Representations of the Lie ring $sl_2(\mathbf{Z})$

June 20, 2002

Alexandr.N.Grishkov

Omsk State University(Russia) and University of Sao Paulo(Brasil)

e-mail: grishkov@ime.usp.br

1 Introduction.

In this work we begin to study the representations of the Lie ring $S = sl_2(\mathbf{Z})$. Denote by $\mathcal{O}$ the category of the torsion free finite-dimensional S-modules. It means that if $V \in \mathcal{O}$, then for $v \in V, n \in \mathbf{Z} : nv = 0 \to n = 0$ or $v = 0$. Moreover, $dimV = dim_{\mathbf{Q}} V \otimes \mathbf{Q} < \infty$. A module $V \in \mathcal{O}$ is called irreducible if the $S \otimes_{\mathbf{Z}} \mathbf{Q}$-module $V \otimes_{\mathbf{Z}} \mathbf{Q}$ is irreducible. Fix a standard basis $\{e, h, f\}$ on S such that $eh = 2e, ef = h, fh = -2f$. For every module $V \in \mathcal{O}$, denote by V_d a submodule $\{v \in V \mid \exists i \in \mathbf{Z}, vh = iv\}$. A module V is said to be *diagonal* if $V = V_d$. A module V is said to be *semisimle* if it is the direct sum of diagonal irreducible submodules. Obviously, every diagonal S-module contains a unique maximal semisimle submodule which we denote by V_s. The following proposition contains a complete description of the irreducible diagonal S-modules.

Proposition 1 ([2]) *Let $\alpha = (\alpha_n, \alpha_{n-2}, \ldots, \alpha_{2-n})$ be a vector from $\mathbf{Z}_+^n$ such that $(i+1)(n-i)/\alpha_{n-2i} = \beta_{n-2i+2} \in \mathbf{Z}$. Denote by $V(\alpha)$ the S-module with the $\mathbf{Z}$-basis $\{v_n, v_{n-2}, \ldots, v_{-n}\}$ such that*

$$v_i h = iv_i, v_i e = \beta_i v_{i+2}, v_i f = \alpha_i v_{i-2}, i = n, n-2, \ldots, -n.$$

Then $V(\alpha)$ is an irreducible diagonal S-module. Every irreducible diagonal S-module has such a form for some α. Moreover, $V(\alpha) \simeq V(\gamma)$ if and only if $\alpha = \gamma$.

A diagonal S-module V is said to be *extremal* if, for every diagonal S-module W such that $V \subseteq W$, we have $V_s \neq W_s$. Denote by $\mathcal{ED}$ the set of the diagonal extremal S-modules. The main problem in the theory of diagonal S-module is to describe the structure of an extremal S-module. Note that $V \oplus W \in \mathcal{ED}$ implies $V \in \mathcal{ED}$ and $W \in \mathcal{ED}$. The converse is not true (if V and W are irreducible then $V, W \in \mathcal{ED}$, but usualy $V \oplus W \notin \mathcal{ED}$). Let us call a S-module V connected if $V \simeq W_s$ for some indecomposible module $W \in \mathcal{ED}$. It is intresting to describe the set of the connected S-modules. Observe that every connected module is semisimple.

Conjecture 1 *For $V, W \in \mathcal{ED}$ we have $V \otimes W \in \mathcal{ED}$.*

2 The tensor product of irredusible diagonal $sl_2(\mathbf{Z})$-modules.

In this section, we describe the structure of semisimple module $(V\otimes W)_s$ and calculate the index $|V\otimes W:(V\otimes W)_s|$ for diagonal irreducible S-modules V and W.

Let $V=V(\alpha)$ and $W=W(\gamma)$ be irreducile diagonal S-modules with bases $\{v_n,v_{n-2},\dots,v_{-n}\}$ and $\{w_m,w_{m-2},\dots,w_{-m}\}$ respectively and with the actions

$$v_ih=iv_i, v_ie=\alpha_iv_{i+2}, v_if=\beta_iv_{i-2}, i=n,n-2,\dots,-n,$$

$$w_ih=iw_i, w_ie=\tau_iw_{i+2}, w_if=\gamma_iw_{i-2}, i=m,m-2,\dots,-m,$$

and

$$\alpha_{n-2i}\beta_{n-2i+2}=-i(n-i+1),\quad \gamma_{m-2i}\tau_{m-2i+2}=-i(m-i+1). \tag{1}$$

We assume that $n\le m$. If $U=V\otimes W$, then $U_s\simeq\sum_{k=0}^{m}\oplus U_k$, where U_k is an irreducible diagonal S-module and $dimU_k=n+m-2k+1$ (see [1]). Let $\{u^k_{n+m-2k},u^k_{n+m-2k-2},\dots,u^k_{-n-m+2k}\}$ denote the standard basis of U_{n+m-2k}. Then $u^k_{n+m-2k}e=0$ and

$$u^k_{n+m-2k}=\sum_{i=0}^{k}x_i^kv_{n-2i}\otimes w_{m-2k+2i},$$

where $x_i^k\in\mathbf{Z}, x_0^k>0$ and $gcd(x_0^k,\dots,x_k^k)=1$. Here $gcd(X), X\subseteq\mathbf{Z}$ means the greatest common divisor of integers in X.

Similarly, $u^k_{2k-m-n}f=0$ and $u^k_{2k-m-n}=\sum_{i=0}^{k}y_i^kv_{-n+2i}\otimes w_{-m-2i+2k}$, where $y_0^k,\dots,y_k^k\in\mathbf{Z}, y_0^k>0$ with $gcd(y_0^k,\dots,y_k^k)=1$. Note that, by the conditions, $x_0^k,\dots,x_k^k$ and $y_0^k,\dots,y_k^k$ are uniquely defined.

For every $u=\sum_{i=n,j=m}^{-n,-m}a_{i,j}v_iw_j\in U$, define $|u|=gcd\{a_{i,j}\mid i=n,\dots,-n, j=m,\dots,-m\}$ and $||u||=u/|u|$.

Denote $a_i^k=u^k_{n+m-2k}f^i$ and $b_i^k=u^k_{2k-m-n}e^i$, $k=0,\dots,m$, $i=0,\dots,n+m-2k$. Clearly $u^k_{n+m-2i}=||a^k_{i-k}||$ and $u^k_{2k-m-n}=||b^k_{i-k}||$.

We use the following notation

$$\begin{aligned}\alpha_i^p&=\textstyle\prod_{j=0}^{p-1}\alpha_{2i+2j-n},\quad \beta_i^p=\prod_{j=0}^{p-1}\beta_{n-2i-2j},\\ \tau_i^p&=\textstyle\prod_{j=0}^{p-1}\alpha_{2i+2j-m},\quad \gamma_i^p=\prod_{j=0}^{p-1}\beta_{m-2i-2j}.\end{aligned} \tag{2}$$

The main result of this section is the following

Theorem 1 *Let V and W be diagonal irreducible S-modules with the bases defined above. Then $(V \otimes W)_s$ has a basis $\{||a_j^i|| \mid i = 0, \dots, m, j = 0, \dots, n+m-2i\}$ and*

$|V \otimes W : (V \otimes W)_s| =$

$(x_0^m)^{n-m+1} \prod_{q=1}^{m-1} (x_0^q)^{n-q+1} \prod_{q=1}^{m-1} (y_0^q)^{m-q} (m!)^{n-m+1} \prod_{q=1}^{m-1} (q!)^{n+m-2q+1}.$

$(\alpha_m^m)^{n-m+1} \prod_{q=1}^{m-1} (\alpha_q^q)^{m-q+1} \prod_{q=1}^{m-1} (\beta_{n-q}^q)^{m-q} \prod_{q=0}^{n-m-2} (\beta_q^{m+1})^{n-m-1-q}.$

$(\prod_{q=1}^{n} \prod_{k=0}^{q} |u_k^q|^{-1})(\prod_{q=0}^{m-1} \prod_{k=0}^{q} |b_k^q|^{-1}).$

$\prod_{p=0}^{2m-2} (n+m-p)^{g(p)},$ (3)

where

$$g(t) = \begin{cases} [(t+2)/2](n+m-3[t/2]-2+(-1)^t), & \text{if } 0 \le t \le m-2, \\ g(2m-4-t)+n-m+1, & \text{if } m-1 \le t \le 2m-4. \end{cases}$$

Here $[x]$ is an integer part of x.

Proof. If $U^p = \{u \in U \mid uh = pu\}$ and $U_s^p = U_s \cap U^p$, then

$$U = \sum_{p=-n-m}^{n+m} \oplus U^p, U_s = \sum_{p=-n-m}^{n+m} \oplus U_s^p.$$

Hence

$$|V \otimes W : (V \otimes W)_s| = \prod_{p=-n-m}^{n+m} |U^p : U_s^p|. \tag{4}$$

Note that $U^p = U_s^p = 0$, if $p \not\equiv n+m \ (mod\ \ 2)$. Fix $q \in \{1, \dots, m\}$. Then the elements $\{a_{q-k}^k, k = 0, \dots, q\}$ form a basis of U_s^q. If $a_{q-k}^k = \sum_{i=0}^q t_k^i v_{n-2i} \otimes w_{m+2i-2q}$ then

$$|U^q : U_s^q| = \triangle_q \prod_{k=0}^{q} |a_{q-k}^k|^{-1}, \tag{5}$$

where $\triangle_q = det||t_k^i||_{i,k=0}^q$. From the definition of t_k^i, we obtain

$$t_k^i = \sum_{j=0}^{i} x_j^k \begin{pmatrix} i-j \\ p-k \end{pmatrix} \beta_j^{i-j} \gamma_{k-j}^{q-k-i+j}. \tag{6}$$

We will prove that

$$\triangle_q = \triangle_{q-1}(\sum_{j=0}^{q}(-1)^j x_j^q \beta_j^{q-j}\gamma_{q-j}^j). \tag{7}$$

Decompose $\triangle_q$ by the last raw which has the form $(x_0^q, \dots, x_q^q)$. Then

$$\triangle_q = \sum_{j=0}^{q}(-1)^j x_j^q \nabla_j \tag{8}$$

Denote by $\bar{\mathbf{w}}_0, \dots, \bar{\mathbf{w}}_{q-1}$ the columns of the determinant $\triangle_{q-1}$ and by $\bar{\mathbf{v}}_0, \dots, \bar{\mathbf{v}}_q$ the columns of $\triangle_q$ without the last line. Then

$$\bar{\mathbf{w}}_i = (\left(\begin{array}{c} i \\ q-1 \end{array}\right)\beta_i\gamma_{q-i-1}, \dots, \sum_{j=0}^{i} x_j^k \left(\begin{array}{c} i-j \\ q-k-1 \end{array}\right)\beta_j^{i-j}\gamma_{k-j}^{q-k-i+j-1}, \dots, x_i^{q-1}).$$

We transform linearly the columns of the determinant ∇_i for a fixed $0 \le i \le q$. Denote by $\nabla_i^{(j)}$ the determinant ∇_i obtained by a substitution of the first $(j+1)$ columns by $\bar{\mathbf{w}}_0, \dots, \bar{\mathbf{w}}_j$. Note that $\bar{\mathbf{v}}_0 = \gamma_{m-2p+2}\bar{\mathbf{w}}_0$. Hence, $\nabla_i = \gamma_{m-2q+2}\nabla_i^{(0)}$. Suppose that, by induction, we have an equality $\nabla_i = \gamma_{q-j}^j \nabla_i^{(j-1)}$ for $0 \le j < i-1$. Then

$$\bar{\mathbf{v}}_j - \beta_{n-2j+2}\bar{\mathbf{w}}_{j-1} = (\dots, \textstyle\sum_{l=0}^{j} x_l^k \left(\begin{array}{c} j-l \\ q-k \end{array}\right)\beta_l^{j-l}\gamma_{k-l}^{q-k-j+l} -$$
$$\beta_{n-2j+2}(\textstyle\sum_{l=0}^{j-1} x_l^k \left(\begin{array}{c} j-l-1 \\ q-k-1 \end{array}\right)\beta_l^{j-l-1}\gamma_{k-l}^{q-k-j+l}), \dots).$$

Since $\beta_{n-2j+2}\beta_l^{j-l-1} = \beta_{n-2l}\dots\beta_{n-2j+4}\beta_{n-2j+2} = \beta_l^{j-l}$, we have

$$\begin{aligned}
&\bar{\mathbf{v}}_j - \beta_{n-2j+2}\bar{\mathbf{w}}_{j-1} = \\
&(\dots, \textstyle\sum_{l=0}^{j-1} x_l^k(\left(\begin{array}{c} j-l \\ q-k \end{array}\right) - \left(\begin{array}{c} j-l-1 \\ q-k-1 \end{array}\right))\beta_l^{j-l}\gamma_{k-l}^{q-k-j+l} + x_j^k\gamma_{k-j}^{q-k}, \dots) = \\
&(\dots, \gamma_{m-2q+2j+2}\textstyle\sum_{l=0}^{j} x_l^k \left(\begin{array}{c} j-l \\ q-k-1 \end{array}\right)\beta_l^{j-l}\gamma_{k-l}^{q-k-j+l-1}, \dots) = \\
&\gamma_{m-2q+2j+2}\bar{\mathbf{w}}_j.
\end{aligned} \tag{9}$$

From (9), it follows that $\nabla_i = \gamma_{p-i}^i \nabla_i^{(i-1)}$. Now denote by $\nabla_i^{[j]}$ the determinant $\nabla_i^{(i-1)}$ obtained by substituting the last j columns by $\bar{\mathbf{w}}_{q-j+1}, \dots, \bar{\mathbf{w}}_q$.

As above, we can see that $\bigtriangledown_i^{(i-1)} = \beta_i^{q-i}\bigtriangledown_i^{[q-i]}$. Since $\bigtriangledown_i^{[q-i]} = \triangle_{q-1}$, we obtain $\bigtriangledown_i = \gamma_{q-i}^i\beta_i^{q-i}\triangle_{q-1}$ and the equality (7) follows from (8). Now we have to calculate

$$S_q = \sum_{j=0}^{q}(-1)^j x_j^q \beta_j^{q-j}\gamma_{q-j}^j.$$

Clearly,

$$u_{n+m-2k}^k e = (\textstyle\sum_{j=o}^k x_j^k v_{n-2j}\otimes w_{m-2k+2j})e =$$

$$\textstyle\sum_{j=o}^k x_j^k(\alpha_{n-2j}v_{n-2j+2}\otimes w_{m-2k+2j} + \tau_{m-2k+2j}v_{n-2j}\otimes w_{m-2k+2j+2} =$$

$$\textstyle\sum_{j=o}^k (x_{j+1}^k\alpha_{n-2j-2} + x_j^k\tau_{m-2k+2j})v_{n-2j}\otimes w_{m-2k+2j+2}.$$

Hence,

$$x_{j+1}^k\alpha_{n-2j-2} + x_j^k\tau_{m-2k+2j} = 0. \tag{10}$$

By (10), we have

$$\begin{aligned} x_{j+1}^k &= -\frac{x_j^p\tau_{m-2k+2j}}{\alpha_{n-2j-2}} = \frac{x_{j-1}^p\tau_{m-2k+2j}\tau_{m-2k+2j-2}}{\alpha_{n-2j-2}\alpha_{n-2j}} = \\ &(-1)^{j+1}\frac{x_0^k\tau_{m-2k+2j}\cdots\tau_{m-2k}}{\alpha_{n-2j-2}\cdots\alpha_{n-2}} = (-1)^{j+1}\frac{x_0^k\tau_{m-k}^{j+1}}{\alpha_{n-j-1}^{j+1}}. \end{aligned} \tag{11}$$

Note that (1) and (2) yield

$$\gamma_{i-j}^j\tau_{m-i}^j = \gamma_{m-2i+2j}\cdots\gamma_{m-2i+2}\tau_{m-2i}\cdots\tau_{m-2i+2j-2} =$$

$$(-1)^j(i-j+1)(m-i+1)\cdot\ldots\cdot(i-j)(m-i+j) =$$

$$(-1)^j(i-j+1)^{[j]}(m-i+1)^{[j]}.$$

Analogously,

$$\alpha_{n-i}^j\beta_{i-j}^j = (-1)^j(i-j+1)^{[j]}(n-i+1)^{[j]}.$$

Therefore,

$$S_q = x_0^q \sum_{j=0}^q \frac{\tau_{m-q}^j \beta_j^{q-j} \gamma_{q-j}^j \alpha_{n-q}^{q-j}}{\alpha_{n-j}^j \alpha_{n-q}^{q-j}} =$$

$$\frac{(-1)^q x_0^q}{\alpha_q^q} \sum_{j=0}^q (q-j+1)^{[j]}(m-q+1)^{[j]}(j+1)^{[q-j]}(n-q+1)^{[q-j]} =$$

$$\frac{(-1)^q x_0^q}{\alpha_q^q} \sum_{j=0}^q \frac{q!}{j!}\frac{q!}{(q-j)!}(m-q+1)^{[j]}(n-q+1)^{[q-j]} =$$

$$\frac{(-1)^q x_0^q q!}{\alpha_q^q} \sum_{j=0}^q \left(\begin{array}{c} q \\ j \end{array} \right) (m-q+1)^{[j]}(n-q+1)^{[q-j]}. \tag{12}$$

Recall the following formula which one can easily prove by induction on q

$$(a+b)^{[q]} = \sum_{j=0}^q \left(\begin{array}{c} q \\ j \end{array} \right) a^{[j]} b^{[q-j]}. \tag{13}$$

From (12) and (13), we conclude

$$S_q = x_0^q q!(n+m-2q+2)^{[q]}/\alpha_q^q.$$

Hence, by (7),

$$\triangle_q = \prod_{p=0}^q x_0^p p!(n+m-2p+2)^{[p]}(\alpha_p^p)^{-1}. \tag{14}$$

Analogously, if $b_{q-k}^k = \sum_{i=0}^q s_k^i v_{2i-n} \otimes w_{2k-2i-m}, k = 0, \ldots q$ and $\triangle^q = det||s_k^i||_{i,k=0}^q$, then

$$s_k^i = \sum_{j=0}^i y_j^k \left(\begin{array}{c} i-j \\ q-k \end{array} \right) \alpha_{i-j}^{i-j} \tau_{k-j}^{q-k-i+j}$$

and

$$|U^{m+n-q} : U_s^{m+n-q}| = \triangle^q \prod_{k=1}^q |b_{q-k}^k|, \tag{15}$$

$0 < q \leq m$.

In the same way as we got the formula (14), we can obtain

$$\triangle^q = \prod_{p=0}^{q-1} y_0^p p!(n+m-2p+2)^{[p]}(\beta_{n-p}^p)^{-1}. \tag{16}$$

Now we have to calculate the indices $|U^p : U^p_s|$ for $m < p < n$. In this case, for a fixed $m < p < n,\ \ p \equiv n+m(mod\ 2)$ we have $a^k_{p-k} = \sum_{i=p-m}^{p} t^i_k v_{n-2i} \otimes w_{m+2i-2p}$ and

$$|U^p : U^p_s| = \triangle_p \prod_{k=0}^{m} |a^k_{p-k}|^{-1}, \tag{17}$$

where $\triangle_p = det||t^i_k||^{p,m}_{i=p-m,k=0}$ Denote by $\mathbf{v}^p_i$ the i-th column of $\triangle_p$. Similarly to (9), we can prove that $\mathbf{v}^{p+1}_i = \beta_{n-2i+2}\mathbf{v}^p_{i-1} + \gamma_{-m-2p+2i}\mathbf{v}^p_i$, so $\triangle_{p+1} = \beta_{n-2p}\beta_{n-2p+2}\dots\beta_{n-2p+2m}\triangle_p = \beta^{m+1}_{p-m}\triangle_p$ and

$$\triangle_p = \beta^{m+1}_{p-m}\beta^{m+1}_{p-m-2}\dots\beta^{m+1}_0\ \triangle_m\ . \tag{18}$$

From (4),(5),(14),(15),(16),(17) and (18), we obtain

$$\begin{aligned}
&|U : U_s| = \textstyle\prod_{q=0}^{n+m} |U^q : U^q_s| = \\
&(\textstyle\prod_{q=1}^{n} \triangle_q \prod_{k=0}^{q} |u^q_k|^{-1})(\prod_{q=0}^{m-1} \triangle^q \prod_{k=0}^{q} |b^q_k|^{-1}) = \\
&(x^m_0)^{n-m+1} \textstyle\prod_{q=1}^{m-1}(x^q_0)^{n-q+1} \prod_{q=1}^{m-1}(y^q_0)^{m-q}(m!)^{n-m+1} \prod_{q=1}^{m-1}(q!)^{n+m-2q+1}. \\
&(\alpha^m_m)^{n-m+1} \textstyle\prod_{q=1}^{m-1}(\alpha^q_q)^{m-q+1} \prod_{q=1}^{m-1}(\beta^q_{n-q})^{m-q} \prod_{q=0}^{n-m-2}(\beta^{m+1}_q)^{n-m-1-q}. \\
&(\textstyle\prod_{q=1}^{n} \prod_{k=0}^{q} |u^q_k|^{-1})(\prod_{q=0}^{m-1} \prod_{k=0}^{q} |b^q_k|^{-1}). \\
&((n-m+2)^{[m]})^{n-m+1} \textstyle\prod_{q=1}^{m-1}((n+m-2q+2)^{[q]})^{n+m-2q+1}.
\end{aligned} \tag{19}$$

Clearly, there exists a function $g(p), p \in \mathbf{Z}$ such that

$$((n-m+2)^{[m]})^{n-m+1} \prod_{q=1}^{m-1} ((n+m-2q+2)^{[q]})^{n+m-2q+1} = \prod_{p=0}^{2m-2} (n+m-p)^{g(p)}.$$

It is not difficult to verify that

$$\begin{aligned}
&g(2t) = (t+1)(n+m-3t-1), && 0 \le 2t < m-1, \\
&g(2t+1) = (t+1)(n+m-3t-3), && 0 \le 2t+1 < m-1, \\
&g(m-2+t) = g(m-2-t), && 0 \le t < m-1.
\end{aligned}$$

From here and (19) we derive the formula (3).

The theorem is proved.

References

[1] N.Jacobson, Lie Algebras, Interscience publishers, New York, 1962.

[2] A.Yuschenko, Representatins of the Lie ring $sl_2(\mathbf{Z})$, Master thesis, Omsk University, 1991.