Lista 6. Gabatitos

Espaços com produto interno

1. Consideremos $w = (x, y)$ tal que $\langle u, w\rangle = -1$ e $\langle v, w\rangle = -1$
$$x + 2y = -1$$
$$-x + y = -1$$
portanto $x = \frac{1}{3}$ e $y = -\frac{2}{3}$.

2. $$4 = \langle u - v, u - v\rangle = \langle u, u\rangle - \langle u, v\rangle - \langle v, u\rangle + \langle v, v\rangle = 1 - 2\langle u, v\rangle + 1$$
Portanto
$$\langle u, v\rangle = -1$$

3. Observe que
$$\langle u + v, u - v\rangle = \langle u, u\rangle - \langle u, v\rangle + \langle v, u\rangle - \langle v, v\rangle = \langle u, u\rangle - \langle u, v\rangle + \langle u, v\rangle - \langle v, v\rangle = \langle u, u\rangle - \langle v, v\rangle$$

4.
   - $\langle x, y\rangle = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2$
     $\langle y, x\rangle = y_1x_1 - 2y_1x_2 - 2y_2x_1 + 5y_2x_2$
   - $\langle x + y, z\rangle = (x_1 + y_1)z_1 - 2(x_1 + y_1)z_2 - 2(x_2 + y_2)z_1 + 5(x_2 + y_2)z_2$
     $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
   - $\langle \alpha x, y\rangle = \alpha x_1y_1 - 2\alpha x_1y_2 - 2\alpha x_2y_1 + 5\alpha x_2y_2 = \alpha\langle x, y\rangle$
   - $\langle x, x\rangle = x_1x_1 - 2x_1x_2 - 2x_2x_1 + 5x_2x_2 = x_1^2 + 5x_2^2 \geq 0$
   - $\langle x, x\rangle = 0 \Leftrightarrow x = 0$
     Com produto interno usual temos $\|u\| = \sqrt{\langle u, u\rangle} = \sqrt{5}$ e em relação ao produto definido no exercío temos $\langle x, x\rangle = 1 + 5(4) = 21$

5. (a) $d(u, v) = \|u - v\| = \sqrt{\langle u - v, u - v\rangle} = \sqrt{2}$.
   (b) $d(u, v) = \|u - v\| = \left\|1 - \frac{1}{4}t - 3t^2\right\| = \sqrt{\int_0^1 \left(1 - \frac{1}{4}t - 3t^2\right)^2 dt} = \frac{227}{240}$
   (c) $d(u, v) = \|u - v\| = \left\| \begin{pmatrix} 0 & 0 & 2 \\ 4 & 5 & 5 \\ -1 & -1 & -1 \end{pmatrix} \right\|$
$$d(u, v) = \sqrt{tr\left(\begin{pmatrix} 0 & 4 & -1 \\ 0 & 5 & -1 \\ 2 & 5 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 4 & 5 & 5 \\ -1 & -1 & -1 \end{pmatrix} \right)} = \sqrt{73}$$

6. $S = [t^2 - 1, t - 1]$, logo usando o processo de ortogonalização de Gram-Schmidt temos a base ortogonal é $\{w_1, w_2\}$, onde
$$w_1 = t - 1$$
$$w_2 = t^2 - 1 - \frac{\langle t^2 - 1, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = t^2 - 1 - \frac{5/12}{1/30}(t - 1) = t^2 - \frac{23}{2}t + \frac{25}{2}$$
Como por teorema $P_2(\mathbb{R}) = S \oplus S^{\perp}$, então $S^{\perp} = [1]$, portanto para qualquer $p(t) = a_0 + a_1t + a_2t^2$ temos que $p = p_1 + p_2$, onde
$$p_1 = a_2(t^2 - 1) + a_1(t - 1)$$
$$p_2 = a_2 + a_1 + a_0$$

7. Considere a função $f(t) = \|u + tv\|$, para $t \in \mathbb{R}$, i.e,

$$f(t) = \sqrt{\langle v, v\rangle t^2 + 2\langle u, v\rangle t + \langle u, u\rangle}$$

logo

$$f'(t) = \frac{\langle v, v\rangle t + \langle u, v\rangle}{\sqrt{\langle v, v\rangle t^2 + 2\langle u, v\rangle t + \langle u, u\rangle}}$$

disso temos que o vetor de menor norma é quando

$$t = -\frac{\langle u, v\rangle}{\langle v, v\rangle}$$

8. Observar que se for um produto interno, então $\langle u, u\rangle > 0$ para qualquer $u \neq 0$, em particular para $(0, 1)$, logo

$$\langle (0, 1), (0, 1)\rangle = t > 0$$

e se $t > 0$ claramente é produto interno.

9. $\langle A, B\rangle = tr\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right) = 0$

$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{tr\left(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right)} = \sqrt{3}$

$\|B\| = \sqrt{\langle B, B\rangle} = \sqrt{tr\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)} = 1$

Observemos que $W = \left[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right]$, para calcular usemos o processo de Ortogonalização de Gram-Schmidt, logo a base ortonormal é $\{w_1, w_2, w_3\}$, onde

$$w_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\tilde{w}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} - \left\langle \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$w_2 = \frac{\tilde{w}_2}{\|\tilde{w}_2\|} = \frac{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}}{\left\| \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\|} = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \end{bmatrix}$$

$$\tilde{w}_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \end{bmatrix} \right\rangle \begin{bmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

$$w_3 = \frac{\tilde{w}_3}{\|\tilde{w}_3\|} = \frac{\begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}}{\left\| \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix} \right\|} = \begin{bmatrix} -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & 0 \end{bmatrix}.$$

10. Basta solucionar a equação

$$\left\langle mt^2 - 1, t\right\rangle = 0$$

i.e.

$$\frac{3}{2} - \frac{15}{4}m = 0$$

11. A base é $\{w_1, w_2, w_3\}$, onde

$$w_1 = \frac{u_1}{\|u_1\|} = \frac{(1,1,1)}{\sqrt{3}}.$$

$$\tilde{w}_2 = u_2 - \langle u_2, w_1\rangle\, w_1 = (1,-1,1) - \frac{1}{3}(1,1,1) = (2/3, -4/3, 2/3).$$

$$w_2 = \frac{\tilde{w}_2}{\|\tilde{w}_2\|} = \frac{(2/3,-4/3,2/3)}{2\sqrt{6}/3} = \frac{1}{\sqrt{6}}(1,-2,1).$$

$$\tilde{w}_3 = u_3 - \langle u_3, w_1\rangle\, w_1 - \langle u_3, w_2\rangle\, w_2 = (-1,0,1)$$

$$w_3 = \frac{\tilde{w}_3}{\|\tilde{w}_3\|} = \frac{-1,0,1)}{\sqrt{2}}.$$

12. $proj_V(u) = \frac{\langle(1,1),(1,3)\rangle}{\langle(1,3),(1,3)\rangle}(1,3) = \frac{2}{5}(1,3)$

13. $proj_U(f(t)) = \frac{\langle 2t-1, t\rangle}{\langle t, t\rangle} t = \frac{\int_0^1 (2t-1)t dt}{\int_0^1 t^2 dt} t = \frac{5}{2} t$

14. A base ortonormal é $\{w_1, w_2, w_3\}$, onde

$$w_1 = 1$$

$$\tilde{w}_2 = 1 + t - \langle 1+t, 1\rangle \cdot 1 = 1 + t - \frac{3}{2} = -\frac{1}{2} + t.$$

$$w_2 = \frac{\tilde{w}_2}{\|\tilde{w}_2\|} = \frac{t - \frac{1}{2}}{\|t - \frac{1}{2}\|} = 2\sqrt{3}(t - \frac{1}{2})$$

$$\tilde{w}_3 = 2t^2 - \left\langle 2t^2, 1\right\rangle \cdot 1 - \left\langle 2t^2, -\frac{1}{2} + t\right\rangle \left(-\frac{1}{2} + t\right) = 2t^2 - \frac{3}{2} - \frac{1/6}{1/12}\left(-\frac{1}{2} + t\right)$$

$$\tilde{w}_3 = -\frac{1}{2} - 2t + 2t^2. \qquad w_3 = \frac{\tilde{w}_3}{\|\tilde{w}_3\|}.$$

Para o $W^\perp$, consideremos $p(t) = a + bt + ct^2$ tal que

$$\langle p(t), 5\rangle = 0$$

$$\langle p(t), 1 + t\rangle = 0$$

logo

$$5a + \frac{5}{2}b + \frac{5}{3}c = 0$$

$$\frac{3}{2}a + \frac{5}{6}b + \frac{7}{12}c = 0$$

assim

$$b = -\frac{2}{3}a\ ,\ c = -4a$$

portanto

$$W^\perp = \left\{p(t) = a + bt + ct^2\ :\ \langle p(t), 5\rangle = 0,\ \langle p(t), 1+t\rangle = 0\right\}$$

$$W^{\perp} = \left\{ p(t) = a + bt + ct^2 \ : \ b = -\frac{2}{3}a \ , \ c = -4a, \ a \in \mathbb{R} \right\}$$

$$W^{\perp} = \left[1 - \frac{2}{3}t - 4t^2 \right]$$

15. Observar que $W = \left[\left(1, 0, 1, \frac{1}{2}\right), \left(0, 1, -1, -\frac{1}{2}\right) \right]$ usemos o processo Gram-Schmidt para achar uma base $\{w_1, w_2\}$ ortogonal de W. Assim

$$w_1 = \left(1, 0, 1, \frac{1}{2}\right)$$

$$w_2 = \left(0, 1, -1, -\frac{1}{2}\right) - \frac{\left\langle \left(0, 1, -1, -\frac{1}{2}\right), \left(1, 0, 1, \frac{1}{2}\right) \right\rangle}{\left\langle \left(1, 0, 1, \frac{1}{2}\right), \left(1, 0, 1, \frac{1}{2}\right) \right\rangle} \left(1, 0, 1, \frac{1}{2}\right)$$

$$w_2 = \left(0, 1, -1, -\frac{1}{2}\right) - \frac{-5/4}{9/4} \left(1, 0, 1, \frac{1}{2}\right) = \left(\frac{5}{9}, 1, -\frac{4}{9}, -\frac{2}{9}\right)$$

Portanto, se $u = (1, 1, 0, -1)$,

$$proj_W(u) = \frac{\left\langle (1, 1, 0, -1), \left(1, 0, 1, \frac{1}{2}\right) \right\rangle}{\left\langle \left(1, 0, 1, \frac{1}{2}\right), \left(1, 0, 1, \frac{1}{2}\right) \right\rangle} \left(1, 0, 1, \frac{1}{2}\right) + \frac{\left\langle (1, 1, 0, -1), \left(\frac{5}{9}, 1, -\frac{4}{9}, -\frac{2}{9}\right) \right\rangle}{\left\langle \left(\frac{5}{9}, 1, -\frac{4}{9}, -\frac{2}{9}\right), \left(\frac{5}{9}, 1, -\frac{4}{9}, -\frac{2}{9}\right) \right\rangle} \left(\frac{5}{9}, 1, -\frac{4}{9}, -\frac{2}{9}\right)$$

$$proj_W(u) = \frac{-1/2}{9/4} \left(1, 0, 1, \frac{1}{2}\right) + \frac{16/9}{14/9} \left(\frac{5}{9}, 1, -\frac{4}{9}, -\frac{2}{9}\right) = \left(\frac{26}{63}, \frac{8}{7}, -\frac{46}{63}, -\frac{23}{63}\right)$$

16. Claramente

$$\int_0^{\pi} \cos(nx) dx = 0$$

usando a identidade

$$\cos(nx)\cos(mx) = \frac{\cos((n+m)x) + \cos((n-m)x)}{2}$$

Temos para $n \neq m$

$$\int_0^{\pi} \cos(nx)\cos(mx) dx = \int_0^{\pi} \frac{\cos((n+m)x) + \cos((n-m)x)}{2} dx = 0$$

17. Claramente para que seja isometria $x \neq y$, caso contrario a matriz nem seria invertivel. Agora, se for isometria temos que

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ x & y & z \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ x & y & z \end{pmatrix}^{t}$$

portanto

$$\begin{pmatrix} \sqrt{2} - \frac{\sqrt{2}x}{y-x} & \frac{z}{y-x} & -\frac{1}{y-x} \\ \frac{\sqrt{2}x}{y-x} & -\frac{z}{y-x} & \frac{1}{y-x} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & x \\ \frac{1}{\sqrt{2}} & 0 & y \\ 0 & 1 & z \end{pmatrix}$$

assim

$$z = 0$$

$$\frac{-1}{y-x} = x$$

$$\frac{1}{y-x} = y$$

logo $y = -x$ o qual contradiz a igualdade

$$\frac{\sqrt{2}x}{y-x} = \frac{1}{\sqrt{2}}$$

Portanto, a aplicação determinada pela matriz

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ x & y & z \end{pmatrix}$$

Não é uma isometria.

18. É facil ver que T é isomorfismo. Por outro lado, observemos que

$$\langle T(A), T(B) \rangle = tr\left(T(B)^t T(A)\right) = tr\left((B^t)^t A^t\right) = tr\left(BA^t\right)$$

$$\langle A, B \rangle = tr\left(B^t A\right)$$

É facil ver que

$$tr\left(BA^t\right) = tr\left(B^t A\right)$$

Portanto T é uma isometria.