

Lista 1

MAT0460/MAT6674 — 2º SEMESTRE DE 2018

Let k be a field. All modules M are left modules.

Exercício 1.

Let Q be the quiver $1 \xrightarrow{\alpha} 2$.

(a) Show that the only indecomposable representations are:

$$S(1) : \quad k \longrightarrow 0$$

$$S(2) : \quad 0 \longrightarrow k$$

$$P(1) : \quad k \xrightarrow{1} k,$$

(b) Show that $P(1)$ is not irreducible (simple).

(c) Suppose that V is a representation of Q with $V_1 = k^n$, $V_2 = k^m$ and $V_\alpha = A : k^n \rightarrow k^m$. Show that $V \cong S(1)^{d_1} \oplus S(2)^{d_2} \oplus P(1)^r$, with d_1 the dimension of the kernel of A , d_2 is the dimension of cokernel of A , and r is a rank of A .

Exercício 2.

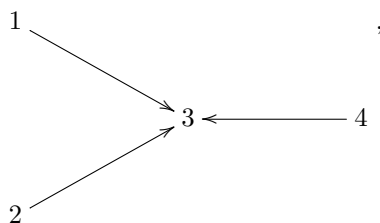
Let Q be the quiver $1 \rightrightarrows 2$, and let M be the representation below:

$$k^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} k^3$$

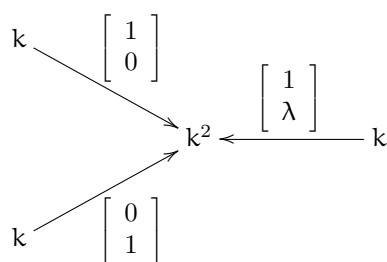
- 1) Compute $\text{End}M$.
- 2) Show that M is indecomposable.

Exercício 3.

Let Q be the quiver



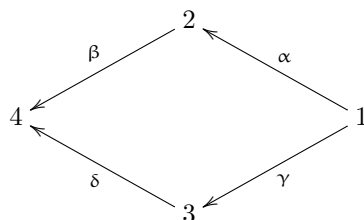
and let M be the representation below:



- 1) Compute $\text{End}M$.
- 2) Show that M is indecomposable if and only if $\lambda \neq 0$.

Exercício 4.

Let Q be the quiver

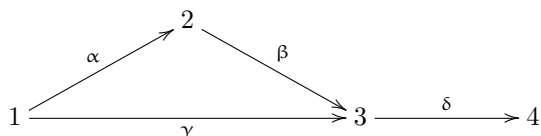


and define two ideals $I_1 = \langle \alpha\beta + \gamma\delta \rangle$ and $I_2 = \langle \alpha\beta - \gamma\delta \rangle$. Show that

- (a) $I_1 \neq I_2$ unless the characteristic of k is 2;
- (b) there exists an isomorphism of algebras kQ/I_1 and kQ/I_2 .

Exercício 5.

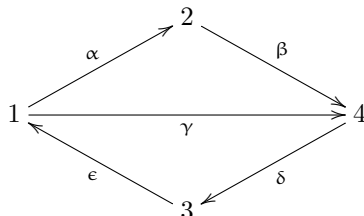
Let Q be the quiver



and define two ideals $I_1 = \langle \gamma\delta \rangle$ and $I_2 = \langle \gamma\delta - \alpha\beta\delta \rangle$. Show that $kQ/I_1 \cong kQ/I_2$.

Exercício 6.

Let Q be the quiver



and define two ideals $I_1 = \langle \gamma\delta, \delta\epsilon \rangle$ and $I_2 = \langle \gamma\delta - \alpha\beta\delta, \delta\epsilon \rangle$. Show that $kQ/I_1 \cong kQ/I_2$.

Exercício 7.

Write down the standard resolution for all 6 indecomposable representations of the quiver

$$1 \longrightarrow 2 \longrightarrow 3 .$$

Exercício 8.

Give an example of a bound quiver algebra of global dimension 3.

Exercício 9.

Give an example of a bound quiver algebra of global dimension 4.

Exercício 10.

Give an example of a bound quiver algebra of infinite global dimension.

Exercício 11.

Let A be a finite dimensional basic and connected algebra. Show that $Q_{A^{op}} = (Q_A)^{op}$ and that there exists an admissible ideal I^{op} of $KQ_{A^{op}}$ such that $A^{op} \cong (KQ_{A^{op}})/I^{op}$.

Exercício 12.

Write a bound quiver presentation of each of the following algebras:

$$\begin{bmatrix} K & 0 & 0 & 0 & 0 \\ K & K & 0 & 0 & 0 \\ K & 0 & K & 0 & 0 \\ K & 0 & K & K & 0 \\ K & K & K & K & K \end{bmatrix}, \begin{bmatrix} K & 0 & 0 & 0 & 0 \\ K & K & 0 & 0 & 0 \\ K & 0 & K & 0 & 0 \\ K & 0 & 0 & K & 0 \\ K & K & K & K & K \end{bmatrix}, \begin{bmatrix} K & 0 & 0 & 0 & 0 \\ 0 & K & 0 & 0 & 0 \\ K & K & K & 0 & 0 \\ K & 0 & 0 & K & 0 \\ K & K & K & K & K \end{bmatrix} .$$

Exercício 13.

Let (Q, I) be a bound quiver and $A = kQ/I$ its bound quiver algebra. Let i be any vertex in Q . Define a representation

$$P(i) = (P(i)_j, \varphi_\alpha)_{j \in Q_0, \alpha \in Q_1}$$

where $P(i)_j$ is the k -vector space with basis the set of all residue classes $c + I$ of paths c from i to j in Q ; and if $\alpha : j \rightarrow l$ is an arrow in Q , then $\varphi_\alpha : P(i)_j \rightarrow P(i)_l$ is the linear map defined on the basis by

composing the paths from i to j with the arrow α that is,

$$\varphi_\alpha(c + I) = c\alpha + I.$$

Show that each $P(i)$ is indecomposable projective kQ/I module.

Exercício 14.

Let (Q, I) be a bound quiver and $A = kQ/I$ its bound quiver algebra. Describe indecomposable injective kQ/I modules.

Exercício 15.

Show that all positive roots of the form $q(x_1, x_2) = x_1^2 + x_2^2 - 3x_1x_2$ are consecutive Fibonacci numbers at odd places (i.e. $(0, 1), (1, 0), (1, 3), (3, 1), (3, 8), (8, 3), \dots$).

Exercício 16.

Let $Q = A_n$. Show that all positive root of quadratic form q_Q are

$$e_i + e_{i+1} + \dots + e_j, \quad 1 \leq i \leq j \leq n$$

where e_i form canonical base in \mathbb{Z}^n . Therefore, conclude that Q has precisely $\frac{n(n-1)}{2}$ positive roots.

Exercício 17.

Let $Q = D_n$. Show that all possible root of quadratic form q_Q are:

$$e_i + e_{i+1} + \dots + e_j, \quad 1 \leq i \leq j \leq n,$$

$$e_1 + e_3 + \dots + e_j, \quad j \geq 3,$$

e

$$e_1 + e_2 + 2(e_3 + \dots + e_i) + e_{i+1} + \dots + e_j, \quad 3 \leq i \leq j \leq n,$$

where e_i form canonical base in \mathbb{Z}^n . Therefore, conclude that $Q = D_n$ has $n(n-1)$ positive roots.

Exercício 18.

Let A be given by $A = k[x_1, x_2]$. Show that

- (a) A is not local;
- (b) elements 0 and 1 are unique idempotents of A ;
- (c) $J(A) = 0$.

Exercício 19.

Let $f : A \rightarrow B$ be algebra homomorphism. Show that if f is surjective then $f(J(A)) \subseteq J(B)$. Give a counter-example when f is not surjective.

Exercício 20.

Let Q uma aljava acíclica. Mostre que $J(kQ)$ é gerado pelas flechas em Q .

Exercício 21.

Construir a aljava do Gabriel da uma álgebra semisimples.

Exercício 22.

Let $A = \begin{bmatrix} k & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$. Construct Gabriel quiver of A .

Exercício 23.

Let $A = \mathbb{U}_3(k)$, and C is a subalgebra of A which consists of all matrix of the forms

$$\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ 0 & \lambda_{22} & \lambda_{23} \\ 0 & 0 & \lambda_{33} \end{pmatrix}$$

with $\lambda_{11} = \lambda_{22} = \lambda_{33}$. Show that C is isomorphic to kQ/I , where $I = \langle \alpha^2, \beta^2, \alpha\beta \rangle$ is an ideal in kQ , and Q is a quiver with one vertex 1 and 2 loops α and β at 1.

Exercício 24.

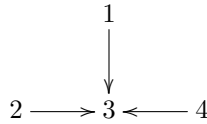
Let $Q = A_3$, $I = \langle \beta\alpha \rangle$ be an admissible ideal in kQ . Let dimension vector $\underline{v} = (1, 1, 1)$. Calculate the orbits of $Gl(\underline{v})$ on the sub-variety of $R_{Q,\underline{v}}$ consisting of bound representation of Q .

Exercício 25.

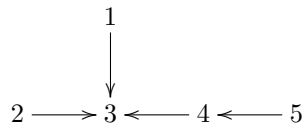
Suppose that $V \in R_{Q,\underline{v}}$. Let $Gl(\underline{v}) \cdot V$ be the orbit of V in $R_{Q,\underline{v}}$. Show that the dimension of codimension of this orbit is exactly $\dim Ext_Q(V, V)$.

Exercício 26.

Find all indecomposable representations of the quivers



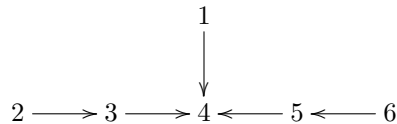
and



using reflection functors.

Exercício 27.

Consider the quiver



Find all dimension vectors of indecomposable representations by applying powers of the Coxeter element c . Find indecomposable representation with dimensions $(1, 1, 1, 2, 1, 0)$, $(1, 1, 2, 3, 2, 1)$, $(1, 0, 1, 2, 2, 1)$.

Exercício 28.

Show that each of the following integral quadratic forms is positive definite:

(a)

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 - x_1x_3 - x_2x_4 - x_3x_4 + x_1x_4.$$

(b)

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 + x_1x_3 - x_1x_4 - x_2x_3 + x_2x_4 - x_3x_4.$$