Recognizable Sets with Multiplicities in the Tropical Semiring

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Abstract

The last ten years saw the emergence of some results about recognizable subsets of a free monoid with multiplicities in the Min-Plus semiring. An interesting aspect of this theoretical body is that its discovery was motivated throughout by applications such as the finite power property, Eggan's classical star height problem and the measure of the nondeterministic complexity of finite automata. We review here these results, their applications and point out some open problems.

1 Introduction

One of the richest extensions of finite automaton theory is obtained by associating multiplicities to words, edges and states. Perhaps the most intuitive appearence of this concept is obtained by counting for every word the number of successful paths spelling it in a (nondeterministic) finite automaton. This is motivated by the formalization of ambiguity in a finite automaton and leads to the theory of recognizable subsets of a free monoid with multiplicities in the semiring of natural numbers. This theory leads, in turn, to the consideration of semigroups of matrices with coefficients in $\mathbb{N}$ and it encounters some classical results from algebra and analysis in the context of representation theory and (formal) power series (in noncommuting variables).

In finite automaton theory this approach was pioneered and vigorously pursued since the early sixties by the "French School" led by Marcel Paul

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Schützenberger. A major step was undertaken by Samuel Eilenberg who sistematized both the formalism and the most important results in his seminal book, [12], published in 1974. In particular he explicited the machinery and this prompted the consideration of multiplicities in any semiring K. The two most important particular cases studied in Eilenberg's book are given by the boolean semiring (leading to classical finite automata) and the semiring of natural numbers (leading to the consideration of ambiguity in finite automata). More recent treatments of the subject can be found in [3, 32].

In 1978 the author was led to the investigation of recognizable sets with multiplicities in another semiring, denoted $\mathcal{M}$, in [36, 33]. This is just the semiring of the natural numbers extended with ∞ under the operations of taking minimums and addition. Such semiring, sometimes called the Min-Plus semiring, is important in operations research where it is used in problems of cost minimization [7]. Here, we shall call it the *tropical semiring*, a suggestion of Christian Choffrut.

Our purpose in this paper is to survey the emerging theory of recognizable subsets of a free monoid with multiplicities in the tropical semiring. We shall also point out, in our way, the applications of this theory to linguistic problems as well as to the capturing of the nondeterministic complexity of finite automata. We shall omit the proofs which can be found elsewhere.

2 The finite section and the limitedness problems

In this section we describe a problem from two different viewpoints.

Let K be a semiring and let M_nK denote the multiplicative monoid of $n \times n$ matrices with coefficients in K. Let S be a subset of M_nK. For $i, j \in [1, n]$ we define the (i,j)-*section* of S as being the set of cofficients (i, s, j), when s runs over S. A subset of K is called a *section* of S if it is an (i,j)-section for some i and j. Clearly, set S is finite if and only if every section of S is finite.

The *finite section problem* (for K) takes a finite subset X of M_nK and a pair (i,j) of indices as input. It consists of deciding whether or not the (i,j)-section of the subsemigroup of M_nK generated by X is finite.

Another, more restricted, problem is given by the *finite closure problem* (for K), which consists of deciding whether or not the subsemigroup of M_nK generated by a given finite set of matrices is finite or not. Clearly, whenever the finite section problem is decidable so is the finite closure problem. But the converse does not hold in general.

It turns out that the finite section problem is equivalent to the limitedness problem in automaton theory which we describe now.

Let $\mathcal{A}$ be a K-A-automaton, that is to say an automaton over the alphabet A with multiplicities in the semiring K. Then the behavior $\|\mathcal{A}\|$ of $\mathcal{A}$ is just a function $\|\mathcal{A}\|\colon A^* \to K$. We recall that, for a path c in $\mathcal{A}$, its label, denoted by $|c|$, is the product of letters of its edges and the multiplicity of c, denoted by $\|c\|$, is the product of multiplicities of its edges. The behavior of $\mathcal{A}$ associates to each word $s \in A^*$ the sum of multiplicities of all successful paths with label s. The family of behaviors of K-A-automata is denoted by $\operatorname{Rec}_K A^*$. Its elements are called recognizable K-subsets of A^*.

We say that a K-subset S, $S\colon X \to K$ is *limited* if its image XS is a finite subset of K. The *limitedness problem* (for K) consists of deciding whether or not the behavior of a given K-A-automaton is limited or not.

We shall be interested in the case when K is the tropical semiring $\mathcal{M}$. In this case, the operations of $\mathcal{M}$ being min and $+$, the multiplicity of a path is the sum of the multiplicities of its edges while the behavior of an $\mathcal{M}$-A-automaton $\mathcal{A}$ is given by

$$s\|\mathcal{A}\| = \min\{\, \|c\| \mid c \text{ is a successful path with label } |c| = s \,\}.$$

Thus, in particular, the behavior of $\mathcal{A}$ is limited if and only if there is a natural m, such that for every $s \in A^*$, either $s\|\mathcal{A}\| = \infty$ or $s\|\mathcal{A}\| \leq m$. A successful path with label s and multiplicity $s\|\mathcal{A}\|$ is called *victorious*. Further details about our notation can be found in [12, 38].

Using a standard construction which associates a subsemigroup of M_nK to every K-A-automaton $\mathcal{A}$ it can be shown that for every semiring K the finite section problem is decidable if and only if the limitedness problem is decidable.

These problems were considered in [28] where their decidability for the semiring of natural numbers was shown. Indeed, in this case, under certain connectivity hypothesis, the finite section problem is equivalent to the finite closure problem whose decidability also follows from work of G. Jacob [22]. Related work can be found in [6, 40].

For the tropical semiring these questions were addressed in 1978 in [36, 33], where we proved that every torsion subsemigroup of $M_n\mathcal{M}$ is locally finite, a Schur-type result. At the same time it was shown that the finite closure problem for the tropical semiring is decidable.

The limitedness problem for the tropical semiring was raised in [9]. It was shown to be decidable in a memorable paper by Kosaburo Hashiguchi,

[15], in 1982. The solution is very complicated and difficult to visualize and this led to further research to find other proofs of this result.

The first attempt in this direction was completed by Hashiguchi himself [18] in 1986. He obtained an improved proof accompanied by a new characterization of the limited recognizable $\mathcal{M}$-subsets of A^*. Unortunately, Hashiguchi's proofs are not completely satisfactory. This is because his method conduces to a bound, say B, such that $\|\mathcal{A}\|$ is limited if and only if the image $A^*\|\mathcal{A}\|$ is contained in the set $Y = [0, B] \cup \infty$. The algorithm then consists of checking whether or not the inverse image $Y\|\mathcal{A}\|^{-1}$ of Y is everything or not. This can be done because for every $y \in \mathcal{M}$, the set $y\|\mathcal{A}\|^{-1}$ is a recognizable subset of A^* which can be computed. Thus, the algorithm neeeds the construction of $B+2$ automata which is just impossible in practice because of the exponential bounds B furnished by the proof.

The second attempt to find an alternate proof of Hashiguchi's theorem was completed by Hing Leung [26] in 1987. Leung visualized the finite section problem as a question of convergence using the one-point compactification of the tropical semiring equiped with the discrete topology. Let us denote by ω the point at infinity and by $\mathcal{T}$ the resulting semiring.

Thus, $\mathcal{T}$ has elements $\mathbb{N} \cup \{\omega, \infty\}$ totally ordered by $0 < 1 < 2 < \cdots < \omega < \infty$. The operations of $\mathcal{T}$ are $\min(a, b)$ and $a+b$, where $a+b = \max(a, b)$ if a or b does not belong to $\mathbb{N}$. A sequence a_n converges to a if and only if either $a_n = a$ for every sufficiently large n or $a = \omega$ and, for every $m \in \mathbb{N} \cup \{\infty\}$, there exists $p \in \mathbb{N}$ for which $m \notin \{a_n \mid n > p\}$.

The topology of $\mathcal{T}$ extends naturally to $M_n\mathcal{T}$ where matrix multiplication results a continuous function. Thus, $M_n\mathcal{T}$ becomes a topological semigroup and the topological closure of any subsemigroup is again a subsemigroup of $M_n\mathcal{T}$.

Leung's solution consists of an algorithm which decides whether or not ω belongs to the (i, j)-section of the topological closure of the subsemigroup of $M_n\mathcal{T}$ generated by a given finite subset of $M_n\mathcal{T}$. This clearly solves the finite section problem. His algorithm is easy to state and can be computed for examples with small n.

Finally, I also attempted to find an alternate proof for Hashiguchi's theorem. Unfortunately my proof, initiated in 1986, is still incomplete but it will be essentially another proof of correctness of Hing Leung's algorithm, obtained independently.

Maybe a word on one of the main differences among the three approaches is in order. All these proofs are built around some Ramsey-type result which serves as a stopping rule for the algorithms. Hashiguchi uses the weakest possible Ramsey type result: the pigeon-hole principle. Leung uses a powerful theorem of T. C. Brown about locally finite semigroups [5, 36, 39]. My own proof uses a new Ramsey-type result, developed in [35, 37], from which Brown's theorem follows easily.

We close this section by stating the algorithm for deciding the finite section problem for the tropical semiring.

Initially we consider another semiring, denoted by $\mathcal{R}$, which has elements $\{0, 1, \omega, \infty\}$ totally ordered by

$$0 < 1 < \omega < \infty.$$

The operations of $\mathcal{R}$ are min and max for addition and multiplication respectively.

Let us consider an idempotent element e of $M_n\mathcal{R}$. The position (i, j) of e is said to be *blind* if $(i, e, j) = 1$ and there is no $k \in [1, n]$ such that $(k, e, k) = 0$ and $(i, e, k), (k, e, j) \in \{0, 1\}$. This is equivalent to saying that $(i, e'^3, j) = 3$, where e' is the matrix e considered as an element of $M_n\mathcal{T}$. It can be shown that position (i, j) of e is blind if and only if the (i, j)-section of the cyclic subsemigroup of $M_n\mathcal{T}$ generated by e' is infinite. To record this situation we make the following definition.

The *perforation* of e is another idempotent matrix in $M_n\mathcal{R}$, denoted $e^{\#}$, given by

$$(i, e^{\#}, j) = \begin{cases} \omega & \text{if } (i,j) \text{ is blind} \\ (i, e, j) & \text{otherwise.} \end{cases}$$

A subset Y of $M_n\mathcal{R}$ is *closed under perforation* if the perforation of every idempotent in Y is also in Y.

The semirings $\mathcal{T}$ and $\mathcal{R}$ are related by the function $\Psi : \mathcal{T} \to \mathcal{R}$, given by

$$x\Psi = \begin{cases} x & \text{if } x \in \mathcal{R} \\ 1 & \text{otherwise.} \end{cases}$$

Function Ψ extends naturally to $M_n\mathcal{T}$ but we warn the reader that it is not continuous.

Having defined this much notation we can state the promised algorithm. Let X be a finite subset of $M_n\mathcal{M}$. Let Y be the least subsemigroup of $M_n\mathcal{R}$ closed under perforation which contains $X\Psi$. Hing Leung [26] showed that the (i, j)-section of the subsemigroup of $M_n\mathcal{M}$ generated by X is finite if and only if the (i, j)-section of Y contains ω.

Since $\mathcal{R}$ is a finite semiring, Y is finite and can be computed by starting with $X\Psi$ and alternately closing it under product and perforation. This yields an algorithm to decide the finite section problem for the tropical semiring.

A word about the complexity of this problem. Leung [26] has shown that the finite section problem for the tropical semiring is PSPACE-hard. Later he proved [25] that the algorithm converges after a polynomial number of closures under product and perforation. It follows that it has polynomial space complexity; hence, the limitedness problem is PSPACE-complete.

3 Applications to linguistic problems

The initial motivation for both the finite closure and the finite section problems came from linguistic considerations.

The original linguistic problem to be solved was posed by John Brzozowski in 1966 during the seventh SWAT (now FOCS) Conference. It asked for an algorithm to decide whether or not a given recognizable subset of A^* possessed the finite power property. Recall that a subset X of A^* has the finite power property if there exists a natural m for which

$$X^* = (1 \cup X)^m.$$

This problem was shown decidable independently by K. Hashiguchi and the author in 1978 [14, 33], while in [20] it is shown that the problem becomes undecidable for context-free languaages. See also [27].

Hashiguchi's solution (see also [34, 31]) is very short and he works directly on the automaton recognizing X. His method is based on an ingenious idea built around a double recurrence.

On the other hand, the author reduced the finite power property to the finite closure problem for the tropical semiring. The basic idea of the reduction is simple and we illustrate it by an example in Figure 1. Let $\mathcal{A}$ be a finite automaton recognizing X and let $\mathcal{B}$ be the automaton recognizing X^*, obtained by the standard construction. Let q be the initial (and only final) state of $\mathcal{B}$. We transform $\mathcal{B}$ into an $\mathcal{M}$-A-automaton by assigning multiplicity 1 to every edge with terminus in q and multiplicity 0 to the remaining edges of $\mathcal{B}$ (on the Figure multiplicity 0 is omitted). Clearly, for every $s \in A^*$, we have

$$s\|\mathcal{B}\| = \begin{cases} \min\{\, m \mid s \in X^m \,\} & \text{if } s \in X^* \\ \infty & \text{otherwise.} \end{cases}$$

It follows that X has the finite power property if and only if $\|\mathcal{B}\|$ is limited.

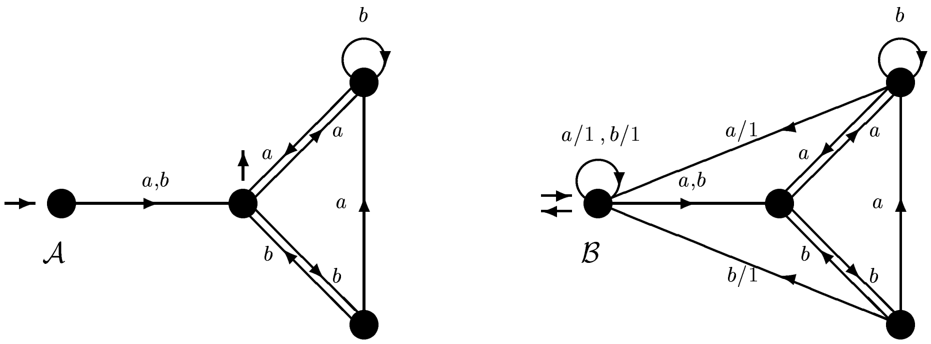

Figure 1: A set $X = |\mathcal{A}|$ with the finite power property: $X^* = (1 \cup X)^4$.

Let S be the standard submonoid of $M_n\mathcal{M}$ associated to $\mathcal{B}$. It turns out that in the particular case of this construction, due to the restricted way of assigning multiplicities to the edges of $\mathcal{B}$, the (q,q)-section of S is finite if and only if S itself is finite. Hence, X has the finite power property if and only if the monoid S is finite.

Hashiguchi extended this idea and developed an algorithm to decide whether a given set belongs to the closure of a given finite family of sets under a given subset of the rational operations. More precisely, let X be a recognizable subset of A^*, let F be a finite family of recognizable subsets of A^*, and let ρ be a subset of the operations in $\{\cup, ., *\}$ (the rational operations). The algorithm decides whether or not the set X belongs to the closure of F under ρ. For instance, X has the finite power property if and only if X^* belongs to the closure of $\{X\}$ under union and concatenation. This algorithm uses, in a very significant way, the finite section problem for the tropical semiring. Actually, we are unaware of any other proof which avoids the finite section problem. For more details consult [17, 30].

Building on this idea, Hashiguchi, in a veritable "tour de force", solved last year the entire star-height problem in [19] after a partial solution he obtained in [16] in 1982. More precisely, he developed an algorithm to compute the star-height of a given recognizable set. It is worth recalling that this classical problem, formulated by Eggan [11] in 1963, remained open for 24 years in spite of the many attempts to solve it. Unfortunately, the ideas in this deep paper are too complex to be reported here.

4 The nondeterministic complexity of finite automata

Another application of the tropical semiring is connected to the capture of the nondeterministic complexity of a finite automaton. The idea here is to associate to every word the minimum number of "decisions" necessary to spell it out in a given nondeterministic automaton. This can be realized by associating multiplicity 1 to every nondeterministic edge. The behavior of the resulting $\mathcal{M}$-A-automaton is precisely the desired series. This idea, for Turing Machines, first appeared in [23] and was later considered for finite automata by J. Goldstine, C. Kintala and D. Wotschke [24].

More precisely, let $\mathcal{A} = (Q, I, T)$ be a (not necessarily deterministic) finite automaton over the alphabet A. We say that edge (p, a, q) of $\mathcal{A}$ is *deterministic* if there are no other edges (p, a, r) in $\mathcal{A}$, with $r \neq q$.

We convert $\mathcal{A}$ into an $\mathcal{M}$-A-automaton by defining the multiplicity of (p, a, q) in $Q \times A \times Q$:

$$(p, a, q)E = \begin{cases} 0 & \text{if } (p, a, q) \text{ is a deterministic edge} \\ 1 & \text{if } (p, a, q) \text{ is not a deterministic edge} \\ \infty & \text{if } (p, a, q) \text{ is not an edge of } \mathcal{A}. \end{cases}$$

The behavior $\|\mathcal{A}\|$ of $\mathcal{A}$ ($\|\mathcal{A}\| \colon A^* \to \mathcal{M}$) is called the *nondeterministic complexity* of the finite automaton $\mathcal{A}$. Thus, $s\|\mathcal{A}\|$ is the minimum number of nondeterministic edges needed to spell s from I to T.

An important aspect of the nondeterministic complexity of automaton $\mathcal{A}$ is the assymptotic behavior of the coefficients in $\|\mathcal{A}\|$. This can be measured by the function sh defined as follows. Let $F \colon A^* \to \mathcal{M}$ be an $\mathcal{M}$-subset of A^*. For F and for $m \geq 0$, we define

$$\mathrm{sh}(F, m) = \min\{\, |s| \mid s \in A^*,\ m \leq sF < \infty \,\}.$$

Thus, $\mathrm{sh}(F, m)$ is the minimum length needed to achieve a finite coefficient which exceeds m. Note that if F is limited then $\mathrm{sh}(F, m)$ is undefined for sufficiently large m and if F is unlimited then $\mathrm{sh}(F, m)$ is always defined and unbounded. In particular, we use the function $\mathrm{sh}(\|\mathcal{A}\|, m)$ to measure the assymptotic behavior of the nondeterministic complexity of automaton $\mathcal{A}$.

It was shown in [38] that the nondeterministic complexity $\|\mathcal{A}_p\|$ of automaton $\mathcal{A}_p$, shown in Figure 2, satisfies

$$\mathrm{sh}(\|\mathcal{A}_p\|, m) \in \Theta(m^p).$$

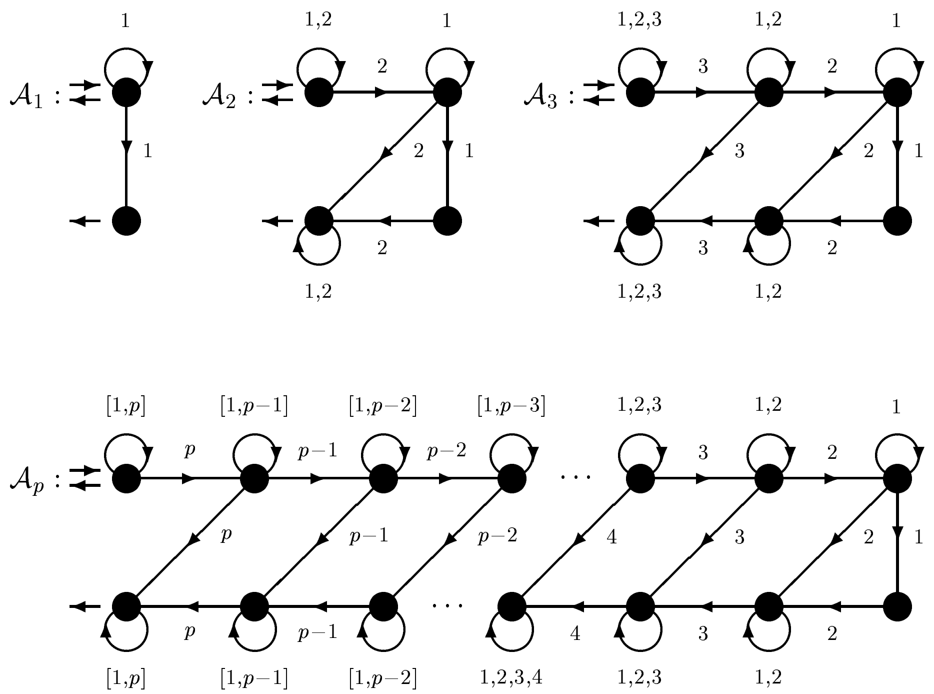

Figure 2: Automata $\mathcal{A}_p$

Inspired by this fact we define, for every p, the family $\mathcal{H}_p$ of $\mathcal{M}$-subsets in $\mathrm{Rec}_{\mathcal{M}}\, A^*$ for which sh grows not faster than a p-degree polynomial. More precisely, we put

$$\mathcal{H}_p = \{\, F \in \mathrm{Rec}_{\mathcal{M}}\, A^* \mid \mathrm{sh}(F, m) \in O(m^p) \,\}.$$

In particular, $\mathcal{H}_0$ results to be the family of limited recognizable $\mathcal{M}$-subsets of A^*.

Using a characterization of limited recognizable subsets of A^* obtained by Hashiguchi in [18] one can prove that the families $\mathcal{H}_p$ exhaust $\mathrm{Rec}_{\mathcal{M}}\, A^*$. In other words, the families $\mathcal{H}_p$ form a proper hierarchy of the recognizable $\mathcal{M}$-subsets of A^*. More details can be found in [38].

We mention that the aluded characterization of Hashiguchi is done in terms of the star-height of rational expressions which do not use unions. Further, the existence of such an expression of height p implies pertinence to $\mathcal{H}_p$. Also, it can be shown that the algorithm of the previous section applied to the automaton $\mathcal{A}_p$ converges with exactly p closures under perforation. These facts hardly happen by coincidence but we do not yet have a satisfactory explanation for them.

5 Further research

We close this paper mentioning some open problems. The first challenge is to obtain a deeper understanding of the existing results. This is needed because the proofs of many of the reported results are utterly intricate and rely on very elaborated combinatorial arguments. Better and more informative proofs have to be found! A brilliant example in this direction is given by the introduction of the topology on the tropical semiring done by Leung. I hope that many other such simplifications will be found which will finally lead to a better understanding of the the star-height problem. Indeed, the existing proof, putting all pieces together, takes more than a hundred pages of very heavy combinatorial reasoning.

The most important open problem seems to be to settle whether the equivalence problem for recognizable $\mathcal{M}$-subsets of the free monoid is decidable or not. In spite of the fact that this problem is dangerously close to known undecidable problems I believe that it is decidable. I offer two (admittedly unrelated) facts to support my belief. Indeed, the same problem is decidable for the semiring $\mathbb{N}$ [12, page 143]; on the other hand, the basic tool [12, page 156] to prove undecidability results related to $M_n\mathbb{N}$ cannot be used for the tropical semiring. This is so because it is easy to see that every finitely generated subsemigroup of $M_n\mathcal{M}$ has polynomial growth function, i.e. for every $f\colon A^* \to M_n\mathcal{M}$, A finite, $|A^m f| \in O(m^{n^2})$. Thus, $M_n\mathcal{M}$ does not contain free subsemigroups generated by at least two letters.

Another open problem is to characterize the classes $\mathcal{H}_p$ in the hierarchy of the previous section. In particular, are there decision procedures for each of those classes? I vagely believe and strongly hope that the p-th family in the hierarchy is intimately related to the family of sets of star height p.

Another problem: I conjecture that the complexity of every recognizable $\mathcal{M}$-subset is basically a polynomial. More precisely, for every recognizable $\mathcal{M}$-subset F of A^* there exists p such that $\mathrm{sh}(F, m) \in \Theta(m^p)$.

Still another one: in [9] Choffrut embedded $\mathcal{M}$ in some other semirings of interest such as the semiring K of the recognizable subsets of $\{ a \}^*$. Mascle showed in [29] that the Schur-type result of [33] extends to this semiring. He also showed that the finite closure problem remains decidable. However, let us take an excursion to the theory of recognizable rational relations [2, 8, 21, 10, 4]. Recently it was proved [13] that it is undecidable whether a rational subset of $A^* \times \{ a \}^*$ is recognizable or not. This was made even more precise in [1] where "tight bounds" are given for the decidability of related questions. But a construction of Choffrut [9, 10] can be used to reduce the above question to the finite section problem for the semiring K. Hence, the finite section problem for K is undecidable. A similar argument can be used to show that the equivalence problem for K-recognizable sets is also undecidable. These facts motivate further investigations to try to make more precise the transition of the finite section problem from decidable to undecidable when extending the tropical semiring. In particular, what happens for the semiring $(Z \cup \{ \infty \}, \min, +)$, where Z is the ring of integers?

A final question: can one characterize the family of victorious paths of $\mathcal{M}$-A-automata, seen as subsets of the free monoid generated by the edge set? These sets seem to be peculiar, whatever that means, and more information on them could well clarify the problems we addressed in this paper.

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