On the toral rank of simple Lie algebra
over a field of characteristic 2

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1 Abstract
We give a short proof of the following result of S.Skryabin [1]: a finite di-
mensional Lie algebra over a field of characteristic 2 and absolute toral rank
one is solvable.

2 Introduction.
In this note we give a short proof of the following theorem of S.Skryabin [1].

Theorem 1 Finite dimensional simple Lie algebra over a field of character-
istic 2 has absolute toral rank at least 2.

Proof. Let $F$ be an algebraically closed field of characteristic 2, let $\hat{L}$ be
a finite dimensional simple Lie algebra over $F$ of toral rank one, let $L$ be a
2-envelope of $\hat{L}$. Then $L = L_0 \oplus L_1$, where $L_0$ is a Cartan subalgebra of $L$,
$[L_0, L_1] \subseteq L_1$ and $[L_1, L_1] \subseteq L_0$. By the definition of absolute toral rank, $L_0$
contains a unique toroidal element $h = h^{[2]}$. We denote

$$\mathcal{N} = \{a \in L | \exists n : a^{[2^n]} = 0\}, \quad \mathcal{T} = \{t \in L | t^{[2]} = t \neq 0\}.$$ 

We need the following simple result.

Lemma 1 Let $n \in \mathcal{N}$, $t, s \in \mathcal{T}$.
For any $a \in L$, if $a + [a, n] = 0$ then $a = 0$. If $[t, s] = 0$ then $t = s$. 

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Note that $L_1 \subseteq \mathcal{N}$. Indeed, if $a \in L_1$, then $a^{[2]} \in L_0$ and if $a^{[2]}$ is not nil then $a^{[2]} = h + n$, where $n$ is a nil element. In this case $[a, a^{[2]}] = a + [a, n] = 0$, hence $a = 0$ by Lemma 1.

If $H = [L_1, L_1]$ then $\bar{L} = H \oplus L_1$ and $H$ is not a nil-subalgebra. Indeed, if $H$ is a nil subalgebra then by the Jacobson-Engel theorem [2] $\bar{L}$ is nilpotent since all elements of $L_1$ are nil. Then $h \in H$.

Let $N$ be a maximal nil-ideal of $L_0$ and $N_0 = Fh \oplus N$. If $L_0 = N_0$, then for some $a, b \in L_1$ we have $[a, b] = h + n$, $n \in N$, since $h \in [L_1, L_1]$. As $N$ is a nil-ideal of $L_0$ we can suppose that $[[a, N], b], [[b, N], a] \in N$. Hence $[a^{[2]}, b^{[2]}] = [[[a, b], b], a] = h + n + [[b, n], a] = h + m$, $m \in N$.

On the other hand $a^{[2]}, b^{[2]} \in N$ as we proved above. Hence, $[a^{[2]}, b^{[2]}] \in N$, it is proved that $L_0 \neq N_0$. Let $F e$ be an ideal of the factor-algebra $L_0/N_0$ and $e$ be some preimage of $\bar{e}$. It is clear that $e^{[2]} \in N_0$. For some $\gamma \in k$ we have $(e + \gamma h)^{[2]} \in N$. Hence we can suppose that $e^{[2]} \in N$. Denote $L_0' = \{x \in L_0 | x, e \in N\}$. If $L_0' = L_0$ then $F e \oplus N$ is an ideal of $L_0$ with nil-radical $F e \oplus N$, contradicting the maximality of $N$. Hence $L_0 = L_0' \oplus F f$ for some $f \in L_0$. Let $X = \{a^{[2]} | a \in L_1\} \subseteq L_0$. Since $[L_1, L_1] = H$ and $L_0 = H + F \{x^{[2]}, i = 1, 2, ... , x \in H\}$, it follows that the $F$-space $F \{X\}$ generated by $X$ is equal to $L_0$. This means that there exists $c \in L_1$ such that $c^{[2]} \notin L_0'$, hence $[c^{2}, e] = \alpha h + m$, where $\alpha \in F^*, m \in N_0$. Now we can choose $a = c, b = \alpha^{-1}[c, e]$ and in this case we have $[a, b] = \alpha^{-1}[c, [c, e]] = \alpha^{-1}[c^{[2]}, e] = h + n, n \in N$.

Let $a^{[2k+1]} = b^{[2k+1]} = 0$ and define

$$x = h + a + a^{[2]} + ... + a^{[2k]}, \quad y = h + b + b^{[2]} + ... + b^{[2k]}.$$ 

Then $x, y \in T$ and if $z = [x, y]$ then $[x, z] = z, [y, z] = z$. We prove that $z$ is nil. Otherwise, $z^{[2]} = p + n$, where $p \in T$, $[p, n] = 0$ and $n \in \mathcal{N}$. We have $[z^{[2]}, x] = 0 = [p, x] + [n, x]$. Hence $[p, [p, x]] + [p, [n, x]] = [p, x] + [[p, n], x] + [n, [p, x]] = [p, x] + [n, [p, x]]$.

But $n$ is nil, hence $[p, x] = 0$ by Lemma 1. But in this case $z^{[2]} = [x + n, z] = z + [n, z] = 0$, hence $z = 0$, by Lemma 1, since $n$ is nil. Hence $0 = z = [x, y]$ and $x = y$ by Lemma 1, a contradiction.

Then $z^{[2r+1]} = 0$ for some $r$. Denote $u = x + y + z + z^{[2]} + ... + z^{[2r]}$. It is easy to see that $u^{[2]} = u$. We have $[u, x] = [y, x] + z = 0$, hence $u = x$ or $u = 0$, by Lemma 1. If $u \neq 0$, then $u = x$, and we can prove analogously that $u = y = x$, a contradiction. Hence $u = 0$. 

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Let $V = L_1N$ be an $N$-submodule of $L_1$. Then $P = V \oplus L_0$ is a proper 2-subalgebra of $L$. We can suppose that $a \notin V$. Indeed, if $a \in V$ and $c \notin V$, $c \in L_1$, then for $\gamma \in k$ we have $f(\gamma) = [(a + \gamma c)^{[2]}, e] = [a^{[2]}, e] + \gamma[a, c, e] + \gamma^2[a^{[2]}, e]$. If $f(\gamma) \in L_0'$ for all $\gamma \in k^*$ then $[a^{[2]}, e] \in L_0'$, contradiction. Then $f(\gamma) \notin L_0'$ for some $\gamma \in k^*$ and we can choose $a + \gamma c$ instead of $a$. But $a + \gamma c \notin V$.

Let us prove that $u \equiv (a + b)(mod(P))$. We have by definition,

$$z = a + b + [a, b^{[2]}] + [b, a^{[2]}] + \sum_{i>0} ([a, b^{[2i]}] + [b, a^{[2i]}]) + z_0,$$

where $z_0 \in L_0$. But

$$[a, b] = h + n, \quad [a, b^{[2]}] = [h + n, b] = b + [b, n], \quad [b, a^{[2]}] = a + [a, n],$$

$$[a, b^{[4]}] = [[b, n], b^{[2]}] = [b, [n, b^{[2]}]] \in V;$$

by induction, for $i > 1$,

$$[a, b^{[2i]}] = [[[n, b^{(2i-1)}], b^{(2i-1)}] \in V.$$

Hence, by (1), $z \in P$.

We have, $u = a + b + z + ... \equiv (a + b)(mod(P))$. But, if $u = 0$, it means that $a \equiv b(modP)$. By choice of $a, b$ we have that $b = [a, e]$ and $[b, e] = [a, e^{[2]}] \in V$, since $e^{[2]} \in N$. As $[a, e] = b \equiv [b, e] \equiv 0(modP)$ we have $a \equiv 0(modP)$, a contradiction.

\[ \square \]

Note that S.Skryabin [1] obtained a much deeper result. He described all simple finite dimensional Lie algebras over a field $F$ that contain some Cartan subalgebra of toral rank 1.

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References
