Exercise 2.3. Let $M_n^c(k)$ be a matrix coalgebra with basis $\{e_ij\}_{1 \leq i,j \leq n}$ (see the class). Prove that the dual algebra $M_n^c(k)^*$ is an (n, n)-matrix algebra over k.

Exercise 2.4. Let A be a finite-dimensional associative unital k-algebra.

(a) Prove that A^* is a coalgebra. (Hint: Use that $(A \otimes A)^* \cong A^* \otimes A^*$.

(b) Prove that B is a subalgebra of A if and only if $B^\perp = \{f : A \to k \mid f(B) = 0\}$ is a coideal of A^*.

(c) Prove that I is a two-sided ideal of A if and only if $I^\perp = \{f : A \to k \,|f(I) = 0\}$ is a subcoalgebra of A^*.

Exercise 2.5. Let V, W be two vector spaces. Show that a natural map $\tau : V^* \otimes W^* \to (V \otimes W)^*$ is always injective. And that τ is isomorphism if either V or W is finite-dimensional.

Exercise 2.6. Show that map m (see the class!) factors as $\Delta^* \circ \tau$.

Exercise 2.7. Show that $m = \Delta^* \circ \tau$ is an associative map, showing that the diagram from the class is commutative.

Exercise 2.8. Let V be a vector space, and let $I, J \subset V^*$ are subspaces. Show:

$$(I \otimes J)^\perp = I^\perp \otimes V + V \otimes J^\perp.$$

Exercise 2.9. Using Exercise 2.8 show the duality between coideals in C and subalgebras in C^* (respectively subcoalgebras in C and ideals in C^*).

Lecture 3

Exercise 3.1. Give a full definition of direct and inverse limit, filling in all the details

Exercise 3.2. Write the coalgebra $k[x]$ as a direct limit of fd coalgebras, dualise the direct system, and check that $k[[x]]$ is its inverse limit.

Exercise 3.3. If each A_i $(i \in I)$ is a finite dimensional algebra, write down an inverse system for $A = \prod_{i \in I} A_i$ and convince yourself that A is its inverse limit.

Exercise 3.4. If $A = \varprojlim \{A_i, \varphi_{ij}\}$ is an inverse limit of finite dimensional vector spaces, the maps φ_{ij} might not be surjective. But check that $\{\varphi_i(A), \varphi_{ij}\}$ is another inverse system with surjective maps, and that A is its limit.

Exercise 3.5. Check that a direct sum of coalgebras is a coalgebra. Convince yourself that if each C_i (resp. A_i) is a finite dimensional coalgebra (resp. algebra), then

$$\left(\bigoplus C_i\right)^* = \prod {C_i}^*, \quad \left(\prod A_i\right)^* = \bigoplus {A_i}^*.$$

Exercise 3.6.
1. Understand $k[[x]]$ as a subalgebra of $\prod_{n\in\mathbb{N}} k[x]/x^n$. Show that it's closed.
2. More generally, convince yourself that the following theorem is true:

   "Every pseudocompact algebra is a closed subalgebra of a direct product of discrete finite dimensional algebras."

Exercise 3.7. Convince yourself that discrete finite dimensional vector spaces are linearly compact.

Lecture 4

Exercise 4.1. Let V be a linearly compact vector space. Show that open subspaces are closed, and that a closed subspace is open if, and only if, it has finite codimension.

Exercise 4.2.
1. Let $\mathcal{I}$ be the set of open ideals of the pseudocompact algebra A. If X is a closed subspace of A, show that

   $$X = \bigcap_{I\in\mathcal{I}} (X + I).$$

   [hint: use linear compactness. If $y \notin X$ then $(y + X) \cap \bigcap_{I\in\mathcal{I}} I = \varnothing$.]
2. Show that maximal closed left ideals have finite codimension.
3. Show that every proper closed (left) ideal is contained in a maximal closed (left ideal).

Exercise 4.3. Use Theorem 4.3 to show that every pseudocompact vector space has the form $\prod_{i\in I} k$ for some set I. So although the open ideal/submodule structure of a pseudocompact algebra/module can be complicated, the underlying vector space is always pretty easy.

Exercise 4.4. A pseudocompact A-module is *semisimple* if it's a direct product of simple A-modules. Show that a pseudocompact algebra A is semisimple if, and only if, every pseudocompact A-module is semisimple.

Lecture 5

Exercise 5.1. Show that the algebra $A = k[x]/(x^m)$ is pointed and its radical $J(A)$ is generated by x.

Exercise 5.2. Show that the algebra $A = \begin{pmatrix} k & k[x]/x^2 \\ 0 & k[x]/x^2 \end{pmatrix}$ is basic and its radical has the form

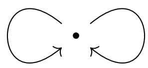
$$J(A) = \begin{pmatrix} 0 & k[x]/x^2 \\ 0 & xk[x]/x^2 \end{pmatrix}.$$

Exercise 5.3. Build the Gabriel quiver of the algebra $A = \mathbb{U}_n(k)$.

Exercise 5.4. Let $A = \mathbb{U}_3(k)$, and C be subalgebra consisting of all matrices

$$\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ 0 & \lambda_{22} & \lambda_{23} \\ 0 & 0 & \lambda_{33} \end{pmatrix}$$

such that $\lambda_{11} = \lambda_{22} = \lambda_{33}$. Show that C is isomorphic to kQ/I, where $I = \langle \alpha^2, \beta^2, \alpha\beta \rangle$ is ideal in kQ, and Q is the following quiver



Exercise 5.5. Check we can multiply $\mathbb{N} \times \mathbb{N}$-lower triangular matrices.

Exercise 5.6. Calculate the quivers of the algebra $k[[x]]$ and of the algebra of $\mathbb{N} \times \mathbb{N}$-lower triangular matrices.

Exercise 5.7. Calculate Q and I such that $k[[x, y]] \cong k[[Q]]/I$.

Exercise 5.8. We described the algebra $k[[Q]]$ when

$$Q = \quad \cdots \xleftarrow{\alpha_3} 3 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_1} 1$$

Describe the algebra $k[[Q]]$ when

$$Q = \quad \cdots \xleftarrow{\alpha_3} 3 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_1} 1 \xleftarrow{\alpha_0} 0 \xleftarrow{\alpha_{-1}} -1 \longleftarrow \cdots$$

Exercise 5.9. There is a description of representations that correspond to arbitrary pseudocompact $k[[Q]]$-modules, but it's a bit more fiddly. Write down the representations corresponding to the $k[[x]]$-module $k[[x]]$ and do the same thing for $\mathbb{N} \times \mathbb{N}$-lower triangular matrices. Do the same thing for the algebra from the previous question.

Note that they're not "nilpotent", but what you might call "pronilpotent".