Coalgebras, pseudocompact algebras and their representations

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Chapter 1

Finite-dimensional algebras and their modules

1.1 Definitions and examples

Let k be a field. An algebra A over k is a k-vector space in which one can multiply the vectors. More precisely there is a bilinear map (=product) m on from $A \times A$ to A and element 1_A so that:

$$\begin{aligned} a \cdot (b \cdot c) &= (a \cdot b) \cdot c, \\ a \cdot 1_A &= a = 1_A \cdot a. \end{aligned}$$

As we will speak about coalgebras tying to dualize this construction, alternatively, we define *algebra* as a k–vector space A together with two linear maps

$$\begin{aligned} m &: A \otimes A \to A \\ \eta &: k \to A \end{aligned}$$

so that the following diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes \mathrm{id}_A} & A \otimes A \\ \downarrow{\scriptstyle \mathrm{id}_A \otimes m} & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow[m]{} & A \end{array} \qquad \begin{array}{ccccc} A \otimes A & \xrightarrow{m} & A & \xleftarrow{m} & A \otimes A \\ {\scriptstyle \mathrm{id}_A \otimes \eta}\uparrow & & {\scriptstyle \mathrm{id}_A}\uparrow & & \uparrow{\scriptstyle \eta \otimes \mathrm{id}_A} \\ A \otimes k & = & A & = & k \otimes A \end{array}$$

which reads off as the axioms:

$$\begin{aligned} m(m \otimes \mathrm{id}_A) &= m(\mathrm{id}_A \otimes m), \\ m(\eta \otimes \mathrm{id}_A) &= \mathrm{id}_A = m(\mathrm{id}_A \otimes \eta). \end{aligned}$$

Indeed writing $m(a \otimes b) = a \cdot b$, $\eta(1) = 1_A$ if is easy to see that this is equivalent to the definition above.

Examples:.

a) $k[x_1, x_2, \ldots, x_n]$ – polinomials with usuals product;

b) Algebra $k[x]/(x^2)$ of 'dual numbers' consists of all pairs of the form $a + bx$, where $a, b \in k$ and x is an element such that $x^2 = 0$;

c) $M_n(A)$ — all $n \times n$ matrices with coefficients in some algebra A;

d)

$$\mathbb{U}_n(k) = \begin{pmatrix} k & k & \dots & k \\ 0 & k & \dots & k \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & k \end{pmatrix}$$

the subset of all upper-triangular matrices $M_n(k)$ is a subalgebra in $M_n(k)$.

f) $A = k\langle x_1, x_2\rangle$ – algebra of all polynomials of two noncommutative variables x_1 and x_2 is a infinite dimensional algebra (called the *free algebra*).

g) Let G be a finite group with identity e. The group algebra $k[G]$ is an algebra whose basis is $\{a_g \mid g \in G\}$ and multiplication (on the basis) $a_g a_h = a_{gh}$ and then extended linearly for the whole algebra.

h) Another important source of algebra arrise from oriented graphs. Let Q be oriented graph (=quiver). The *path algebra* kQ of Q is an k-algebra whose basis is formed by all oriented paths in Q (including trivial path $p_i, i \in Q_0$), and multiplication is defined by concatenation of paths. If two paths cannot be concatenated, then their product is defined as 0. For instanse if

$$1 \xrightarrow{h} 2$$

has a basis of 3 elements p_1, p_2 (trivial paths on the vertices) and h (path of length 1), endowed by multiplication

$$\begin{aligned} p_1^2 &= p_1, \quad p_2^2 = p_2, \\ p_1 p_2 &= p_2 p_1 = 0, \\ p_1 h &= h p_2 = h, \\ h p_1 &= p_2 h = h^2 = 0. \end{aligned}$$

One checks that there exists an isomorphism $kQ \cong \mathbb{U}_2(k)$, given by

$$p_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad p_2 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad h \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

More generally, if $Q = A_n$ the following quiver

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \dots \longrightarrow n-1 \longrightarrow n$$

Then kQ is isomorphic to $\mathbb{U}_n(k)$.

1.2 Modules over algebras

To study algebras it is convinient to study their modules (=representation). Given a k-algebra A, by *left module* over A we mean a k-vector space M together with the (left) action of A on M so that

$$a(bx) = (ab)x, \qquad 1_A x = x,$$

for all $a, b \in A$ and all $x \in M$. If M, N are left A–modules, then a module map $f : M \to N$ is a linear map which commutes with the A–actions, in the sense that

$$f(ax) = af(x), \quad \text{for all } a \in A, x \in M.$$

Definition 1.1. A module M is called *simple* if the only submodules of M are 0 and M.

Every module can be built up out of simple modules in the following sense. Given a module M a *composition series* of M is a sequence of submodules:

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M,$$

such that the modules M_i/M_{i-1} are simple for all i. It is easy to see (exersise!) that any finite-dimensional module M has a composition series. And, moreover, such series are unique in a certain way:

Theorem 1.1 (Jordan-Hölder Theorem)**.** *Any two composition series of a module M have the same length and the same composition factors, up to isomorphism.*

Proposition 1.1 (Schur's Lemma)**.** *. If S_1 and S_2 are simple modules over a finite-dimensional k-algebra A then:*

$$\mathrm{Hom}_A(S_1, S_2) = \begin{cases} D & \text{if } S_1 \cong S_2 \\ 0 & \text{otherwise} \end{cases},$$

where D is a division algebra over k.

Simple modules are important due to the following property:

Proof. . Let $f : S_1 \to S_2$. Then if $f \neq 0$, then $\ker(f)$ is a proper submodule of S_1, so $\ker(f) = 0$. Also, $\mathrm{im}(f)$ is a nonzero submodule of S_2, so $\mathrm{im}(f) = S_2$. Therefore, f is bijective, so is an isomorphism. □

Another kind of module which can serve as a building block for all modules are the indecomposable modules. These are more general than simple modules, but the way in which every module is built up from indecomposables is much simpler. A module M is called *indecomposable* if whenever $M = N_1 \oplus N_2$ for some submodules N_1, N_2, we have $N_1 = 0$ or $N_2 = 0$.

Every simple module is indecomposable, but not-vice versa. For example, let $A = k[x]/(x^2)$ and let $M = A$. Then M is indecomposable (show!) but not simple because xM is a proper nonzero submodule of M.

Theorem 1.2 (Krull-Schmidt). *Every finite-dimensional module M can be written as*

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n,$$

where the M_i are indecomposable. Furthermore, if

$$M_1 \oplus M_2 \oplus \cdots \oplus M_n \cong N_1 \oplus N_2 \oplus \cdots \oplus N_p$$

where the N_i are indecomposable, then $n = p$ and the M_i are isomorphic to the $N_{s(i)}$, for some permutation s.

And, therefore, the following problem is crucial:

Problem. *Given a finite-dimensional algebra A, classify (up to isomorphism) its finite-dimensional indecomposable modules.*

1.3 Semisimple modules and algebras

Definition 1.2. A module M is called *semisimple* if it decomposes as a direct sum of simple modules.

Definition 1.3. An algebra A is called *semisimple* if A-as semisimple as module over itself (equivalently if every left ideal in A has a complement, se Exercise 1.2)

Typical example of s.s. algebra is $A = M_n(A)$. And a complete classification is similar. Indeed, as A is semisimple hence decomposes as a FINITE! (see Exercise 1.4) direct sum of simples $A = \bigoplus_{i=1}^{m} S_i$. Some of simples can be isomorphim but we can reorded the simples so that

$$A = \bigoplus_{i=1}^{n} S_i^{d_i}$$

Now, taking endomorphism algebra of A and keeping is mind Schur's Lemma we have

$$\begin{aligned}
\text{End}_A &= \text{End}_A \bigoplus_{i=1}^{n} (S_i^{d_i}) \\
&\overset{\text{Schur's lemma}}{=\!=} \bigoplus_{i=1}^{n} \text{End}_A(S_i^{d_i}) \\
&\overset{\text{Ex. 1.5}}{=\!=} \bigoplus_{i=1}^{n} M_{d_i}(\text{End}_A(S_i)) \\
&\overset{\text{Schur's lemma again!}}{=\!=} \bigoplus_{i=1}^{n} M_{d_i}(D_i)
\end{aligned}$$

As $A^{op} \cong \text{End}_A$ we have that

$$A = \bigoplus_{i=1}^{n} M_{d_i}(D_i^{op})$$

and this proves the following, fundamental theorem

Theorem 1.3 (Wedderburn's structure theorem). *Given a finite-dimensional algebra A, the following conditions are equivalent:*

(i) *A is semi-simple.*

(ii) $A \cong M_{n_1}(D_1) \times \dots M_{n_r}(D_r)$

The $D_1, \dots, D_r$ above are finite dimensional division k-algebras. The product is unique up to permutation of the factors. In particular, if k is algebraically closed field then $D_i \cong k$ for all i.

1.4 Jacobson radical

Another key concept in the structural theory of associative algebras is *Jacobson radical* $J(A)$ which is defined as the intersection of all maximal left ideals in A. The following characterization is quite useful.

Proposition 1.2. *One has:*

$$\begin{aligned} J(A) &= \bigcap (\text{ maximal left ideals of } A) \\ &= \{x \in A \mid 1 - \ ax \text{ has a left inverse for all } a \in |\} \end{aligned}$$

Examples:.

a) $J(k[x_1, x_2, \dots, x_n]) = 0$

b) $J(k[x]/x^2) = \langle x \rangle$

c) $J(M_n(A)) = 0$

d) $J(kQ) = \langle$arrows in $Q\rangle$, if Q is acyclic.

Proposition 1.3. *For a finite-dimensional k-algebra A the following conditions are equivalente:*

(i) $J(A) = 0$

(ii) *A is semisimple;*

Proof. $(ii) \Rightarrow (i)$ follows directly from Wedderburn's theorem. We prove that $(i) \Rightarrow (ii)$.

By Exercise 1.8 if $J(A) = 0$ and $\dim_k A = 0$ then there exists finite set I of maximal left ideals M_i, so that $J(A) = \bigcap_{i \in I} M_i = 0$. Therefor the natural map

$$A \to \bigoplus_{i \in I} A/M_i$$

is injective. But $\bigoplus_{i \in I} A/M_i$ is semisimples module hence A too! (see Exercise 1.3). ☐

Therefore, for any finite-dimensional algebra A we have that

$$A/J(A) \cong M_{n_1}(D_1) \oplus \dots \oplus M_{n_r}(D_r),$$

since $A/J(A)$ is always semi-simple. It turns out that for algebraically closed field the projection map $A \mapsto A/J(A)$ splits as algebra map so that

$$A = J(A) \oplus A/J(A) \cong J(A) \oplus M_{n_1}(D_1) \oplus \dots \oplus M_{n_r}(D_r)$$

1.5 Exercices

Exercise 1.1. Prove Schur's Lemma.

Exercise 1.2. Prove that any submodule of a semisimple module has complement.

Exercise 1.3. Show that module is semisimple iff it is a sum of simple modules. Conclude that each submodule of a semisimples module is semisimples.

Exercise 1.4. Suppose that module M is finitely-generated and semisimple. Show that M is a direct sum of finitely many simples.

Exercise 1.5. Let S be a simple module. Show that $\mathrm{End}_A(S^n) \cong M_n(\mathrm{End}_A(S))$.

Exercise 1.6. Show that $M_n(D)^{op} \cong M_n(D^{op})$

Exercise 1.7. Given a finite-dimensional algebra A show that $J^n(A) = 0$ for some n.

Exercise 1.8. Suppose that A is f.d. algebra with $J(A) = 0$, show that there is a finite set X so that

$$J(A) = \bigcap_{i \in X} M_i$$

(*Hint:* Consider all possible finite intersection:

$$I_\Phi := \bigcap_{M \in \Phi} M.$$

The collection of all such ideals I_Φ has a minimal element I_X. Prove that this element has a desired property).

Exercise 1.9. Let $A = \mathbb{U}_2(K) = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$. Show that

$$J(A) = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix},$$

and describe radical of algebra $\mathbb{U}_n(k)$.

Exercise 1.10. Suppose that char$k = 0$. Show that the group algebra $k[G]$ of finite group G is semi-simple.

Exercise 1.11. Let $f : A \to B$ be a surjective homomorphism of finite-dimensional algebras show that $f(J(A)) = J(B)$. (*Hint:* Treat B as an A-module via f. Then use the property that $\mathrm{rad}_A(M) = J(A)M$ for each A-module).

Exercise 1.12. Show that the algebra kQ is generated by $p_i, i \in Q_0$ and $a_h, h \in Q_1$ with the following relations:

1) $p_i^2 = p_i$, $p_i p_j = 0$ if $i \neq j$;

2) $a_h p_{s(h)} = a_h$ $a_h p_j = 0$ if $j \neq s(h)$;

2) $p_{t(h)} a_h = a_h$ $p_i a_h = 0$ if $i \neq t(h)$.

Exercise 1.13. Supposing that Q is acyclic, show that the radical of the algebra kQ is generated by all the arrows in Q.

Exercise 1.14. Using Exercise 1.13, show that $kQ/J(kQ) \simeq \prod_{i \in Q_0} k$.

Chapter 2

Coalgebras

2.1 Definitions and examples

Reversing the arrows in diagramatic definition of an algebra leads to a notion of *coalgebra* which seems much less intuitive then an algebra at first. But what we will see is that in many example such notion appears naturally.

Definition 2.1. Suppose that C is a k-vector space together with a k-linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$. Then C is called a *coalgebra*, and the maps Δ and ε are called the *comultiplication* (or coproduct) and the *counit* respectively, if the following two diagrams are commutative:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \mathrm{id}_C \otimes \Delta} \\ C \otimes C & \xrightarrow[\Delta \otimes \mathrm{id}_C]{} & C \otimes C \otimes C \end{array} \qquad \begin{array}{ccccc} C \otimes C & \xleftarrow{\Delta} & C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle \mathrm{id}_C \otimes \varepsilon}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_C} & & \downarrow{\scriptstyle \varepsilon \otimes \mathrm{id}_C} \\ C \otimes k & = & C & = & k \otimes C \end{array}$$

Observe that in many common examples of product operations defining usual algebras involve combining two elements in a some (natural) way such as multiplication of numbers or composition of functions. In coalgebra world, many natural coproducts take an element and pull it apart into two pieces in all possible ways, summing over all the possibilities. Below we provide typical examples.

Example 2.1. The field k is a coalgebra in a canonical way, where $\Delta : k \to k \otimes k$ and ϵ are both the identity maps under the natural identification of $k \otimes k$ with with k. This is called the *trivial coalgebra*.

Example 2.2. Algebra of polynomials $C = k[x]$ is also a coalgebra with coproduct and counit given (on the basis $1, x, x^2, \dots$) by:

$$\Delta(x^n) = \sum_{i+j=n} x^i \otimes x^j, \qquad \varepsilon(x^n) = \delta_{0n}.$$

Example 2.3. Let $C = M_n(k)$ be the k-algebra of $n \times n$-matrices with entries in k. For $1 \leq i, j \leq n$ let e_{ij} be matrix units. It is easy to check that $e_{ij}e_{st} = \delta_{js}e_{it}$. We can give C a coalgebra structure by defining:

$$\Delta(e_{ij}) = \sum_{1 \leq s \leq n} e_{is} \otimes e_{sj}$$

and

$$\varepsilon(e_{ij}) = \delta_{ij},$$

and then extend operation linearly to C.

Example 2.4. Let $C = kQ$ be a path algebra of a quiver (finite or infinite). C is a coalgebra with the structure defined by:

$$\Delta(p) = \sum_{p_1 p_2 = p} p_1 \otimes p_2$$

for any path p and counit ε is given by $\varepsilon(v) = 1$ on each vertex v and $\varepsilon(p)$ for any path of positive length.

Example 2.5.

2.2 Algebras and coalgebras

The connection between algebras and coalgebras can be explained precisely using the notion of dual vector space. Recall that given a vector space V over the field k its *dual space* is

$$\begin{aligned} V^* &= \mathrm{Hom}_k(V, k) \\ &= \{\text{all } k\text{-linear maps } V \text{ to } k\} \\ &= \{\text{linear functionals on } V\}. \end{aligned}$$

When V is finite-dimensional then we have a canonical isomorphism between V and V^{**}. In what follows it is useful to use how duals interact with the tensor product. For vector spaces V and W we always have a canonical linear transformation

$$\tau : V^* \otimes W^* \to (V \otimes W)^*,$$

where $\tau(f \otimes g)(v \otimes w) = f(v)g(w)$. One may check that τ is always an injective linear transformation, and that when either V or W is finite dimensional, then τ is an isomorphism, but not when both V and W are infinite dimensional.

Let (C, Δ, ε) be a coalgebra over k, we claim that the dual space C^* has a natural k-algebra structure. Indeed for $f, g \in C^* = \mathrm{Hom}_k(C, k)$, we can define a product $fg \in C^*$ by

$$[fg](c) = \sum f(c_{(1)}) \otimes g(c_{(2)}) = (f \otimes g) \circ \Delta(c), \tag{2.1}$$

where

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)}.$$

This defines a map $m : C^* \otimes C^* \to C^*$, which can be factorized (see Exercise ?) as $m = \Delta^* \circ \tau$:

$$m : C^* \otimes C^* \xrightarrow{\tau} (C \otimes C)^* \xrightarrow{\Delta^*} C^*.$$

Using this factorization and coassociativity of Δ one checks that m is defining an associative product on C^*. Indeed, we have by coassociativity, so the diagram

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\otimes C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_C\otimes\Delta} \\ C\otimes C & \xrightarrow[\Delta\otimes \mathrm{id}_C]{} & C\otimes C\otimes C \end{array}$$

commutes. Dualizing we get that

$$\begin{array}{ccc} C^* & \xleftarrow{\Delta^*} & (C\otimes C)^* \\ {\scriptstyle\Delta^*}\uparrow & & \uparrow{\scriptstyle (\mathrm{id}_C\otimes\Delta)^*} \\ (C\otimes C)^* & \xleftarrow[(\Delta\otimes \mathrm{id}_C)^*]{} & (C\otimes C\otimes C)^* \end{array}$$

commutes. Expanding, we consider the diagram

$$\begin{array}{ccccc} C^* & \xleftarrow{\Delta^*} & (C\otimes C)^* & \xleftarrow{\tau} & C^*\otimes C^* \\ {\scriptstyle\Delta^*}\uparrow & & \uparrow{\scriptstyle (\mathrm{id}_C\otimes\Delta)^*} & & \uparrow{\scriptstyle (\Delta^*\circ\tau)\otimes \mathrm{id}_C} \\ (C\otimes C)^* & \xleftarrow[(\Delta\otimes \mathrm{id}_C)^*]{} & (C\otimes C\otimes C)^* & & \\ {\scriptstyle\tau}\uparrow & & & \nwarrow{\scriptstyle j} & \\ C^*\otimes C^* & & \xleftarrow[\mathrm{id}_C\otimes(\Delta^*\circ\tau)]{} & & C^*\otimes C^*\otimes C^* \end{array}$$

where $j : C^*\otimes C^*\otimes C^* \to (C\otimes C\otimes C)^*$ is the canonical map for three vector spaces. One shows (exercise!) that the outside square commutes, and the outside map $C^*\otimes C^*\otimes C^* \to C^*$ says that $m = \Delta^*\circ i$ is associative; in other words the multiplication map diagram commutes.

Moreover a dual to a counit map ε defines a unit $u = (\varepsilon)^*$, which makes $(C^*, \Delta^*\circ\tau, (\varepsilon)^*)$ to be an associative algebra.

One can also ask whether or not the dual A to an associative algebra A is always a coalgebra? When A is a finite-dimensional the answer is positive, while in general one should restrict himself to the class of pseudocompact algebras which we will discuss on the next lecture.

Example 2.6. Let $C = M_n(k)$ be a coalgebra as in example ?.

Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be two coalgebras. A linear map $f : C \to D$ is a *morphism* of coalgebras if the following diagrams are commutative:

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle\Delta_C}\downarrow & & \downarrow{\scriptstyle\Delta_D} \\ C\otimes C & \xrightarrow{f\otimes f} & D\otimes D \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{f} & D \\ & {\scriptstyle\varepsilon_C}\searrow \quad \swarrow{\scriptstyle\varepsilon_D} & \\ & k & \end{array}$$

With such morphisms the k-coalgebras form a category denoted by $\mathbf{Cog}_k$, respectively category of finite dimensional coalgebras form a category $\mathbf{cog}_k$. Now the discussion above can be formilized as

Theorem 2.1. *Passing to dual vector spaces $(-)^*$ gives rise a duality*

$$(-)^* : \mathbf{alg}_k \to \mathbf{cog}_k$$

2.3 Coideals, subcolagebras and correspondence between structures

Let C be a coalgebra. A subspace $I \subset C$ is a *coideal* if

$$\Delta(I) \subset I \otimes C + C \otimes I$$

and $\varepsilon(I) = 0$. A subspace $D \subset C$ is a *subcoalgebra* of C if

$$\Delta(D) \subset D \otimes D.$$

Proposition 2.1. *Let C and D be coalgebras, and let $f : C \to D$ be a morphism of coalgebras.*

(1) If E is a subcoalgebra of C, then E is a coalgebra with $\Delta_E = \Delta_{C|E}$ and $\varepsilon_E = \varepsilon_{C|V}$.

(2) If I is a coideal of C, then C/I is a factor coalgebra with

$$\Delta_{C/I}(c + I) = \sum (c(1) + I) \otimes (c(2) + I),$$
$$\varepsilon_{C/I}(c + I) = \varepsilon_C(c).$$

(3) $\operatorname{Ker} f$ is a coideal of C and $\operatorname{Im} f = f(C)$ is a subcoalgebra of D.

(4)

$$\tilde{f} : C/\operatorname{Ker} f \to f(C)$$
$$c + \operatorname{Ker} f \mapsto f(c)$$

is an isomorphism of coalgebras

For any subset $X \subseteq V$ define

$$X^{\perp} = \{f \in V^* \mid f(x) = 0, \quad x \in X\}.$$

Theorem 2.2. *Let C be a coalgebra, and let $A = C^*$ be the dual algebra. Then:*

(1) If I is an ideal of $A = C^*$, then $I^{\perp}$ is a subcoalgebra of C.

(2) If B is a subalgebra of $A = C^*$, then $B^{\perp}$ is a coideal of C.

(3) If J is a coideal of C, then $J^{\perp}$ is a subalgebra of $A = C^*$.

(4) If D is a subcoalgebra of C, then $D^{\perp}$ is an ideal of $A = C^*$.

Corollary 2.1. *If C is a finite dimensional coalgebra, then there are bijections:*

$$\{\textit{ideals of } C^*\} \longleftrightarrow \{\textit{subcoalgebras of } \mathrm{C}\}$$
$$\{\textit{subalgebras of } C^*\} \longleftrightarrow \{\textit{coideals of } \mathrm{C}\}$$

given by $(\cdot)^{\perp}$.

Example 2.7. Let $C = M_n(k)$ be the matrix coalgebra. Let

$$I = \operatorname{span}\{e_{ij} \mid i > j\} = \{\text{strictly lower triangular matrices}\},$$

It is easy to check that

$$\Delta(e_{ij}) \subseteq I \otimes C + C \otimes I$$

and $\varepsilon(e_{ij}) = 0$. Therefore I is a coideal. By Theorem **??**

$$I^{\perp} = \{e^*_{ij} \mid i \leq j\}$$

is a subalgebra of $C^* \cong M_n(k)$, and hence $I^{\perp}$ is the subalgebra of upper triangular matrices in $M_n(k)$. On the other hand the algebra $C^* \cong M_n(k)$ has no nontrivial ideals (being simple) hence by the Theorem above C has only trivial subcoalgebras.

2.4 Fundamental theorem for coalgebras

Theorem 2.3. *Let (C, Δ, ε) be a coalgebra over a field k and $x \in C$. Then there exists subcoalgebra $D \subseteq C$ such that $x \in D$ and $\dim_k D < \infty$.*

Proof. Let

$$\Delta(x) = \sum_i b_i \otimes c_i.$$

Consider the element

$$\Delta_2(x) = \sum_i \Delta(b_i) \otimes c_i = \sum_{i,j} a_j \otimes b_{ij} \otimes c_i$$

Note that we may assume that (a_j) are linearly independent and so are (c_i). Let D be a subspace spanned by (b_{ij}). Of course $\dim_k D < \infty$. Furthermore $x \in D$, because

$$x = \sum_{i,j} \varepsilon(a_j)\varepsilon(c_i)b_{ij}$$

It reminds to show that D is a subcoalgebra, i.e. $\Delta(D) \subseteq D \otimes D$. Indeed, note that

$$\sum_{i,j} \Delta(a_j) \otimes b_{ij} \otimes c_i = \sum_{i,j} a_j \otimes \Delta(b_{ij}) \otimes c_i,$$

an since c_i are linearly independent, for all i we have

$$\sum_j \Delta(a_j) \otimes b_{ij} = \sum_j a_j \otimes \Delta(b_{ij}).$$

Thus

$$\sum_j a_j \otimes \Delta(b_{ij}) \subset C \otimes C \otimes D$$

and since a_j are linearly independent, we obtain that $\Delta(b_{ij}) \in C \otimes D$ for all i, j. Analogously we show that $\Delta(b_{ij}) \in D \otimes C$. Thus

$$\Delta(b_{ij}) \in C \otimes D \cap D \otimes C = D \otimes D.$$

□

Corollary 2.2. *Every coalgebra is the sum of its finite-dimensional subcoalgebras.*

The Fundamental Theorem of Coalgebras is major diference between algebras and coalgebras. For example consider the algebra $k[x]$. Whenever $f \in k[x]$ is such that $\deg(f) > 0$, then a subalgebra generated by f is always infinite dimensional (if $\deg(f) = 0$ then subalgebra generated by f is k). This can never occur in coalgebras. One can try to "dualise" seeking for the dual property that would say, if A is an algebra and $x \in A$ then there is an ideal I of finite codimension (i.e. $\dim_k A/I < \infty$) that does not contain x. But, this fails for general algebras, e.g. consider $A = k(x)$ the field of rational functions. Algebra A has no ideals I with A/I finite dimensional except $I = A$. (Note: Any simple infinite dimensional algebra would have the same problem.) But such property is true (after proper interpretation) for a special class of algebras called pseudocompact algebras (we aim to discuss this in the next lecture).

Corollary 2.3. *Let C be a coalgebra of dimension n. Then C is isomorphic to a quotient of matrix coalgebra $M_n(k)$.*

2.5 Exercices

Exercise 2.1. Let C be a coalgebra and let $c \in C$ such that $\Delta(c) = c \otimes c$. Prove that if $\varepsilon(c) \neq 0$, then $\varepsilon(c) = 1$

Exercise 2.2. Let C be a k-vector space with basis $\{s, c\}$. Prove that C is a coalgebra with comultiplication Δ and counit ε defined by:

$$\Delta(s) = s \otimes c + c \otimes s, \qquad \varepsilon(s) = 0,$$
$$\Delta(c) = c \otimes c - s \otimes s, \qquad \varepsilon(s) = 1.$$

Exercise 2.3. Let $M_n^c(k)$ be a matrix coalgebra with basis $\{e_ij\}_{1\leq i,j\leq n}$ (see the class). Prove that the dual algebra $M_n^c(k)^*$ is an (n, n)-matrix algebra over k.

Exercise 2.4. Let A be a finite-dimensional associative unital k-algebra.

(a) Prove that A^* is a coalgebra. (Hint: Use that $(A \otimes A)^* \cong A^* \otimes A^*$.

(b) Prove that B is a subalgebra of A if and only if $B^\perp = \{f : A \to k \mid f(B) = 0\}$ is a coideal of A^*.

(c) Prove that I is a two-sided ideal of A if and only if $I^\perp = \{f : A \to k \mid f(I) = 0\}$ is a subcoalgebra of A^*.

Exercise 2.5. Let V, W be two vector spaces. Show that a natural map $\tau : V^* \otimes W^* \to (V \otimes W)^*$ is always injective. And that τ is isomorphism if either V or W is finite-dimensional.

Exercise 2.6. Show that map m (see the class!) factors as $\Delta^* \circ \tau$.

Exercise 2.7. Show that $m = \Delta^* \circ \tau$ is an associative map, showing that the diagram from the class is commutative.

Exercise 2.8. Let V be a vector space, and let $I, J \subset V^*$ are subspaces. Show:

$$(I \otimes J)^\perp = I^\perp \otimes V + V \otimes J^\perp.$$

Exercise 2.9. Using Exercise 2.8 show the duality between coideals in C and subalgebras in C^* (respectively subcoalgebras in C and ideals in C^*).

Chapter 3

Pseudocompact algebras

Let k be a field. Remember from the last lecture the:

- *Fundamental Theorem of coalgebras*:Every coalgebra is a union of finite dimensional coalgebras.

3.1 Motivation

The Fundamental Theorem says that although coalgebras can be very big, they can be understood very well by studying their finite dimensional subcoalgebras. This is a very powerful and unusual property.

We also saw that finite dimensional coalgebras and comodules are "the same thing" as finite dimensional algebras and modules (formally: the respective categories are *dual*).

We also saw that for *any* coalgebra C, the dual space C^* is an algebra. The task today is to understand what algebras we get.

Example 3.1. I claim that $k[x]^* = k[[x]]$: a map $k[x] \to k$ is freely determined by where we send the basis $\{1, x, x^2, \ldots\}$, so as a vector space we can write it as the product

$$k[x]^* = k_1 \times k_x \times k_{x^2} \times \ldots$$

where the vector $(\lambda_1, \lambda_x, \lambda_{x^2}, \ldots)$ is the linear map sending $1 \mapsto \lambda_1$, $x \mapsto \lambda_x$, etc. What's the multiplication? Denote by f_{x^n} the map sending x^n to 1 and the other x_i to 0. The product is

$$C^* \otimes C^* \xrightarrow{\tau} (C \otimes C)^* \xrightarrow{-\circ\Delta} C^*$$

and so the product of f_x with f_x is by definition

$$(f_x \otimes f_x)\Delta.$$

Applying to x^2 we get

$$(f_x \otimes f_x)\Delta(x^2) = (f_x \otimes f_x)(1 \otimes x^2 + x \otimes x + x^2 \otimes 1) = 0 + 1 + 0 = 1,$$

and

$$(f_x \otimes f_x)\Delta(x^n) = 0 \quad \forall n \neq 2,$$

so that $f_x \cdot f_x = f_{x^2}$. This can be formalized to get the isomorphism $k[x]^* = k[[x]]$.

Q: More generally, what algebras do we get by dualizing coalgebras?

3.2 Direct and inverse limits

Definition 3.1. A partially ordered set I is

- *directed above* if for every $x, y \in I$ there is z with $z \geqslant x, y$.
- *directed below* if for every $x, y \in I$ there is z with $z \leqslant x, y$.

If I is a partially ordered set, we can think of I as a category, with objects the elements of I and a map $x \to y$ whenever $x \geqslant y$.

Definition 3.2. If $\mathcal{C}$ is a category (eg. of algebras or coalgebras or modules or comodules),

- a *direct system* $\{C_i, \psi_{ij}, I\}$ in $\mathcal{C}$ is the image of a (covariant) functor $F : I \to \mathcal{C}$, where I is directed below.
- an *inverse system* $\{D_i, \varphi_{ij}, I\}$ in $\mathcal{C}$ is the image of a functor $F : I \to \mathcal{C}$, where I is directed above.

Example 3.2. Let C be a coalgebra and $\mathcal{C}$ be the category of k-coalgebras. The set of finite dimensional subcoalgebras of C with inclusion maps forms a direct system: it's directed below because if X, Y are finite dimensional subcoalgebras, then $X + Y$ is a finite dimensional subcoalgebra that contains them both.

Definition 3.3.

- The *direct limit* $\varinjlim C_i$ of a direct system $\{C_i, \psi_{ij}, I\}$ is the categorical colimit: it's an object C of $\mathcal{C}$ together with maps $\psi_i : C_i \to C$ for every $i \in I$ such that all triangles commute, and universal as such.
- The *inverse limit* $\varprojlim D_i$ of an inverse system $\{D_i, \varphi_{ij}, I\}$ is the categorical limit: it's an object D of $\mathcal{C}$ together with maps $\varphi_i : D \to D_i$ for every $i \in I$ such that all triangles commute, and universal as such.

Exercise 1: give a full definition of direct and inverse limit, filling in all the details

Example 3.3. The direct limit of the direct system from the last example is precisely the *union* of the finite dimensional subcoalgebras of C. The fundamental theorem of coalgebras says *exactly* that $C = \varinjlim C_i$!

So now:

- A coalgebra C is the direct limit of its finite dimensional subcoalgebras $\{C_i, \psi_{ij}\}$, where $\psi_{ij} : C_i \to C_j$ is inclusion.
- Apply duality. This is a *contravariant* functor, so the arrows swap direction. So we get an *inverse system* of finite dimensional *algebras* $\{{C_i}^*, \varphi_{ij} = {\psi_{ij}}^*\}$.
- Duality commutes with (co)limits, and so

$$C^* = \varprojlim {C_i}^*.$$

  That is, the algebras that come from dualizing coalgebras are precisely *inverse limits of finite dimensional algebras*.

Exercise 2: write $k[x]$ as a direct limit of fd coalgebras, dualise the direct system, and check that $k[[x]]$ is its inverse limit.

Example 3.4. Let I be a set and for each $i \in I$, let A_i be a finite dimensional algebra. Then

$$A = \prod_{i \in I} A_i$$

is an inverse limit of finite dimensional algebras.

Exercise 3: to check this!

3.3 Topologies

The true power of pseudocompact objects comes from topology. To get the idea, consider vector spaces:

Example 3.5. k – field, V – k-vector space. As we all know:

$$V \cong V^{**} \iff \dim(V) < \infty.$$

It's annoying that $V \ncong V^{**}$ when $\dim(V)$ is infinite!

But look what happens when we work in the category of *topological* vector spaces (k has the discrete topology).

Suppose V has basis $B = \{b_1, b_2, b_3, \ldots\}$, then

$$V = \bigoplus_{b_i \in B} k.$$

Like in Example 3.1,

$$V^* = \prod_{b_i \in B} k.$$

These are *topological* vector spaces, so the product should be a *topological product*. That means V^* has the *product topology*. A basis of open neighbourhoods of 0 is given by

$$B_n = \{(0, \ldots, 0, \lambda_n, \lambda_{n+1}, \ldots) \mid \lambda_i \in k\}$$

Now $V^{**} = \mathrm{Hom}_k(V^*, k)$. But these are topological vector spaces, so "Hom" means *continuous* homs. k is discrete so if $f \in V^{**}$ then $\mathrm{Ker}(f) = f^{-1}(0)$ is open. This means that f has to send *almost every* coordinate to 0. In other words

$$V^{**} \cong \bigoplus_{b_i \in B} k = V!$$

Definition 3.4. k – field, given the discrete topology. A *pseudocompact algebra* is an inverse limit of discrete finite dimensional associative k-algebras, taken in the category of topological k-algebras.

Proposition 3.1. *A topological algebra A is pseudocompact if, and only if, it has a basis B of open neighbourhoods of 0 consisting of finite codimension ideals such that*

$$\bigcap_{I \in B} I = 0 \quad \textit{and} \quad A = \varprojlim_{I \in B} A/I.$$

Example 3.6.
- Fd algebras are pseudocompact.
- Products of fd algebras (with the product topology) are pseudocompact.
- $k[[x]]$, with basis of neighbourhoods of 0 given by the ideals (x^n) ($n \in \mathbb{N}$) is pseudocompact.
- G – profinite group, $\mathcal{N}$ – set of open normal subgroups of G. Then

  $$k[[G]] := \varprojlim_{N \in \mathcal{N}} k[G/N]$$

  is pseudocompact. This is the algebra to study to understand the representation theory of G.

Theorem 3.1. *The category of coalgebras and coalgebra homomorphisms is dual to the category of pseudocompact algebras and continuous algebra homomorphisms. In both directions, the duality is given by $X \mapsto X^* = \mathrm{Hom}_k(X, k)$ (where* Hom *always means* continuous *homs!).*

Exercise 6:
"Every pseudocompact algebra is a closed subalgebra of a direct product of discrete finite dimensional algebras."

If you know about profinite groups, the best thing about them is that they are *compact*. Pseudocompact algebras are not usually compact. For instance $A = k$ is discrete, so already isn't compact when k is infinite. But they sort of morally are:

Definition 3.5. An *affine subspace* of a vector space is a coset of a subspace. A topological vector space V is *linearly compact* if whenever W_i ($i \in I$) are closed affine subspaces of V such that $\bigcap_{i \in I} W_i = \varnothing$, then $W_{i_1} \cap \ldots \cap W_{i_n} = \varnothing$ for some $\{i_1, \ldots, i_n\} \subseteq I$.

Exercise 7: Discrete finite dimensional vector spaces are linearly compact.

Proposition 3.2. *Pseudocompact algebras are linearly compact.*

Proof. Products of linearly compact spaces are linearly compact (this is just like Tychonoff's theorem) and closed subspaces of linearly compact spaces are linearly compact (this is easy). □

3.4 Exercises

Exercise 3.1. Give a full definition of direct and inverse limit, filling in all the details

Exercise 3.2. Write the coalgebra $k[x]$ as a direct limit of fd coalgebras, dualise the direct system, and check that $k[[x]]$ is its inverse limit.

Exercise 3.3. If each A_i ($i \in I$) is a finite dimensional algebra, write down an inverse system for $A = \prod_{i \in I} A_i$ and convince yourself that A is its inverse limit.

Exercise 3.4. If $A = \varprojlim \{A_i, \varphi_{ij}\}$ is an inverse limit of finite dimensional vector spaces, the maps φ_{ij} might not be surjective. But check that $\{\varphi_i(A), \varphi_{ij}\}$ is another inverse system with surjective maps, and that A is its limit.

Exercise 3.5. Check that a direct sum of coalgebras is a coalgebra. Convince yourself that if each C_i (resp. A_i) is a finite dimensional coalgebra (resp. algebra), then

$$\Big(\bigoplus C_i\Big)^* = \prod {C_i}^*, \quad \Big(\prod A_i\Big)^* = \bigoplus {A_i}^*.$$

Exercise 3.6. 1. Understand $k[[x]]$ as a subalgebra of $\prod_{n\in\mathbb{N}} k[x]/x^n$. Show that it's closed.

2. More generally, convince yourself that the following theorem is true:

   "Every pseudocompact algebra is a closed subalgebra of a direct product of discrete finite dimensional algebras."

Exercise 3.7. Convince yourself that discrete finite dimensional vector spaces are linearly compact.

Chapter 4

Basic structure of pseudocompact algebras

We'll see that the structure of pseudocompact algebras is just as well behaved as that of finite dimensional algebras! Let k be a field and A a pseudocompact k-algebra.

4.1 The Jacobson radical and semisimple algebras

Definition 4.1. The (topological) *Jacobson radical* $J(A)$ of A is the intersection of the maximal closed left ideals of A.

Exercise:

1. Let $\mathcal{I}$ be the set of open ideals of A. If X is a closed subspace of A, show that

$$X = \bigcap_{I \in \mathcal{I}} X + I.$$

[hint: use linear compactness. If $y \notin X$ then $(y + X) \cap \bigcap_{I \in \mathcal{I}} I = \varnothing$.]

2. Show that maximal closed left ideals have finite codimension.

3. Show that every proper closed (left) ideal is contained in a maximal closed (left ideal).

Example 4.1. It's *not* true that every maximal ideal of finite codimension is closed! Let A be the $\mathbb{F}_p$-algebra $\prod_{\mathbb{N}} \mathbb{F}_p$. Let I be the (topologically dense!) ideal $\bigoplus_{\mathbb{N}} \mathbb{F}_p$ and let M be a maximal ideal containing I. Since A is commutative, A/M is a field in which $x^p = x \,\forall x$, so $A/M \cong \mathbb{F}_p$ has codimension 1. But it's not closed, because it's dense!

Example 4.2. 1. But $J\left(\prod_{\mathbb{N}} k\right) = 0$ because

$$I_n = \underbrace{k \times \ldots \times k}_{n \text{ times}} \times 0 \times k \times k \times \ldots$$

is a maximal closed ideal, and $\bigcap_n I_n = 0$.

2. $J(k[[x]]) = (x)$ – because $k[[x]]$ is local! Note that

$$k[[x]]/J(k[[x]]) = k[[x]]/(x) \cong k$$

is a semisimple algebra. This is different from the algebra $k[x]$: the intersection $J(k[x])$ of its maximal ideals is 0, so $k[x]/J(k[x])$ is *not* semisimple!

Since there are more abstract maximal ideals than closed maximal ideals, it might be surprising that:

Proposition 4.1. *The topological Jacobson radical is equal to the abstract Jacobson radical.*

Proof. Call them $tJ(A)$ and $aJ(A)$ respectively. Then $aJ(A) \subseteq tJ(A)$ is clear so we do the other. If $x \notin aJ(A)$ then by Lecture 1, $1 - yx$ is not left invertible for some y, so that $A(1 - yx) \subsetneq A$, which is closed so contained in a maximal closed left ideal M. But then

$$1 - yx \in M \implies 1 \in yx + M \implies yx \notin M \implies x \notin M \implies x \notin tJ(A).$$

□

Definition 4.2. The pseudocompact algebra A is (topologically) semisimple if for every closed left ideal I of A, there's a closed left ideal L of A such that $A = I \oplus L$.

Theorem 4.1. *The following are equivalent for a pseudocompact algebra A:*

1. *A is semisimple*
2. $J(A) = 0$
3. *A is an inverse limit of finite dimensional semisimple algebras*
4. *A is a direct product of algebras $M_n(\Delta)$, for Δ a finite dimensional k-division algebra.*

Proof.

$1 \implies 2$) If A is semisimple and $J(A) \neq 0$ then $A = J(A) \oplus I$ for I closed and proper, so $I \subseteq M$ for M a maximal closed left ideal. But then $J(A) \nsubseteq M$.

$2 \implies 3$) For I an open ideal, let $\pi_I : A \to A/I$ be the projection. Now

$$J(A/I) = \pi_I(J(A)) = \pi_I(0) = 0,$$

so each A/I is semisimple.

$3 \implies 4$) This is a bit of work to make precise but morally: by 3 and an exercise from the last lecture, A is an inverse limit of finite products of matrices with surjective maps φ_{ij} between them. Because matrices are simple, the kernel of φ_{ij} on a factor is either 0 or everything, so φ_{ij} just kills some factors. This is the inverse system whose limit is the product, so A is a product of matrices.

$4 \implies 1)$ Fix a closed ideal I of A and some matrix factor $A_i = M_n(\Delta)$ in the product. If $\pi : A \to M_n(\Delta)$ is the projection then $\pi(I)$ is a left ideal so it has some complement L_i. The product of these L_i is a complement to I in A.

□

Example 4.3. The topology is important! The algebra $A = \prod_{\mathbb{N}} k$ is topologically semisimple by the theorem, but it's not semisimple as an abstract algebra. For instance, the (non-closed!) ideal $I = \bigoplus_{\mathbb{N}} k$ intersects every non-zero ideal of A.

Example 4.4. Let G be a profinite group and suppose $\operatorname{char}(k) = 0$. By Maschke's Theorem, every finite dimensional $k[G/N]$ is semisimple, so by the theorem above, $k[[G]]$ is just some massive product of matrix algebras. This implies that the characteristic 0 representation theory of profinite groups is essentially the same as for finite groups.

Example 4.5. If A is pseudocompact, $J(A/J(A)) = 0$, so $A/J(A)$ is semisimple. We saw this for $k[[x]]$ above and also saw that it is false in general, because $J(k[x]) = 0$.

4.2 Modules for a pseudocompact algebra

If A is a pseudocompact algebra, a *pseudocompact A-module* is an inverse limit of discrete finite dimensional topological A-modules.

Theorem 4.2. *The category of (left) pseudocompact A-modules is dual to the category of (left) A^*-comodules.*

We can't quite have a Krull-Schmidt Theorem for pseudocompact modules: for instance if $A = \prod_{\mathbb{N}} k$, then A as an A-module is an infinite product of simple modules. But we get very close:

Theorem 4.3. *If M is a finitely generated pseudocompact A-module, then M is a direct product of indecomposable pseudocompact modules, and the decomposition is essentially unique.*

Proof. Just the idea. We need:

Proposition 4.2. *If M, X are pseudocompact A-modules and M is finitely generated, then* $\operatorname{Hom}_A(M, X)$ *is pseudocompact.*

Proof. If $X = \varprojlim X_i$, by the definition of inverse limit we have

$$\operatorname{Hom}_A(M, X) = \operatorname{Hom}_A(M, \varprojlim X_i) = \varprojlim \operatorname{Hom}_A(M, X_i).$$

But $\operatorname{Hom}_A(M, X_i)$ is *finite dimensional* because X_i is finite dimensional and a hom is determined by where I sent the finitely many generators of M. □

"So" $E = \operatorname{End}_A(M) = \operatorname{Hom}_A(M, M)$ is a pseudocompact algebra. The functor $\operatorname{Hom}_A(M, -)$ sends pseudocompact A-modules to pseudocompact E-modules. But $\operatorname{Hom}_A(M, M)$ is *free*, so (result of Gabriel) a product of indecomposable projectives. So M is also a product of indecomposables! □

Exercise: Use the theorem to show that every pseudocompact vector space has the form $\prod_{i\in I} k$ for some set I. So although the open ideal/submodule structure of a pseudocompact algebra/module can be complicated, the underlying vector space is always pretty easy.

Exercise: A pseudocompact A-module is *semisimple* if it's a direct product of simple A-modules. Show that a pseudocompact algebra A is semisimple if, and only if, every pseudocompact A-module is semisimple.

4.3 Exercises

Exercise 4.1. Let V be a linearly compact vector space. Show that open subspaces are closed, and that a closed subspace is open if, and only if, it has finite codimension.

Exercise 4.2.

1. Let $\mathcal{I}$ be the set of open ideals of the pseudocompact algebra A. If X is a closed subspace of A, show that

$$X = \bigcap_{I\in\mathcal{I}} (X + I).$$

   [hint: use linear compactness. If $y \notin X$ then $(y + X) \cap \bigcap_{I\in\mathcal{I}} I = \varnothing$.]

2. Show that maximal closed left ideals have finite codimension.

3. Show that every proper closed (left) ideal is contained in a maximal closed (left ideal).

Exercise 4.3. Use Theorem 4.3 to show that every pseudocompact vector space has the form $\prod_{i\in I} k$ for some set I. So although the open ideal/submodule structure of a pseudocompact algebra/module can be complicated, the underlying vector space is always pretty easy.

Exercise 4.4. A pseudocompact A-module is *semisimple* if it's a direct product of simple A-modules. Show that a pseudocompact algebra A is semisimple if, and only if, every pseudocompact A-module is semisimple.

Chapter 5

Representations of pseudocompact algebras

5.1 Basic algebras

Recall, that indecomposable modules are sort of building blocks for any module over f.d. algebra.

In what follows we need the notion of basic and pointed algebras. We call the algebra A *basic* if $A/J(A)$ is isomorphic to $\prod_i D_i$ with D_i are finite dimensional division k-algebras. Algebra A is called *pointed* if $A/J(A)$ is isomorphic to $\prod_i k$.

Basic algebras play a fundamental role in the theory of representations of finite-dimensional algebras due to the following theorem

Theorem 5.1. *For any finite-dimensional algebra A exists a basic finite dimensional algebra B so that there are equivalences of categories:*

$$A\text{-}\mathbf{Mod} \cong B\text{-}\mathbf{Mod}, \qquad A\text{-}\mathbf{mod} \cong B\text{-}\mathbf{mod}.$$

So the study of the representations of all finite-dimensional algebras "reduces" to the study of the representations of basic algebras. In what follows we show that for any arbitrary basic algebra A its representations can be studied in a combinatorial way (reduces the problem to some linear algebra problems) via representations of oriented graphs (=quivers).

5.2 Quivers and path algebras

A *quiver* Q is an oriented graph. We define a quiver Q as a tuple (Q_0, Q_1, s, t) where Q_0 is the set of vertices, Q_1 is a set of edges (arrows), and for a given arrow $h \in Q_1$, denote by $s(h), t(h) \in Q_0$ its initial and terminal vertex:

$$s(h) \xrightarrow{h} t(h).$$

A *representation* of a quiver Q is the setting a vector space V_i for each vertex $i \in Q_0$, and a linear mapping $V_h : V_{s(h)} \to V_{t(h)}$ for each arrow $h \in Q_1$. They form an abelian category $\mathrm{Rep}(Q)$ (resp. $\mathrm{rep}(Q)$ when all V_i are finite-dimensional) with naturally defined morfisms.

The *path algebra* kQ of a quiver Q is an k-algebra whose basis is formed by all oriented paths in Q (including trivial paths $p_i, i \in Q_0$), and multiplication is defined by concatenation of paths. If two paths cannot be concatenated, then their product is defined as 0.

Example 5.1. The path algebra of a quiver

$$1 \xrightarrow{h} 2$$

has a basis of 3 elements p_1, p_2 (trivial paths on the vertices) and h (path of length 1), endowed by multiplication

$$\begin{aligned} p_1^2 = p_1, \quad p_2^2 = p_2, \\ p_1p_2 = p_2p_1 = 0, \\ p_1h = hp_2 = h, \\ hp_1 = p_2h = h^2 = 0. \end{aligned}$$

One checks that there exists an isomorphism $kQ \cong \mathbb{U}_2(k)$, given by

$$p_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad p_2 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad h \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Given a module over path algebra $M \in kQ$-**Mod** one associates the representation V^M of the quiver Q in the following way:

$$\begin{aligned} V_i^M &:= e_iV, \\ V_h^M(x) &:= hx = e_{t(h)}hx \in V_{t(h)}^M, \qquad x \in V_{s(h)}. \end{aligned}$$

And vice-versa if V is a representation of the quiver Q, one cheks that $M^V := \bigoplus_{i \in Q_0} V_i$ has a structure of kQ-**Mod**. More precisely the correspondense

$$V \mapsto M^V, \qquad M \mapsto V^M,$$

gives the equivalences of categories:

$$kQ\text{-}\mathbf{Mod} \cong \mathrm{Rep}(Q), \qquad kQ\text{-}\mathbf{mod} \cong \mathrm{rep}(Q).$$

Another fundamental result is the following.

Theorem 5.2 (P. Gabriel)**.** *Let A be a pointed finite dimensional algebra over algebraically closed field. There exists a quiver Q_A and an admissible ideal I in kQ_A such that A is isomorphic to kQ_A/I.*

Where by *admissible* ideal I we mean a two-sided ideal $I \subset kQ$ so that

$$R_Q^m \subseteq I \subseteq R_Q^2,$$

for a $m \geq 2$, and R_Q is the two-sided ideal in algebra kQ generated by all arrows of Q. In other words, I is admissible, if it does not contain arrows of Q and includes all paths of length $\geq m$. Therefore, if field is algebraically closed, for each finite-dimensional algebra A we have

$$A\text{-}\mathbf{mod} \cong B\text{-}\mathbf{mod} \cong kQ_B/I\text{-}\mathbf{mod} \cong \mathrm{rep}(Q_B, I).$$

Construction of Q_A

- "vertices" in Q_A are basis in $A/J(A)$
- "arrows" are basis in J/J^2

5.3 Pseudocompact algebras

All this works very well for pseudocompact algebras. Let k be algebraically closed. Then we know from the last lecture that $A/J(A) = \prod_{i\in I} M_{n_i}(k)$ and we say that a pseudocompact algebra B is *basic* if $B/J(B) = \prod k$.

Proposition 5.1. *A is Morita equivalent to a basic pseudocompact algebra.*

Proof. Just like fd algebras:

- if $A/J(A) = \prod_{i\in I} M_{n_i}(k)$ then as a left module, $A = \prod_{i\in I} P_i^{n_i}$ with each P_i indecomposable.
- The module $X = \prod_{i\in I} P_i$ is finitely generated, so the algebra $B = \mathrm{End}_A(X)$ is pseudocompact.
- B is basic and Morita equivalent to A.

□

Let R be a finite quiver. The path algebra kR might not be finite dimensional, but for any n, the subspace $kR_{>n}$ with basis the paths of length bigger than n is an ideal and the algebra $kR/kR_{>n}$ is finite dimensional. We get an inverse system of finite dimensional algebras and we define

$$k[[R]] = \varprojlim_{n\in\mathbb{N}} kR/kR_{>n},$$

the *completed path algebra of R.*

Example 5.2. If $R = \bullet \circlearrowleft x$, then $kR/kR_{>n} \cong k[x]/(x^n)$ and so

$$k[[R]] = \varprojlim kR/kR_{>n} \cong \varprojlim k[x]/(x^n) = k[[x]].$$

Now let Q be an arbitrary quiver. Then Q is the union (= direct limit) of its finite subquivers: $Q = \bigcup R_i$. If $R_i \subseteq R_j$, then the inclusion yields an algebra homomorphism $k[[R_j]] \to k[[R_i]]$ (order swapped!) by sending a path of R_j to itself if it's in R_i, or to 0 otherwise. So the direct system of quivers gives an inverse system of pseudocompact algebras, and we define

$$k[[Q]] := \varprojlim k[[R_i]],$$

the *completed path algebra of Q.*

Example 5.3. The infinite quiver

$$Q = \quad \cdots \xleftarrow{\alpha_3} 3 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_1} 1$$

is the union of the finite quivers

$$R_n = \quad n \longleftarrow \cdots \xleftarrow{\alpha_3} 3 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_1} 1 .$$

For each n, $k[R_n] = k[[R_n]]$ is isomorphic to lower triangular $n \times n$-matrices:

$$k[R_3] = \begin{pmatrix} e_1 & 0 & 0 \\ \alpha_1 & e_2 & 0 \\ \alpha_2\alpha_1 & \alpha_2 & e_3 \end{pmatrix}.$$

The inclusions yield obvious projections, for instance

$$2 \xleftarrow{\alpha_1} 1 \quad \hookrightarrow \quad 3 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_1} 1$$

yields the projection

$$\begin{pmatrix} e_1 & 0 & 0 \\ \alpha_1 & e_2 & 0 \\ \alpha_2\alpha_1 & \alpha_2 & e_3 \end{pmatrix} \twoheadrightarrow \begin{pmatrix} e_1 & 0 \\ \alpha_1 & e_2 \end{pmatrix}.$$

The inverse limit $k[[Q]]$ is the algebra of "lower triangular $\mathbb{N} \times \mathbb{N}$-matrices"

$$k[[Q]] \quad = \quad \begin{pmatrix} e_1 & 0 & 0 & 0 & \\ \alpha_1 & e_2 & 0 & 0 & \cdots \\ \alpha_2\alpha_1 & \alpha_2 & e_3 & 0 & \\ \alpha_3\alpha_2\alpha_1 & \alpha_3\alpha_2 & \alpha_3 & e_4 & \\ & \vdots & & & \ddots \end{pmatrix}.$$

Theorem 5.3. *Let B be a basic pseudocompact algebra. There is a quiver Q and a closed ideal I of $k[[Q]]$ contained in $J^2(k[[Q]])$ such that*

$$B \cong k[[Q]]/I.$$

Proof. Just like for fd algebras, but being careful that the functor $R \to k[[R]]$ is contravariant, so some dualities appear:

- $B/J(B) = \prod_{i \in I} k$ since it's basic. The vertices of Q are the primitive idempotents $\{e_i \,|\, i \in I\}$.
- For each i, j we have a linearly compact vector space $e_j(J(B)/J^2(B))e_i$. It is dual to a discrete vector space V_{ji}. The arrows from e_i to e_j are a basis of V_{ji}.
- Just like for fd algebras, we get a continuous surjective algebra map $k[[Q]] \twoheadrightarrow B$, and its kernel I is contained in $J^2(k[[Q]])$.

□

Representations

Definition 5.1. Let Q be a quiver. A *representation* of Q is a vector space for every vertex, and a linear map for every arrow.

To simplify the conversation, let's focus on finite dimensional representations: so that's a finite dimensional vector space in finitely many vertices. Not every finite dimensional representation will correspond to a topological $k[[Q]]$-module:

Example 5.4. If $Q = \bullet \circlearrowleft x$, then $k[[Q]] = k[[x]]$. The representation $[a] \circlearrowleft \mathrm{id}$ does *not* give a $k[[x]]$-module: consider $y = \sum_{i=0}^{\infty} x^i \in k[[x]]$. Then

$$y \cdot a = a + a + a + \dots \qquad \text{– doesn't make sense in } k!$$

Proposition 5.2. *The category of finite dimensional pseudocompact $k[[Q]]$-modules is equivalent to the category of finite dimensional* nilpotent *representations. This means that there exists $n \in \mathbb{N}$ such that the LT corresponding to any path of length (at least) n is* 0.

Example 5.5. So by linear algebra, the representations of the indecomposable finite dimensional pseudocompact $k[[x]]$-modules look like this:

$$k^n \circlearrowleft \begin{pmatrix} 0 & 0 & & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & 1 & 0 \end{pmatrix}.$$

5.4 Exercises

Exercise 5.1. Show that the algebra $A = k[x]/(x^m)$ is pointed and its radical $J(A)$ is generated by x.

Exercise 5.2. Show that the algebra $A = \begin{pmatrix} k & k[x]/x^2 \\ 0 & k[x]/x^2 \end{pmatrix}$ is basic and its radical has the form

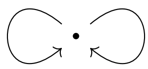
$$J(A) = \begin{pmatrix} 0 & k[x]/x^2 \\ 0 & xk[x]/x^2 \end{pmatrix}.$$

Exercise 5.3. Build the Gabriel quiver of the algebra $A = \mathbb{U}_n(k)$.

Exercise 5.4. Let $A = \mathbb{U}_3(k)$, and C be subalgebra consisting of all matrices

$$\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ 0 & \lambda_{22} & \lambda_{23} \\ 0 & 0 & \lambda_{33} \end{pmatrix}$$

such that $\lambda_{11} = \lambda_{22} = \lambda_{33}$. Show that C is isomorphic to kQ/I, where $I = \langle \alpha^2, \beta^2, \alpha\beta \rangle$ is ideal in kQ, and Q is the following quiver



Exercise 5.5. Check we can multiply $\mathbb{N} \times \mathbb{N}$-lower triangular matrices.

Exercise 5.6. Calculate the quivers of the algebra $k[[x]]$ and of the algebra of $\mathbb{N} \times \mathbb{N}$-lower triangular matrices.

Exercise 5.7. Calculate Q and I such that $k[[x, y]] \cong k[[Q]]/I$.

Exercise 5.8. We described the algebra $k[[Q]]$ when

$$Q = \quad \cdots \xleftarrow{\alpha_3} 3 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_1} 1$$

Describe the algebra $k[[Q]]$ when

$$Q = \quad \cdots \xleftarrow{\alpha_3} 3 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_1} 1 \xleftarrow{\alpha_0} 0 \xleftarrow{\alpha_{-1}} -1 \longleftarrow \cdots$$

Exercise 5.9. There is a description of representations that correspond to arbitrary pseudocompact $k[[Q]]$-modules, but it's a bit more fiddly. Write down the representations corresponding to the $k[[x]]$-module $k[[x]]$ and do the same thing for $\mathbb{N} \times \mathbb{N}$-lower triangular matrices. Do the same thing for the algebra from the previous question.

Note that they're not "nilpotent", but what you might call "pronilpotent".