Trabalho – (Panorama de Matemática)

MAT0554 — 2° SEMESTRE DE 2018

Solve any 4 exercises in Block A + any 2 in Block B.

Block A

Ex. 1. Let $\mathcal{C}$ be an arbitrary category. Show the following:

a) identical morphism $\mathrm{id}_X : X \to X$ is unique for each object $X \in \mathcal{C}$;

b) an arbitrary isomorphism $\mathcal{C}$ has unique inverse;

c) Let $f : X \to Y$ and $f : Y \to Z$ be two morphisms. Show that if two of the morphisms f, g and $f \circ g$ are isomorphisms, then the third is also isomorphism. (This property is called *two of three*).

Ex. 2. Let I be an arbitrary partially ordered set in which a partial order is given by $\preceq$. With I we associate the category $\mathcal{C}_I$, in which objects are elements of the set I, and for two arbitrary $i, j \in I$, $\mathbf{Mor}(i, j)$ — empty set if $i \not\preceq j$ and has one element if $i \preceq j$. Using reflexivity and transitivity of relation $\preceq$, one determines the composition of morphisms in $\mathcal{C}_I$ and shows that the construction above defines the category $\mathcal{C}_I$.

Ex. 3. Let I and J be two posets. Show that an arbitrary functor between the categories $\mathcal{C}_I$ and $\mathcal{C}_J$ is given by poset homomorphism (i.e. by the function that preserves order) $f : I \to J$.

Ex. 4. Let X be an arbitrary topological space and $I(X)$ is the set of all closed subsets of X. Show that $I(X)$ is a poset (with the order given by the inclusion of subsets). Specify category $\mathbf{Top}(X)$ as $\mathcal{C}_{I(X)}$ from exercise 2.

Ex. 5. Let V be vector space over $\mathbb{R}$. Show that the *complexification* $V \mapsto V \otimes_{\mathbb{R}} \mathbb{C}$ defines the functor $\mathbf{Vect}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{C}}$.

Ex. 6. Let $\mathcal{C}, \mathcal{D}$ be arbitrary categories, and $F : \mathcal{C} \to \mathcal{D}$ be a functor. Show the following:

a) F maps isomorphisms to isomorphisms;

b) if F is full and faithful, and $F(f) : F(X) \to F(Y)$ is an isomorphism in $\mathcal{D}$, then $f : X \to Y$ is an isomorphism in $\mathcal{C}$.

Ex. 7. Let G be an arbitrary group and $[G, G]$ its commutator. Show that $G/[G, G]$ is an abelian group, and the map $G \mapsto G/[G, G]$ defines the functor $Ab : \mathbf{Group} \to \mathbf{AbGroup}$.

Ex. 8. Define $\mathbf{Top}_*$ as a category whose objects are the pairs (X, x_0), X is topological space, $x_0 \in X$ is a fixed point and morphisms $f : (X, x_0) \to (Y, y_0)$ are continuous maps $f : X \to Y$ such that $f(x_0) = y_0$. Make sure that $\mathbf{Top}_*$ is a category, and the map $\pi_1 : \mathbf{Top}_* \to \mathbf{Group}$, which to the pair (X, x_0) connects its fundamental group $\pi_1(X, x_0)$, defines covariant functor.

Ex. 9. Define **CHaus** a category whose objects are compact Hausdorff spaces and morphisms are continuous maps. And define the map β : **Top** $\to$ **CHaus**, which to an arbitrary topological space X associates its Stone-Cech compactification βX (i.e., "maximum" compact Hausdorff space "generated" by X). Show that β defines a functor.

Ex. 10. Let R, S be rings (not necessarily commutative). Consider the category of right modules over these rings $\mathcal{C} = \text{Mod}_R$ and $\mathcal{D} = \text{Mod}_S$ (in which morphisms are homomorphisms of modules). Show that, fixing (R, S)-bimodule X, one can define two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ as follows:

$$F(Y) = Y \otimes_R X, \qquad Y \in \mathcal{D},$$
$$G(Z) = \text{Hom}_S(X, Z), \qquad Z \in \mathcal{C}.$$

Block B

Ex. 11. Suppose we are given three functor $F, G, H : \mathcal{C} \to \mathcal{D}$. Show that if $\alpha : F \to G$ and $\beta : G \to H$ are natural transformation, then the composition $\beta\alpha : F \to H$ is also natural. Show that the functors $F : \mathcal{C} \to \mathcal{D}$ form a category of $\mathbf{F}(\mathcal{C}, \mathcal{D})$, where the set of morphisms between any functors is given by natural transformations. Which natural transformation defines identical morphism?

Ex. 12. Let X be a topological space. Define a poset $I(X)$ as the set of all closed subsets of X (see Exercise 2 and 4). and poset $J(X)$ as the set of all subsets of X. Show that the functor of embedding of category $\mathcal{C}_{I(X)} \hookrightarrow \mathcal{C}_{J(X)}$ has left adjoint functor that maps an arbitrary subset in A to its closure $\bar{A}$.

Ex. 13. Let I be the set of all ideals in commutative ring $\mathbb{C}[x_1, \dots, x_n]$, and J is the set of all subsets in $\mathbb{C}^n$. Define $f : I \to J^{op}$ as a function that takes the ideal to its zero set in $\mathbb{C}^n$, and $f : J^{op} \to I$ which maps any subset of $\mathbb{C}^n$ into an ideal of polynomials which annihilates it . Show that f and g form Galois correspondence. Conclude that they form a pair of adjoint functors between the respective categories.

Ex. 14. Show that the complexifiaction functor $\mathbf{Vect}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{C}}$ (see. Exercise 5) is left to adjoint to functor which restricts the scalars $\mathbf{Vect}_{\mathbb{C}} \to \mathbf{Vect}_{\mathbb{R}}$.

Ex. 15. Let $G : \mathbf{Alg}_k \to \mathbf{Vect}_k$ be forgetful functor. Describe its left adjoint. (*Hint:* Given a vector space V build tensor algebra $T(V) = k \oplus V \oplus (V \otimes V) \oplus \dots$ and show that the correspondence $V \mapsto T(V)$ determines left adjoint functor to G).

Ex. 16. Describe the unit and counit of adjunction between **AbGroup** and **Group**.