Lista 2

MAT0460/MAT6674 — 2° SEMESTRE DE 2018

Let k be a field. All modulos M are left modules.

Exercício 1.
Let A be a $\mathbb{F}_q$-algebra and let S be a simple A module so that $\mathrm{End}_A(S) \cong \mathbb{F}_{q^d}$. For a positive integer n, put $M = nS$. Show

(a) The number of composition series of M is $[n]_{q^d}!$

(b) For each $0 \leqslant r \leqslant n$, the number of submodules X of M with $X \cong rS$ is $\binom{n}{r}_{q^d}$.

Exercício 2.
Let A be a $\mathbb{F}_q$-algebra and let M, N be finite-dimensional A-modules. Suppose that $\mathrm{Hom}_A(N, M) = 0$. Show that $F^{M \oplus N}_{M,N} = 1$.

Exercício 3.
Let $Q = 1 \to 2$ be the Dynkin quiver of type A_2. Let $S(1)$ and $S(2)$ be the two simple modules of the path algebra.

(i) Compute $[S(1)] * [S(2)]$ and $[S(2)] * [S(1)]$ in twisted Hall algebra.

(ii) Show that the quantum Serre relations hold.

Exercício 4.
Let Q be a finite quiver with no oriented cycles. Let i and j be two vertices i and j in Q such that there is no arrow between i and j, and $n-1$ arrows from j to i. We want to show the quantum Serre relation

$$\sum_{t=0}^{n} (-1)^t \binom{n}{r}_q q^{t(t-1)/2} [S(i)]^t * [S(j)] * [S(i)]^(n-t) = 0$$

in the twisted Hall algebra of $\mathbb{F}_q Q$, where $S(i)$ denotes the simple 1- dimensional representation associated to i.

(i) Let V be an n-dimensional $\mathbb{F}_q$-vector space. For $d \leqslant n$, show that the number of flags $0 \subset V_1 \subset V_2 \subset \cdots \subset V_d \subset V$ such that $\dim V_i = i$ is equal to

$$[n]_q \cdot [n-1]_q \cdot \cdots \cdot [n-d+1]_q = \frac{[n]_q!}{[n-d]_q!}$$

Show that it is also the number of flags $V \supset V_1 \supset V_2 \supset \cdots \supset V_d \supset 0$ such that $\mathrm{codim} V_i = i$.

(ii) Let R be the full subquiver of Q with vertices $\{i,j\}$. Let M be a representation of R such that $\dim M_i = n$ and $dimM_j = 1$. Show that there is an indecomposable representation N of R such that $M = N \oplus S(i)^d$.

(iii) Let M be a representation of Q such that the coefficient of $[M]$ in $[S(i)]^t * [S(j)] * [S(i)]^(n-t)$ is non-zero. Show that $M_k = 0$ for $k \in \{i,j\}$ so that M can be viewed as a representation of R of the form $N \oplus S(i)^d$ with N indecomposable. Show that $1 \leqslant d$ and $t \leqslant d$.

(iv) The coefficient of $[M]$ in $[S(i)]^t * [S(j)] * [S(i)]^(n-t)$ counts the number of composition series $M = M_0 \supset M_1 \supset \cdots \supset M_{n+1} = 0$ such that $M_{s-1}/M_s \cong S(i)$ for $s \neq t+1$ and $M_t/M_{t+1} \cong S(j)$.

(v) Deduce from (i) and (iv) that the coefficient of $[M]$ in $[S(i)]^t * [S(j)] * [S(i)]^(n-t)$ equals

$$\frac{[d]_q![n-t]_q!}{[d-t]q!}.$$

(vi) Conclude.

Exercício 5.
Describe Coxeter matrix (the one that corresponds for Coxeter element) for all Euclidian quivers canonically orientated (orientation is toward unique root vertex).

Exercício 6.
Let $Q = \widetilde{D}_4$ canonically orientated. Describe possible defects of indecomposable kQ-modules. Descibe all possible dimensions of indecomposable modules. Do the same for $Q = \widetilde{E}_6$ canonically orientated.

Exercício 7.
Let $Q = \widetilde{D}_4$ canonically orientated. Describe preprojective modules.

Exercício 8.
Describe simple regular representations of period 1 for $Q = \widetilde{E}_6$ canonically orientated.

Exercício 9.
Let A, B be finite-dimensional algebras and $F : \mathrm{mod}(A) \to \mathrm{mod}(B)$ be immersion functor. Show that F preserves indecomposability and reflects isomorphisms.

Exercício 10.
Let A be a finite-dimensional algebra. Show that the canonical projection functor $\mathrm{mod}A \to \mathrm{mod}A/J(A)$ preserves indecomposability and reflects isomorphisms.

Exercício 11.
Show that there exists a functor $F : \mathrm{rep}Q^{(2)} \to \mathrm{mod}(k[x,y])$ which preserves indecomposability and reflects isomorphisms. (*Hint:* Define ideal $I = \langle \alpha_1\alpha_2 - \alpha_2\alpha_1 \rangle$ of the infinite dimensional algebra $kQ^{(2)}$. For a representation V of $Q^{(2)}$ define representation $F(V)$ of $Q^{(2)}$ by setting $F(V)_0 = V_0^3$ and

$$F(V)_{\alpha_i} = \begin{bmatrix} 0 & \mathrm{id}_{V_0} & V_{\alpha_i} \\ 0 & 0 & \mathrm{id}_{V_0} \\ 0 & 0 & 0 \end{bmatrix}$$

Show that $F(V)$ satisfies the ideal I and define F on morphisms to get a functor. Show that the functor satisfies the required properties).

Exercício 12.
Show (constructing embedding of $\mathrm{mod}(k\langle x, y\rangle$ into $\mathrm{mod}(kQ)$) that kQ is wild if Q is given by

$$\begin{tikzcd}
 & & 1 \arrow[d] & & & \\
 & & 2 \arrow[d] & & & \\
3 \arrow[r] & 4 \arrow[r] & 5 & 6 \arrow[l] & 7 \arrow[l] & 8 \arrow[l]
\end{tikzcd}$$

Exercício 13.
Show (constructing embedding of $\mathrm{mod}(k\langle x, y\rangle$ into $\mathrm{mod}(kQ)$) that kQ is wild if Q is given by

$$\begin{tikzcd}
1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & 2
\end{tikzcd}$$

Exercício 14.
Show (constructing embedding of $\mathrm{mod}(k\langle x, y\rangle$ into $\mathrm{mod}(kQ)$) that kQ is wild if Q is given by

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[loop above]
\end{tikzcd}$$