Lista 1

MAT0460/MAT6674 — 2° SEMESTRE DE 2018

Let k be a field. All modulos M are left modules.

Exercício 1.

Let Q be the quiver $1 \xrightarrow{\alpha} 2$.

(a) Show that the only indecomposable representations are:

$$S(1): \quad k \longrightarrow 0$$

$$S(2): \quad 0 \longrightarrow k$$

$$P(1): \quad k \xrightarrow{1} k,$$

(b) Show that $P(1)$ is not irreducible (simple).

(c) Suppose that V is a representation of Q with $V_1 = k^n$, $V_2 = k^m$ and $V_\alpha = A : k^n \to k^m$. Show that $V \cong S(1)^{d_1} \oplus S(2)^{d_2} \oplus P(1)^r$, with d_1 the dimension of the kernel of A, d_2 is the dimension of cokernel of A, and r is a rank of A.

Exercício 2.

Let Q be the quiver $1 \rightrightarrows 2$, and let M be the representation below:

$$k^2 \underset{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}{\overset{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}{\rightrightarrows}} k^3$$

1) Compute $\mathrm{End}M$.

2) Show that M is indecomposable.

Exercício 3.
Let Q be the quiver

$$\begin{tikzcd}
1 \arrow[dr] & & \\
& 3 & 4 \arrow[l] \\
2 \arrow[ur] & &
\end{tikzcd},$$

and let M be the representation below:

$$\begin{tikzcd}
k \arrow[dr, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] & & \\
& k^2 & k \arrow[l, swap, "\begin{bmatrix} 1 \\ \lambda \end{bmatrix}"] \\
k \arrow[ur, swap, "\begin{bmatrix} 0 \\ 1 \end{bmatrix}"] & &
\end{tikzcd}$$

1) Compute $\mathrm{End}M$.

2) Show that M is indecomposable if and only if $\lambda \neq 0$.

Exercício 4.
Let Q be the quiver

$$\begin{tikzcd}
& 2 \arrow[dl, swap, "\beta"] & \\
4 & & 1 \arrow[ul, swap, "\alpha"] \arrow[dl, "\gamma"] \\
& 3 \arrow[ul, "\delta"] &
\end{tikzcd}$$

and define two ideals $I_1 = \langle \alpha\beta + \gamma\delta \rangle$ and $I_2 = \langle \alpha\beta - \gamma\delta \rangle$. Show that

(a) $I_1 \neq I_2$ unless the characteristic of k is 2;

(b) there exists an isomorphism of algebras kQ/I_1 and kQ/I_2.

Exercício 5.
Let Q be the quiver

$$\begin{tikzcd}
& 2 \arrow[dr, "\beta"] & & \\
1 \arrow[ur, "\alpha"] \arrow[rr, swap, "\gamma"] & & 3 \arrow[r, "\delta"] & 4
\end{tikzcd}$$

and define two ideals $I_1 = \langle \gamma\delta \rangle$ and $I_2 = \langle \gamma\delta - \alpha\beta\delta \rangle$. Show that $kQ/I_1 \cong kQ/I_2$.

Exercício 6.
Let Q be the quiver

$$\begin{tikzcd}
 & 2 \arrow[dr, "\beta"] & \\
1 \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & & 4 \arrow[dl, "\delta"] \\
 & 3 \arrow[ul, "\epsilon"] &
\end{tikzcd}$$

and define two ideals $I_1 = \langle \gamma\delta, \delta\varepsilon \rangle$ and $I_2 = \langle \gamma\delta - \alpha\beta\delta, \delta\varepsilon \rangle$. Show that $kQ/I_1 \cong kQ/I_2$.

Exercício 7.
Write down the standard resolution for all 6 indecomposable representations of the quiver

$$1 \longrightarrow 2 \longrightarrow 3 \ .$$

Exercício 8.
Give an example of a bound quiver algebra of global dimension 3.

Exercício 9.
Give an example of a bound quiver algebra of global dimension 4.

Exercício 10.
Give an example of a bound quiver algebra of infinite global dimension.

Exercício 11.
Let A be a finite dimensional basic and connected algebra. Show that $Q_{A^{op}} = (Q_A)^{op}$ and that there exists an admissible ideal I^{op} of $KQ_{A^{op}}$ such that $A^{op} \cong (KQ_{A^{op}})/I^{op}$.

Exercício 12.
Write a bound quiver presentation of each of the following algebras:

$$\begin{bmatrix} K & 0 & 0 & 0 & 0 \\ K & K & 0 & 0 & 0 \\ K & 0 & K & 0 & 0 \\ K & 0 & K & K & 0 \\ K & K & K & K & K \end{bmatrix}, \begin{bmatrix} K & 0 & 0 & 0 & 0 \\ K & K & 0 & 0 & 0 \\ K & 0 & K & 0 & 0 \\ K & 0 & 0 & K & 0 \\ K & K & K & K & K \end{bmatrix}, \begin{bmatrix} K & 0 & 0 & 0 & 0 \\ 0 & K & 0 & 0 & 0 \\ K & K & K & 0 & 0 \\ K & 0 & 0 & K & 0 \\ K & K & K & K & K \end{bmatrix}.$$

Exercício 13.
Let (Q, I) be a bound quiver and $A = kQ/I$ its bound quiver algebra. Let i be any vertex in Q. Define a representation

$$P(i) = (P(i)_j, \varphi_\alpha)_{j \in Q_0, \alpha \in Q_1}$$

where $P(i)_j$ is the k-vector space with basis the set of all residue classes $c + I$ of paths c from i to j in Q; and if $\alpha : j \to l$ is an arrow in Q, then $\varphi_\alpha : P(i)_j \to P(i)_l$ is the linear map defined on the basis by composing the paths from i to j with the arrow α that is,

$$\varphi_\alpha(c + I) = c\alpha + I.$$

Show that each $P(i)$ is indecomposable projective kQ/I module.

Exercício 14.
Let (Q, I) be a bound quiver and $A = kQ/I$ its bound quiver algebra. Describe indecomposable injective kQ/I modules.

Exercício 15.
Show that all positive roots of the form $q(x_1, x_2) = x_1^2 + x_2^2 - 3x_1x_2$ are consecutive Fibonacci numbers at odd places (i.e. $(0, 1), (1, 0), (1, 3), (3, 1), (3, 8), (8, 3), \dots$).

Exercício 16.
Let $Q = A_n$. Show that all positive root of quadratic form q_Q are

$$e_i + e_{i+1} + \cdots + e_j, \qquad 1 \leqslant i \leqslant j \leqslant n$$

where e_i form canonical base in $\mathbb{Z}^n$. Therefore, conclude that Q has precisely $\frac{n(n-1)}{2}$ positive roots.

Exercício 17.
Let $Q = D_n$. Show that all possible root of quadratic form q_Q are:

$$e_i + e_{i+1} + \cdots + e_j, \qquad 1 \leqslant i \leqslant j \leqslant n,$$

$$e_1 + e_3 + \cdots + e_j, \qquad j \geqslant 3,$$

e

$$e_1 + e_2 + 2(e_3 + \cdots + e_i) + e_{i+1} + \cdots + e_j, \qquad 3 \leqslant i \leqslant j \leqslant n,$$

where e_i form canonical base in $\mathbb{Z}^n$. Therefore, conclude that $Q = D_n$ has $n(n-1)$ positive roots.

Exercício 18.
Let A be given by $A = k[x_1, x_2]$. Show that

(a) A is not local;

(b) elements 0 and 1 are unique idempotents of A;

(c) $J(A) = 0$.

Exercício 19.
Let $f : A \to B$ be algebra homomorphism. Show that if f is surjective then $f(J(A)) \subseteq J(B)$. Give a counter-example when f is not surjective.

Exercício 20.
Let Q uma aljava acíclica. Mostre que $J(kQ)$ é gerado pelas flechas em Q.

Exercício 21.
Construir a aljava do Gabriel da uma álgebra semisimples.

Exercício 22.
Let $A = \begin{bmatrix} k & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$. Construct Gabriel quiver of A.

Exercício 23.
Let $A = \mathbb{U}_3(k)$, and C is a subalgebra of A which consists of all matrix of the forms

$$\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ 0 & \lambda_{22} & \lambda_{23} \\ 0 & 0 & \lambda_{33} \end{pmatrix}$$

with $\lambda_{11} = \lambda_{22} = \lambda_{33}$. Show that C is isomorphic to kQ/I, where $I = \langle \alpha^2, \beta^2, \alpha\beta \rangle$ is an ideal in kQ, and Q is a quiver with one vertex 1 and 2 loops α and β at 1.

Exercício 24.
Let $Q = A_3$, $I = \langle \beta\alpha \rangle$ be an admissible ideal in kQ. Let dimension vector $\underline{v} = (1, 1, 1)$. Calculate the orbits of $Gl(\underline{v})$ on the sub-variety of $R_{Q,\underline{v}}$ consisting of bound representation of Q.

Exercício 25.
Suppose that $V \in R_{Q,\underline{v}}$. Let $Gl(\underline{v}) \cdot V$ be the orbit of V in $R_{Q,\underline{v}}$. Show that the dimension of codimension of this orbit is exactly $\dim \mathrm{Ext}_Q(V, V)$.

Exercício 26.
Find all indecomposable representations of the quivers

$$\begin{tikzcd} & 1 \arrow[d] & \\ 2 \arrow[r] & 3 & 4 \arrow[l] \end{tikzcd}$$

and

$$\begin{tikzcd} & 1 \arrow[d] & & \\ 2 \arrow[r] & 3 & 4 \arrow[l] & 5 \arrow[l] \end{tikzcd}$$

using reflection functors.

Exercício 27.
Consider the quiver

$$\begin{tikzcd} & & 1 \arrow[d] & & \\ 2 \arrow[r] & 3 \arrow[r] & 4 & 5 \arrow[l] & 6 \arrow[l] \end{tikzcd}$$

Find all dimension vectors of indecomposable representations by applying powers of the Coxeter element c. Find indecomposable representation with dimensions $(1, 1, 1, 2, 1, 0)$, $(1, 1, 2, 3, 2, 1)$, $(1, 0, 1, 2, 2, 1)$.

Exercício 28.
Show that each of the following integral quadratic forms is positive definite:

(a)
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 - x_1x_3 - x_2x_4 - x_3x_4 + x_1x_4.$$

(b)
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 + x_1x_3 - x_1x_4 - x_2x_3 + x_2x_4 - x_3x_4.$$