Deformation of complex spaces: with a view to Kodaira-Spencer and Kuranishi theory

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Abstract

Despite of the wide range of applications of deformation theory, here, we are mainly concerned with the moduli problems, however the author must mention that this theory has applications in sigularity theory, conformal field theory, superstringtheory and complex dynamics.

The problem of moduli dates back to Riemman, when trying to find the number of parameters of the space of conformal equivalence classes of compact one-dimensional complex manifolds of genus g , namely $\mathscr{M}_g$. Riemann proved that $\mathscr{M}_g$ could be discribed by a "continuum" of $3g-3$, 1 and 0 parameters for the cases, respectively, $g \geq 2$, $g = 1$ and $g = 0$. Nowadays, we call these number of parameters the moduli of a give space.

The complex space $\mathscr{M}_g$ was extensively studied in the following years by Riemann, Teichmüller, Rauch, Ahlfors and Bers, culminating in the creation of the theory of quasi-conformal maps and Teichmüller theory. After these developments, Grothendieck extended the Teichmüller theory to a more categorial context (without the using of quasi-conformal maps). However the problem was solved in the geometrical algebraic just in 1969, when Deligne and Mumford proved that the moduli variety $\mathscr{M}_g$ of non-singular complex algebraic curves of genus g is, in fact, a quasi-projective variety of dimension $3g-3$. So, we may ask:"... and what about the higher dimensional case ?" and thiis is the moduli problem.

In the differential complex geometric setting, the problem of moduli consist in finding a "natural" complex analytic structure (maybe with singularities) for the space of isomorphic classes of a given even dimensional compact oriented manifold. More generally, this problem can be extended to the context of schemes and quasi-coherent sheaves with some minor changes.

In this article, however, we focus mainly on the theory developed Kodaira, Spencer and Kuranishi on local deformations (deformation on the germ of a fixed point). The author should mention too that the theory can be developed globally. For the global setting there are three major approachs: Mumford's geometric invariant theory, Artin's algebraic stacks and Griffiths' theory of period matrix domains.

1 Kodaira and Spencer first steps in the theory

Along the article all manifolds and spaces are assumed to be paracompact and connected. Furthermore, unless mentioned, we shall assume that the spaces being deformed are always compact.

First we need to define what's a deformation. We should think of a deformation as a continuous variation of a given compact oriented real even-dimensional manifold as in the case of a bundle. So we should first define the notion of a family of manifolds varying "continuously".

The "continuous" variation of a manifold can be stated formally as simpleness of a map. the first two definitions turns out to be equivalent in the assumptions of a deformation, while the "flatness" is a generalization for complex spaces, allowing the fibers to have singularities. Under the assumption that the fibers are smooth, submersion and "flatness" are assumed by specialists to coincide, though the author does not guarantee this equivalence.

Definition 1. *An analytic proper surjective morphism between complex spaces $f : X \twoheadrightarrow Y$ is called simple if one of the following three holds:*

- *The morphism is a submersion, when the complex spaces are complex manifolds;*
- *The morphism is locally trivial, i.e, there exists a fixed complex space X_0, such that for every point $f(x) \in Y$, there exists an open neighborhood U such that the following diagram commutes in the category of topological spaces (or smooth manifolds, for the smooth case).*

$$\begin{tikzcd} U \times_Y X_0 \arrow[r, hook] \arrow[d, two heads] & X \arrow[d, two heads] \\ U \arrow[r, hook] & Y \end{tikzcd}$$

- *The morphism is flat, i.e., $(f^\sharp)_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat for all points $x \in X$.*

Definition 2. *An analytic family of complex spaces (or analytic deformation of complex spaces) is a proper surjective simple morphism of complex spaces $\varpi : \mathscr{M} \twoheadrightarrow \mathscr{P}$.*

$\mathscr{P}$ is called the parameter space and ,for a fixed point $0 \in \mathscr{P}$, $\mathscr{M} \times_{\mathscr{P}} * = \varpi^{-1}(t) = M_t$ is called a deformation of M_0

$$\begin{tikzcd} \mathscr{M} \times_{\mathscr{P}} * \arrow[r, hook] \arrow[d, two heads] & \mathscr{M} \arrow[d, two heads] \\ * \arrow[r, hook, "0"] & \mathscr{P} \end{tikzcd}$$

Remark 1. *The fibers of a proper morphism are always compact, hence M_t is always a compact manifold. Furthermore, we could have weaken the notion of properness by demanding just the fibers being compact, however these two notions turn out to be equivalent if ϖ is closed and $\mathscr{P}$ is compactly generated Hausdorff.*

Moreover, for categorial aspects, properness is equivalent to universally closed when $\mathscr{M}$ is Hausdorff and $\mathscr{P}$ is locally compact Hausdorff, which is the case in question.

At first a deformation seems to be a kind of isotopy on some ambient space, but this is totally incorrect, since the fibers are always diffeomorphic by the following theorem.

Theorem 1. *(*Ehsresmann's fibration theorem*) A proper surjective submersion between smooth manifolds $f : M \twoheadrightarrow N$ is a locally trivial fibration.*

Proof. (*Sketch*) We have by the inverse function theorem that locally f is a projection. Covering by such charts, we can define a global transversal vector field by patching $(\frac{\partial}{\partial t^k})_i$ the horizontal vector fields on U_i with a partition of unity. Using this approach in $f^{-1}(\overline{U})$ we can cover it by finitely many charts satisfying the previous condition, then projecting the charts and picking the intersection gives the necessary neighborhood that satisfies the locally triviality. □

Remark 2. *In Ehresmann's fibration theorem, the properness of the map is not dispensable in any case. For non-compact fibers a neighborhood resembling a "tube" around a fixed fiber could collapses when going to the top" of the fiber as in a draw of a cone.*

Now that we have defined what's a deformation, it's reasonable to ask:"What's "$\frac{\partial M_t}{\partial t}(t)$"?". The answer is simple and follows from the following observation.

Let $\mathcal{U} = \{U_i\}$ be a covering of $\mathscr{M}$ with coordinates $w_i = (z_i^1, z_i^2, ..., z_i^n, t_1, t_2, ..., t_m) = (z_i(t), t)$ such that $\varpi(w_i) = t$. Then, defining f_{ij} as the change of coordinates between U_j and U_i in the first coordinate vector z_i, we have that in $U_{ij} = U_i \cap U_j$

$$f_{ij}(w_j) = z_i$$

holds, therefore, in U_{ijk},

$$z_i = f_{ij}(f_{jk}(w_k))$$

, then differentiating by t is a good guess to know how the holomorphic structure depends on $t \in \mathscr{P}$

$$\frac{\partial w_i^l}{\partial t^q} = \frac{\partial f_{ik}^l(w_k)}{\partial t^q} = \frac{\partial f_{ij}^l(f_{jk}, t)}{\partial t^q} = \frac{\partial f_{ij}^l}{\partial t^q} + \sum_r \frac{\partial f_{ij}^l}{\partial f_{jk}^r}\frac{\partial f_{jk}^r}{\partial t^q} = \frac{\partial f_{ij}^l}{\partial t^q} + \sum_r \frac{\partial z_i^l}{\partial z_j^r}\frac{\partial f_{jk}^r}{\partial t^q}$$

Writing this as a vector field

$$\sum_l \frac{\partial f^l_{ik}(w_k)}{\partial t^q}\frac{\partial}{\partial z^l_i} = \sum_l \frac{\partial f^l_{ij}}{\partial t^q}\frac{\partial}{\partial z^l_i} + \sum_{r,l} \frac{\partial f^r_{jk}}{\partial t^q}\frac{\partial z^l_i}{\partial z^r_j}\frac{\partial}{\partial z^l_i} = \sum_l \frac{\partial f^l_{ij}}{\partial t^q}\frac{\partial}{\partial z^l_i} + \sum_r \frac{\partial f^r_{jk}}{\partial t^q}\frac{\partial}{\partial z^r_i}$$

Hence $\theta^q_{ik} = \sum_l \frac{\partial f^l_{ik}(w_k)}{\partial t^q}\frac{\partial}{\partial z^l_i}$ would be a good guess. In fact, by the previous equality, it turns out to satisfy the cocyle condition

$$\theta^q_{ik} = \theta^q_{ij} + \theta^q_{jk}$$

, therefore we have the following map.

Definition 3. *$\mathcal{KS}_t : \mathcal{T}_{\mathscr{P},t} \longrightarrow \boldsymbol{H}^1(M_t, \mathcal{T}_{M_t})$, such that it's $\mathbb{C}$-linear and $\mathcal{KS}(\frac{\partial}{\partial t^q}) = [\theta^q_{ij}]$ for all p, is called the Kodaira-Spencer map. We can write it globally in a fancier way: $\mathcal{KS} : \mathcal{T}_{\mathscr{P}} \longrightarrow \boldsymbol{R}^1\varpi_*(\mathcal{T}_{\mathscr{M}/\mathscr{P}})$.*

Remark 3. *Notice that $\mathcal{T}_{\mathscr{M}/\mathscr{P}}$ is well defined by*

$$0 \to \mathcal{T}_{\mathscr{M}/\mathscr{P}} \to \mathcal{T}_{\mathscr{M}} \to \varpi^*\mathcal{T}_{\mathscr{P}} \to 0$$

since every vector field on $\mathscr{M}$ can be projected to $\mathscr{P}$ by ϖ_*.

Moreover, the Kodaira-Spencer map, turns out to be so natural that it arises in a long exact sequence of sheaves.

Proposition 1. *The Kodaira-Spencer map is equal to the map induced by the long exact sequence of sheaves*

$$0 \to \varpi_*\mathcal{T}_{\mathscr{M}/\mathscr{P}} \to \varpi_*\mathcal{T}_{\mathscr{M}} \to \varpi_*\varpi^*\mathcal{T}_{\mathscr{P}} \cong \mathcal{T}_{\mathscr{P}} \longrightarrow \boldsymbol{R}^1\varpi_*(\mathcal{T}_{\mathscr{M}/\mathscr{P}})$$

Proof. (*Sketch*) Firstly, it's known that $\mathbf{R}^1\varpi_*(\mathcal{T}_{\mathscr{M}/\mathscr{P}})$ is canonically isomorphic to $\varpi_*\mathbf{H}^1(\mathscr{M}, \mathcal{T}_{\mathscr{M}/\mathscr{P}})$ (because both are universal cohomological functors), therefore it's enough to compute the long exact sequence of

$$0 \to \mathcal{T}_{\mathscr{M}/\mathscr{P}} \to \mathcal{T}_{\mathscr{M}} \to \varpi^*\mathcal{T}_{\mathscr{P}} \to 0$$

induced by the usual Čech cohomology. But the map

$$\mathbf{H}^0(\mathscr{M}, \varpi^*\mathcal{T}_{\mathscr{P}}) \to \mathbf{H}^1(\mathscr{M}, \mathcal{T}_{\mathscr{M}/\mathscr{P}})$$

is given by $(f(\frac{\partial}{\partial t^k}))_{ij} = \delta(((\xi_l)_l))_{ij} = ((\xi_i - (\frac{\partial}{\partial t^k})_i) - (\xi_j - (\frac{\partial}{\partial t^k})_j))_{ij} = ((\frac{\partial}{\partial t^k})_j - (\frac{\partial}{\partial t^k})_i)) = (\sum_l (\frac{\partial}{\partial t^k})_j f^l_{ij}(w_j)\frac{\partial}{\partial z^l_i})_{ij} = \theta^k_{ij}$, such that $d\varpi(\xi) = \frac{\partial}{\partial t^k}$. □

Remark 4. *acyclic The proposition above shows that using the fact that, for a locally finite covering $\mathcal{U}$, there is a "canonical" morphism $\mathbf{H}^1(\mathcal{U}, \mathcal{T}_M) \to \mathbf{H}^{(0,1)}_{\overline{\partial}}(M)$ (see [**Manetti**] at page 8), it's possible to describe the Kodaira-Spencer map as $\mathcal{KS}(\gamma) = [-\overline{\partial}\xi]$, such that $d\varpi(\xi) = \gamma$. Furthermore, the later can be seen as an obstruction to the lifting of vector fields in a holomorphic way.*

2 In a fancier categorial setting

In the categorial setting, the moduli problem can be stated more naturally by defining the moduli functor.

An will be the category of complex spaces.

Definition 4. *We define a presheaf*

$$F : \mathbf{An}^{op} \longrightarrow \mathbf{Set}$$

, such that $F(S)$ is the set of equivalence classes of analytic families parametrized by $S \in \mathbf{An}$.

Following the Grothendieck's approach to the functor of points, a nice question to do in this case would be to ask if the moduli functor F is representable by some object $\mathscr{M}$. But what is exactly the meaning of this?

If $F \cong \mathrm{Hom}(-, \mathscr{M})$ for some $\mathscr{M} \in \mathbf{An}$, then there exists a natural transformation (in this case, an isomorphism)

$$\alpha : F \Rightarrow \mathrm{Hom}(-, \mathscr{M})$$

Therefore the morphism $\alpha(S) : F(S) \to \mathrm{Hom}(S, \mathscr{M})$ sends each equivalence class $[f : X \twoheadrightarrow S]$ to a morphism $\varphi : S \to \mathscr{M}$. Hence, for the case $S = *$, we have $F(*) \cong \mathrm{Hom}(*, \mathscr{M}) \cong \mathbf{Forget}(\mathscr{M})$, which means that complex spaces are in bijective corresponce to points in $\mathscr{M}$. Then finding such $\mathscr{M}$ is equivalent to finding a complex analytic structure to the space of all complex spaces.

So, for instance, if $\mathscr{M} = \mathscr{M}_g$ is the space of all non-singular complex algebraic curves of a given genus g, then α could be defined by

$$\alpha(S)([f : X \twoheadrightarrow Y])(s) = X_s$$

.

The above observations motivates the following definition

Definition 5. *The presheaf F is called a fine moduli functor if is representable, i.e, $F \cong \mathrm{Hom}(-, \mathscr{M}) = h_{\mathscr{M}}$ for some $\mathscr{M} \in \mathbf{An}$. In this case, $\mathscr{M}$ is called a fine moduli space.*

The previous definition seems a good one. However, it turns out that almost all moduli problems cannot have a fine moduli space. The problem is the non-trivial automorphism of the fibers, so if, for instance, a complex manifold has any biholomorphism which is not the identity, the fine moduli space of the complex structures over it will not exist.

We can illustrate the above obstruction by the following example. Let's find for instance the fine moduli space $\mathscr{V}$ of all real vector spaces of dimension 1. We know that $\mathscr{V} = *$, such that the point corresponds to $\mathbb{R}$. But what are the deformation of a real vector space of dimension 1? The answer is simple: it's a line bundle.

Until now, things are going well, but since $\mathbb{R}$ has non-trivial automorphisms, for instance multiplication by -1, we can form the Möbius band (real tautological line bundle) γ_1. Therefore $\alpha(\mathbb{S}^1)(\gamma_1) \in \mathrm{Hom}(\gamma_1, *) = *$, then $\alpha(\mathbb{S}^1)(\gamma_1) = \alpha(\mathbb{S}^1)(\mathbb{R} \times \mathbb{S}^1)$. But $\alpha(\mathbb{S}^1)$ is a bijection, so the Möbius band would be parallelizable, an absurd.

So, even for the most stupid moduli problem, the fine moduli space does not exist. This motivates the next definition.

Definition 6. *The presheaf F is called a coarse moduli functor ids there exists an object $\mathscr{M} \in \mathbf{An}$ and a natural transformation $\alpha \Rightarrow h_{\mathscr{M}}$, such that*

- *α is the reflector of F in the of representable presheaves;*
- *$\alpha(*)$ is a bijection for any $* \in \mathbf{An}$ with a point as the underlying topological space.*

In this case, $\mathscr{M}$ is called a coarse moduli space.

Coarse moduli space turns out to exits in most of the cases, but it still not the better definition. Nowadays, the notion of moduli stack is the one used to describe a moduli problem. Roughly explaining, a moduli stack is a presheaf with values in grupoids, such that it's a sheaf that "behaves" as a "space".

Now, let's interpret the Kodaira-Spencer map in categorial terms. For this, we need to talk about infinitesimal deformation, then our category will not be $\mathbf{An}$, but instead we will need to use germs of complex spaces.

Definition 7. *Let $*/\mathbf{An}$ be the co-slice category, where $*$ is a point and a initial object (for instance* $\mathrm{Specan}(\mathbb{C})$*). We define the category $\mathbf{Germs}(\mathbf{An})$ of germs of complex spaces to the "localization" of the category $*/\mathbf{An}$, where morphisms in $*/\mathbf{An}$ are considered equivalent when they coincide in a neighborhood of $*$.*

Remark 5. *One should $f : * \to X \in \mathbf{Germs}(\mathbf{An})$ as a skyscraper sheaf with underlying topological space X and a sheaf $\mathcal{O}_{X,x}$ above the point $f(*) = x$. Hence, we will denote a germ by (X, x). Furthermore, the category $\mathbf{Germs}(\mathbf{An})$ is clearly equivalent to the category of $\mathbb{C}$-analytic algebras.*

We are only interested in deformations of a fixed initial complex space, so it's reasonable to change to moduli functor F and consider similar functor $A_{(X,x)} : \mathbf{Germs}(\mathbf{An}) \to \mathbf{Grpd}$, such that $A(S, s)$ is the category of deformations

$$(X, x) \hookrightarrow (Y, y) \twoheadrightarrow (S, s)$$

such that the special fiber Y_s is isomorphic to (X, x).

Remark 6. *The functor $A_{(X,x)}$, indeed, has its values in* **Grpd***, since any morphism between deformation will be finite, therefore using the caracterization of flatness by changing the base, it's possible to define an isomorphism of the special fibers. This says that special fibers being isomorphic is equivalent to a morphism between deformations been an isomorphism. But since the special fiber dos not change in our case, we have that the morphisms are alwys isomorphisms.*

Now, let $T^1(X, x) = [A_{(X,x)}(\mathrm{Specan}(\mathbb{C}[t]/(t^2)))]$, where the bracket "$[,]$" means the set of equivalence classes. Then we define the Kodaira-Spencer map in a more general form.

Definition 8. *Let $\varpi : (Y, y) \twoheadrightarrow (S, s)$ be a deformation of germs. The Kodaira-Spencer map*

$$\mathcal{KS} : \mathcal{T}_{(S,s)} \cong \mathrm{Hom}((S, s), \mathrm{Specan}(\mathbb{C}[t]/(t^2))) \longrightarrow T^1(X, x)$$

is defined by $\mathcal{KS}(\phi) = [\phi^*\varpi]$

Actually, these two definitions coincides in the smooth case and that the above map is the same thing as $v \mapsto v\varphi(t)$, where $\varphi(t)$ is an invariant of the deformation as we shall see later. By the previous interpretation of the Kodaira-spencer map, we can see that deformations over $\mathrm{Specan}(\mathbb{C}[t]/(t^2))$ corresponds to first order deformations.

Analogously, we can define higher order deformations.

Definition 9. *An n-th order deformation is any analytic family over* $\mathrm{Specan}(\mathbb{C}[t]/(t^{n+1}))$

The n-th order deformation will correspond to $[\varphi_n] \in \mathbf{H}^n(M, \mathcal{T}_M)$, where $\varphi(t)$ is a $(0, 1)$- form that we will see later.

3 Local deformation in the setting of Kodaira-Spencer and Kuranishi

In the theory of Kodaira-Spencer and Kuranishi, deformation is taken in a sufficiently small polydisk of $\mathbb{C}^m$ around the origin $D = D_\varepsilon$. Now, let $\mathcal{U}_0 = \{U_i\}$ be a covering of $M = M_0$. If ε is sufficiently small, we can treat $U_i \times D$ as a covering of $\mathscr{M}$, such that $(\zeta(z_i, t), t)$ are coordinate charts in $U_i \times \{t\}$. Now, we want to know how is the complex structure in M_t. There are two approaches to this problem: by the almost complex structure or by the complex structure itself. We shall begin with the second one.

For $t = 0$, we know that f being holomorphic on U_i is equivalent to

$$\frac{\partial f}{\partial \overline{z_i}^k} = 0$$

for all k. But what's the condition for non-zero t in terms of $\frac{\partial}{\partial \overline{z_i}^k}$?

Usually, in the literature ([**Kod**], [**UenoShim**], for instance) it's assumed that if "t is close enough to 0" then

$$\frac{\partial \zeta^j(z_i,t)}{\partial \overline{z_i}^k} = \sum_l \varphi(t)^l_{ij} \frac{\partial}{\partial z^l_i}$$

holds. Therefore, we will assume this fact and explain, why it holds in the case of Riemann surfaces.

In general, we can treat $\varphi(t) = \sum_{k,l} \varphi^i_{kl}(t) \frac{\partial}{\partial z^k_i} \otimes dz^l_i$ as a section of $\mathscr{A}^{(0,1)}(\mathcal{T}_M)$. For the case of Riemann surfaces, instead of φ, we use the notation μ and call this section of $\mathcal{T}_M \otimes \overline{\mathcal{K}}_M$ a Beltrami differential if $||\mu||_\infty \leq 1$ holds. Given a 2-dimensional smooth manifolds with local coordinates (x, y) and metric ds^2, finding a complex structure is the same to finding isothermal coordinates (u, v), such that $ds^2 = \rho(du^2+dv^2)$. But this problem reduce to

$$ds^2 = \rho|dw^2|^2 = \rho|\frac{\partial}{\partial z}|^2|dz + \frac{\frac{\partial w}{\partial \overline{z}}}{\frac{\partial w}{\partial z}} d\overline{z}|^2$$

, where $z = x + iy$ and $w = u + iv$. Therefore finding a complex structure is equivalent to find a diffeomorphic solution to the Beltrami differential equation

$$\frac{\partial w}{\partial \overline{z}} = \mu \frac{\partial w}{\partial z}$$

, which had be proven to be solvable for the case of Beltrami differentials ($||\mu||_\infty \leq 1$). This explains the assumption for the case $n = 1$.

Now let's answer when a function is holomorphic in $U_i \times \{t\}$.

Proposition 2. *A smooth function on M_t is holomorphic on an open set iff*

$$(\overline{\partial} - \varphi(t))f = 0$$

holds.

Proof. f being holomorphic on $U_i \times \{t\}$ is equivalent to

$$\frac{\partial f}{\partial \overline{\zeta_i(z_i,t)}^k} = 0$$

for all k. But $(\overline{\partial} - \varphi(t))f = 0$ is equivalent to

$$\frac{\partial f}{\partial \overline{z_i}^k} - \sum_l \varphi^i_{kl}(t) \frac{\partial f}{\partial z^l_i} = 0$$

for all k. Now, expanding in terms of ζ^k_i and using the caracterization

$$\frac{\partial \zeta^j(z_i,t)}{\partial \overline{z_i}^k} = \sum_l \varphi(t)^l_{ij} \frac{\partial}{\partial z^l_i}$$

we arrive to the equation

$$\sum_s \frac{\partial f}{\partial \overline{\zeta_i}^s}(\frac{\partial \overline{\zeta_i}^s}{\partial \overline{z_i}^l} - \sum_k \varphi^i_{lk}(t)\frac{\partial \overline{\zeta_i}^s}{\partial z_i^k}) = 0$$

for all l. But the second term is equals to a multiplication by an invertible matriz for t close enough to 0. □

Remark 7. *Notice that, by the previous proposition, $\varphi(t)$ determines entirely the complex structure of M_t, since it determines its sheaf of holomorphic functions.*

We should mention to that $\varphi(t)$ satisfies the Maurer-Cartan equation up to a normalization. We will see later that this equation is equivalent to the integrability of the almost complex structure given by $\varphi(t)$. So, in our case, we are just saying that a complex structure is integrable.

Proposition 3. *$\varphi(t)$ satisfies the Maurer-Cartan equation*

$$\overline{\partial}\varphi(t) - \frac{1}{2}[\varphi(t), \varphi(t))]$$

We now follow the first approach, we view the complex structure throught the tangent bundle point of view: dealing with the almost complex structure $J(t)$ of M_t. We know that a complex structure on M corresponds to a splitting $T_{\mathbb{C}} = T_t^{(1,0)} \oplus T_t^{(0,1)}$. So it's natural to measure the "error" of the deviation of this splitting for small t. More specifically, we want to know how big is $T_t^{(0,1)} \cap T_0^{(1,0)}$.

In the literature ([**Huy**], for instance), the following approach is done: if t is closer enough to 0, the restriction of canonical projection $pr^{(0,1)}|_{T_t^{(0,1)}}$: $T_t^{(0,1)} \to T^{(0,1)}$ defines an isomorphism, so we can define

$$\varphi(t) = pr^{(1,0)} \circ (pr^{(0,1)}|_{T_t^{(0,1)}})^{-1} : T^{(0,1)} \longrightarrow T^{(1,0)}$$

and it satisfies $(1+\varphi(t))(v) \in T_t^{(0,1)}$ for all $v \in T^{(0,1)}$ according to [**Huy**]. It'easy to check that $(1+\varphi(t))(v) \in T_t^{(0,1)}$ for all v is equivalent to defining $\varphi(t)$ as $\varphi(t) = pr^{(1,0)} \circ (pr^{(0,1)}|_{T_t^{(0,1)}})^{-1} : T^{(0,1)} \longrightarrow T^{(1,0)}$.

The map φ can be expanded in power series around 0

$$\varphi(t) = \sum_i t^i \varphi_i$$

, such that $\varphi_0 = 0$ (because $J(0) = J$). The author must mention, too, that φ_1 is the Kodaira-Spencer map. Now we must know when we can integrate the almost complex structure to a well defined complex structure. This is given by the following theorem.

Theorem 2. *Given a smooth manifold M with an almost complex structure J, then J is integrable iff $[T^{(0,1)}M, T^{(0,1)}M] \subset T^{(0,1)}$.*

However, it turns out that the integrability condition is equivalent to the Maurer-Cartan equation.

Proposition 4. *The integrability of $J(t)$ is equivalent to the equation*

$$\overline{\partial}\varphi(t) + [\varphi(t), \varphi(t)] = 0$$

This motivates our following definition

Definition 10. *An infinitesimal deformation of M_0 is a section $\varphi(t)$ of $\mathscr{A}^{(0,1)}(\mathcal{T}_M)$ that satisfies the Maurer-Cartan equation. Namely,*

$$\varphi(t) = \sum_{k,l} \varphi^i_{kl}(t) \frac{\partial}{\partial \overline{z_i}^k} \otimes dz^l_i$$

on U_i.

A natural question that can arise is when a given element of the cohomology class $[\varphi_1] \in$ can be integrated to a $\varphi(t)$. By the power series expansion around 0, the integrability conditions turns into the equation

$$\overline{\partial}\varphi_j = -\sum_{0<i<j} [\varphi_i, \varphi_{j-i}]$$

, such that $\overline{\partial}\varphi_1 = 0$. Therefore a natural obstruction to the existence of the second order deformation is to the form $[\varphi_1, \varphi_1]$ be $\overline{\partial}$-closed. Therefore we have the following corollary.

Corrolary 1. *A cohomology class $\xi \in \mathbf{H}^1(M, \mathcal{T}_M)$, such that $[\xi, \xi] \in \mathbf{H}^1(M, \mathcal{T}_M)$ does not vanish, cannot be extended to a $\varphi(t)$.*

The author must mention that each φ_i correspond to a deformation of i-th order of M. With this in mind, we have to following generalization of the above corollary.

Theorem 3. *A first order deformation $\varpi : \mathscr{M} \twoheadrightarrow \mathrm{Spec}(\mathbb{C}[t]/(t^2))$ can be extended to a third order deformation iff $(KS)(\frac{\partial}{\partial t}) = \eta$ satisfies $[\eta, \eta] = 0$.*

Actually, these two approachs of defining $\varphi(t)$ can be unified by considering a one parameter family of diffeomorphisms of M and expanding the new coordinates in power series around 0. We will not follow this approach, but, for the interested one, we refer to [**Huy**].

4 The Kuranishi family

We will now define the important notion of a Kuranishi family. A Kuranishi family corresponds in some sense to a neighborhood of the moduli space of complex structures of a given manifold M.

Definition 11. *An analytic family $\varpi : \mathscr{M} \twoheadrightarrow \mathscr{P}$ is complete at the point $0 \in \mathscr{P}$ if any other family $\pi : \mathscr{V} \twoheadrightarrow \mathscr{W}$ with $V_s \cong M_0$ satisfies, for a neighborhood U of s, the following diagram*

$$\begin{array}{ccc} \mathscr{M} \times_{\mathscr{P}} U & \xhookrightarrow{f} & \mathscr{M} \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ U & \xhookrightarrow{g} & \mathscr{P} \end{array}$$

for some g, such that $g(s) = 0$ and $f(V_s) = M_0$. If $(dg)_s$ in unique for all g satisfying the above property, then the family is called semi-universal or versal at the point 0. Moreover, if g is unique then the family is called universal.

Remark 8. *The universality condition is equivalent to the representability of the functor $A_{(M_0,0)}$ is equivalent to a "local" existence of a fine moduli space in some sense.*

The following difficult theorem is very useful for the characterization of universal families.

Theorem 4. *If a family $\varpi : \mathscr{M} \twoheadrightarrow \mathscr{P}$ is complete at every point around a neighborhood of 0, versal at 0 and $\mathbf{H}^0(M, \mathcal{T}_M) = 0$, then the family is universal at 0.*

Remark 9. *Even if $\mathbf{H}^0(M, \mathcal{T}_M) \neq 0$, the family may have a universal family. For instance, the family of n-dimensional complex tori is universal at every point.*

It's possible to give characterizations of versality and completness in terms of the Kodaira-Spencer map.

Theorem 5. *If $\mathcal{KS}_t$ is surjective, then the family is complete at t. Furthermore, a family is versal at 0 iff it's complete in a neighborhood of 0 and $\mathcal{KS}_0$ is an isomorphism.*

Definition 12. *A Kuranishi family $\varpi : \mathscr{M} \twoheadrightarrow \mathscr{P}$ is a complete family at every point of $\mathscr{P}$ and versal at 0. In this case, $\mathscr{P}$ is called the Kuranishi space.*

Now we give shall "construct" locally the coarse moduli space of the complex structures on the manifold M. However, before this, we need the following theorem.

Theorem 6. *If $\varpi : \mathscr{M} \twoheadrightarrow \mathscr{P}$ is a Kuranishi family, then a finite subgroup $G \subset \mathrm{Aut}_{an}(M)$ acts on a neighborhood U of 0, such that 0 is the unique fixed point of the action.*

Proof. (*Sketch*) Extending $f_0 \in G$ to the diagram in a neighborhood U of 0

$$\begin{tikzcd} \mathscr{M} \times_{\mathscr{P}} U \arrow[r, hook, "f"] \arrow[d, two heads] & \mathscr{M} \arrow[d, two heads] \\ U \arrow[r, hook, "g"] & \mathscr{P} \end{tikzcd}$$

, we've got that $g(0) = 0$, so 0 is a fixed point. Furthermore, by the universality, g is uniquely determined at 0. Now, let $V = \bigcap_{g \in G} g(U)$. Then the family $\varpi^{-1}(V) \twoheadrightarrow V$ satisfies the condition. $\square$

By the previous theorem, if $G = \mathrm{Aut}_{an}(M)$ is finite, then in a small neighborhood U of 0, we can assume that G acts on U, therefore U/G corresponds to differento complex structures close to the one of M. So, in some sense, U/G a neighborhood of the coarse moduli space of M. If furthermore, $\mathbf{H}^0(M, \mathcal{T}_M) = 0$ holds, then by the upper-semicontinuity of the cohomology groups, we have in a neighborhood of 0 that $\mathbf{H}^0(M_t, \mathcal{T}_{M_t}) = 0$. In general, if a deformation is a Kuranishi family on 0, then it will be a Kuranishi family for every point t close to 0. So, assuming these two supositions, we would have a universal family in a neighborhood of 0, therefore U/G would correspond to a neighborhood of the parameter space of the fine moduli space.

To finish this article we state the two main theorem, for which the proof requires convergence of the power series defined by $\varphi(t)$ and solves the obstruction for creating $\varphi(t)$ a given. But, before this, we must defined the so called number of moduli.

Definition 13. *A given compact complex manifold $M = M_0$ has a number of moduli if it has a deformation $\varpi : \mathscr{M} \twoheadrightarrow D$ which is complete and effectivelly parametrized, i.e, the Kodaira-Spencer map does not vanish in any non-zero vector at 0 . In this case, $m(M) = \dim_{\mathbb{C}} D$ is called the number of moduli.*

This motivates the following conjecture

Conjecture 1. *If the moduli number is defined, then $m(M) = \dim_{\mathbb{C}} \mathbf{H}^1(M, \mathcal{T}_M)$.*

In order to solve the conjecture, theorems of existence regarding the number of moduli have been created. We now give the first weaker version of the existence theorem.

Theorem 7. *(*Theorem of Existence*) A compact complex manifold M with vanishing $\mathbf{H}^2(M, \mathcal{T}_M)$ has deformation such that $\mathcal{KS}_0$ is an isomorphism.*

Now, we state the following stronger theorem proved by Kuranishi.

Theorem 8. *For every compact complex manifold M there exists a complete deformation at* 0*.In this case, the parameter space can have singularities.*

In the end, the conjectured turned to be false. Mumford found a counter example in 1962 ,using a 3-dimensional complex manifold. Later , in 1967, Kas found a complex elliptic suface that violates the conjecture.

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