# KEY MATHEMATICAL CONCEPTS IN THE TRANSITION FROM SECONDARY SCHOOL TO UNIVERSITY

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This report<sup>1</sup> from the ICME12 Survey Team 4 examines issues in the transition from secondary school to university mathematics with a particular focus on mathematical concepts and aspects of mathematical thinking. It comprises a survey of the recent research related to: calculus and analysis; the algebra of generalised arithmetic and abstract algebra; linear algebra; reasoning, argumentation and proof; and modelling, applications and applied mathematics. This revealed a multi-faceted web of cognitive, curricular and pedagogical issues both within and across the mathematical topics above. In we conducted an international survey of those engaged in teaching in university mathematics departments. Specifically, we aimed to elicit perspectives on: what topics are taught, and how, in the early parts of university-level mathematical studies; whether the transition should be smooth; student preparedness for university mathematics studies; and, what university departments do to assist those with limited preparedness. We present a summary of the survey results from 79 respondents from 21 countries.

Keywords: Transition, mathematics, secondary, university, survey.

### BACKGROUND

Changing mathematics curricula and their emphases, lower numbers of student enrolments in undergraduate mathematics programmes (Barton & Shervn. 2009: andhttp://www.mathunion.org/icmi/other-activities/pipeline-project/) and changes due to an enlarged tertiary entrant profile (Hoyles, Newman, & Noss, 2001; Hockman, 2005), have provoked some international concern about the mathematical ability of students entering university (PCAST, 2012; Smith, 2004) and the traumatic effect of the transition on some of them (Engelbrecht, 2010). Decreasing levels of mathematical competency have been reported with regard to essential technical facility, analytical powers, and perceptions of the place of precision and proof in mathematics (Gill, O'Donoghue, Faulkner & Hannigan, 2010; Hourigan & O'Donoghue, 2007; Kajander & Lovric, 2005; Luk, 2005; Selden, 2005). The shifting profile of students who take service mathematics courses has produced a consequent decline in mathematical standards (Gill, O'Donoghue, Faulkner & Hannigan, 2010; Jennings, 2009). However, not all studies agree on the extent of the problem (Engelbrecht & Harding,

<sup>&</sup>lt;sup>1</sup> A longer version of this report is available from

http://faculty.math.tsinghua.edu.cn/~jxie/papers/icme2012\_ST4.pdf or

2008; Engelbrecht, Harding & Potgieter, 2005) and James, Montelle and Williams (2008) found that standards had been maintained.

This situation has to be put in the context of the report of the President's Council of Advisors on Science and Technology (PCAST) (2012). This states that in the USA alone there is a need to produce, over the next decade, around 1 million more college graduates in Science, Technology, Engineering, and Mathematics (STEM) fields than currently expected and recommends funding around 200 experiments at an average level of \$500,000 each to address mathematics preparation issues. This emphasises the importance of addressing these transition issues.

The Survey Team 4 brief was restricted to a consideration of the role of mathematical thinking and concepts related to transition and we found relatively few papers in the recent literature dealing directly with this. Hence we also reviewed literature analysing the learning of mathematics on one or both sides of the transition boundary. To achieve this we formed the, somewhat arbitrary, division of this mathematics into: calculus and analysis; the algebra of generalised arithmetic and abstract algebra; linear algebra; reasoning, argumentation and proof; and modelling, applications and applied mathematics, and report findings related to each of these fields. We were aware that other fields such as geometry and statistics and probability probably should have been included, but were not able to do so.

## THE SURVEY

We considered it important to obtain data on transition from university mathematics departments. We wanted to know what topics are taught and how, if the faculty think the transition should be smooth, or not, their opinions on whether their students are well prepared mathematically, and what university departments do to assist those who are not. Hence, we constructed an anonymous questionnaire on transition using an Adobe Acrobat pdf form and sent it internationally by email to members of mathematics departments. The 79 responses from 21 countries were collected electronically. The sample comprised 56 males and 23 females with a mean of 21.9 years of academic teaching. Of these 45 were at the level of associate professor, reader or full professor, and 30 were assistant professors, lecturers or senior lecturers. There were 5 or more responses from each of South Africa, USA, New Zealand and Brazil.

Clearly the experience for beginning university students varies considerably depending on the country and the university that they attend. For example, while the majority teaches pre-calculus (53, 67.1%), calculus (76, 96.2%) and linear algebra (49, 62%) in their first year, minorities teach complex analysis (1), topology (3), group theory (1), real analysis (5), number theory (9), graph theory (12), logic (15), set theory (17) and geometry (18), among other topics. Further, in response to 'Is the approach in **first** year mathematics at your university: Symbolic, Procedural; Axiomatic, Formal; Either, depending on the course.' 21 (26.6%) answered that their departments introduce symbolic and procedural approaches in first year mathematics (50, 63.3%) replied that their approach depended on the course.

When asked 'Do you think students have any problems in moving from school to university mathematics?' 72 (91.1%) responded "Yes" and 6 responded "No". One third of those who answered "Yes" described these problems as coming from a lack of preparation in high school, supported by comments such as "They don't have a sufficiently good grasp of the expected school-mathematics skills that they need." Further, two thirds of those who answered "Yes" described the problems as arising from the differences between high school classes and university (including more than 50% of the respondents from those sending at least 5 responses), such as differences in class size and work load, with many specifically citing the conceptual nature of university mathematics as being different from the procedural nature of high school mathematics. Comments here included "university is much more theoretical" and "Move from procedural to formal and rigourous [sic], introduction to proof, importance of definitions and conditions of theorems/rules/statements/formulas." There is also a need to "...deal with misconceptions which students developed in secondary school...We also have to review secondary school concepts and procedures from an adequate mathematical point of view." Other responses cited: students' weak algebra skills (12.5%); that university classes are harder (5%); personal difficulties in adjusting (10%); poor placement (3%); and, poor teaching at university (1%).

Looking at specific mathematical knowledge, we enquired 'How would you rate first year students' mathematical understanding of each of the following on entry to university?' With a maximum score of 5 for high, the mean scores of the responses were algebra or generalised arithmetic (3.0), functions (2.8), real numbers (2.7), differentiation (2.5), complex numbers (1.9), definitions (1.9), vectors (1.9), sequences and series (1.9), Riemann integration (1.8), matrix algebra (1.7), limits (1.7) and proof (1.6). The mathematicians were specifically asked whether students were well prepared for calculus study. Those whose students did not study calculus at school rated their students' preparation for calculus at 2.1 out of 5. Those whose students did, rated secondary school calculus as preparation to study calculus at university at 2.4, and as preparation to study analysis at university at 1.5. These results suggest that there is some room for improvement in school preparation for university study of calculus and analysis.

Taken all together, these responses indicate that university academics do perceive both some inadequacies in students' knowledge and difficulties in transition.

Since there has been some literature (e,g., Clark & Lovric, 2009) indicating that, rather than being 'smooth', the transition to university should require some measure of struggle by students, we asked 'Do you think the transition from secondary to university education in mathematics should be smooth?' Here, 54 (68.4%) responded "Yes" and 22 (27.8%) responded "No". Of those who responded "No", many of the comments were similar to the following, expressing the belief that change is a necessary part of the transition: "Not necessarily smooth, because it is for most students a huge change to become more independent as learners." and "To learn mathematics is sometimes hard." Those who answered yes were then asked 'what could be done to make the transition from secondary to university education in mathematics smoother?' The majority of responses mentioned changes that could be made at the high school level, such as: encourage students to think

independently and abstractly; change the secondary courses; have better trained secondary teachers; and, have less focus in secondary school on standardised tests and procedures. A few mentioned changes that could be made at the university, such as: better placement of students in classes; increasing the communication between secondary and tertiary teachers; and, addressing student expectations at each level. This lack of communication between the two sectors was also highlighted as a major area requiring attention by the two-year study led by Thomas (Hong, Kerr, Klymchuk, McHardy, Murphy, Spencer, & Thomas, 2009).

Since one would expect that, seeing students with difficulties in transition, universities might respond in an appropriate manner, we asked 'Does your department periodically change the typical content of your first year programme?' 33 (41.8%) responded "Yes" and 44 (55.7%) responded "No". The responses to the question 'How does your department decide on appropriate content for the first year mathematics programme for students?' by those who answered yes to the previous question showed that departments change the content of the first year programme based on the decision of committees either on university level or on department level. Some respondents said that they change the course based on a decision by an individual member of faculty who diagnoses students' need and background to change the course content for the first year students. 15 of the 35 responded that their universities try to integrate student, industry, and national needs into first year mathematics courses. The follow-up question 'How has the content of your first year mathematics courses changed in the last 5 years?' showed that 35 had changed their courses in the last 5 years, but 10 of these said that the change was not significant. 17 out of the 35 respondents reported that their departments changed the first year mathematics courses by removing complex topics, or by introducing practical mathematical topics. In some of the courses, students were encouraged to use tools for calculation and visualisation. However, there were also 6 departments that increased the complexity and the rigour of their first year mathematics courses.

The survey considered the notion of proof in several questions. In response to 'How important do you think definitions are in **first** year mathematics?' 52 (65.8%) replied that definitions are important in first year mathematics, while 15 presented their responses as neutral. Only 8 respondents replied that definitions are not important in first year mathematics. Responses to the question 'Do you have a course that explicitly teaches methods of proof construction?' were evenly split with 49.4% answering each of "Yes" and "No". Of those who responded "Yes", 15 (38.4%) replied that they teach methods of proof construction during the first year, 23 (58.9%) during the second year and 5 (12.8%) in either third or fourth year. While some had separate courses (e.g. proof method and logic course) for teaching methods of proofs, many departments teach methods of proofs traditionally, by introducing examples of proof and exercises in mathematics class. Some respondents replied that they teach methods of proof construction in interactive contexts, citing having the course taught as a seminar, with students constructing proofs, presenting them to the class, and discussing/critiquing them in small size class. One respondent used the modified Moore method in interactive lecture. Looking at some specific methods of introducing students to proof construction was the question 'How useful do you think that a course that includes assistance with the following would be for students?' Four possibilities were listed, with mean levels of agreement out of 5 (high) being: Learning how to read a proof, 3.7; Working

on counterexamples, 3.8; Building conjectures, 3.7; Constructing definitions, 3.6. These responses appear to show a good level of agreement with employing the suggested approaches as components of a course on proof construction. It may be that these are ideas that the 49.4% of universities that currently do not have a course explicitly teaching proof construction could consider implementing as a way to assist transition.

Mathematical modelling in universities was another topic our survey addressed. In response to the questions "Does your university have a mathematical course/activity dedicated to mathematical modeling and applications?" and "Are mathematical modelling and applications contents/activities integrated into other mathematical courses?", 44 replied that their departments offer dedicated courses for modelling, while 41 said they integrate teaching of modelling into mathematics courses such as calculus, differential equations, statistics, etc and 7 answered that their university does not offer mathematics courses for mathematical modelling and applications. Among the reasons given for choosing dedicated courses were that: the majority of all mathematics students will end up doing something other than mathematics so applications are far more important to them than are detailed theoretical developments; most of the mathematics teaching is service teaching for non-majoring students so it is appropriate to provide a course of modelling and applications that is relevant to the needs of the target audience; and if modelling is treated as an add-on then students do not learn the methods of mathematical modeling. Those who chose integrated courses did so because: for modeling, students need to be equipped with a wide array of mathematical techniques and solid knowledge base. Hence it is appropriate for earlier level mathematics courses to contain some theory, proofs, concepts and skills, as well as applications.

Considering what happens in upper secondary schools, 26 (33%) reported that secondary schools in their location have mathematical modelling and applications integrated into other mathematical courses, with only 4 having dedicated courses. 44 (56%) said that there were no such modelling courses in their area. When asked for their opinion on how modelling should be taught in schools, most of the answers stated that it should be integrated into other mathematical courses. The main reasons presented for this were: the many facets of mathematics; topics too specialised to form dedicated courses; to allow cross flow of ideas, avoid compartmentalization; and students need to see the connection between theory and practice, build meaning, appropriate knowledge. The question 'What do you see as the key differences between the teaching and learning of modelling and applications in secondary schools and university, if any?' was answered by 33 (42%) of respondents. The key differences pointed out by those answering this question were: at school, modelling is poor, too basic and mechanical, often close implementation of simple statistics tests; students have less understanding of application areas; university students are more independent; they have bigger range of mathematical tools, more techniques; they are concerned with rigour and proof. Asked 'What are the key difficulties for student transition from secondary school to university in the field of mathematical modelling and applications, if any?' the 35 (44%) university respondents cited: lack of knowledge (mathematical theory, others subjects such as physics, chemistry, biology, ecology); difficulties in formulating precise mathematical problems/interpreting word problems/understanding processes, representations, use of parameters; poor mathematical skills, lack of logical thinking; no experience from secondary

schools; and lack of support. One message for transition is to construct more realistic modeling applications for students to study in schools.

In order to investigate how universities respond to assist students with transition problems we enquired "Do you have any academic support structures to assist students in the transition from school to university? (e.g., workshops, bridging courses, mentoring, etc).", and 56 (71%) replied 'Yes' and 22 'No'. Of those saying yes, 34% have a bridging course, 25% some form of tutoring arrangement, while 23% mentioned mentoring, with one describing it as a "Personal academic mentoring program throughout degree for all mathematics students" and another saying "We tried a mentoring system once, but there was almost no uptake by students." Other support structures mentioned included 'study skills courses', 'maths clinics', 'support workshops', 'pre-course', 'remedial mathematics unit', and a 'Mathematics Learning Service (centrally situated), consulting & assignment help room (School of Maths). The MLS has a drop-in help room, and runs a series of seminars on Maths skills. These are also available to students on the web.' Others talked of small group peer study, assisted study sessions, individual consultations, daily help sessions, orientation programmes and remedial courses. There is some evidence that bridging courses can assist in transition (Varsavsky, 2010), by addressing skill deficiencies in basic mathematical topics (Tempelaar, Rienties, Giesbers & Schim van der Loeff, 2012) and building student confidence (Carmichael & Taylor, 2005). Other successful transition courses (e.g., Leviatan, 2008; Oates, Paterson, Reilly & Statham, 2005) have introduced students to the mathematical "culture" and its typical activities (generalizations, deductions, definitions, proofs, etc.), as well as central concepts and tools, or comprise a first year programme of tutor training and collaborative tutorials. While most universities have such courses it appears that establishment of one by those who do not would assist students with transition.

Overall the survey confirmed that students do have some problems in transition and these are sometimes related to a mathematical preparation that could be improved. However, there are also a number of areas that universities could address to assist students, such as adjusting the content of first year courses, and instituting a course on proving and proof (where this doesn't exist) and constructing a bridging course.

### LITERATURE REVIEW

### **Theoretical Perspectives in the transition literature**

A number of different lenses have been used to analyse the mathematical transition from school to university. Some have been summarised well elsewhere (see e.g., Winsløw, 2010) but we preface our discussion with a brief list of the major theoretical perspectives we found in the transition-related literature. One theory that is in common use is the Anthropological Theory of Didactics (ATD) based on the ideas of Chevallard (1985), with its concept of a *praxeology* comprising task, technique, technology, theory. ATD focuses on analysis of the organisation of praxeologies relative to institutions and the diachronic development of didactic systems. A second common perpective is the Theory of Didactical Situations (TDS) of Brousseau (1997), which employs *didactical situations* in which the teacher orchestrates elements of the didactical milieu under the constraints of a dynamic didactical contract. Other

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research uses the action-process-object-schema (APOS) framework of Dubinsky (e.g. Dubinsky & McDonald, 2001) for studying learning. This describes how a process may be constructed from actions by reflective abstraction, and subsequently an object is formed by encapsulation of the process. Other authors find the Three Worlds of Mathematics (TWM) framework of Tall (2008) useful. This describes thinking and learning as taking place in three worlds: the embodied; the symbolic; and the formal. In the embodied world we build mental conceptions using visual and physical attributes of concepts, along with enactive sensual experiences. The symbolic world is where the symbolic representations of concepts are acted upon, or manipulated, and the formal world is where properties of objects are formalized as axioms, and learning comprises the building and proving of theorems by logical deduction from these axioms. We use the acronyms above to refer to each of these frameworks in the text below.

### **Calculus and Analysis**

A number of studies have focused on the problems of transition from calculus to analysis, considering real numbers (Bergé, 2008, 2010; Ghedamsi, 2008; Mamona-Downs, 2010), functions (Dias, Artigue, Jahn & Campos, 2008; Vandebrouk, 2010), limits (Bloch et al. 2006; Bloch & Ghedamsi, 2004, 2010), continuity (Artigue, 2008) and open and closed sets (Bridoux, 2010) and sequences and series (González-Martin, 2009; Gyöngyösi, Solovej & Winsløw, 2011). From these some key areas giving rise to epistemological and mathematical obstacles have been identified.

*Functions:* Students have a limited understanding of the concept of function (Junior, 2006) and need to be able to switch between local and global perspectives (Artigue, 2009; Rogalski, 2008; Vandebrouck, 2010, 2011). Using a TWM lens suggests a need to reconceptualise the concept of function in terms of its multiple registers and process-object duality. The formal axiomatic world of university mathematics requires students to adopt a local perspective on functions, whereas only pointwise (functions considered as a correspondence between two sets of numbers) and global points of view (representations are tables of variation) are constructed at secondary school (Vandebrouck, 2011). An ATD-based study of the transition from concrete to abstract perspectives in real analysis by Winsløw (2008) suggests that in secondary schools the focus is on practical-theoretical blocks of concrete analysis, while at university level the focus is on more complex praxeologies of concrete analysis and on abstract analysis. He considers two kinds of transitions in the student's mathematical activity: from activity mainly centred on practical blocks to that of working with more comprehensive and structured mathematical organisations; and to tasks with theoretical objects. Since the second kind of transition presupposes the first an incomplete achievement of the first transition produces an obstacle for the second one by making the tasks to be worked on inaccessible.

*Limits:* Students need to work with limits, especially of infinite sequences or series. Two obstacles regarding the concept of infinite sum are the intuitive and natural idea that the sum of infinity of terms should also be infinite, and the conception that an infinite process must go through each step, one after the other and without stopping, which leads to the potential infinity concept (González-Martin, 2009; González-Martín, Nardi, & Biza, 2011). According

to Oehrtman (2009), students' reasoning about limit concepts appears to be influenced by metaphorical application of experiential conceptual domains, including collapse, approximation, proximity, infinity as number and physical limitation metaphors. However, only physical limitation metaphors were consistently detrimental to students' understanding. One approach to building thinking about limits, suggested by Mamona-Downs (2010), is the set-oriented characterization of convergence behaviour of sequences of that supports the metaphor of 'arbitrary closeness' to a point. Employing a TDS framework Ghedamsi (2008) developed two situations that allowed students to connect productively the intuitive, perceptual and formal dimensions of the limit concept. Two approximation methods were used as experimental situations: the construction of the better rational approximation of  $\sqrt{2}$  and, if possible, its generalisation to other irrationals; and the cosine fixed point, which gives access to real numbers that we cannot make explicit and consequently requires the implementation of formal procedures.

Institutional factors; An aspect of transition highlighted by the ATD is that praxeologies exist in relation to institutions. Employing the affordances of ATD, Praslon (2000) showed that by the end of high school in France a substantial institutional relationship with the concept of derivative is already established. Hence, for this concept, he claims that the secondary-tertiary transition is not about intuitive and proceptual perspectives moving towards formal perspectives, as TWM might suggest, but is more complex, involving an accumulation of micro-breaches and changes in balance according several dimensions (tool/object dimensions, particular/general objects, autonomy given in the solving process, role of proofs, etc). Building on this work Bloch and Ghedamsi (2004) identified nine factors contributing to a discontinuity between high school and university in analysis and Bosch, Fonseca and Gascón (2004) show the existence of strong discontinuities in the praxeological organization between high school and university, and build specific tools for qualifying and quantifying these. Also employing an institutional approach, Dias, Artigue, Jahn and Campos (2008) conducted a comparative ATD study of the secondary-tertiary transition in Brazil and France, using the concept of function as a filter. They conclude that although contextual influences tend to remain invisible there is a need for those inside a given educational system to become aware of them in order to envisage productive collaborative work and evolution of the system. One crucial aspect of the institution studied by Smida and Ghedamsi (2006) is the teaching practice of the lecturers. They distinguish two kinds of teaching projects leading to two different models of teaching practices: those where axiomatic, structures and formalism are the discourse that justify and generate the expected knowledge and know-how; and projects where the intent is to enrol in a constructivist setting. Further they highlight three groups of lecturers: those with a logico-theoretical profile, who do not take into account cognitive demands; those with a logico-constructivist profile, who have some cognitive concern; and those who take into account cognitive demands.

*Other areas*: One TDS-based research project examined a succession of situations for introducing the notions of interior and closure of a set and open and closed set (Bridoux, 2010), using meta-mathematical discourse and graphical representations to assist students to develop an intuitive insight that allowed the teacher to characterise them in a formal language.

Another examined the notion of completeness (Bergé, 2008, 2010), analysing whether students have an operational or conceptual view, or if it is taken for granted. The conclusion was that many students have a weak understanding that does not include ideas such as:  $\mathbf{R}$  is the set that contains all the suprema of its bounded above subsets; Cauchy sequences come from the necessity of characterizing the kind of sequences that 'must' converge; and completeness is related to the issue whether a limit is guaranteed to lie in  $\mathbf{R}$ .

Some possible ways to assist the calculus-analysis transition have also been considered. For example, Gyöngyösi, Solovej and Winsløw (2011) report an experiment using Maple CAS-based work to ease the transition from calculus to real analysis. Using a combination of theoretical frameworks to study transition they conclude that the use of instruments changes the kinds of mathematics students do, and those with an overall lower performance also commit more errors when using instrumented techniques. A similar use of graphing calculator technology in consideration of the Fundamental Theorem of Calculus by Scucuglia (2006) made it possible for the students to become gradually engaged in deductive mathematical discussions based on results obtained from experiments. In addition, Biehler, Fischer, Hochmuth and Wassong (2011) propose that blending traditional course attendance with systematic e-learning study can facilitate the bridging of school and university mathematics.

## Abstract Algebra

Understanding the constructs, principles, and eventually axioms, of the algebra of generalised arithmetic could be a way to assist students in the transition to study of more general algebraic structures. With a focus on students' work on solving a parametric system of simultaneous equations and the difficulties they experience with working with variables, parameters and unknowns, Stadler (2011) describes students' experience of the transition from school to university mathematics as an often perplexing re-visiting of content and ways of working that seems simultaneously both familiar and novel. Using a perspective that is discursive and enculturative, largely based on Sfard's commognition, the study showed that constructs of number, symbolic literals, operators, the '=' symbol itself, and the formal equivalence relation, as well as the principles of arithmetic, all contribute to building a deep understanding of equation. This agrees with the observations of Godfrey and Thomas (2008), who, using the TWM framework, provided evidence that many students have a surface structure view of equation and fail to integrate the properties of the object with that surface structure. For example, an embodied input-output, procedural, or operational, view of equation persists for approximately 25% of secondary school students, even when they reach university level, and equivalence is not well understood by school students.

Students' encounter with abstract algebra at university marks a significant point in the transition to advanced mathematical formalism and abstraction. Topics such as group theory are characterised by deeper levels of insight and sophistication (Barbeau, 1995) and ask of students a commitment to what is often a fast-paced first encounter in lectures (Clark et al., 1997). As Hazzan (1999) notes, students' difficulty with abstract algebra can be attributed to the novelty of dealing with concepts that are introduced abstractly, defined and presented by their properties, along with an examination of what facts can be determined from these

properties alone. The role of verbalisation in this process, as a semantic mediator between symbolic and visual mathematical expression, may require a level of verbalisation skills that Nardi (2011) notes is often lacking in first year undergraduates.

Studies that focus on the student experience in their first encounters with key concepts in abstract algebra describe a number of difficulties. One identified by Hart (1994) is an over-reliance upon concrete examples of groups leading to a lack of skills in proof production. Another is the prerequisite of students' understanding of the process-object duality of mathematical concepts to understand Group Theory (Asiala et al., 1994), with cosets, normality and isomorphism identified as stumbling blocks in the early stages.

One method employed to assist students with these is to reduce group theory's high levels of abstraction (Hazzan, 1999, 2001), for example, by asking students to construct the operation table for low order groups. This follows the principles of Burn (1996, 1998) who recommended reversing the order of presentation, using examples and applications to stimulate the discovery of definitions and theorems through permutation and symmetry. This idea was implemented by Larsen (2009) in the form of a series of tasks exploring the symmetries of an equilateral triangle, constructing multiplication tables for groups of low order and culminating in negotiating preliminary understandings of group structure, the order of a group and isomorphism.

In an analysis of student responses to introductory Group Theory problem sheets, Nardi (2000) identified students': difficulties with the static and operational duality within the concept of order of an element as well as the semantic abbreviation contained in |q|; often problematic use of 'times' and 'powers of' in association with the group operation; ambivalent use of geometric images as part of meaning bestowing processes with regard to the notion of coset; and problematic conceptualisation of multi-level abstractions embedded in the concept of isomorphism. The duality underlying the concept of group and its binary operation, were also discussed by Iannone and Nardi (2002), who offered evidence of students' tendency to: consider a group as a special kind of set, often ignoring the binary operation that is fundamental to its entity; consider the group axioms as properties of the group elements rather than the binary operation; and omit checking axioms that they perceive as obvious, such as associativity. In addition, research by Ioannou, (see Ioannou & Nardi, 2008, 2009; 2010; Ioannou & Iannone, 2011) considers students' first encounter with abstract algebra, focusing on the Subgroup Test, symmetries of a cube, equivalence relations, and employing the notions of kernel and image in the First Isomorphism Theorem. Provisional conclusions are that students' overall problematic experience of the transition to abstract algebra is characterised by the strong interplay between strictly conceptual matters, affective issues and those germane to the wider study skills and coping strategies that students arrive at university with.

### Linear Algebra

A sizeable amount of research in linear algebra has documented students' transition difficulties, particularly as these relate to students' intuitive or geometric ways of reasoning and the formal mathematics of linear algebra (Dogan-Dunlap, 2010; Gueudet-Chartier, 2004;

Harel, 1990). Related to this work, Hillel (2000) constructed a theoretical framework for understanding student reasoning in linear algebra, identifying geometric, algebraic, and abstract modes of description, with the limited opportunities for generalisation in the first two a potential obstacle for understanding the abstract mode. According to Dorier, Robert, Robinet and Rogalski (2000a) difficulties with formal, abstract aspects of linear algebra arise from a strong emphasis on algebraic concepts in linear algebra that leaves little room for set theory and elementary logic. The approach of Dorier, Robert, Robinet, and Rogalski (2000b) and Rogalski (2000) to dealing with these problems involves teaching linear algebra as a long term strategy, having students revisit problems in a variety of different settings—geometric, algebraic, and formal.

The relationship between linear algebra and geometry were at the core of the ten years of research by Gueudet (Gueudet, 2004, 2008) that identified specific views on students' difficulties, in the secondary-tertiary transition in linear algebra, resulting from different theoretical perspectives. The epistemological view leads to a focus on linear algebra as an axiomatic theory, which is very abstract for the students. Focusing on reasoning modes leads her to identify the need, in linear algebra, for various forms of flexibility, in particular flexibility between dimensions. Also working at the geometry-formalism boundary Portnoy, Grundmeier and Graham (2006) demonstrated that pre-service teachers who had been utilizing transformations as processes that transformed geometric objects into other geometric objects had difficulty writing proofs involving linear transformations. Although understanding of transformation contributed to understanding of the concept in general, they may not have developed the necessary object understanding for writing correct proofs. Britton and Henderson (2009) demonstrated students' difficulty in moving between a formal understanding of subspace and the algebraic mode in which a problem on closure was stated. These authors argue that such difficulties stem from an insufficient understanding of the various symbols used in the questions and in the formal definition of subspace.

Employing a framework using APOS theory in conjunction with TWM, Stewart and Thomas (2007, 2009, 2010, and Thomas & Stewart, 2011) analysed student understanding of various concepts in linear algebra, including linear independence and dependence, eigenvectors, span and basis. The authors found that generally students do not think of these concepts from an embodied standpoint, but instead rely upon a symbolic, process-oriented matrix manipulation way of reasoning. However, employing a course that introduced students to embodied, geometric representations in linear algebra along with the formal and the symbolic appeared to enrich student understanding of the concepts and allowed them to bridge between them more effectively than with just symbolic processes. An emphasis on working with symbolic expressions, losing sight of the mathematical objects that the symbols represent, can give rise to the formalism obstacle (Corriveau, 2009), since a new algebra requires students to accept delegation of parts of the control of validity and meaning to this algebra, leading to a loss of control and meaning.

Employing computers in attempts to address students' difficulties in bridging the many representational forms and the variety of concepts present in linear algebra goes back to the work of Sierpinska, Dreyfus, and Hillel (1999) and more recently Berry, Lapp and Nyman

(2008) and Dogan-Dunlap and Hall (2004) have used this approach. Sierpinska, Dreyfus and Hillel (1999) found that the use of computer environment tasks enabled students to develop a dynamic understanding of transformation, but hindered their ability to understand transformation as relating a general vector to its image under the transformation and that students made determinations about a transformation's linearity based upon a single example. Recently, Meel and Hern (2005) found that using interactive applets helped students recognise misinterpretation or mis-generalization through examination of additional examples.

Another aspect that has been investigated is students' intuitive thinking in linear algebra. Working with Models and Modeling (Lesh & Doerr, 2003) and APOS frameworks Possani, Trigueros, Preciado, and Lozano (2010) leveraged students' intuitive ways of thinking. They did this by utilising a genetic composition of linear independence and dependence and systems of equations in order to aid in the creation of a task sequence that presented students with a problem to mathematise and then use to understand linear independence and dependence. Student use of different modes of representation in making sense of the formal notion of subspace was analysed by Wawro, Sweeney, and Rabin (2011), and their results suggest that in generating explanations for the definition, students rely on their intuitive understandings of subspace, which can be problematic but can also be very powerful in developing a more comprehensive understanding of subspace.

Some research teams have been spearheading innovations in the teaching and learning of linear algebra. Cooley, Martin, Vidakovic, and Loch (2007) developed a linear algebra course that combines the teaching of linear algebra with learning about APOS theory. By focusing on a theory for how mathematical knowledge is generated, students were made aware of their own thought processes and could then enrich their understanding of linear algebra accordingly. In the United States, another group of researchers has used a design research approach (Kelly, Lesh, & Baek, 2008), simultaneously creating instructional sequences and examining students' reasoning about key concepts such as eigenvectors and eigenvalues, linear independence, linear dependence, span, and linear transformation (Henderson, Rasmussen, Zandieh, Wawro, & Sweeney, 2010; Larson, Zandieh, & Rasmussen, 2008; Sweeney, 2011). They argue that knowledge of student thinking prior to formal instruction is essential for developing thoughtful teaching that builds on and extends student thinking. In a study on tasks for developing student reasoning they report (Wawro, Zandieh, Sweeney, Larson, & Rasmussen, 2011) student reasoning during reinvention of the concepts of span and linear independence, guided by an innovative instructional sequence beginning with vector equations rather than systems of equations. This successfully leveraged students' intuitive imagery of vectors as movement to develop formal definitions. This challenges the view that students' intuitive ways of reasoning are an obstacle to induction into formal mathematics.

### **Logic and Proof**

Some well-recognised differences in the transition to university mathematics include the requirement for: abstraction and logical deductive reasoning (Engelbrecht, 2010); rigour (Leviatan, 2008); a context of mathematical theory (De Vleeschouwer, 2010); and proof. A

number of French researchers (e.g. Durand-Guerrier, 2003; Deloustal-Jorrand, 2004; Rogalski & Rogalski, 2004) have pointed out the importance of taking in account quantification matters in order to analyse difficulties related to implication, and more generally mathematical reasoning. An example highlighted by Chellougui (2009) working with new university students involved didactic phenomena related to the alternation of the two types of quantifiers and difficulties in mobilizing the definition of the objects and the structures, illustrating a major problem in the conceptualisation process. Durand-Guerrier and Arsac (2005) highlight the fact that a major challenge for learners is to develop simultaneously mathematical knowledge and logical skills, which are closely intertwined. A semantic and dialogic perspective introduced by Barrier (2009) highlights the importance of moving back and forth between syntax and semantics in the proving process in advanced mathematics (e.g. Blossier, Barrier & Durand-Guerrier, 2009). Such research suggests a need to develop programmes allowing new university students to master the logical competencies required for the learning of advanced mathematics, possibly similar to second language learning (Durand-Guerrier & Njomgang Ngansop, 2011).

A previous ICME survey report on proof (Mariotti et al., 2004) raised a number of questions that relate to transition issues and suggested that a key difference between school and university is that schools focus on argumentation while universities consider deductive proof. However, while many of their ideas have been taken up by researchers there appear to have been few studies directly addressing proof as an issue of transition (at the time of writing the book *Proof and Proving in Mathematics Education: The 19<sup>th</sup> ICMI study*–Hanna & de Villiers, 2012, was still in press). In spite of this the research conducted points out some key differences between approaches to proof in school and in university and makes suggestions for pedagogical approaches that might assist in the transition. These aspects are considered here.

One recommendation for pedagogical change that would have implications for transition is the need for more explicit teaching of proof, both in school and university (Balacheff, 2008; Hanna & de Villiers, 2008; Hemmi, 2008), with some (e.g., Stylianides & Stylianides, 2007; Hanna & Barbeau, 2008) arguing for it to be made a central topic in both institutions. One possible introduction to proof in schools, suggested by Harel (2008) and Palla, Potari and Spyrou (2011) involves proof by mathematical induction. However, this topic should not be considered too quickly. Rather a slower approach, stressing and valuing both ways of understanding and ways of thinking that the DNR framework describes (Harel, 2008), and distinguishing between proof schemes and proofs, is necessary for understanding. According to Solomon (2006), enabling students to access academic proof processes in the transition from pre-university to undergraduate mathematics is a question of understanding and building on students' own pre-existing epistemological resources in order to foster an epistemic fluency that will allow them to recognise, and engage in, the process of creating and validating mathematical knowledge.

A number of potential difficulties in any attempt to place proving and proof more prominently in the transition years have been identified. These include the role of definitions, and the problem of student met-befores (Tall & Mejia-Ramos, 2006). A desire to use definitions as the basis of deductive reasoning in schools is likely to meet serious problems, since, according to Harel (2008), this form of reasoning is generally not available to school students. A study by Hemmi (2008) agrees that students have difficulties understanding the role of definitions in proofs. She advocates a style of teaching that uses the principle of *transparency*, making the difference between empirical evidence and deductive argument visible to students. In addition, the cognitive influence of student met-befores can be strong, with Cartiglia et al. (2004) showing that the most recent met-before for university students, namely a formal approach, had a strong influence on their reasoning. A further difficulty, highlighted by Iannone and Inglis (2011), was a range of weaknesses in beginning university mathematics students' ability to produce a deductive argument, even when they were aware they should do so. Another obstacle could be the form of teaching in schools, where mostly argumentation skills are advanced with little or no deductive reasoning. A potential way forward here, proposed by Inglis, Mejia-Ramos, and Simpson (2007), is the use of the full Toulmin argumentation scheme, including its modal qualifier and rebuttal, since then it would not be necessary for teaching to go straight to the use of formal deductive warrants.

One pedagogical strategy to address the teaching of proving researchers propose is student construction and justification of conjectures. This approach highlights the use of open problems that ask for a conjecture, and appears to be an effective way to introduce the learning of proof. Pedemonte (2007, 2008) also discusses the relationship between argumentation and proof in terms of the construct of *structural distance*, moving from abductive, or plausible, argumentation to a deductive proof, arguing for an abductive step in the structurant argumentation in order to assist transition by decreasing the gap between the arithmetic field in argumentation and the algebraic field in proof. The approach by Kondratieva (2011) uses the idea of an interconnecting problem to get students to construct and justify conjectures. Conjecture production may also have a role during production of indirect proofs (Antonini & Mariotti, 2008), such as by contradiction and contraposition. Using a Cognitive Unity approach (Garuti, Boero, Lemut & Mariotti, 1996), Antonini and Mariotti (2008) show that the production of indirect argumentation can hide some significant cognitive processes that may be activated and then bridged by conjecture production.

Some researchers propose the idea of pivotal, bridging or counter examples could assist students with proof ideas (Stylianides & Stylianides, 2007; Zazkis & Chernoff, 2008). One potential benefit of a counterexample is to produce cognitive conflict in the student, and a pivotal example is designed to create a turning point in the learner's cognitive perception. Counterexamples can also foster deductive reasoning, since we make deductions by building models and looking for counterexamples. For Zazkis and Chernoff (2008) a counterexample is a mathematical concept, while a pivotal example is a pedagogical concept, and pivotal examples should be within, but pushing the boundaries of, the set of examples students have experienced. The role of examples also arose in research by Weber and Mejia-Ramos (2011) on reading proofs, and suggests, based on a consideration of proof reading by mathematicians, that students might be taught how to use examples to increase their conviction in, or understanding of, a proof. In order to know what skills to teach students, Alcock and Inglis (2008, 2009) maintain identifying the different strategies of proof construction among experts, will grow knowledge of what skills to teach students, and how they can be employed. They

suggest there is a need for large-scale studies to investigate undergraduate proof production, and an extension of this to include upper secondary school could be beneficial for transition.

### **Mathematical Modelling and Applications**

Mathematical modelling and applications continues to be a central theme in mathematics education (Blum et al., 2002), with a growing research literature and several international conferences/events dedicated to its teaching and learning. The primary focus of much research is on practice activities. However, it appears that no literature exists explicitly discussing these topics with a focus on the 'transition' from the secondary to the university levels. One reason might be that there have been no roadmaps to sustained implementation of modelling education at all levels. The role of applications and mathematical modelling in everyday teaching practice is somewhat marginal for all levels of education and could be better integrated into all levels of mathematics education.

There is recent literature partially relevant to the secondary-tertiary transition issue and this is briefly considered here. One crucial duality, mentioned by Niss et al. (2007), is the difference between 'applications and modelling for the learning of mathematics' and 'learning mathematics for applications and modelling'. This duality is seldom made explicit in lower secondary school, and instead both orientations are simultaneously insisted on. However, at upper secondary or tertiary level the duality is often a significant one.

The close relationship between modelling and problem solving is taken up by a number of authors. For example, English and Sriraman (2010) suggest that mathematical modelling is a powerful option for advancing the development of problem solving in the curriculum. However, according to Petocz et al. (2007), there are distinct advantages to using real world tasks in problem solving in order to model the way mathematicians work. One difficulty described by Ärlebäck and Frejd (2010) is that upper secondary students have little experience working with real situations and modelling problems, making the incorporation of real problems from industry problematic. One possible solution is closer collaboration, with representatives from industry working directly with classroom teachers. A second possible difficulty (Gainsburg, 2008) is that teachers tend not to make many real-world connections in teaching. One possible solution to this, suggested by the German experience, is to bring together combinations of students, teachers and mathematicians to work on modelling problems (Kaiser & Schwarz, 2006). Occasionally the opportunity is created through a "modelling week" (Göttlich, 2010; Heilio, 2010; Kaland et al., 2010), where small groups of school or tertiary students work intensely for a week, in a supported environment, on selected, authentic modelling problems. Other proposals to assist teachers include a scheme for modelling tasks that provides an overview of their different features, thus offering guidance in task design and selection processes for specific aims and predefined objectives and target groups (Maaß, 2010) and the integration of mathematical modelling into pre-service teacher education (Bracke, 2010).

Some differences between problem solving at school and university are identified by Perrenet and Taconis (2009), who investigated changes in mathematical problem-solving beliefs and behaviour of mathematics students during the years after entering university. Significant

shifts in the growth of attention to metacognitive aspects in problem-solving or the growth of the belief that problem-solving is not only routine but has many productive aspects were explained by the change to authentic mathematics problems at university compared to secondary school mathematics problems.

There is some agreement that there is a need to target curriculum changes in the upper secondary school to include more modelling activities, although recognising that high-stakes assessment at the upper secondary-tertiary interface is an unresolved problem in any implementation (Stillman, 2007). Other possible initiatives for embedding modeling in the curriculum suggested by Stillman and Ng (2010) are: a system-wide focus emphasising an applications and modelling approach to teaching and assessing all mathematics subjects in the last two years of pre-tertiary schooling; and interdisciplinary project work from upper primary through secondary school, with mathematics as the anchor subject. Another initiative presented by Maaß and Mischo (2011) is the framework and methods of the project STRATUM (Strategies for Teaching Understanding in and through Modelling), whose aim is to design and evaluate teaching units for supporting the development of modelling competencies in low-achieving students at the German Hauptschule. Also, in the USA, Leavitt and Ahn (2010) have provided a teacher's guide to implementation strategies for Model Eliciting Activities (MEAs), which are becoming more popular in secondary schools. Another arena that might prove helpful to students making the secondary-tertiary transition in mathematical modelling and applications is entry to contests in mathematical modelling and applications (Xie, 2010).

### Conclusion

The literature review presented here reveals a multi-faceted web of cognitive, curricular and pedagogical issues, some spanning across mathematical topics and some intrinsic to certain topics – and certainly exhibiting variation across the institutional contexts of the many countries our survey focused on. For example, most of the research we reviewed discusses the students' limited cognitive preparedness for the requirements of university-level formal mathematical thinking (whether this concerns the abstraction, for example, within Abstract Algebra courses or the formalism of Analysis). Within other areas, such as discrete mathematics, much of the research we reviewed highlighted that students may arrive at university with little or no awareness of certain mathematical fields.

The literature review presented in this report, or the longer version<sup>1</sup>, is certainly not exhaustive. However we believe it is reasonable to claim that the bulk of research on transition is in a few areas (e.g. calculus, proof) and that there is little research in other areas (e.g. discrete mathematics). While this might simply reflect curricular emphases in the various countries that our survey focused on, it also indicates directions that future research may need to pursue. Furthermore across the preceding sections a pattern seems to emerge with regard to *how*, not merely *what*, students experience in their first encounters with advanced mathematical topics, whether at school or at university. Fundamental to addressing issues of transition seems also to be the coordination and dialogue across educational levels – here mostly secondary and tertiary – and our survey revealed that at the moment this appears largely absent.

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