On the Kodaira Vanishing Theorem

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Abstract

We review the proof and certain consequences of the vanishing theorem due to Kodaira.

1 Introduction

When considering classical geometrical objects such as connections or (almost) complex structures, one realizes that studying forms with coefficients is a natural thing to do. Indeed, as we saw, a connection on a vector bundle E can be seen as a 1-form with coefficients in $End(E)$. On the other hand, a complex structure can be seen as a form with coefficients in the tangent bundle. Studying the behaviour of the cohomology defined for this forms is not only a natural following step, but also a fruitful one. As a matter of fact using certain cohomology with coefficients we approached Calabi's conjecture, and during the latter lectures, we approached the Embedding theorem also due to Kodaira as well. Further, when studying deformations of complex structures the one parameter deformations are controlled by the so-called Kodaira-Spencer classes which are first cohomology classes with coefficients in the holomorphic tangent bundle (see [3], Proposition 6.1.5).
It is in this context that Kodaira's vanishing theorem appears. This theorem states that under reasonable assumptions certain higher cohomology groups vanish, providing us with certain control over the cohomology with coefficients. In its current version it says that the inverse of an ample invertible sheaf on a projective complex manifold X over an algebraically closed field has no cohomology below the dimension of X. Nonetheless, we are to follow an approach which is closer to the original article of 1953 in which we prove the same result but in rather differential geometric terms; namely, instead of a general projective manifold, we consider a compact Kähler manifold, and instead of an ample invertible sheaf, we consider a *positive* line bundle.

We are to follow the notation used throughout the course, namely the one in [4]. Recall then that given a complex n-dimensional manifold X, we write

$$\Lambda^k_{\mathbb{C}} X := \Lambda^k (T_{\mathbb{C}} X)^* = \bigoplus_{p+q=k} \Lambda^{p,q} X$$

for the k-th exterior power of the complexified cotangent bundle, its respective decomposition $\Lambda^{p,q} X := \Lambda^p (T^{1,0} X)^* \otimes \Lambda^q (T^{0,1} X)^*$, and the sheaf of sections

$\mathcal{A}^k_{\mathbb{C},X}$ or complex k-forms.
On the other hand, given a holomorphic vector bundle $E \longrightarrow X$ we write

$$\Lambda^k_{\mathbb{C}} E := \Lambda^k_{\mathbb{C}} X \otimes_{\mathbb{C}} E = \bigoplus_{p+q=k} \Lambda^{p,q} E$$

for the k-th exterior power of the complexified cotangent bundle with coefficients in E and its respective decomposition $\Lambda^{p,q}E := \Lambda^{p,q}X \otimes_{\mathbb{C}} E$.
Finally, we write

$$\mathcal{A}^{p,q}(E) := \Gamma(-, \Lambda^{p,q}E)$$

for the sheaf of sections, whose sections are (p,q)-forms on X with coefficients in E.
Notice that since for a holomorphic vector bundle the sheaf of sections is monoidal

$$\mathcal{A}^{p,q}(E)(U) = \mathcal{A}^{p,q}_X(U) \otimes_{\mathcal{O}_X(U)} \mathcal{A}^0(E)(U),$$

that is, a section σ can be written as a $\mathcal{O}_X(U)$-linear combination of elements of the form $\alpha_i \otimes s_i$, where α_i is a (p,q)-form over U and s_i is a section of E over U.

This description allows us to introduce the $\bar{\partial}_E$ operator. Given by

$$\bar{\partial}_E := \bar{\partial} \otimes id_E : \mathcal{A}^{p,q}(E)(X) \longrightarrow \mathcal{A}^{p,q+1}(E)(X) : \Sigma\alpha_i \otimes s_i \longmapsto \Sigma\bar{\partial}(\alpha_i) \otimes s_i.$$

on E-valued (p,q)-forms.

We are to conclude this section with a rather central object to this document, namely the E-valued cohomology, also refered to as cohomology with coefficients. Fix $p \geq 0$, then the $\bar{\partial}_E$ operator yields a complex of sheaves

$$\mathcal{A}^{p,0}(E) \longrightarrow \mathcal{A}^{p,1}(E) \longrightarrow \mathcal{A}^{p,2}(E) \longrightarrow \mathcal{A}^{p,3}(E) \longrightarrow \dots \qquad (1.1)$$

As a consequence of the $\bar{\partial}$ lemma, this sequence is exact in sheaves and induces cohomology groups $H^{p,q}(X,E)$ defined as the cohomology of this complex.

2 Hodge Theory for E-valued forms

Recall that whenever E is endowed with a hermitian structure h, the hermitian metric defines a $\mathbb{C}$-antilinear isomorphism of the underlying real bundles E and its dual complex E^*. Further, if X is endowed with a hermitian structure g as well, there is an induced hermitian structure on $\Lambda^{p,q}E$ denoted by that allows us to define

$$\bar{*}_E : \mathcal{A}^{p,q}(E)(X) \longrightarrow \mathcal{A}^{n-p,n-q}(E^*)(X) : \alpha \otimes s \longmapsto *(\bar{\alpha}) \otimes h(s),$$

where $*$ is the usual Hodge star operator. This star operator satisfies an involution type equality, namely $\bar{*}_E \bar{*}_E = (-1)^{p+q}$.
As the techniques used in this section are going to be exactly the ones used to

prove the results for both real (deRham) and complex (Dolbeult) Hodge analogues, Serre duality will still hold using the following pairing as isomorphism, and with the obvious adjustment:

$$(,) : H^{p,q}(X,E) \times H^{n-p,n-q}(X,E^*) \longrightarrow \mathbb{C} : (\sigma,\sigma') \longmapsto \int_X \sigma \wedge \sigma',$$

where the wedge product is defined by $(\alpha \otimes s) \wedge (\beta \otimes \xi) := (\alpha \wedge \beta) \otimes \xi(s)$. This hints the reason because of which the star operator is defined to have values in the space of forms with coefficients in the dual of E. Just as before, this pairing is non-degenerate, for $(\sigma, \bar{*}_E \sigma) \neq 0$.
Analogous to the real Hodge theory, this star operator is going to define an inner product on the ring of E-valued forms, where different degree forms are orthogonal by definition, and

$$\langle \sigma_1, \sigma_2 \rangle = \int_X \sigma_1 \wedge \bar{*}_E \sigma_2,$$

whenever σ_1 and σ_2 are of the same degree. This inner product helps us making sense of dual operators which turn out to be given (in general) by the following equation

$$\bar{\partial}_E^* = -\bar{*}_{E^*} \bar{\partial}_E \bar{*}_E,$$

and in so, of the laplacian as well

$$\Delta_{\bar{\partial}_E} := \bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E.$$

Note that, just as in the real and Dolbeault cases, the laplacian commutes with the star, that is

$$\bar{*}_E \Delta_{\bar{\partial}_E} = \Delta_{\bar{\partial}_E} \bar{*}_E.$$

This, in order, makes sense of the harmonic E-valued forms $\mathcal{H}^{p,q}(X,E)$ which is the kernel of the E-laplacian operator.

Theorem 2.1. *Let X be a compact, complex manifold endowed with a hermitian metric, and E be a hermitian, holomorphic vector bundle over X. Then $\mathcal{H}^{p,q}(X,E)$ is finite dimensional and*

$$\mathcal{A}^{p,q}(E)(X) = \mathcal{H}^{p,q}(X,E) \oplus \bar{\partial}_E^* \mathcal{A}^{p,q+1}(E)(X) \oplus \bar{\partial}_E \mathcal{A}^{p,q-1}(E)(X)$$

To close this section we provide an E-valued version of Hodge theorem, which follows as a corollary of the latter decomposition

Corollary 2.2.

$$\mathcal{H}^{p,q}(X,E) \cong H^{p,q}(X,E)$$

Proof. Using the Hodge decomposition we see that there is a morphism from the space of harmonic forms to the cohomology, namely the restriction of the natural projection from the space of closed forms. Indeed, notice that any harmonic form is $\bar{\partial}_E$-closed. Now, suppose we are given a non-zero element in the second term of our decomposition $\bar{\partial}_E^* \alpha$, then $\bar{\partial}_E \bar{\partial}_E^* \alpha \neq 0$, for $\langle \bar{\partial}_E \bar{\partial}_E^* \alpha, \alpha \rangle = \langle \bar{\partial}_E^* \alpha, \bar{\partial}_E^* \alpha \rangle \neq 0$; therefore, the space of closed forms is $\mathcal{H}^{p,q}(X,E) \oplus \bar{\partial}_E \mathcal{A}^{p,q-1}(E)(X)$. Finally, the projection is surjective, and clearly the kernel is by definition $\bar{\partial}_E \mathcal{A}^{p,q-1}(E)(X)$, so the result follows.

□

Moral: Every cohomology class has got a unique harmonic representative; therefore, show that any computation holds for harmonic forms suffices to conclude the same result in cohomology.

3 Kähler identities for E-valued forms

Throughout this section X denotes a compact connected Kähler manifold with fundamental form $\omega \in \mathcal{A}^{1,1}_X(X)$. We define a generlized Lefchetz operator for E-valued forms

$$L_E : \mathcal{A}^{p,q}(E)(X) \longrightarrow \mathcal{A}^{p+1,q+1}(E)(X) : \Sigma\alpha_i \otimes s_i \longmapsto \Sigma\omega \wedge \alpha_i \otimes s_i.$$

A straightforward computation shows that $[L^*_E, L_E] = n - (p+q)$.

In order to deduce the version of the Kähler identities that we are to use, we need to introduce a connection on E, that is a $\mathbb{C}$-linear morphism of sheaves

$$\nabla : \mathcal{A}^0(E) \longrightarrow \mathcal{A}^1_{\mathbb{C}}(E) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E)$$

which satisfies Leibnitz identity $\nabla(f \cdot s) = df \otimes s + f \cdot \nabla s$. Now, due to the splitting in the codomain of the connection, it clearly decomposes as well as $\nabla = \nabla^{1,0} + \nabla^{0,1}$.

Definition 3.1. *A connection ∇ on a holomorphic vector bundle E is said to be compatible with the holomorphic structure whenever $\nabla^{0,1} = \bar{\partial}_E$.*

We proved proposition 5.25 in [4], which asserts the existence of the Chern connection. That is a connection that is compatible with the holomorphic structure of a holomorphic vector bundle endowed with a hermitian structure.

The coefficient version of the Kähler identity $[L^*, \bar{\partial}] = -i\partial^*$ is also known as the Nakano identity, and it reads as follows

Lemma 3.2. *Let ∇ be the Chern connection on E. Then*

$$[L^*_E, \bar{\partial}_E] = -i(\nabla^{1,0})^*$$

NOTE: For this compactness is not required.

4 The Vanishing Theorem

Theorem 4.1. *Let E is a positive line bundle over a compact Kähler manifold X, then*

$$H^q(X, \mathcal{A}^p_X \otimes E) = 0 \qquad for \quad p+q > n.$$

Definition 4.2. *A line bundle E is called positive if its first Chern class $c_1(E) \in H^2(X, \mathbb{R})$ can be represented by a closed positive real $(1,1)$-form.*

Notice that a compact complex manifold that admits a positive line bundle L is automatically Kähler, for then $c_1(L)$ is itself the Kähler form. Further, recall that we saw that every closed real $(1,1)$-form representing the first Chern class is the curvature of a Chern connection, so we can restate the definition as follows

Definition 4.3. *A line bundle E over X is called positive if there exist hermitian structures h_E, h_X on E and X respectively, such that*

$$\omega = \frac{i}{2\pi} F_\nabla,$$

where ω is the fundamental form associated to h_X, and ∇ is the Chern connection associated to h_E.

Remark: The various notions of positivity of vector bundles found throughout the literature usually coincide for line bundles. Also this condition is stated in terms of the 'ampleness' of the sheaf of sections of the bundle when treated in more algebraic geometric contexts.
We are to give a simple proof of the Vanishing theorem as a consequence of the following lemma (apparently also due to Nakano)

Lemma 4.4. *For $\xi \in \mathcal{H}^{p,q}(X,E)$,*

- $\frac{i}{2\pi} \langle F_\nabla \wedge L_E^*(\xi), \xi \rangle \leq 0$;
- $\frac{i}{2\pi} \langle L_E^*(F_\nabla \wedge \xi), \xi \rangle \geq 0$.

Proof. First, notice that since $\xi \in \mathcal{H}^{p,q}(X,E)$, $\Delta_{\bar\partial_E}(\xi) = 0$, and consequently $\bar\partial_E(\xi) = \bar\partial_E^*(\xi) = 0$. On the other hand, $F_\nabla = d^\nabla \circ \nabla = \nabla^{1,0} \circ \bar\partial_E + \bar\partial_E \circ \nabla^{1,0}$; thus, $F_\nabla(\xi) = \bar\partial_E \nabla^{1,0}(\xi)$. This yields

$$\begin{aligned} i\|(\nabla^{1,0})^*(\xi)\|^2 &= i\left\langle (\nabla^{1,0})^*(\xi), (\nabla^{1,0})^*(\xi) \right\rangle \\ &= \left\langle -[L_E^*, \bar\partial_E]\xi, (\nabla^{1,0})^*(\xi) \right\rangle \\ &= \left\langle \bar\partial_E L_E^*(\xi), (\nabla^{1,0})^*(\xi) \right\rangle \\ &= \left\langle L_E^*(\xi), \bar\partial_E^*(\nabla^{1,0})^*(\xi) \right\rangle \\ &= \left\langle L_E^*(\xi), (\bar\partial_E \circ \nabla^{1,0})^*(\xi) \right\rangle \\ &= \langle F_\nabla \wedge L_E^*(\xi), \xi \rangle . \end{aligned}$$

Then, multiplying by $\frac{i}{2\pi}$, one gets

$$\frac{i}{2\pi} \langle F_\nabla \wedge L_E^*(\xi), \xi \rangle = \frac{-1}{2\pi} \|(\nabla^{1,0})^*(\xi)\|^2 \leq 0$$

as desired. Similarly,

$$\begin{aligned} i\|\nabla^{1,0}(\xi)\|^2 &= i\left\langle \nabla^{1,0}(\xi), \nabla^{1,0}(\xi) \right\rangle \\ &= i\left\langle (\nabla^{1,0})^* \nabla^{1,0}(\xi), \xi \right\rangle \\ &= \left\langle -[L_E^*, \bar\partial_E] \nabla^{1,0}(\xi), \xi \right\rangle \\ &= \left\langle -L_E^*(\bar\partial_E \nabla^{1,0}(\xi)), \xi \right\rangle + \left\langle \bar\partial_E L_E^* \nabla^{1,0}(\xi), \xi \right\rangle \\ &= \langle -L_E^*(F_\nabla(\xi)), \xi \rangle + \left\langle L_E^* \nabla^{1,0}(\xi), \bar\partial_E^* \xi \right\rangle \\ &= -\langle L_E^*(F_\nabla \wedge \xi), \xi \rangle . \end{aligned}$$

This time, multiplying by $\frac{-i}{2\pi}$, one gets

$$\frac{i}{2\pi}\langle L_E^*(F_\nabla \wedge \xi), \xi\rangle = \frac{1}{2\pi}\|\nabla^{1,0}(\xi)\|^2 \geq 0$$

□

We are ready to prove the vanishing theorem

Proof. First, as indicated by an above mentioned remark, our generalized version of the Hodge theorem implies that it suffices to prove that the space of harmonic forms is zero, that is $\mathcal{H}^{p,q}(X,E) = 0$. Then, let $\xi \in \mathcal{H}^{p,q}(X,E)$. Now, since E is positive, there exists a hermitian structure h_E on E such that for the Chern connection $\nabla = \nabla^{1,0} + \bar{\partial}_E$, the fundamental form is given by $\omega = \frac{i}{2\pi}F_\nabla$. Substracting the two inequalities in the previous lemma, we get

$$\begin{aligned}\frac{i}{2\pi}(\langle L_E^*(F_\nabla \wedge \xi), \xi\rangle - \langle F_\nabla \wedge L_E^*(\xi), \xi\rangle) &= \langle L_E^*(\omega \wedge \xi) - \omega \wedge L_E^*(\xi), \xi\rangle \\ &= \langle L_E^*(L_E(\xi)) - L_E(L_E^*(\xi)), \xi\rangle \\ &= \langle [L_E^*, L_E](\xi), \xi\rangle \quad \geq 0.\end{aligned}$$

By the generalized Kähler identities $[L_E^*, L_E](\xi) = (n - (p+q))(\xi)$; therefore, for $p + q > n$, $\|\xi\| = 0$ and the result follows.

□

To close this section we list without prove some consequences that can be proved easily using the Vanishing theorem. Needless to say, there are numerous consequences that were already mentioned such as Kodaira embedding theorem, and its own consequences such as the fact that any line bundle is associated to a divisor.

- Cohomology of projective spaces with coefficients in line bundles: Since $\mathbb{P}^n$ is a Kähler manifold and the curvature of $\mathcal{O}(1)$ is a multiple of the Fubini-Study form, the vanishing theorem will allow us to compute $H^q(\mathbb{P}^n, \mathcal{O}(m))$ for $m > 0$. Indeed, we know that the canonical bundle is given by $\mathcal{O}(-n-1)$; therefore, for $p = n$ we conclude that

  $$H^q(X, \mathcal{O}(m)) = 0 \qquad for \quad q > 0 \quad and \quad m > -n-1$$

  In particular, using Serre duality the cohomology vanishes for $0 < q < n$.

- Weak Lefschetz or the cohomology of hyperspaces is determined by the whole space: To be precise, the canonical restriction $H^k(X, \mathbb{C}) \longrightarrow H^k(Y, \mathbb{C})$ is a bijection for X Kähler, Y a hyperspace with $\mathcal{O}(Y)$ positive, and $k \leq n-2$.

- Grothendieck's lemma or the only interesting bundles on the Riemann sphere are in Pic: Although this uses the so-called Serre's vanishing theorem, this can be proven mimicking the proof above. The lemma says that any vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles $\mathcal{O}(a_i)$, and that the a_i's are determined uniquely up to permutation.

References

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