

Lista 2

MAT0460/MAT6674 — 2º SEMESTRE DE 2018

Let k be a field. All modules M are left modules.

Exercício 1.

Let A be a \mathbb{F}_q -algebra and let S be a simple A module so that $\text{End}_A(S) \cong \mathbb{F}_{q^d}$. For a positive integer n , put $M = nS$. Show

- (a) The number of composition series of M is $[n]_{q^d}!$
- (b) For each $0 \leq r \leq n$, the number of submodules X of M with $X \cong rS$ is $\binom{n}{r}_{q^d}$.

Exercício 2.

Let A be a \mathbb{F}_q -algebra and let M, N be finite-dimensional A -modules. Suppose that $\text{Hom}_A(N, M) = 0$. Show that $F_{M, N}^{M \oplus N} = 1$.

Exercício 3.

Let $Q = 1 \rightarrow 2$ be the Dynkin quiver of type A_2 . Let $S(1)$ and $S(2)$ be the two simple modules of the path algebra.

- (i) Compute $[S(1)] * [S(2)]$ and $[S(2)] * [S(1)]$ in twisted Hall algebra.
- (ii) Show that the quantum Serre relations hold.

Exercício 4.

Let Q be a finite quiver with no oriented cycles. Let i and j be two vertices i and j in Q such that there is no arrow between i and j , and $n - 1$ arrows from j to i . We want to show the quantum Serre relation

$$\sum_{t=0}^n (-1)^t \binom{n}{t}_q q^{t(t-1)/2} [S(i)]^t * [S(j)] * [S(i)]^{n-t} = 0$$

in the twisted Hall algebra of $\mathbb{F}_q Q$, where $S(i)$ denotes the simple 1- dimensional representation associated to i .

- (i) Let V be an n -dimensional \mathbb{F}_q -vector space. For $d \leq n$, show that the number of flags $0 \subset V_1 \subset V_2 \subset \dots \subset V_d \subset V$ such that $\dim V_i = i$ is equal to

$$[n]_q \cdot [n-1]_q \cdot \dots \cdot [n-d+1]_q = \frac{[n]_q!}{[n-d]_q!}$$

Show that it is also the number of flags $V \supset V_1 \supset V_2 \supset \dots \supset V_d \supset 0$ such that $\text{codim} V_i = i$.

- (ii) Let R be the full subquiver of Q with vertices $\{i, j\}$. Let M be a representation of R such that $\dim M_i = n$ and $\dim M_j = 1$. Show that there is an indecomposable representation N of R such that $M = N \oplus S(i)^d$.
- (iii) Let M be a representation of Q such that the coefficient of $[M]$ in $[S(i)]^t * [S(j)] * [S(i)]^{(n-t)}$ is non-zero. Show that $M_k = 0$ for $k \in \{i, j\}$ so that M can be viewed as a representation of R of the form $N \oplus S(i)^d$ with N indecomposable. Show that $1 \leq d$ and $t \leq d$.
- (iv) The coefficient of $[M]$ in $[S(i)]^t * [S(j)] * [S(i)]^{(n-t)}$ counts the number of composition series $M = M_0 \supset M_1 \supset \dots \supset M_{n+1} = 0$ such that $M_{s-1}/M_s \cong S(i)$ for $s \neq t+1$ and $M_t/M_{t+1} \cong S(j)$.
- (v) Deduce from (i) and (iv) that the coefficient of $[M]$ in $[S(i)]^t * [S(j)] * [S(i)]^{(n-t)}$ equals

$$\frac{[d]_q! [n-t]_q!}{[d-t]_q!}.$$

- (vi) Conclude.

Exercício 5.

Describe Coxeter matrix (the one that corresponds for Coxeter element) for all Euclidian quivers canonically orientated (orientation is toward unique root vertex).

Exercício 6.

Let $Q = \tilde{D}_4$ canonically orientated. Describe possible defects of indecomposable kQ -modules. Describe all possible dimensions of indecomposable modules. Do the same for $Q = \tilde{E}_6$ canonically orientated.

Exercício 7.

Let $Q = \tilde{D}_4$ canonically orientated. Describe preprojective modules.

Exercício 8.

Describe simple regular representations of period 1 for $Q = \tilde{E}_6$ canonically orientated.

Exercício 9.

Let A, B be finite-dimensional algebras and $F : \text{mod}(A) \rightarrow \text{mod}(B)$ be immersion functor. Show that F preserves indecomposability and reflects isomorphisms.

Exercício 10.

Let A be a finite-dimensional algebra. Show that the canonical projection functor $\text{mod}A \rightarrow \text{mod}A/J(A)$ preserves indecomposability and reflects isomorphisms.

Exercício 11.

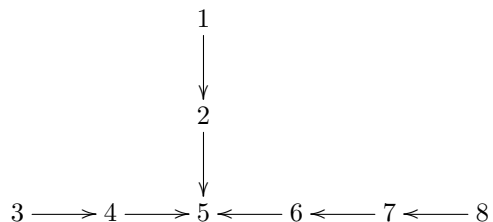
Show that there exists a functor $F : \text{rep}Q^{(2)} \rightarrow \text{mod}(k[x, y])$ which preserves indecomposability and reflects isomorphisms. (*Hint:* Define ideal $I = \langle \alpha_1 \alpha_2 - \alpha_2 \alpha_1 \rangle$ of the infinite dimensional algebra $kQ^{(2)}$. For a representation V of $Q^{(2)}$ define representation $F(V)$ of $Q^{(2)}$ by setting $F(V)_0 = V_0^3$ and

$$F(V)_{\alpha_i} = \begin{bmatrix} 0 & \text{id}_{V_0} & V_{\alpha_i} \\ 0 & 0 & \text{id}_{V_0} \\ 0 & 0 & 0 \end{bmatrix}$$

Show that $F(V)$ satisfies the ideal I and define F on morphisms to get a functor. Show that the functor satisfies the required properties).

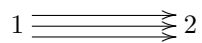
Exercício 12.

Show (constructing embedding of $\text{mod}(k\langle x, y \rangle)$ into $\text{mod}(kQ)$) that kQ is wild if Q is given by



Exercício 13.

Show (constructing embedding of $\text{mod}(k\langle x, y \rangle)$ into $\text{mod}(kQ)$) that kQ is wild if Q is given by



Exercício 14.

Show (constructing embedding of $\text{mod}(k\langle x, y \rangle)$ into $\text{mod}(kQ)$) that kQ is wild if Q is given by

