On Semigroups of Matrices over the Tropical Semiring

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Abstract

The tropical semiring $\mathcal{M}$ consists of the set of natural numbers extended with infinity, equipped with the operations of taking minimums (as semiring addition) and addition (as semiring multiplication). We use factorization forests to prove finiteness results related to semigroups of matrices over $\mathcal{M}$. Our method is used to recover results of Hashiguchi, Leung and the author in a unified combinatorial framework.

1 Introduction

In 1978 [6] we characterized finitely generated finite semigroups of matrices over the tropical semiring $\mathcal{M}$. ($\mathcal{M}$ is $\mathbb{N} \cup \{\infty\}$ equipped with the operations of minimum and addition.) The results obtained were applied to solve a long standing open problem of John Brzozowski.

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In 1982 K. Hashiguchi [2] proved the decidability of a more general problem. Let us take a finitely generated semigroup of matrices over the tropical semiring. Is the set of all coefficients in a given row and column finite or not? Hashiguchi's method consisted in finding an upper bound for the coefficients which holds whenever the set in question is finite. That upper bound can be used to synthetize an algorithm which decides the proposed problem. Later on, in 1986, Hashiguchi improved both his results and his upper bound but the resulting algorithm is still impractical.

In 1987 H. Leung [4, 5] published an algorithm to decide the same problem based on an elaborate extension of our method in [6]. Leung used topological arguments and consequently, while giving a much better upper bound on the complexity of the problem, he lost the upper bound on the coefficients.

In 1986 the author discovered independently the same algorithm found by Leung and began building a combinatorial framework for the study of the structure of finitely generated semigroups of matrices over the tropical semiring. This paper contains our ideas to solve the proposed problem using factorization forests [8]. Our method gives simultaneously H. Leung's algorithm and an upper bound on the coefficients.

For further motivation, applications and many remaining open problems we refer the reader to our survey article [7].

Finally, we mention that using Lemmas 10 and 7 it is easy to recover Hashiguchi's Main Lemma of [3] with validity for any finitely generated semigroup of matrices over the tropical semiring. This solves an open problem stated in [3].

2 Semirings and idempotent matrices

We introduce initially the semirings of our interest. The *tropical semiring*, denoted $\mathcal{M}$, has support $\mathcal{M} = \mathbb{N} \cup \{\infty\}$ and operations $a \oplus b = \min\{a, b\}$ and $a \otimes b = a + b$. The operations of $\mathbb{N}$ are extended to $\mathcal{M}$ in the usual way and the identities of $\oplus$ and $\otimes$ are, respectively, ∞ and 0. Notice that $\mathcal{M}$ is a complete positive commutative semiring [1].

We shall also need an extension of $\mathcal{M}$ obtained by introducing a new element ω for which a topological interpretation can be given. See [5] for details. This semiring shall be denoted by $\mathcal{T}$, its support is $\mathcal{T} = \mathbb{N} \cup \{\omega, \infty\}$

totally ordered by the relation

$$0 < 1 < 2 < \cdots < \omega < \infty.$$

We extend the operations of $\mathcal{M}$ by defining, for $x \in \mathcal{T}$,

$$\omega + x = x + \omega = \max\{\omega, x\}.$$

Clearly, $\mathcal{M}$ is a subsemiring of $\mathcal{T}$.

Our results for the semirings $\mathcal{M}$ and $\mathcal{T}$ are obtained through the consideration of finite projections as follows. Let $\mathcal{R}$ be the semiring with support $\mathcal{R} = \{\,0, 1, \omega, \infty\,\}$, totally ordered by the relation $0 < 1 < \omega < \infty$,equipped with the operations $a \oplus b = \min\{a, b\}$ and $a \otimes b = \max\{a, b\}$.

The semirings $\mathcal{T}$ and $\mathcal{R}$ are related by the projection function $\Psi\colon\ \mathcal{T} \to \mathcal{R}$, given by

$$x\Psi = \begin{cases} x & \text{if } x \in \mathcal{R} \\ 1 & \text{otherwise.} \end{cases}$$

The subsemiring $\mathcal{M}\Psi$ of $\mathcal{R}$ is denoted by $\mathcal{N}$; its support is $\mathcal{N} = \{\,0, 1, \infty\,\}$. We shall use a projection $\pi\colon\ \mathcal{R} \to \mathcal{N}$ given by

$$x\pi = \begin{cases} x & \text{if } x \in \mathcal{N} \\ 1 & \text{if } x = \omega. \end{cases}$$

For a semiring K we denote by M_nK the multiplicative monoid of $n \times n$ matrices with coefficients in K. For $x \in M_nK$ and $i, j \in [1, n]$ the coefficient of x in row i and column j is denoted by (i, x, j). The pair (i, j) shall be called a *position*; we shall say that (i, x, j) is the coefficient of x in position (i, j).

In the case of our semirings the functions Ψ and π are extended to the corresponding matrix monoids in the natural way.

Now we characterize idempotent matrices in $M_n\mathcal{R}$.

Lemma 1 *For every matrix e in $M_n\mathcal{R}$ the following statements are equivalent.*

(i) e is idempotent;

(ii) for every $p \geq 1$ and for every $k_0, k_1, \ldots, k_p \in [1, n]$, $(k_0, e, k_p) \leq \max\{\,(k_{q-1}, e, k_q) \mid q \in [1, p]\,\}$ and for every $i, j \in [1, n]$, there exists a $k \in [1, n]$ such that $(k, e, k) \leq (i, e, j) = \max\{\,(i, e, k), (k, e, j)\,\}$;

(iii) for every $i, j, k \in [1, n]$, $(i, e, j) \leq \max\{ (i, e, k), (k, e, j) \}$ and for every $i, j \in [1, n]$, $(i, e, j) = \max\{ (i, e, k), (k, e, j) \}$ for some $k \in [1, n]$.

Proof. (i) implies (ii). Assume that e is idempotent and let $i, j \in [1, n]$. Then $(i, e, j) = (i, e^2, j) = \min\{ \max\{ (i, e, k), (k, e, j) \} \mid k \in [1, n] \}$, hence, for every $k \in [1, n]$, $(i, e, j) \leq \max\{ (i, e, k), (k, e, j) \}$. By induction on p, for every $k_0, k_1, \ldots, k_p \in [1, n]$, $(k_0, e, k_p) \leq \max\{ (k_{q-1}, e, k_q) \mid q \in [1, p] \}$. On the other hand, since $e = e^2$ implies that $e = e^n$, there exist $i = k_0, k_1, \ldots, k_n = j$ such that $(i, e, j) = \max\{ (k_{q-1}, e, k_q) \mid q \in [1, n] \}$. Since $k_q \in [1, n]$ for each one of the $n + 1$ q's, there exist $0 \leq r < r' \leq n$, such that $k_r = k_{r'} = k$. Using what we just proved, for any $0 \leq s < t \leq n$, $(k_s, e, k_t) \leq \max\{ (k_{q-1}, e, k_q) \mid q \in [s + 1, t] \} \leq \max\{ (k_{q-1}, e, k_q) \mid q \in [1, n] \} = (i, e, j)$, hence, $(k_s, e, k_t) \leq (i, e, j)$. Thus,

$$(k, e, k) \leq (i, e, j) \leq \max\{ (i, e, k), (k, e, j) \} \leq (i, e, j);$$

where the second inequality follows from the first part of statement (ii). Hence, $(k, e, k) \leq (i, e, j) = \max\{ (i, e, k), (k, e, j) \}$, as required.

(ii) implies (iii). This is clear.

(iii) implies (i). Choose $i, j \in [1, n]$. The two conditions in (iii) are equivalent to saying that $(i, e, j) = \min\{ \max\{ (i, e, k), (k, e, j) \} \mid k \in [1, n] \}$. Hence, $(i, e, j) = (i, e^2, j)$ and e is idempotent. ∎

Condition (ii) above suggests the following definition. Let $e \in M_n\mathcal{R}$ be idempotent. We say that position (i, j) is *anchored in* e if there exists a $k \in [1, n]$ such that $0 = (k, e, k) \leq (i, e, j) = \max\{ (i, e, k), (k, e, j) \}$.

Lemma 2 *Let $e \in M_n\mathcal{R}$ be idempotent. If $(i, e, j) = 0$ then (i, j) is anchored. Let $k_0, k_1, \ldots, k_p \in [1, n]$ be such that $(k_{q-1}, e, k_q) \leq (k_0, e, k_p)$ for every q. Then, if (k_{r-1}, k_r) is anchored for some r then so is (k_0, k_p).*

Proof. The first assertion is an immediate consequence of (ii) in Lemma 1. To see the second one let $l \in [1, n]$ be such that $0 = (l, e, l) \leq (k_{r-1}, e, k_r) = \max\{ (k_{r-1}, e, l), (l, e, k_r) \}$. Successively using Lemma 1(ii) and the facts that $(k_{r-1}, e, l) \leq (k_{r-1}, e, k_r)$ and that $(k_{q-1}, e, k_q) \leq (k_0, e, k_p)$, for every q, we have that

$$\begin{aligned} (k_0, e, l) &\leq \max\{ (k_0, e, k_1), \ldots, (k_{r-2}, e, k_{r-1}), (k_{r-1}, e, l) \} \\ &\leq \max\{ (k_0, e, k_1), \ldots, (k_{r-2}, e, k_{r-1}), (k_{r-1}, e, k_r) \} \\ &\leq (k_0, e, k_p), \end{aligned}$$

i.e. $(k_0, e, l) \le (k_0, e, k_p)$. Similarly, $(l, e, k_p) \le (k_0, e, k_p)$; hence, $\max\{ (k_0, e, l), (l, e, k_p) \} \le (k_0, e, k_p)$. Then, applying Lemma 1 in the third inequality we have that $0 = (l, e, l) \le (k_{r-1}, e, k_r) \le (k_0, e, k_p) \le \max\{ (k_0, e, l), (l, e, k_p) \} \le (k_0, e, k_p)$. This implies that (k_0, k_p) is anchored in e. ∎

The next definitions and Lemma are essential in the sequel. Let $e \in M_n\mathcal{R}$ be idempotent. Assume that $(i, e, j) = 1$. Position (i, j) is *stable in* e if it is anchored, otherwise it is *unstable*. Thus, (i, j) is unstable if and only if $(k, e, k) = 1$ whenever $k \in [1, n]$ and $\max\{ (i, e, k), (k, e, j) \} = 1$. The *stabilization* of e is the matrix $e^{\#} \in M_n\mathcal{R}$ with coefficients

$$(i, e^{\#}, j) = \begin{cases} \omega & \text{if } (i, e, j) = 1 \text{ and } (i, j) \text{ is unstable} \\ (i, e, j) & \text{otherwise.} \end{cases}$$

If $e = e^{\#}$ then e is *stable*, otherwise e is *unstable*. A set $X \subseteq M_n\mathcal{R}$ is *stable* if the stabilization $e^{\#}$ of any idempotent e in X also belongs to X.

For $a, b \in M_n\mathcal{R}$ we define $a \le b$ if, for every $i, j \in [1, n]$, $(i, a, j) \le (i, b, j)$. It is easy to see that this partial order is compatible with matrix multiplication, i.e., if $a \le b$ and $c \le d$ then $ac \le bd$.

Lemma 3 *Let $e \in M_n\mathcal{R}$ be idempotent. Then $e^{\#}$ is a stable idempotent for which $ee^{\#} = e^{\#} = e^{\#}e$. Further, if e is unstable then $e^{\#} <_{\mathcal{J}} e$, where $<_{\mathcal{J}}$ denotes the usual ordering of the $\mathcal{J}$-classes of $M_n\mathcal{R}$.*

Proof. Let i, j and k be in $[1, n]$. Observe initially that $(i, e, j) \le (i, e^{\#}, j)$. Also, $(i, e, j) \ne (i, e^{\#}, j)$ implies that $(i, e, j) = 1$, $(i, e^{\#}, j) = \omega$ and that position (i, j) is not anchored in e. Finally, if $0 = (k, e, k) \le (i, e, j) = \max\{ (i, e, k), (k, e, j) \}$, then we also have that $0 = (k, e^{\#}, k) \le (i, e^{\#}, j) = \max\{ (i, e^{\#}, k), (k, e^{\#}, j) \}$. Indeed, the assumption implies that $(k, k), (i, j)$, (i, k) and (k, j) are all anchored; hence, the respective coefficients in e and $e^{\#}$ are the same.

To prove that $e^{\#}$ is idempotent we first claim that, for every $i, j, k \in [1, n]$, $(i, e^{\#}, j) \le \max\{ (i, e^{\#}, k), (k, e^{\#}, j) \}$. Indeed, if we assume the contrary then, by Lemma 1,

$$(i, e, j) \le \max\{ (i, e, k), (k, e, j) \} \le \max\{ (i, e^{\#}, k), (k, e^{\#}, j) \} < (i, e^{\#}, j).$$

We can conclude that $(i, e, j) = 1$, $\max\{ (i, e^{\#}, k), (k, e^{\#}, j) \} = 1$ and $(i, e^{\#}, j) = \omega$. If $(i, e^{\#}, k) = 1$ then (i, k) is anchored in e; hence, by

Lemma 2, (i,j) is anchored in e, a contradiction. Analogous conclusion holds if $(k, e^{\#}, j) = 1$ and this proves the claim.

Next we claim that for every $i, j \in [1, n]$ there exists a $k \in [1, n]$, such that $(i, e^{\#}, j) = \max\{(i, e^{\#}, k), (k, e^{\#}, j)\}$. Indeed, choose a k for which $(i, e, j) = \max\{(i, e, k), (k, e, j)\}$ and (k, e, k) is minimum. Then, using what we already proved,

$$\max\{(i, e, k), (k, e, j)\} = (i, e, j) \leq (i, e^{\#}, j) \leq \max\{(i, e^{\#}, k), (k, e^{\#}, j)\}.$$

If the last inequality is an equality then we are done, otherwise we can conclude that $(i, e, j) = (i, e^{\#}, j) = 1$ while $\max\{(i, e^{\#}, k), (k, e^{\#}, j)\} = \omega$. Then, (i,j) is anchored in e and the choice of k implies that $(k, e, k) = 0$. Thus, from the initial observations, $0 = (k, e^{\#}, k) \leq (i, e^{\#}, j) = \max\{(i, e^{\#}, k), (k, e^{\#}, j)\}$; a contradiction which establishes the claim. Thus, $e^{\#}$ is indeed idempotent by Lemma 1.

Now we prove that $(e^{\#})^{\#} = e^{\#}$. To see this assume that $(i, e^{\#}, j) = 1$. Then, $(i, e, j) = 1$ and (i,j) is stable in e. Let $k \in [1, n]$ be such that $0 = (k, e, k) \leq (i, e, j) = \max\{(i, e, k), (k, e, j)\}$. From the initial observations, $0 = (k, e^{\#}, k) \leq (i, e^{\#}, j) = \max\{(i, e^{\#}, k), (k, e^{\#}, j)\}$; hence, (i,j) is stable in $e^{\#}$ and $(i, (e^{\#})^{\#}, j) = (i, e^{\#}, j)$.

Now we claim that $ee^{\#}e = e^{\#}$ which implies that $ee^{\#} = e^{\#}e = e^{\#}$. Indeed, $e \leq e^{\#}$ and the idempotence of e and $e^{\#}$ imply that $e \leq ee^{\#}e \leq e^{\#}$. Assume, for a contradiction, that $(i, ee^{\#}e, j) \neq (i, e^{\#}, j)$ for some $i, j \in [1, n]$. Then, $(i, ee^{\#}e, j) = (i, e, j) = 1$ and $(i, e^{\#}, j) = \omega$. Let $k, l \in [1, n]$ be such that $1 = (i, ee^{\#}e, j) = \max\{(i, e, k), (k, e^{\#}, l), (l, e, j)\}$. Then, $(k, e^{\#}, l) \leq 1$; hence (k, l) is anchored in e. By Lemma 2 (i,j) is anchored in e implying that $(i, e^{\#}, j) = 1$, a contradiction which establishes the claim.

Finally assume that e is unstable. From $ee^{\#}e = e^{\#}$ we conclude that $e^{\#} \leq_{\mathcal{J}} e$. Assume that $e \,\mathcal{J}\, e^{\#}$. Since $S = M_n\mathcal{R}$ is finite, we have $e \,\mathcal{D}\, e^{\#} = ee^{\#}$. Then, $e \,\mathcal{R}\, ee^{\#} = e^{\#}$. By a dual argument, $e \,\mathcal{L}\, e^{\#}$; hence, $e \,\mathcal{H}\, e^{\#}$. Being both e and $e^{\#}$ idempotents, we conclude that $e = e^{\#}$, a contradiction of the stability of e. This concludes the proof of Lemma 3. ∎

3 Bounding the coefficients

In this section we derive some bounds on the size of coefficients of matrices over the tropical semiring. Initially we introduce measures of this size. Let

$Y \subseteq M_n\mathcal{T}$, $z \in M_n\mathcal{R}$ and $r \in \mathcal{R}$. We define

$$\mathbf{s}_r(Y,z) = \min\{\,(i,y,j) \mid y \in Y \text{ and } (i,z,j) = r\,\},$$

$$\mathbf{S}_r(Y,z) = \max\{\,(i,y,j) \mid y \in Y \text{ and } (i,z,j) = r\,\},$$

assuming that the min and max of an empty set are, respectively, ∞ and 0. Thus, $\mathbf{s}_r(Y,z)$ ($\mathbf{S}_r(Y,z)$) is the least (greatest) coefficient in Y in positions whose coefficient in z is r.

Let $y \in M_n\mathcal{M}$, $Y \subseteq M_n\mathcal{M}$ and $z \in M_n\mathcal{R}$. We say that y *agrees with* z if $y\Psi{=}z\pi$. Also, Y *agrees with* z if every matrix in Y agrees with z. Note that if y_i agrees with z_i, for $i = 1, 2$, then y_1y_2 agrees with z_1z_2. Also, if z is idempotent then Y agrees with z if and only if it agrees with $z^{\#}$, since $z\pi = z^{\#}\pi$. The definitions above on the size will be used mainly when Y agrees with z.

Finally recall that if Y is a subset of a semigroup S then Y^+ denotes the subsemigroup of S generated by Y.

Lemma 4 *Let $y_i \in M_n\mathcal{M}$ agree with $z_i \in M_n\mathcal{R}$, for $i = 1, 2$. Then, $\mathbf{S}_1(y_1y_2, z_1z_2) \leq \mathbf{S}_1(y_1, z_1) + \mathbf{S}_1(y_2, z_2)$ and $\min\{\,\mathbf{s}_\omega(y_1, z_1), \mathbf{s}_\omega(y_2, z_2)\,\} \leq \mathbf{s}_\omega(y_1y_2, z_1z_2)$.*

Proof. Assume that $(i, z_1z_2, j) = 1$ and let $k \in [1, n]$ be such that $(i, z_1z_2, j) = \max\{\,(i, z_1, k), (k, z_2, j)\,\} = 1$. If $(i, z_1, k) = 0$ then $(i, y_1, k) = 0$, since y_1 agrees with z_1. Thus, $(i, y_1, k) \leq \mathbf{S}_1(y_1, z_1)$. Similarly, $(k, y_2, j) \leq \mathbf{S}_1(y_2, z_2)$. It follows that $(i, y_1y_2, j) \leq (i, y_1, k)+(k, y_2, j) \leq \mathbf{S}_1(y_1, z_1)+\mathbf{S}_1(y_2, z_2)$. Thus, $\mathbf{S}_1(y_1y_2, z_1z_2) = \max\{\,(i, y_1y_2, j) \mid (i, z_1z_2, j) = 1\,\} \leq \mathbf{S}_1(y_1, z_1) + \mathbf{S}_1(y_2, z_2)$.

Assume that $(i, z_1z_2, j) = \omega$ and let $k \in [1, n]$ be such that $(i, y_1y_2, j) = (i, y_1, k) + (k, y_2, j)$. Then, $\omega = (i, z_1z_2, j) \leq \max\{\,(i, z_1, k), (k, z_2, j)\,\}$. If $(i, z_1, k) = \infty$ then $(i, y_1, k) = \infty$, since y_1 agrees with z_1; thus $(i, y_1y_2, j) = \infty$ and, since y_1y_2 agrees with z_1z_2, we conclude that $(i, z_1z_2, j) = \infty$: a contradiction. It follows that $(i, z_1, k) < \infty$ and similarly $(k, z_2, j) < \infty$. Thus, $\max\{\,(i, z_1, k), (k, z_2, j)\,\} < \infty$ and we conclude that $\omega = (i, z_1z_2, j) = \max\{\,(i, z_1, k), (k, z_2, j)\,\}$. If $(i, z_1, k) = \omega$ then $\mathbf{s}_\omega(y_1, z_1) \leq \mathbf{s}_\omega(y_1, z_1) + (k, y_2, j) \leq (i, y_1, k) + (k, y_2, j) = (i, y_1y_2, j)$. Similarly, if $(k, z_2, j) = \omega$ then $\mathbf{s}_\omega(y_2, z_2) \leq (i, y_1y_2, j)$. Altogether, $\min\{\,\mathbf{s}_\omega(y_1, z_1), \mathbf{s}_\omega(y_2, z_2)\,\} \leq (i, y_1y_2, j)$. Thus, $\min\{\,\mathbf{s}_\omega(y_1, z_1), \mathbf{s}_\omega(y_2, z_2)\,\} \leq \min\{\,(i, y_1y_2, j) \mid (i, z_1z_2, j) = \omega\,\} = \mathbf{s}_\omega(y_1y_2, z_1z_2)$. ∎

Lemma 5 *Let $e \in M_n\mathcal{R}$ be idempotent and assume that $Y \subseteq M_n\mathcal{M}$ agrees with e. Then,* $\mathbf{S}_1(Y^+, e^\#) \leq 2\mathbf{S}_1(Y, e)$ *and* $\min\{\, \mathbf{s}_\omega(Y, e), q \,\} \leq \mathbf{s}_\omega(Y^q, e^\#)$, *for every $q \geq 1$.*

Proof. Assume that $(i, e^\#, j) = 1$. Let $y_1, y_2, \ldots, y_q$ be in Y, for $q \geq 1$. If $q = 1$ we have nothing to prove, so assume that $q \geq 2$. Now, $(i, e^\#, j) = 1$ implies that $(i, e, j) = 1$ and that (i, j) is anchored in e. Let $k \in [1, n]$ be such that $0 = (k, e, k) \leq (i, e, j) = \max\{\, (i, e, k), (k, e, j) \,\} = 1$. If $(i, e, k) = 0$ then $(i, y_1, k) = 0$ since y_1 agrees with e. Thus, $(i, y_1, k) \leq \mathbf{S}_1(Y, e)$. Similarly, $(k, y_q, j) \leq \mathbf{S}_1(Y, e)$. On the other hand, since each y_r agrees with e, $(k, e, k) = 0$ implies that $(k, y_r, k) = 0$ for every $r \in [1, q]$. Thus, $(i, y_1 \cdots y_q, j) \leq (i, y_1, k) + (k, y_2, k) + \cdots + (k, y_q, j) \leq 2\mathbf{S}_1(Y, e)$. It follows that $\mathbf{S}_1(Y^+, e^\#) = \max\{\, (i, y, j) \mid y \in Y^+ \text{ and } (i, e^\#, j) = 1 \,\} \leq 2\mathbf{S}_1(Y, e)$.

Let now $q \geq 1$ and assume that $(i, e^\#, j) = \omega$. Let $y_1, y_2, \ldots, y_q \in Y$ and let $i = k_0, k_1, k_2, \ldots, k_q = j$ in $[1, n]$ be such that $(i, y_1 \cdots y_q, j) = (k_0, y_1, k_1) + \cdots + (k_{q-1}, y_q, k_q)$. Since $y_1 \cdots y_q$ agrees with $e^\#$ we conclude that $0 < (i, y_1 \cdots y_q, j) < \infty$, and, consequently, for each r, $(k_{r-1}, y_r, k_r) < \infty$. Since each y_r agrees with e we conclude that for each r, $(k_{r-1}, e, k_r) \leq \omega$; hence, $\max\{\, (k_{r-1}, e, k_r) \mid r \in [1, q] \,\} \leq \omega$. Assume initially that $(k_{r-1}, e, k_r) = \omega$ for some $r \in [1, q]$. Then,

$$\mathbf{s}_\omega(Y, e) \leq (k_{r-1}, y_r, k_r) \leq \sum_{r=1}^{q} (k_{r-1}, y_r, k_r) = (i, y_1 \cdots y_q, j). \tag{1}$$

Assume now that $(k_{r-1}, e, k_r) < \omega$ for every $r \in [1, q]$. From $(i, e^\#, j) = \omega$ we conclude that $1 \leq (i, e, j)$. Using Lemma 1 we have:

$$1 \leq (i, e, j) \leq \max\{\, (k_{r-1}, e, k_r) \mid r \in [1, q] \,\} \leq 1. \tag{2}$$

Now we claim that $(k_{r-1}, e, k_r) = 1$ for every r. Indeed, assume that $(k_{r-1}, e, k_r) = 0$ for some r. Then, by Lemma 2, (k_{r-1}, k_r) is anchored in e and so is $(k_0, k_q) = (i, j)$. Since (2) implies that $(i, e, j) = 1$ it follows that $(i, e^\#, j) = 1$: a contradiction which establishes the claim. Now, since each y_r agrees with e we can conclude that $1 \leq (k_{r-1}, y_r, k_r)$ for every r; hence,

$$q \leq \sum_{r=1}^{q} (k_{r-1}, y_r, k_r) = (i, y_1 \cdots y_q, j). \tag{3}$$

Since either (1) or (3) holds, it follows that $\min\{\, \mathbf{s}_\omega(Y,e), q \,\} \le (i, y_1 \cdots y_q, j)$; hence, $\min\{\, \mathbf{s}_\omega(Y,e), q \,\} \le \min\{\, (i,y,j) \mid y \in Y^q \text{ and } (i, e^{\#}, j) = \omega \,\} = \mathbf{s}_\omega(Y^q, e^{\#})$. This concludes the proof of Lemma 5. ∎

4 Basic objects and their properties

In this section we set up the notations and the major definitions needed to prove the main result. Let A be a nonempty finite alphabet and let $\varphi\colon A^+ \to M_n\mathcal{T}$ be a morphism such that $A\varphi \subseteq M_n\mathcal{M}$. In other words, $A^+\varphi$ is just a finitely generated subsemigroup of $M_n\mathcal{M}$. We shall denote by Δ the maximum of the nonnull finite coefficients in $A\varphi$, i.e. $\Delta = \max\{\, \mathbf{S}_1(a\varphi, a\varphi\Psi) \mid a \in A \,\}$. Note that, even though we shall not do it, there is no loss of generality if we assume that $\Delta = 1$.

Let S be the least stable subsemigroup of $M_n\mathcal{R}$ which contains $A\varphi\Psi$. In order to precise a generating set for S we define an alphabet B given by

$$B = A \cup \{\, b_e \mid \ e \text{ is an unstable idempotent in } S \,\},$$

where the union is disjoint. Let $f\colon B^+ \to M_n\mathcal{R}$ be the morphism defined by $af = a\varphi\Psi$, for $a \in A$, and $b_e f = e^{\#}$, for every unstable idempotent e in S. Clearly, $B^+ f = S$.

Now we define the principal object used in our proof. A *tropical tree* T consists of a rooted plane tree with vertex set V and a labeling $\lambda\colon V \to B^+ \times S$ which satisfies:

- no vertex in V has outdegree one;
- every vertex of outdegree 0 has label (b, bf), for some $b \in B$;
- every vertex v of outdegree two has label (x_1x_2, z_1z_2), where (x_1, z_1) and (x_2, z_2) are, respectively, the labels of the direct left and right descendants of v;
- for every vertex v of outdegree $p > 2$ the labels of the direct descendants of v are (x_i, e), for $i \in [1, p]$, and the label of v is $(x_1 x_2 \cdots x_p, e^{\#})$, where e is some idempotent in S which will be called the idempotent of vertex v.

The *label* of the tropical tree T is the label of the root of T.

It will be convenient to classify the vertices of a tropical tree as continuous or discontinuous. Continuity here will be meant with respect to the product in S. More precisely, we say that vertex v is *discontinuous* if it has outdegree $p > 2$ and its idempotent is unstable. A vertex is *continuous* if it is not discontinuous.

All paths considered in T will be directed away from the root. A path c in T is *continuous* if every internal vertex of c is continuous. Note that we allow for discontinuous vertices at the extremities of c. The *span* of a tropical tree T is the length of a longest continuous path in T.

Lemma 6 *Let T be a tropical tree of height h and span s. Let q be the cardinality of a maximum chain of principal ideals generated by unstable idempotents of S. Then, $h \leq (1+q)s \leq |S|s$.*

Proof. Let $c = (v_0, v_1, \ldots, v_r)$ be a continuous path beginning and ending on discontinuous vertices v_0 and v_r. Let v_{r+1} be a descendant of v_r; the existence of such a vertex is guaranteed by the descontinuity of v_r. Let (x_i, z_i) be the label of vertex v_i. Then, z_1 and z_{r+1} are unstable idempotents such that $z_0 = z_1^{\#}$ and $z_r = z_{r+1}^{\#}$. Since c is continuous, $z_1 \in S^1 z_r S^1$. On the other hand, from Lemma 3, $S^1 z_r S^1$ is properly contained in $S^1 z_{r+1} S^1$. Thus, z_1 and z_{r+1} are unstable idempotents such that $S^1 z_1 S^1 \subset S^1 z_{r+1} S^1$.

It follows that if c has p unstable vertices then S has a chain of p principal ideals generated by unstable idempotents. From the definition of span and the choice of q it follows that $h \leq (1+q)s$. Clearly, $1 + q \leq |S|$ and this completes the proof. ∎

A *multiplicative rational expression over* A is a rational expression which uses only the operations of concatenation and star. To such an expression R we associate a function $R \colon \mathbb{N} \to A^*$, where kR is the word obtained by substituting by k each occurrence of $*$ in R. For example, the words associated to $c(ba^*)^*c$ are $cc, cbac, c(ba^2)^2c, c(ba^3)^3c, \ldots$.

The connection between tropical trees and multiplicative rational expressions is given by the next Lemma. Let T be a tropical tree labeled by (x, z), with $x \in A^+$ and let R be a multiplicative rational expression over A. We say that R is a *witness* for T if $0R = x$, $\mathbb{N}R\varphi$ agrees with z, $\mathbf{S}_1(\mathbb{N}R\varphi, z)$ is finite and, for every $k > 0$, $\mathbf{s}_\omega(kR\varphi, z) \geq k$. We alert the reader that the definition of a witness, as well as the next Lemma, refer exclusively to tropical trees with label in $A^+ \times S$.

Lemma 7 *For every tropical tree T labeled by (x,z), with $x \in A^+$, there exists a multiplicative rational expression R over A which is a witness for T and such that $\mathbf{S}_1(\mathbb{N}R\varphi, z) \leq 2^h\Delta$, where h is the height of T.*

Proof. We proceed by induction on the height h of T. If $h = 0$ then the only vertex of T is its root and the label of T is (a, af) for some $a \in A$. Let $R = a$; since $kR = a$ for every k and $af = a\varphi\Psi$, we have that

$$\mathbf{S}_1(\mathbb{N}R\varphi, af) = \mathbf{S}_1(a\varphi, af) = \mathbf{S}_1(a\varphi, a\varphi\Psi) \leq \Delta.$$

Assume now that the root v of T has outdegree 2 and that the tropical trees associated to the direct descendants of v are T_i with label (x_i, z_i), for $i = 1, 2$. Then, the label of T is (x,z), with $x = x_1x_2$ and $z = z_1z_2$. By the induction hypothesis T_i has witness R_i such that $\mathbf{S}_1(\mathbb{N}R_i\varphi, z_i) \leq 2^{h-1}\Delta$. We claim that the multiplicative rational expression $R = R_1R_2$ satisfies the Lemma for T. Indeed, $0R = (0R_1)(0R_2) = x_1x_2 = x$. Also, $\mathbb{N}R\varphi$ agrees with $z_1z_2 = z$. Now, for each $k \geq 0$, we have, by Lemma 4,

$$\mathbf{S}_1(kR\varphi, z) = \mathbf{S}_1((kR_1\varphi)(kR_2\varphi), z_1z_2) \leq \mathbf{S}_1(kR_1\varphi, z_1) + \mathbf{S}_1(kR_2\varphi, z_2).$$

Hence,

$$\begin{aligned}
\mathbf{S}_1(\mathbb{N}R\varphi, z) &= \max\{\, \mathbf{S}_1(kR\varphi, z) \mid k \in \mathbb{N}\,\} \\
&\leq \max\{\, \mathbf{S}_1(kR_1\varphi, z_1) + \mathbf{S}_1(kR_2\varphi, z_2) \mid k \in \mathbb{N}\,\} \\
&\leq \max\{\, \mathbf{S}_1(kR_1\varphi, z_1) \mid k \in \mathbb{N}\,\} + \max\{\, \mathbf{S}_1(kR_2\varphi, z_2) \mid k \in \mathbb{N}\,\} \\
&\leq 2^{h-1}\Delta + 2^{h-1}\Delta = 2^h\Delta.
\end{aligned}$$

Now, for every $k > 0$, using Lemma 4 and the induction hypothesis we have that

$$\begin{aligned}
\mathbf{s}_\omega(kR\varphi, z) &= \mathbf{s}_\omega((kR_1\varphi)(kR_2\varphi), z_1z_2) \\
&\geq \min\{\, \mathbf{s}_\omega(kR_1\varphi, z_1), \mathbf{s}_\omega(kR_2\varphi, z_2)\,\} \geq k.
\end{aligned}$$

This concludes the proof when the root of T has outdegree 2.

Assume finally that the root v of T has outdegree $p > 2$. Let $T_1, T_2, \ldots, T_p$ be the tropical trees associated to the direct descendants of v; let $e =$

$e^2 \in M_n\mathcal{R}$ and $x_i \in A^+$ be such that (x_i, e) is the label of T_i. Then, $x = x_1x_2\cdots x_p$ and $z = e^\#$, where (x, z) is the label of T. By the induction hypothesis T_i has a witness R_i such that $\mathbf{S}_1(\mathbb{N}R_i\varphi, e) \leq 2^{h-1}\Delta$. We claim that $R_1\cdots R_p(R_1\cdots R_p)^*$ satisfies the Lemma for T. Initially we note that $0R = 0R_1\cdots 0R_p = x_1x_2\cdots x_p = x$. Also, for every $k \geq 0$, $kR\varphi$ agrees with z. Indeed, $kR\varphi = (kR_1\cdots kR_p(kR_1\cdots kR_p)^k)\varphi$; hence, since $kR_q\varphi$ agrees with e and $e = e^2$ we have that $kR\varphi$ agrees with e. Then $kR\varphi$ agrees with $z = e^\#$, since $e\pi = e^\#\pi$. Let now $Y = \cup_{q=1}^{p}\mathbb{N}R_q\varphi$. Then Y agrees with e and, for every $k \geq 0$, $kR\varphi \in Y^+$. Also,

$$\mathbf{S}_1(Y, e) = \max\{\, \mathbf{S}_1(\mathbb{N}R_q\varphi, e) \mid q \in [1, p]\,\} \leq 2^{h-1}\Delta.$$

By Lemma 5,

$$\mathbf{S}_1(\mathbb{N}R\varphi, e^\#) \leq \mathbf{S}_1(Y^+, e^\#) \leq 2\mathbf{S}_1(Y, e) \leq 2.2^{h-1}\Delta = 2^h\Delta.$$

Finally, let us fix a $k > 0$. Let $Y = \{\, kR_q\varphi \mid q \in [1, p]\,\}$. Again, Y agrees with e and for every $k > 0$, $kR\varphi \in Y^{p+kp}$, since $kR = kR_1\cdots kR_p(kR_1\cdots kR_p)^k$; thus, $\mathbf{s}_\omega(kR\varphi, e^\#) \geq \mathbf{s}_\omega(Y^{p+kp}, e^\#)$. Also, since $\mathbf{s}_\omega(kR_q\varphi, e) \geq k$, for every q, we have that $\mathbf{s}_\omega(Y, e) = \min\{\, \mathbf{s}_\omega(y, e) \mid y \in Y\,\} \geq k$. Thus, by Lemma 5,

$$\mathbf{s}_\omega(kR\varphi, e^\#) \geq \mathbf{s}_\omega(Y^{p+kp}, e^\#) \geq \min\{\, \mathbf{s}_\omega(Y, e), p+kp\,\} \geq \min\{\, k, p+kp\,\} = k,$$

since $p \geq 3$. This concludes the proof of Lemma 7. ∎

We close this section with a simple property.

Lemma 8 *If a tropical tree is labeled by (x, z) then $xf\pi = z\pi$.*

Proof. This is proved by a straightforward induction on the height of the tree, after observing that f and π are morphisms and that for every idempotent $e \in M_n\mathcal{R}$, $e\pi = e^\#\pi$, since $1\pi = \omega\pi = 1$. ∎

5 Construction of tropical trees

In this section we construct tropical trees needed to show the main result.

We shall need some definitions and results from [8]. For an alphabet A we denote the free semigroup generated by A either by A^+, as usual, or by $A\mathcal{F}$. In the second notation the elements of $A\mathcal{F}$ will be represented as $(a_1, a_2, \ldots, a_p)$, where $a_i \in A$.

A *factorization forest* $F = (X, d)$ *over* A consists of a subset X of A^+ together with a function $d \colon X \to \mathcal{F}X$ such that, for every $x \in X$, $xd = (x_1, x_2, \ldots, x_p)$ implies that $x = x_1 x_2 \cdots x_p$; i.e. xd is a factorization of x whose factors belong to X. The *external set* of F is the set $\{\, x \in X \mid |xd| = 1 \,\}$.

Given F we associate to each $x \in X$ a rooted plane tree xF whose vertices are labeled by elements of X. If $|xd| = 1$ then xF consists just of the root labeled x. If $xd = (x_1, x_2, \ldots, x_p)$, with $p > 1$, then the root of xF has outdegree p and a copy of $x_i F$ is associated to the i-th direct descendant of the root. This allows us to speak of the outdegree of vertices of xF, and of paths in xF. The height of xF is denoted xh and the height of F is $h = \sup\{\, xh \mid x \in X \,\}$.

Let $f \colon B^+ \to S$ be a semigroup morphism, with S finite. Factorization forest F is *Ramseyan mod* f if for every x of outdegree $p \geq 3$, $xd = (x_1, x_2, \ldots, x_p)$ implies that there exists an idempotent $e \in S$ such that $e = xf = x_1 f = \cdots = x_p f$. We say that f *admits a Ramseyan factorization forest* if there exists a factorization forest $F = (B^+, d)$ over B with external set B which is Ramseyan mod f.

We recall the main result of [8].

Theorem 9 *Every morphism $f \colon B^+ \to S$, from a free semigroup to a finite one, admits a Ramseyan factorization forest of height at most $9|S|$.*

In what follows we consider the data defined in Section 4 and fix a factorization forest $F = (B^+, d)$ with the properties asserted by Theorem 9. We call H the height of F and note, for future use that

$$H \leq 9|S|.$$

The following Lemma is the main result of this section.

Lemma 10 *For every $x \in B^+$ there exists a tropical tree of span at most H labeled by (x, z), for some $z \in S$.*

Proof. Based on the factorization forest F, Ramseyan mod f, we define $P \subseteq B^+$ as follows. Let P be the least subset of B^+ which satisfies the following properties:

- B is contained in P;
- if $x \in B^+$, $xd \in P\mathcal{F}$ and $|xd| = 2$ then $x \in P$;
- if $x \in B^+$, $xd \in P\mathcal{F}$, $|xd| > 2$ and xf is stable then $x \in P$.

The set P enjoys the following properties.

Assertion 1 *For every $x \in P$ there exists a tropical tree of height at most H labeled by (x, xf) whose vertices are all continuous.*

Proof. Using the factorization forest F we have associated a tree xF to x. Vertices of this tree are labeled by certain factors of x; since $x \in P$, the definition of P implies that all these labels belong to P. Let v be a vertex of the tree xF and let $y \in B^+$ be its label. We extend the labeling of v to (y, yf). Note that, F being Ramseyan mod f, $x \in P$ implies that whenever the outdegree of v is at least 3 the matrix yf is a stable idempotent. Thus, the tree xF becomes a tropical tree of label (x, xf) with no discontinuous vertices. This tree has height at most H which is the height of F. ∎

Assertion 2 *Let $x \in B^+ \setminus P$ be such that $xd \in P\mathcal{F}$. There exists a tropical tree of height at most H labeled by $(x, xf^{\#})$ whose root is its unique discontinuous vertex.*

Proof. The hypothesis imply that $|xd| > 2$, and that xf is an unstable idempotent. We proceed exactly as in Assertion 1 with the exception that we substitute the label of the root by $(x, xf^{\#})$, which is necessary now to get a tropical tree. ∎

Assertion 3 *Every $x \in B^+ \setminus P$ has a factorization $x = yx'y'$ such that $x' \in B^+ \setminus P$ and $x'd \in P\mathcal{F}$.*

Proof. Let x' be a shortest segment of x which is not in P; then, $x = yx'y'$ for appropriate $y, y' \in B^*$. Let $x'd = (x_1, \ldots, x_p)$. Note that the choice of x' guarantees that every proper nonempty segment of x' is in P; in particular, $x'd \in P\mathcal{F}$. From the definition of P we conclude that $|x'| > 1$. ∎

The construction of our tropical trees will be done by a sequence of substitutions. One instance of this operation is given by the following Assertion.

Assertion 4 *Let $x = yby'$, with $b \in B \setminus A$ and $y, y' \in B^*$. Let T_1 be a tropical tree with label (x, z) and span s_1. Let T_2 be a tropical tree with label (x', bf) and span s_2 whose root is discontinuous. Then there exists a tropical tree T with label $(yx'y', z)$ and span $s = \max\{ s_1, s_2 \}$.*

Proof. Let v_1 be the (external) vertex of tree T_1 whose label is (b, bf) and which corresponds to the factorization yby' of x. Tree T is constructed by substituting vertex v_1 in T_1 by the tree T_2. We maintain the labeling of the vertices in T_2. As for T_1, we consider the path in T_1 from the root to v_1. Vertices of T_1 not on this path maintain their labels. Let (x_i, z_i) be the label in T_1 of vertex v_i on this path. The vertex v_1, which lies on the tree with root v_i, determines a factorization $y_i b y'_i$ of x_i. The label of v_i in T will be $(y_i x' y'_i, z_i)$.

Noting that the label of T_2 is (x', bf), the reader will verify without difficulty that T is a tropical tree with label $(yx'y', z)$. Since the root of T_2 is discontinuous, the span of T is the maximum of the spans of T_1 and T_2. This completes the proof of the Assertion. ∎

We are ready to prove Lemma 10 by induction on $|x|$. For $x \in P$ the tropical tree given by Assertion 1 satisfies the Lemma with $z = xf$. Let then $x \in B^+ \setminus P$ and let us apply Assertion 3. Since $x' \in B^+ \setminus P$ but $x'd \in P\mathcal{F}$ the definition of P implies that $|x'd| > 2$ and that $x'f$ is an unstable idempotent. Let $e = x'f$ and let b be the letter in $B \setminus A$ for which $bf = e^{\#}$. By the induction hypothesis there exists a tropical tree T_1 with label (yby', z), for some $z \in S$, with span s_1 at most H. By Assertion 2 there exists a tropical tree T_2 with label $(x', e^{\#})$ and height (hence span s_2) at most H whose root is discontinuous. The tropical tree T given by Assertion 4 has label $(yx'y', z) = (x, z)$ and span $s = \max\{ s_1, s_2 \} \leq H$. This completes the proof. ∎

Lemma 11 *For every $z \in S$ there exists an $x \in A^+$ and a tropical tree labeled by (x, z).*

Proof. Initially we need a closer look at the generation of S. Let $B_0 = A$. Given $B_k \subseteq B$, $k \geq 0$, let $S_k = (B_k f)^+$. Given S_k, $k \geq 0$, let $B_{k+1} = A \cup \{ b_e \mid e = e^2 \in S_k \text{ and } e \neq e^{\#} \}$. By construction, $B_k \subseteq B_{k+1} \subseteq B$, and $S_k \subseteq S_{k+1} \subseteq S$. Since S is finite, there exists $l \leq |S|$, such that $S_l = S_{l+1} \subseteq$

S. Since S is the least stable subsemigroup of $M_n\mathcal{R}$ which contains A, we conclude that $S_l = S$ and $B_{l+1} = B$.

Now, for every $z \in S$ there exists a least k such that $z \in S_k$. We shall proceed by induction on such a k. Initially we observe that for $z \in B_0 f = Af$ we have $a \in A$ such that $af = z$; hence it is sufficient to consider the one vertex tropical tree with label (a, af).

Consider now $k \geq 0$ and assume that for every $z \in B_k f$ there exists a tropical tree T_z labeled by (x_z, z), with $x_z \in A^+$. Let $z \in S_k$. By construction, there exists $y \in B_k^+$ such that $yf = z$. Our tropical tree is obtained by induction on $|y|$ by repeating the following construction. Let T_i be a tropical tree labeled by (x_i, z_i), with $x_i \in A^+$ and $z_i \in S_k$, for $i = 1, 2$. Let T be the tree whose root has outdegree two and T_1 and T_2 are the tropical trees of the direct left and right descendants of the root. Then, the label of T is $(x_1 x_2, z_1 z_2)$.

Finally, assume that for every $z \in S_k$ we have a tropical tree T_z labeled by (x_z, z), with $x_z \in A^+$. We claim that the same is true for $z \in B_{k+1} f$. Indeed, if $z \in B_k f$ the claim follows at once from the induction hypothesis; otherwise there exists $e = e^2 \in S_k$ such that $b_e f = e^\# = z$. By the induction hypothesis, there exists a tropical tree T_e labeled by (x_e, e), for some $x_e \in A^+$. Consider now the tree whose root has outdegree three and every one of the direct descendants of the root have tropical trees identical to T_e. Then, T is a tropical tree labeled by $(x_e^3, e^\#)$ and consequently it justifies our claim. This concludes the proof of Lemma 11. ∎

6 Main result

In order to precise our problem we need some definitions. Let $X \subseteq M_n K$, for some semiring K. For $I \in K^{1\times n}$ and $J \in K^{n\times 1}$ the (I, J)-*section* of X is the subset of K given by $\{\, IxJ \mid x \in X \,\}$. The case when I and J have exactly one coefficient which is 1_K all others being 0_K is particularly interesting. In this case, assuming that the nonnull coefficients are $(1, I, i)$ and $(j, J, 1)$, the (I, J)-section of X is the set of all coefficients of matrices in X in row i and column j; this set is usually called the (i, j)-*section* of X.

Next we prove the main theorem which we announced in [9]. Note that the equivalence of (i) and (iii) has been proved by K. Hashiguchi in [3] and the equivalence of (i) and (ii) has been proved by H. Leung in [5]. All these proofs were obtained pairwise independently but our presentation certainly was influenced by the work of both K. Hashiguchi and H. Leung.

Theorem 12 *Let $\varphi\colon A^+ \to M_n\mathcal{M}$ be a morphism with A finite; let S be the least stable subsemigroup of $M_n\mathcal{R}$ which contains $A\varphi\Psi$ and let $I \in \{0,\infty\}^{1\times n}$ and $J \in \{0,\infty\}^{n\times 1}$ be matrices. The following statements are equivalent:*

(i) the (I,J)-section of $A^+\varphi$ is infinite;

(ii) the (I,J)-section of S contains ω;

(iii) there exists a multiplicative rational expression R over A such that, for every $k>0$, $k \le I(kR\varphi)J < \infty$.

Proof. (i) implies (ii). Assume, for a contradiction, that the (I,J)-section of S does not contain ω. Let $Z \subseteq \mathcal{M}$ be the (I,J)-section of $A^+\varphi$. Let F be a factorization forest with the properties asserted in Theorem 9; let H be the height of F. Let q be the cardinality of a maximum chain of principal ideals generated by unstable idempotents of S. Let $u = 2^{(1+q)H}\Delta$, where $\Delta = \max\{\mathbf{S}_1(a\varphi, a\varphi\Psi) \mid a \in A\}$. Then, $H \le 9|S|$, $q < |S|$ and $u \le 2^{9|S|^2}\Delta$. We shall show that $Z \subseteq [0,u] \cup \{\infty\}$. This implies that Z is finite, a contradiction that establishes the result.

To see the claim, let $x \in A^+$. By Lemma 10 there exist a tropical tree T of span at most H labeled by (x,z), for some $z \in S$. By Lemma 6 the height h of T satisfies $h \le (1+q)H$. By Lemma 7 T has a witness R such that $\mathbf{S}_1(\mathbb{N}R\varphi, z) \le u$, since $2^h\Delta \le 2^{(1+q)H}\Delta = u$.

Let $m = I(x\varphi)J$. If $m \in \{0,\infty\}$ then we have nothing to prove, otherwise, $0 < m < \infty$. Taking projections by $\Psi\pi$ we have that $m\Psi\pi = I(xf\pi)J = 1$, since $I = I\Psi\pi$, $J = J\Psi\pi$ and, x belonging to A^+, $x\varphi\Psi = xf$. By Lemma 8, $I(z\pi)J = 1$. Now, from the definition of π, $IzJ \in \{1,\omega\}$; the assumption that the (I,J)-section of S does not contain ω implies that $IzJ = 1$. Hence, there exist $i,j \in [1,n]$ such that $(1,I,i) = 0 = (j,J,1)$ and $(i,z,j) = 1$. Now, R being a witness for T, we have that $0R = x$; hence, from $\mathbf{S}_1(\mathbb{N}R\varphi, z) \le u$ we conclude that $(i, x\varphi, j) \le u$. Consequently, $m = I(x\varphi)J \le (1,I,i) + (i,x\varphi,j) + (j,J,1) \le u$ and this proves the claim.

(ii) implies (iii). Let $z \in S$ be such that $IzJ = \omega$. By Lemma 11 there exists an $x \in A^+$ and a tropical tree T labeled by (x,z). By Lemma 7 T has a witness R. Let us fix $k > 0$ and let $m = I(kR\varphi)J$. Since $kR\varphi \in M_n\mathcal{M}$, since it agrees with z and since $IzJ = \omega$, we conclude that $0 < m < \omega < \infty$.

Since $m = I(kR\varphi)J$, there exist $i, j \in [1, n]$ such that $(1, I, i) = 0 = (j, J, 1)$ and $(i, kR\varphi, j) = m$. Since $kR\varphi$ agrees with z and $IzJ = \omega$ it follows that $(i, z, j) = \omega$. Thus, since R is a witness for T we have that $k \leq \mathbf{s}_\omega(kR\varphi, z) \leq (i, kR\varphi, j) = m$. Altogether, $k \leq I(kR\varphi)J < \infty$, as required.

(iii) implies (i). This is clear. ∎

Since $M_n\mathcal{R}$ is a finite semigroup it follows that S is effectively computable. Hence, a direct consequence of Theorem 12 is a result proved by K. Hashiguchi in [2].

Corollary 13 *It is decidable whether or not a given section of a finitely generated subsemigroup of $M_n\mathcal{M}$ is finite.*

The following result, proved in [6], is also easily deduced from Theorem 12.

Corollary 14 *Let $\varphi\colon A^+ \to M_n\mathcal{M}$ be a morphism with A finite. Then $A^+\varphi$ is finite if and only if every idempotent in $A^+\varphi\Psi$ is stable.*

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References

[1] S. Eilenberg. *Automata, Languages, and Machines, Volume A*. Academic Press, New York, 1974.

[2] K. Hashiguchi. Limitedness theorem on finite automata with distance functions. *J. Comput. Syst. Sci.*, 24:233–244, 1982.

[3] K. Hashiguchi. Improved limitedness theorems on finite automata with distance functions. *Theoretical Comput. Sci.*, 72, 1990.

[4] H. Leung. *An Algebraic Method for Solving Decision Problems in Finite Automata Theory.* PhD thesis, Department of Computer Science, The Pennsylvania State University, 1987.

[5] H. Leung. On the topological structure of a finitely generated semigroup of matrices. *Semigroup Forum*, 37:273–287, 1988.

[6] I. Simon. Limited subsets of a free monoid. In *Proc. 19th Annual Symposium on Foundations of Computer Science*, pages 143–150, Piscataway, N.J., 1978. Institute of Electrical and Electronics Engineers.

[7] I. Simon. Recognizable sets with multiplicities in the tropical semiring. In M. P. Chytil, L. Janiga, and V. Koubek, editors, *Mathematical Foundations of Computer Science 1988*, pages 107–120, Berlin, 1988. Springer-Verlag. Lecture Notes in Computer Science, 324.

[8] I. Simon. Factorization forests of finite height. *Theoretical Comput. Sci.*, 72:65–94, 1990.

[9] I. Simon. The nondeterministic complexity of a finite automaton. In M. Lothaire, editor, *Mots - mélanges offerts à M.P. Schützenberger*, pages 384–400. Hermes, Paris, 1990.