Lecture Notes on Compact Lie Groups and
Their Representations

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Preliminary version: use with extreme caution!

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Contents

1 Compact topological groups 1
   1.1 Topological groups and continuous actions ............. 1
   1.2 Representations ........................................ 4
   1.3 Adjoint action ........................................... 7
   1.4 Averaging method and Haar integral on compact groups .. 9
   1.5 The character theory of Frobenius-Schur .................. 13
   1.6 Problems .................................................. 20

2 Review of Lie groups 21
   2.1 Basic definition .......................................... 21
   2.2 Lie algebras .............................................. 22
   2.3 The exponential map ..................................... 25
   2.4 Lie subgroups and homomorphisms ....................... 26
   2.5 The adjoint representation ................................ 29
   2.6 Covering Lie groups ...................................... 31
   2.7 Problems .................................................. 31
Compact topological groups

In this introductory chapter, we essentially introduce our very basic objects of study, as well as some fundamental examples. We also establish some preliminary results that do not depend on the smooth structure, using as little as possible machinery. The idea is to paint a picture and plant the seeds for the later development of the heavier theory.

1.1 Topological groups and continuous actions

A topological group is a group $G$ endowed with a topology such that the group operations are continuous; namely, we require that the multiplication map and the inversion map

$$
\mu : G \times G \to G, \quad \iota : G \to G
$$

be continuous maps.

A continuous action of a topological group $G$ on a topological space $X$ is a continuous map

$$
\Phi : G \times X \to X
$$

such that

$$
\Phi(1, x) = 1, \\
\Phi(g_2, \Phi(g_1, x)) = \Phi(g_2g_1, x),
$$

where $g_1, g_2 \in G, x \in X$ and $1$ denotes the identity element of $G$. We shall write the above relations when there is no ambiguity as

$$
1 \cdot x = x \quad \text{and} \quad g_2 \cdot (g_1 \cdot x) = (g_2g_1)x.
$$

Let $\Phi$ be a continuous action of $G$ on $X$ and set $\varphi_g(x) = \Phi(g, x)$. Then $\varphi_g : X \to X$ is a homeomorphism with inverse $\varphi_g^{-1}$ and $g \mapsto \varphi_g$ defines a homomorphism $\varphi$ from $G$ into the group $\text{Homeo}(X)$ of homeomorphisms
of $X$. If, instead, if we fix $x \in X$ and let $g$ vary in $G$, we obtain the orbit of the action through $x \in X$, namely,

$$G(x) = \{ g \cdot x \mid g \in G \}.$$  

The “orbit map” $g \mapsto g \cdot x$, $G \to X$ has image $G(x)$ and sets up a bijection

(1.1.1) \hspace{1cm} G/G_x \cong G(x),

where

$$G_x = \{ g \in G \mid g \cdot x = x \}$$

is the isotropy group or stabilizer of $x$. Note that $G_{g \cdot x} = gG_xg^{-1}$ for all $g \in G$. It is not difficult to see that (1.1.1) is a homeomorphism, where $G/G_x$ carries the quotient topology.

If $\Psi : G \times Y \to Y$ is another continuous action, a continuous map $f : X \to Y$ is called equivariant or a $G$-map if

$$f(\Phi(g, x)) = \Psi(g, f(x))$$

for all $g \in G$, $x \in X$, or, equivalently, $f(g \cdot x) = g \cdot f(x)$. An equivariant map clearly maps orbits to orbits.

A continuous action $\varphi : G \to \text{Homeo}(X)$ is called:

- **linear** if $X$ is a vector space $V$ and $\varphi_g$ is a linear map for all $g \in G$;
- **orthogonal** if $X$ is an Euclidean space and $\varphi_g$ is an orthogonal transformation for all $g \in G$;
- **unitary** if $X$ is a complex vector space equipped with an Hermitian inner product and $\varphi_g$ is an unitary transformation for all $g \in G$;
- **isometric** if $X$ is a Riemannian manifold and $\varphi_g$ is an isometry for all $g \in G$.

A linear action is also called a representation. We talk of a real representation in case $V$ is a real vector space and of a complex representation in case $V$ is a complex vector space.

**1.1.2 Examples**

(i) The real line $\mathbb{R}$ with its additive group structure is an example of commutative or Abelian topological group. Similarly, $\mathbb{R}^n$ and more generally any vector space $V$ together with addition of vectors can be considered as an Abelian topological group.

(ii) The complex numbers $\mathbb{C}$ with its multiplicative structure is an Abelian topological group. The (closed) subset of $\mathbb{C}$ consisting of unit complex numbers, which is homeomorphic to the circle $S^1$, is closed under multiplication and hence inherits the structure of (compact) Abelian topological group. More generally, a finite product

$$T^n = S^1 \times \cdots \times S^1 \quad (n \text{ factors}),$$

called a torus, is a compact Abelian topological group.
1.1. TOPOLOGICAL GROUPS AND CONTINUOUS ACTIONS

(iii) Any group can be considered as a topological group in a trivial way just by endowing it with the discrete topology. An important class of such examples consist of the finite groups. Indeed, many results about compact (Lie) groups that we will study in this book can be considered as generalizations of easier constructions for finite groups.

(iv) A fundamental class of examples of mostly non-Abelian topological groups consist of the classical matrix groups; they are:

\[
\begin{align*}
GL(n, \mathbb{R}) &= \{ A \in M(n, \mathbb{R}) \mid \det(A) \neq 0 \} \quad \text{(real general linear group)}, \\
GL(n, \mathbb{C}) &= \{ A \in M(n, \mathbb{C}) \mid \det(A) \neq 0 \} \quad \text{(complex general linear group)}, \\
GL(n, \mathbb{H}) &= \{ A \in M(n, \mathbb{H}) \mid A \text{ is invertible} \} \quad \text{(quaternionic general linear group)}, \\
O(n) &= \{ A \in GL(n, \mathbb{R}) \mid AA^t = I \} \quad \text{(orthogonal group)}, \\
U(n) &= \{ A \in GL(n, \mathbb{C}) \mid AA^* = I \} \quad \text{(unitary group)}, \\
Sp(n) &= \{ A \in GL(n, \mathbb{H}) \mid AA^* = I \} \quad \text{(symplectic group)}, \\
SL(n, \mathbb{R}) &= \{ A \in GL(n, \mathbb{R}) \mid \det(A) = 1 \} \quad \text{(real special linear group)}, \\
SL(n, \mathbb{C}) &= \{ A \in GL(n, \mathbb{C}) \mid \det(A) = 1 \} \quad \text{(complex special linear group)}, \\
SO(n) &= \{ A \in O(n) \mid \det(A) = 1 \} \quad \text{(special orthogonal group)}, \\
SU(n) &= \{ A \in U(n) \mid \det(A) = 1 \} \quad \text{(special unitary group)}.
\end{align*}
\]

Here \( M(n, \mathbb{F}) \) denotes the real vector space of \( n \times n \) matrices with entries in \( \mathbb{F} \), where \( \mathbb{F} = \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \) (the reader which is unfamiliar with the quaternion algebra can now safely ignore such examples; we will come back to quaternions below); also, \( A^t \) denotes the transpose matrix of \( A \) and \( A^* \) denotes its conjugate transpose. It is easy to check that these are groups with respect to matrix multiplication. Each group \( G \) is considered with the topology induced from its inclusion in the corresponding vector space \( M(n, \mathbb{F}) \). Plainly, multiplication of matrices

\[
M(n, \mathbb{F}) \times M(n, \mathbb{F}) \to M(n, \mathbb{F})
\]

is a bilinear map and thus continuous; it follows that the induced multiplication on \( G \) is also continuous. The inversion map is only defined on \( GL(n, \mathbb{F}) \), and its continuity follows from the explicit formula for the inverse of a matrix given in terms of the determinant and the cofactors. Since each \( G \) is a topological subspace of an appropriate \( GL(n, \mathbb{F}) \), this proves that \( G \) is a topological group. We want to point out that the compact groups in the list are precisely \( O(n), U(n), Sp(n), SO(n) \) and \( SU(n) \). For instance, viewing \( O(n) \) as the subset of \( M(n, \mathbb{R}) = \mathbb{R}^{n^2} \) consisting of matrices whose columns form an orthonormal basis of \( \mathbb{R}^n \) yields that \( O(n) \) is a closed and bounded (with respect to the Euclidean norm) and hence compact. The other groups are similarly shown to be compact.

(v) There are also five isolated examples

\[
E_6, E_7, E_8, F_4, G_2
\]
of compact simple groups that are called exceptional, in the sense that they do not fit into classical families like in (iv) and involve somehow exotic algebraic structures. We will say more about them in later chapters in the context of Lie theory.

Compact topological groups form a broad class that includes examples such as infinite products of circles (with the product topology) and $p$-adic integers (as inverse limits of finite groups). Beginning in the next chapter, we shall restrict our discussion to compact Lie groups, namely, those endowed with a compatible manifold structure.

1.2 Representations

Representation theory is one of the main topics in these lecture notes. If $\varphi : G \to GL(V)$ is a representation of a topological group, we will also say that $V$ is a linear $G$-space.

Two representations $\varphi : G \to GL(V)$ and $\psi : G \to GL(W)$ of the same group $G$ are considered equivalent if there exists an equivariant linear isomorphism $A : V \to W$, that is $A$ is an isomorphism and

$$A \circ \varphi(g) = \psi(g) \circ A$$

for all $g \in G$.

Fix a representation $\varphi : G \to GL(V)$. A subspace $U$ of $V$ is called invariant if

$$\varphi(g)U \subset U$$

for all $g \in G$. In this case $\varphi$ restricts to a representation of $G$ on $U$, which is called a subrepresentation (or a component) of $\varphi$. It is clear that $\{0\}$ and $V$ are always invariant subspaces of $\varphi$. The representation $\varphi$ is called irreducible if these are the only invariant subspaces that it admits. Finally, $\varphi$ is called completely reducible if $V$ can be written as a non-trivial direct sum of invariant, irreducible subspaces.

1.2.1 Examples  
(i) Let $G$ be $SO(n)$ and $V = \mathbb{R}^n$. Then left-multiplication

$$\varphi(g) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = g \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

defines a real representation. For obvious reasons, this representation is called natural or canonical. Another name for it is vector representation. This set of names also apply to the next example.
(ii) Let $G = SU(n)$ and $V = \mathbb{C}^n$. Then left-multiplication

$$\varphi(g) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = g \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

defines a complex representation. By ignoring the complex structure on $V$ and viewing $\mathbb{C}^n$ as a $2n$-dimensional real vector space, we get a real representation. This process is called realification.

(iii) Let $G = SO(n)$ and let $V$ consist of real symmetric $n \times n$ matrices. Then conjugation

$$\varphi(g)X = gXg^{-1} = gXg^t$$

for $g \in G$ and $X \in V$ defines a representation. This representation is not irreducible, as the subspace of scalar matrices is invariant; further, the subspace of traceless matrices $V_0$ is an invariant complement. It is not hard to see that $V = R \cdot I \oplus V_0$ is a decomposition into irreducible components.

(iii) Let $G$ be the additive group of the real numbers $(\mathbb{R}, +)$. Then

$$\varphi : G \to GL(\mathbb{R}^2), \quad \varphi(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

is a representation which is not completely reducible, as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ spans an invariant subspace which does not admit an invariant complement.

(iv) Let $G = SO(n)$ and let $V_d$ be the space of complex homogeneous polynomials of degree $d$ on $x_1, \ldots, x_n$. Then

$$\varphi_d(g)p\left( \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right) = p(g^{-1}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix})$$

defines a representation of $SO(n)$ on $V_d$. Consider the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

acting on $V_d$. Then $\Delta(V_d) \subset V_{d-2}$. Recall that $\Delta$ is the divergent of the gradient, so it is intrinsically associated to the metric structure of $\mathbb{R}^n$. Since $SO(n)$ acts on $\mathbb{R}^n$ by isometries, it commutes with $\Delta$: $\Delta \pi(g) = \pi(g)\Delta$ for $g \in SO(n)$. It follows that $\mathcal{H}_d := \ker \Delta \subset V_d$ and $\text{im}\Delta \subset V_{d-2}$ are invariant subspaces; the elements of $\mathcal{H}_d$ are called harmonic polynomials of degree $d$. 

Construction of representations

Let $V$ and $W$ be linear $G$-spaces. Then the following spaces carry naturally induced structures of linear $G$-spaces ($g \in G$):

- **a.** direct sum $V \oplus W$: $g \cdot (x, y) = (g \cdot x, g \cdot y)$ ($x \in V, y \in W$);
- **b.** dual space $V^*$: $(g \cdot \lambda)(x) = \lambda(g^{-1} \cdot x)$ ($\lambda \in V^*, x \in V$);
- **c.** tensor product $V \otimes W$: $g \cdot (x \otimes y) = g \cdot x \otimes g \cdot y$ ($x \in V, y \in W$);
- **d.** space of homomorphisms (linear maps) $\text{Hom}(V, W)$: $(g \cdot A)x = g \cdot A(g^{-1} \cdot x)$ ($A \in \text{Hom}(V, W), x \in V$);
- **e.** exterior square $\Lambda^2(V)$: $g \cdot (x \wedge y) = g \cdot x \wedge g \cdot y$ ($x, y \in V$);

to name a few of the most common instances.

Schur’s lemma

Despite the simplicity of its proof, the following result has powerful applications.

1.2.2 Lemma (Schur) Let $V$ and $W$ be irreducible $G$-spaces. If $A : V \to W$ is an equivariant linear map, then $A$ is an isomorphism or $A = 0$.

Proof. Since $A$ is equivariant, its kernel $\ker A \subset V$ and its image $\text{im} A \subset W$ are invariant subspaces. By the irreducibility we have that

$$\ker A = \begin{cases} \{0\} & \text{or} \\ V, & \end{cases} \quad \text{and} \quad \text{im} A = \begin{cases} \{0\} & \text{or} \\ W, & \end{cases}$$

Suppose that $A \neq 0$. Then $\ker A = \{0\}$ and $\text{im} A = W$. Hence $A$ is an isomorphism. \qed

1.2.3 Corollary Let $V$ be an irreducible $G$-space over $\mathbb{R}$. Then the space $\text{End}_G(V)$ of $G$-equivariant endomorphisms of $V$ is a real division algebra.

1.2.4 Remark Frobenius’ theorem says that every (associative, finite-dimensional) real division algebra is one of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Accordingly, in view of Corollary 1.2.3, a real irreducible representation of a topological group $G$ is called of real type, complex type or quaternionic type whether $\text{End}_G(V)$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, respectively.

1.2.5 Corollary Let $V$ be an irreducible linear $G$-space over $\mathbb{C}$. If $A : V \to V$ is an equivariant linear map, then $A = \lambda_0 I$ for some $\lambda_0 \in \mathbb{C}$.

Proof. We have $A - \lambda I$ is also equivariant for all $\lambda \in \mathbb{C}$. Let $\lambda_0$ be an eigenvalue of $A$. Then $A - \lambda_0 I$ is not invertible, so $A - \lambda_0 I = 0$. \qed

1.2.6 Corollary A complex irreducible representation of an Abelian group $G$ must be one-dimensional.
Proof. Let \( \varphi : G \to \text{GL}(V) \) be a complex irreducible representation. Since \( G \) is Abelian,
\[
\varphi(g') \circ \varphi(g) = \varphi(g'g) = \varphi(gg') = \varphi(g) \circ \varphi(g')
\]
for all \( g, g' \in G \). So for all \( g \in G \), \( \varphi(g) : V \to V \) is equivariant. Therefore
\( \varphi(g) = \lambda_g I \) where \( \lambda_g \in \mathbb{C} \). Now every subspace of \( V \) is invariant, hence
\( \dim V = 1 \) by irreducibility. \( \square \)

Main problems in representation theory

We already have elements to formulate some problems in representation theory.

\begin{itemize}
  \item [a.] What types of groups have the property that all of their representations are completely reducible?
  \item [b.] How to efficiently decide whether a given representation is irreducible?
  \item [c.] How to classify irreducible representations of a given group up to equivalence?
\end{itemize}

We will solve (a) and (b) for compact topological groups in this chapter. In later chapters, we will solve (c) for compact Lie groups.

1.3 Adjoint action

Let \( G \) be a topological group. For each \( g \in G \), the inner automorphism defined by \( g \) is the automorphism \( \text{Inn}_g \) of \( G \) given by \( \text{Inn}_g(x) = gxg^{-1} \). Now \( G \) acts on itself by inner automorphisms; this is the adjoint action of \( G \) on itself:
\[
\tilde{\text{Ad}} : G \to \text{Aut}(G) \quad g \mapsto \text{Inn}_g
\]

Of course, the adjoint action is trivial (\( \tilde{\text{Ad}}_g = I \) for all \( g \in G \)) if and only if \( G \) is Abelian, so we can view the adjoint action of \( G \) on itself as a way of organizing geometrically the non-commutativity of \( G \).

In case of matrix groups, \( \tilde{\text{Ad}} \) is \( G \)-conjugation of matrices of \( G \), and the orbits of this action are the conjugacy classes of matrices in \( G \).

The case of \( S^3 \)

Recall the (associative) real division algebra of quaternions
\[
\mathbb{H} = \{ a + bi + cj + dk \mid a, b, c, d \in \mathbb{R} \}.
\]

It consists of a four-dimensional vector space over \( \mathbb{R} \) with basis \( \{1, i, j, k\} \) and the non-commutative multiplication rules
\[
i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.
\]
If \( q = a + bi + cj + dk \), its conjugate is \( \bar{q} = a - bi - cj - dk \), the real and imaginary parts of \( q \) are, respectively, \( \Re q = a \) and \( \Im q = q - \Re q \), and the norm of \( q \) is \( |q| = \sqrt{\bar{q}q} = a^2 + b^2 + c^2 + d^2 \). It follows that every \( q \neq 0 \) has the inverse element \( q^{-1} = \bar{q}/|q|^2 \) (division algebra property) and thus the nonzero quaternions form a multiplicative group \( \mathbb{H}^\times \).

Since \( |qq'| = |q||q'| \) for all \( q, q' \in \mathbb{H} \), the unit sphere \( S^3 = \{ q \in \mathbb{H} : |q| = 1 \} \) is a closed subgroup. It acts on \( \mathbb{R}^4 \) as follows. Identify \( \mathbb{H} = \mathbb{R}^4 \) and consider the standard inner product \( \langle x, y \rangle = \Re(x\bar{y}) \). Define

\[
S^3 \times \mathbb{H} \to \mathbb{H}, \quad q \cdot x = qxq^{-1}.
\]

This action is \( \mathbb{R} \)-linear and orthogonal, as \( |qxq^{-1}| = |q||x||q|^{-1} = 1 \cdot |x| \cdot 1 = |x| \) for \( q \in S^3 \). This defines a homomorphism \( \tilde{\psi} : S^3 \to SO(4) \).

It is clear that the real line \( \mathbb{R} \cdot 1 = \{ a : a \in \mathbb{R} \} \subset \mathbb{H} \) is fixed by \( \tilde{\psi} \). Therefore \( \tilde{\psi} \) decomposes as \( 1 \oplus \psi \), and \( \mathbb{H} \) decomposes into \( \mathbb{R} \cdot 1 \oplus \mathfrak{M} \) where \( \mathfrak{M} \) is the space of imaginary quaternions.

Now \( \psi : S^3 \to SO(3) \) is a homomorphism. It is easy to check that \( \ker \psi = \{ \pm 1 \} \). The induced map

(1.3.1) \[ S^3/\{ \pm 1 \} \to SO(3) \]

is continuous and injective. Since the domain is compact, it is a homeomorphism onto its image. Every element of \( SO(3) \) is a product \( R_i R_j R_k \) where \( R_i \) (resp. \( R_j, R_k \)) is a rotation around the \( i \)- (resp. \( j, k \)-) axis (Euler decomposition). Since \( \psi(e^{i\theta}) \) (resp. \( \psi(e^{j\theta}), \psi(e^{k\theta}) \)) is a rotation around the \( i \)- (resp. \( j, k \)-) axis, we deduce that \( \psi \) is surjective. Now (1.3.1) is a homeomorphism. Note that \( S^3/\{ \pm 1 \} \cong \mathbb{R}P^3 \) and \( \psi : S^3 \to SO(3) \) is a double covering.

It is now easy to understand geometrically the adjoint action of \( S^3 \) on itself. For \( x = a + bi + cj + dk \in S^3 \), there is \( q \in S^3 \) such that \( q \cdot x = a + b'i \), where \( b' = \pm \sqrt{b^2 + c^2 + d^2} \). Thus we see that each orbit intersects the circle

(1.3.2) \[ S^1 = \{ \cos \theta + \sin \theta i : \theta \in \mathbb{R} \} \subset S^3 \]

in precisely two points. The adjoint orbit of \( e^{i\theta} \in S^1 \) is a round 2-sphere \( S^2(\sin \theta) \). It is also easy to see that the circle (1.3.2) meets the adjoint orbits perpendicularly.
1.4. AVERAGING METHOD AND HAAR INTEGRAL ON COMPACT GROUPS

Incidentally, we can also view \( \mathbb{H} \) as a right \( \mathbb{C} \)-module. In this case \( \{ 1, j \} \) is a basis and
\[
\mathbb{H} = \{ \alpha + j\beta : \alpha, \beta \in \mathbb{C} \}.
\]
The action of \( S^3 \) on \( \mathbb{H} \) by left multiplication, namely,
\[
(\alpha + j\beta) \cdot 1 = \alpha + j\beta, \\
(\alpha + j\beta) \cdot j = \alpha j + j\beta j = -\bar{\beta} + j\bar{\alpha},
\]
defines a 2-dimensional complex representation of \( S^3 \) on \( \mathbb{C}^2 \),
\[
\alpha + j\beta \mapsto \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix},
\]
which we denote by \( \varphi_{-1} \). Note that \( \text{im}\varphi_{-1} \subset SU(2) \), that is, \( \varphi_1 \) is unitary with respect to the standard Hermitian product on \( \mathbb{C}^2 \). It is easy to check that \( \varphi_{-1} \) in fact defines a topological isomorphism (that is, a group homomorphism and a homeomorphism) between \( S^3 \) and \( SU(2) \).

1.4 Averaging method and Haar integral on compact groups

If \( G \) is a finite group and \( V \) is a linear \( G \)-space, then we can produce \( G \)-fixed points as follows. For each \( x \in V \) the center of mass of the orbit \( G(x) \)
\[
\bar{x} = \frac{1}{|G|} \sum_{g \in G} g \cdot x,
\]
where \(|G|\) denotes the cardinality of \( G \), is a \( G \)-fixed point, namely
\[
h \cdot \bar{x} = \frac{1}{|G|} \sum_{g \in G} h \cdot (g \cdot x) = \frac{1}{|G|} \sum_{g \in G} (hg) \cdot x = \bar{x}
\]
for all \( h \in G \). In terms of coordinates with respect to a fixed coordinate system, the coordinates of \( \bar{x} \) turn out to be simply the average of the corresponding coordinates of the points in \( G(x) \). In order to generalize this procedure to nonfinite groups, we need a notion of average value of a continuous function. We can establish such a notion in the general setting of compact topological groups as follows.

Let \( G \) be a compact topological group and let \( C(G) \) denote the space of all real valued continuous functions on \( G \). Due to compactness, every \( f \in C(G) \) is uniformly continuous; it is convenient to express this condition by saying that given \( \epsilon > 0 \) there exists a neighborhood \( U \) of the identity \( 1 \in G \) such that for all \( x, y \in G \),
\[
xy^{-1} \in U \Rightarrow |f(x) - f(y)| < \epsilon.
\]
Consider the continuous action of \( G \times G \) on \( G \) on the left and on the right given by
\[
(G \times G) \times G \to G, \quad (g_1, g_2) \cdot g = g_1 g_2 g_1^{-1}.
\]
We consider \( C(G) \) with the sup norm topology. Then there is an induced continuous action of \( G \times G \) on \( C(G) \) given by
\[
(G \times G) \times C(G) \to C(G), \quad [(g_1, g_2) \cdot f](x) = f((g_1, g_2)^{-1} \cdot x) = f(g_1^{-1} g_2).
\]

1.4.2 Theorem There exists a unique \( G \times G \)-invariant positive continuous linear form \( I : C(G) \to \mathbb{R} \) such that \( I(1) = 1 \); here, positive means we have that \( I(f) \geq 0 \) if \( f \geq 0 \), and \( I(f) > 0 \) if \( f \geq 0 \) and \( f \) is not identically zero.

1.4.3 Remark By the Riesz-Markov-Kakutani Representation Theorem, every positive continuous linear form \( I \) on \( C(G) \) arises as integration with respect to a unique regular Borel measure \( d\mu \). In view of that, \( I(f) = \int_G f(g) \, d\mu(g) \) is called the Haar integral of \( f \).

Proof of Theorem 1.4.2. We first prove existence. For each multiset (i.e., finite subset with multiplicities) \( A \subset G \times G \) set
\[
\Gamma(A, f) = \frac{1}{|A|} \sum_{a \in A} m(a)(a \cdot f) \in C(G).
\]
The following multiplicative property for \( \Gamma \) holds:
\[
\Gamma(A, \Gamma(B, f)) = \frac{1}{|A||B|} \sum_{a \in A} \sum_{b \in B} m(a)m(b)(ab) \cdot f = \frac{1}{|AB|} \sum_{c \in AB} m(c)(c \cdot f) = \Gamma(AB, f),
\]
where \( AB \subset G \times G \) is the multiset given by \( \{ab : a \in A \text{ and } b \in B\} \) and \( m(c) = \sum_{ab=c} m(a)m(b) \) for \( c \in AB \).

For each \( h \in C(G) \), define the variation of \( h \) to be
\[
\omega(h) = \max\{h(x) : x \in G\} - \min\{h(x) : x \in G\}.
\]
Of course we can write
\[
\max\{h(x) : x \in G\} = ||h||_\infty \text{ and } \min\{h(x) : x \in G\} = ||-h||_\infty.
\]
Set
\[
\Delta_f = \{\Gamma(A, f) : A \subset G \times G \text{ is a finite multiset}\},
\]
and
\[\Omega_f = \inf \{ \omega(h) : h \in \Delta_f \} .\]

Note that \( \Delta_f \) is an equibounded family of functions, that is,
there exists \( M > 0 \) such that \( |h(x)| \leq M \) for all \( h \in \Delta_f \).
This follows simply because \( f \) is bounded. Moreover, we claim that \( \Delta_f \) is an equicontinuous family of functions, that is, given \( \epsilon > 0 \) there exists a neighborhood \( V \) of the identity such that
\[xy^{-1} \in V \Rightarrow |h(x) - h(y)| < \epsilon \]
for all \( h \in \Delta_f \). In order to show this, first consider the neighborhood \( U \) of 1 such that (1.4.1) holds and note that by compactness of \( G \) we can find another neighborhood \( V \) of 1 contained in \( U \) such that
\[(1.4.4) \quad g^{-1}Vg \subset U \text{ for all } g \in G .\]
Now if \( xy^{-1} \in V \) and \( a = (g_1, g_2) \in A \), then
\[|a \cdot f(x) - a \cdot f(y)| = |f(g_1^{-1}xg_2) - f(g_1^{-1}yg_2)| < \epsilon \]
because
\[g_1^{-1}xg_2(g_1^{-1}yg_2)^{-1} = g_1^{-1}(xy^{-1})g_1 \in U ,\]
by (1.4.1) and (1.4.4). Therefore for \( xy^{-1} \in V \) and \( h = \Gamma(A, f) \in \Delta_f \) we have
\[|h(x) - h(y)| \leq \frac{1}{|A|} \sum_{a \in A} m(a) |a \cdot f(x) - a \cdot f(y)| < \epsilon ,\]
which proves the claim.

Let \( \{h_n\} \) be a minimizing sequence in \( \Delta_f \), namely, \( \lim \omega(h_n) = \Omega_f \). Owing to the Arzelà-Ascoli theorem, there exists a convergent subsequence which we still denote with \( \{h_n\} \). Put \( \bar{h} = \lim h_n \in C(G) \). Then \( \omega(\bar{h}) = \Omega_f \), and we claim that \( \omega(\bar{h}) = 0 \). Suppose, on the contrary, that \( \omega(\bar{h}) > 0 \). Then there exists \( M < ||h||_\infty \) and an open subset \( U \) of \( G \) such that
\[\bar{h}(x) \leq M \]
for all \( x \in U \). By compactness of \( G \) we can write \( G = \bigcup_{i=1}^n g_iU \) for some \( g_i \in G \). Set \( A = \{(g_i, 1) : i = 1, \ldots, n\} \) (all multiplicities 1). Then, if \( x \in g_iU \), we have
\[\{(g_i, 1) \cdot \bar{h}\}(x) = \bar{h}(g_i^{-1}x) \leq M,\]
and this implies that
\[ \| - h \|_{\infty} \leq \Gamma(A, \bar{h}) \leq \frac{M + (n - 1)\|h\|_{\infty}}{n} < \|h\|_{\infty}. \]

Now \( \omega(\Gamma(A, \bar{h})) < \omega(\bar{h}) = \Omega_f. \) Since
\[ \lim \Gamma(A, h_n) = \lim \Gamma(A, \bar{h}) \text{ and } \lim \omega(\Gamma(A, h_n)) = \omega(\Gamma(A, \bar{h})), \]
for \( n \) sufficiently big, we now get that
\[ \omega(\Gamma(A, h_n)) < \Omega_f, \]
a contradiction to the fact that \( \Gamma(A, h_n) \in \Delta_f \) (by the multiplicative property of \( \Gamma \)). Hence \( \omega(\bar{h}) = 0. \)

Now \( \bar{h} \) is a constant, and we set \( I(f) = \bar{h}. \) Notice that \( I \) is linear because \( \Gamma(A, f) \) is linear on \( f \) for fixed \( A; \) \( I \) is positive because \( \Gamma(A, f) \geq 0 \) for \( f \geq 0; \) and \( I \) is equivariant because \( \Delta_{b\cdot f} = \Delta_f \) for all \( b \in G \times G. \) Further, \( I \) is continuous as \( |I(f)| \leq \|f\|_{\infty} \) for all \( f \in C(G). \) This completes the proof of existence.

Suppose now \( I' \) is another \( G \times G \)-invariant positive linear form on \( C(G) \) with \( I'(1) = 1. \) Given \( f \in C(G) \) and \( \epsilon > 0, \) there exists a multiset \( A \subset G \times G \) with
\[ |\Gamma(A, f) - I(f)| < \epsilon. \tag{1.4.5} \]

By linearity and equivariance of \( I', \)
\[ I'((\Gamma(A, f))) = \Gamma(A, I'(f)) = I'(f). \]

Further,
\[ I'(I(f)) = I(f)I'(1) = I(f). \]

Finally, combining (1.4.5) with the positivity of \( I' \) yields
\[ |I'(f) - I(f)| = |I'(\Gamma(A, f)) - I'(I(f))| < \epsilon. \]

Since \( \epsilon > 0 \) is arbitrary, it follows that \( I'(f) = I(f). \) \( \square \)

**Existence of invariant inner (Hermitian) product and complete reducibility**

**1.4.6 Proposition** Let \( G \) be a compact topological group and let \( V \) be a real (resp. complex) linear \( G \)-space. Then there exists a \( G \)-invariant inner (resp. Hermitian) product \( \langle \cdot, \cdot \rangle \) on \( V, \) namely
\[ \langle g \cdot u, g \cdot v \rangle = \langle u, v \rangle \]
for \( u, v \in V \) and \( g \in G. \)
Proof. Let \( \langle \cdot, \cdot \rangle_0 \) be an arbitrary inner (resp. Hermitian) product on \( V \). Set

\[
\langle u, v \rangle = \int_G \langle g \cdot u, g \cdot v \rangle_0 \ d\mu(g).
\]

Then \( \langle \cdot, \cdot \rangle \) is an inner (resp. Hermitian) product on \( V \) and, for \( h \in G \),

\[
\langle h \cdot u, h \cdot v \rangle = \int_G \langle g \cdot (h \cdot u), g \cdot (h \cdot v) \rangle_0 \ d\mu(g)
\]

\[
= \int_G \langle (gh) \cdot u, (gh) \cdot v \rangle_0 \ d\mu(g)
\]

\[
= \int_G f(gh) \ d\mu(g)
\]

\[
= \int_G f(g) \ d\mu(g)
\]

\[
= \langle u, v \rangle,
\]

which completes the proof. \( \square \)

1.4.7 Corollary Every real (resp. complex) representation of a compact topological group is equivalent to an orthogonal (resp. unitary) one.

1.4.8 Theorem Every representation of a compact topological group is completely reducible.

Proof. Let \( V \) be a real (resp. complex) linear \( G \)-space. If it is not irreducible, there is a proper invariant subspace \( U \). Equip \( V \) with an invariant inner (resp. Hermitian) product. Then the orthogonal complement \( U^\perp \) is also invariant and \( V = U \oplus U^\perp \). Since the dimensions of \( U \) and \( U^\perp \) are strictly smaller than that of \( V \), the result now follows from an induction on the dimension of \( V \). \( \square \)

1.5 The character theory of Frobenius-Schur

Let \( G \) be a compact topological group. We consider the following inner product on the space \( C(G, \mathbb{C}) \) of complex-valued continuous functions on \( G \):

\[
(1.5.1) \quad (f_1, f_2) = \int_G f_1(g)\overline{f_2(g)} \, dg \quad (f_1, f_2 \in C(G, \mathbb{C})).
\]

If \( \varphi : G \to U(V) \) is a unitary representation and \( u, v \in V \), define the matrix coefficient

\[
\varphi_{u,v}(g) = \langle \varphi(g)v, u \rangle \quad (g \in G).
\]
Then \( \varphi_{u,v} \in C(G, \mathbb{C}) \). If \( \{ e_1, \ldots, e_n \} \) is an orthonormal basis of \( V \) then

\[
\varphi(g)e_j = \sum_{i=1}^{n} \varphi_{e_i,e_j}(g)e_i
\]

and the \( \varphi_{e_i,e_j}(g) \) are the usual matrix coefficients of \( \varphi(g) \) with respect to the fixed orthonormal basis.

The character of \( \varphi \) is the element of \( C(G, \mathbb{C}) \) defined by

\[
\chi_\varphi(g) = \text{trace } \varphi(g) = \sum_{i=1}^{n} \varphi_{ii}(g).
\]

1.5.2 Remark

(i) It follows from the invariance of the trace under conjugation that the character is constant in each conjugacy class, namely,

\[
\chi_\varphi(hgh^{-1}) = \chi_\varphi(g) \quad \text{for all } g, h \in G.
\]

(ii) For the same reason as in (i), we see that equivalent representations have the same character.

(iii) If \( \varphi = \varphi_1 \oplus \varphi_2 \) then \( \chi_\varphi = \chi_{\varphi_1} + \chi_{\varphi_2} \).

1.5.3 Proposition (orthogonality relations)

(i) If \( \varphi : G \to U(V) \) and \( \varphi' : G \to U(V') \) are inequivalent irreducible unitary representations, and \( u, v \in V \), and \( u', v' \in V' \), then

\[
(\varphi_{u,v}, \varphi'_{u',v'}) = 0.
\]

(ii) If \( \varphi : G \to U(V) \) is an irreducible unitary representation, and \( u, v, u', v' \in V \), then

\[
(\varphi_{u,v}, \varphi'_{u',v'}) = \frac{1}{\dim V} \langle u, u' \rangle \langle v, v' \rangle.
\]

Proof. (i) Let \( A_0 \in \text{Hom}(V, V') \). Define

\[
A = \int_G \varphi'(g)A_0\varphi(g)^{-1}d\mu(g).
\]

Then \( A : V \to V' \) is a linear \( G \)-map, which we write \( A \in \text{Hom}_G(V, V') \). Schur’s lemma implies that \( A = 0 \). Hence

\[
0 = \langle Au, u' \rangle = \int_G \langle \varphi'(g)A_0\varphi(g)^{-1}u, u' \rangle d\mu(g)
\]

\[
= \int_G \langle A_0\varphi(g)^{-1}u, \varphi'(g)^{-1}u' \rangle d\mu(g).
\]
Choose now $A_0(w) = \langle w, v \rangle v'$. Then

$$0 = \int_G \langle \varphi(g)^{-1} u, v' \rangle \langle \varphi'(g)^{-1} u', v \rangle d\mu(g),$$

$$= \int_G \langle u, \varphi(g) v \rangle \langle \varphi'(g) v', u' \rangle d\mu(g),$$

$$= \int_G \varphi'_u, v' \varphi_u, v d\mu(g),$$

$$= \langle \varphi'_u, v', \varphi_u, v \rangle.$$

(ii) Let $A_0 \in \text{End}(V) = \text{Hom}(V, V)$ and define

$$A = \int_G \varphi(g) A_0 \varphi(g)^{-1} d\mu(g).$$

Now $A \in \text{End}_G(V)$ and Schur’s lemma implies that $A = \lambda I$ for some $\lambda \in \mathbb{C}$. Then

$$\lambda \dim V = \text{trace } A = \int_G (\text{trace } A_0) d\mu(g) = \text{trace } A_0.$$

Choose $A_0$ as in (i). Then $\text{trace } A_0 = \langle v', v \rangle$ and a computation similar to the one in (i) shows that

$$\langle \varphi'_u, v', \varphi_u, v \rangle = \langle Au, u' \rangle = \lambda \langle u, u' \rangle = \frac{1}{\dim V} \langle u, u' \rangle \langle v, v' \rangle,$$

and this completes the proof. \qed

1.5.4 Corollary

(i) If $\varphi$ is irreducible, then $(\chi_\varphi, \chi_\varphi) = 1$.

(ii) If $\varphi$ and $\varphi'$ are irreducible and inequivalent, then $(\chi_\varphi, \chi_{\varphi'}) = 0$.

Proof. (i) Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $V$. Then

$$(\chi_\varphi, \chi_\varphi) = \sum_{i=1}^n \sum_{j=1}^n (\varphi_{ii}, \varphi_{jj}) = \sum_{i=1}^n \sum_{j=1}^n \frac{1}{n} \langle e_i, e_j \rangle \langle e_i, e_j \rangle = 1.$$

(ii) This is immediate. \qed

Let $\hat{G}$ denote the set of equivalence classes of complex (unitary) irreducible representations of $G$. Every complex representation $\rho$ can be expressed as the direct sum of irreducible ones, namely

$$\rho = \sum_{\varphi \in \hat{G}} m(\rho ; \varphi) \varphi$$

(finite sum)

where $m(\rho ; \varphi)$ is the multiplicity of the equivalence class of $\varphi$ in $\rho$. Now

$$\chi_\rho = \sum_{\varphi \in \hat{G}} m(\rho ; \varphi) \chi_\varphi$$

which implies that

$$m(\rho ; \varphi) = (\chi_\rho, \chi_\varphi).$$
1.5.5 Theorem Let $G$ be a compact topological group. Then two complex irreducible representations $\rho$ and $\rho'$ are equivalent if and only if their characters coincide as functions, namely $\chi_\rho = \chi_{\rho'}$. A complex irreducible representation $\rho$ is irreducible if and only if $(\chi_\rho, \chi_\rho) = 1$.

Proof. We have that $\chi_\rho = \chi_{\rho'}$ implies that $(\chi_\rho, \chi_\varphi) = (\chi_{\rho'}, \chi_\varphi)$ and therefore $m(\rho; \varphi) = m(\rho'; \varphi)$ for every $\varphi \in \hat{G}$. This shows that $\rho$ and $\rho'$ are equivalent. The other direction was already checked in Remark 1.5.2.

The last assertion follows from the fact that if $\rho = \sum_{\varphi \in \hat{G}} m(\rho; \varphi) \chi_\varphi$ then $(\chi_\rho, \chi_\rho) = \sum_{\varphi \in \hat{G}} m(\rho; \varphi)^2$ and this finishes the proof. \hfill $\Box$

1.5.6 Corollary Let $G$ and $H$ denote compact topological groups.

a. If $\varphi : G \to U(V)$ and $\psi : H \to U(W)$ are irreducible representations, then the outer tensor product $\varphi \otimes \psi : G \times H \to U(V \otimes W)$, given by $\varphi \otimes \psi(g, h)(v, w) = \varphi(g)v \otimes \psi(h)w$, is also an irreducible representation.

b. Every irreducible representation of $G \times H$ arises as in part (i).

Proof. For part (i), we just note that $\chi_{\varphi \otimes \psi}(g, h) = \chi_\varphi(g) \cdot \chi_\psi(h)$ and then

$$
\int_{G \times H} |\chi_{\varphi \otimes \psi}(g, h)|^2 d\mu(g, h) = \int_G |\chi_\varphi(g)|^2 d\mu(g) \int_H |\chi_\psi(h)|^2 d\mu(h) = 1 \cdot 1 = 1.
$$

For part (ii), let $U$ be an irreducible representation of $G \times H$. Restrict $U$ to the subgroup $G \cong G \times \{1\}$ of $G \times H$. Let $V$ be a $G$-irreducible component of $U$. Now $W := \text{Hom}_G(V, U) \neq 0$ and this space carries an $H$-action, since the $G$- and $H$-actions on $U$ commute. Define a linear map

$$
\Phi : V \otimes W \to U, \quad \Phi(u \otimes f) = f(u).
$$

Then $\Phi$ is a $G \times H$-equivariant. Due to the irreducibility of $U$, $\Phi$ is surjective. Moreover, by Schur’s lemma $\dim W$ is the number of components of the $G$-space $U$ that are isomorphic to $V$, so $\dim W \leq \dim U / \dim V$. It follows that $\Phi$ is injective, hence defines a $G \times H$-equivariant isomorphism $V \otimes W \cong U$.

Finally, $W$ must be $H$-irreducible, for an $H$-invariant decomposition $W = W_1 \oplus W_2$ yields a $G \times H$-invariant decomposition $V \otimes W = V \otimes W_1 \oplus V \otimes W_2$. This completes the proof. \hfill $\Box$
1.5.7 Example (Representations of the circle) Consider
\[ G = S^1 = \{ z \in \mathbb{C} : |z| = 1 \}. \]
Then for each \( n \in \mathbb{Z} \) we define a one-dimensional representation
\[ \varphi_n : S^1 \to U(1); \; z \mapsto z^n. \]
Notice that \( \chi_n(z) := \chi_{\varphi_n}(z) = \varphi_n(z) = z^n. \)
Write \( z = e^{i\theta} \) for \( \theta \in \mathbb{R} \). The Frobenius-Schur orthogonality relations reduce to the Fourier orthogonality relations and they say that
\[ \{ e^{ni\theta} : n \in \mathbb{Z} \} \]
is an orthonormal set in \( C(G, \mathbb{C}) \). We also know from Fourier analysis that this set in fact spans a dense subset of \( C(G, \mathbb{C}) \) with regard to the sup-norm topology. It follows from the orthogonality relations that every complex irreducible continuous representation \( \varphi \) of \( S^1 \) is equivalent to \( \varphi_n \) for some \( n \in \mathbb{Z} \). Hence \( \hat{S}^1 = \mathbb{Z} \).

Representations of \( SU(2) \)
Recall the topological isomorphism \( \varphi_{-1} : S^3 \to SU(2) \) given by
\[ \varphi_{-1}(\alpha + j\beta) = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \]
where \( \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \). We think of \( \varphi_{-1} \) as the vector representation of \( SU(2) \) and we produce a series \( \{ \varphi_k \} \) of unitary irreducible representations of \( SU(2) \) indexed by a non-negative integer \( k \). Consider the space of complex polynomials in two variables
\[ \mathbb{C}[z_1, z_2] = \sum_{k=0}^{+\infty} V_k \]
where \( V_k \) is the subspace of homogeneous polynomials of degree \( k \). Note that \( \dim V_k = k + 1 \). Define
\[ \varphi_k(\alpha + j\beta) \cdot p_k(z_1, z_2) = p_k \left( \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right) \]
for \( p_k \in V_k \). Then \( \varphi_k \) is a representation of \( S^3 \) on \( V_k \).

1.5.8 Theorem \( \varphi_k \) is irreducible and \( \hat{S}^3 = \{ \varphi_k \}_{k=0}^{\infty} \).
Proof. Let \( \chi_k = \chi \varphi_k \). We will show that \( (\chi_k, \chi_k) = 1 \) and use Theorem 1.5.5 to deduce that \( \varphi_k \) is irreducible.

Notice that
\[
\varphi_1(e^{i\theta})z_1 = e^{-i\theta}z_1 \quad \text{and} \quad \varphi_1(e^{i\theta})z_2 = e^{i\theta}z_2,
\]
so
\[
\varphi_k(e^{i\theta})z^j_1z^k_j = (e^{-i\theta}z_1)\varphi_1(e^{i\theta}z_2)^{k-j} = e^{(k-2j)i\theta}z^j_1z^k_j.
\]
This implies that
\[
\chi_k(e^{i\theta}) = \sum_{j=0}^{k} e^{(k-2j)i\theta} = \frac{e^{(k+1)i\theta} - e^{-(k+1)i\theta}}{e^{i\theta} - e^{-i\theta}}.
\]
Note that \( \chi_1 \) is also the character of \( \varphi_{-1} \), so Theorem 1.5.5 says that \( \varphi_1 \) and \( \varphi_{-1} \) are equivalent representations. \( \varphi_1 \) is the representation on the dual space to \( \varphi_{-1} \), and we define \( \varphi_{-k} \) to be the representation on the dual space to \( \varphi_k \). Then Theorem 1.5.5 says that \( \varphi_k \) and \( \varphi_{-k} \) are equivalent representations.

Let \( d\sigma \) be the volume element in the metric induced on \( S^3(1) \) from \( \mathbb{R}^4 \). It is clear that \( d\sigma \) is defines a bi-invariant measure. By the uniqueness part of Theorem 1.4.2 we have that \( d\mu \) and \( d\sigma \) must be multiples of each other. Since the volume of \( S^3(1) \) with respect to \( d\sigma \) is \( 2\pi^2 \), we deduce that
\[
d\mu = \frac{1}{2\pi^2}d\sigma.
\]
Recall that the circle \( \{e^{i\theta} : 0 \leq \theta \leq \pi \} \) meets all adjoint orbits of \( S^3 \) in \( S^3 \), and those orbits are round spheres \( S^2(\sin \theta) \), hence of volume \( 4\pi \sin^2 \theta \).

We are now ready to compute:
\[
(\chi_k, \chi_k) = \frac{1}{2\pi^2} \int_{S^3(1)} \chi_k(g)\overline{\chi_k(g)}d\sigma
\]
\[
= \frac{1}{2\pi^2} \int_{0}^{\pi} \chi_k(e^{i\theta})\overline{\chi_k(e^{i\theta})}4\pi \sin^2 \theta d\theta
\]
\[
= \frac{1}{4\pi} \int_{0}^{\pi} \chi_k(e^{i\theta})\overline{\chi_k(e^{i\theta})}|e^{i\theta} - e^{-i\theta}|^2 d\theta
\]
\[
= \frac{1}{4\pi} \int_{0}^{\pi} |e^{(k+1)i\theta} - e^{-(k+1)i\theta}|^2 d\theta
\]
\[
= \frac{1}{4\pi} \int_{0}^{\pi} |2i \sin (k+1)\theta|^2 d\theta
\]
\[
= \frac{1}{2\pi} \int_{0}^{2\pi} (1 - \cos 2(k+1)\theta)d\theta
\]
\[
= 1.
\]
Hence \( \varphi_k \) is irreducible.

Finally, suppose that \( \varphi \in \hat{S}^3 \), \( \deg \varphi = k + 1 \) but \( \varphi \) is not equivalent to \( \varphi_k \). Then

\[
0 = \left( \chi_\varphi, \chi_l \right) = \frac{1}{4\pi} \int_0^{2\pi} \chi_\varphi(e^{i\theta})\overline{\chi_l(e^{i\theta})}|e^{i\theta} - e^{-i\theta}|^2d\theta
\]

for all \( l = 0, 1, 2, \ldots \). Now \( \chi_\varphi(e^{i\theta})(e^{i\theta} - e^{-i\theta}) \) is odd in \( \theta \) and orthogonal to \( \sin((l+1)i\theta) \) for all \( l = 0, 1, 2, \ldots \), and this is a contradiction. Hence \( \varphi \) cannot exist. This proves that \( \{ \varphi_k \}_{k=0,1,2,\ldots} \) is a complete set of representatives of complex irreducible representations of \( S^3 \). □

### The Peter-Weyl theorem for compact matrix groups

A representation \( \varphi : G \to GL(V) \) is called faithful if \( \varphi \) is an injective homomorphism. A compact topological group admitting a faithful representation can also be called a compact matrix group.

#### 1.5.9 Theorem (Peter-Weyl, special case)

Let \( G \) be a compact matrix group. Then the matrix coefficients

\[
\{ \varphi_{ij} : \varphi \in \hat{G}, 1 \leq i, j \leq \deg \varphi \}
\]

span a dense subset of \( C(G, \mathbb{C}) \) in the sup-norm topology.

**Proof.** Let \( A \) be the vector subspace of \( C(G, \mathbb{C}) \) spanned by the matrix coefficients. Note that:

- \( A \) is a subalgebra of \( C(G, \mathbb{C}) \). Indeed, for if \( \varphi, \varphi' \in \hat{G} \) then the matrix coefficients of \( \varphi \otimes \varphi' \) are precisely the products \( \varphi_{ij} \cdot \varphi'_{kl} \). Since \( \varphi \otimes \varphi' \) decomposes into a direct sum of irreducible representations, \( \varphi_{ij} \cdot \varphi'_{kl} \) is a linear combination of matrix coefficients of the components.
- \( A \) is closed under complex conjugation, for the matrix coefficients of \( \varphi^* \) are the \( \overline{\varphi_{ij}} \).
- \( A \) contains the constants, as the trivial representation has one matrix coefficient function constant and equal to 1.
- \( A \) separates points. Here we use the existence of a faithful representation \( \varphi_0 \), which we can assume to be complex, unitary and irreducible. Given \( g, g' \in G \) with \( g \neq g' \), we have \( \varphi_0(g) \neq \varphi_0(g') \) and then some matrix coefficient of \( \varphi_0 \) will take distinct values on \( g \) and \( g' \). The Stone-Weierstrass theorem now implies that \( A \) is dense in \( C(G, \mathbb{C}) \) in the sup-norm. □

It turns out every compact matrix group is a Lie group. We will extend this theorem to arbitrary compact Lie groups and present applications in Chapter ??.
1.6 Problems

1. a. Let $G$ be a compact subgroup of $GL(n, \mathbb{R})$ (resp. $GL(n, \mathbb{C})$). Prove that it is conjugate to a subgroup of $O(n)$ (resp. $U(n)$), namely, there exists $A \in GL(n, \mathbb{R})$ (resp. $A \in GL(n, \mathbb{C})$) such that $AGA^{-1} \subset O(n)$ (resp. $AGA^{-1} \subset U(n)$).

b. Show that $O(n)$ (resp. $U(n)$) is a maximal compact subgroup of $GL(n, \mathbb{R})$ (resp. $GL(n, \mathbb{C})$) and that any two maximal compact subgroups of $GL(n, \mathbb{R})$ (resp. $GL(n, \mathbb{C})$) are conjugate.

2. Let $G$ be a topological group and let $H$ be a subgroup (resp. normal subgroup, Abelian subgroup) of $G$. Check that the closure $\bar{H}$ is also a subgroup (resp. normal subgroup, Abelian subgroup).

3. Let $G$ be a connected topological group. Prove that any neighborhood $U$ of the identity generates $G$ as a group. (Hint: Take $U = V \cap V^{-1}$ and show that $\cup_{n \geq 1} U^n$ is an open subgroup of $G$.)

4. Let $V$ and $W$ be linear $G$-spaces. Check that the canonical isomorphisms
   (i) $V \otimes W \cong W \otimes V$;
   (ii) $(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$;
   (iii) $V^* \otimes W \cong \text{Hom}(V, W)$; are $G$-equivariant.

5. Check that the circle $(1.3.2)$ meets the adjoint orbits of $S^3$ perpendicularly.

6. Let $\varphi, \psi$ be complex representations of a compact topological group $G$. Show that $\chi_{\varphi \otimes \psi} = \chi_\varphi \cdot \chi_\psi$.

7. Let $V$ be a real $G$-space with an invariant inner product. Show that $V^* \cong V$ as $G$-spaces.

8. Determine all the irreducible representations of $SO(3)$, up to equivalence. (Hint: Use the double covering $S^3 \to SO(3)$.)

9. Prove that a discrete normal subgroup of a Lie group $G$ centralizes the identity component $G^0$. Conclude that every discrete normal subgroup of a connected Lie group is central.

10. Let $V$ be an irreducible real (resp. complex) $G$-space. Prove that any two $G$-invariant inner products (resp. Hermitian inner products) differ by a multiplicative constant. (Hint: Diagonalize $\langle , \rangle_2$ with respect to $\langle , \rangle_1$, that is, find a $\langle , \rangle_1$-orthonormal basis of $V$ whose Gram matrix of $\langle , \rangle_2$-inner products is diagonal, and use an argument similar to the proof of Schur’s lemma.)
CHAPTER 2

Review of Lie groups

This chapter is a quick review of the basic theory of Lie groups.

2.1 Basic definition

A Lie group is a smooth manifold with an additional, compatible structure of group. More precisely, a Lie group $G$ is a smooth manifold endowed with a group structure such that the group operations are smooth, namely, the multiplication map $\mu : G \times G \to G$ and the inversion map $\iota : G \to G$ are required to be smooth.

In this book we will only study compact Lie groups. It turns out that every compact Lie group is isomorphic to a group of matrices (this follows e.g. from the Peter-Weyl theorem). Therefore matrix groups supply (almost) all the intuition we need to understand compact Lie groups. Nevertheless, we cultivate the idea that the reader should know that the concept is larger than matrix groups, namely, indeed there exist Lie groups which are not isomorphic to a Lie group of matrices (e.g. the universal covering of $SL(2, \mathbb{R})$).

All topological groups listed in Examples 1.1.2 are in fact Lie groups with respect to the standard smooth structures. This is very easy to see in cases (i), (ii) and (iii). In the sequel we make some comments in case (iv).

2.1.1 Examples  (i) Multiplication of matrices

$$M(n, \mathbb{F}) \times M(n, \mathbb{F}) \to M(n, \mathbb{F})$$

is a bilinear map, hence smooth. Further, the restriction

$$(2.1.2) \quad U \times U \to U,$$

where $U$ is any open subset of $M(n, \mathbb{F})$ for which (2.1.2) makes sense, is also smooth. Note that we can take $U = GL(n, \mathbb{F})$. Consider now the inversion map

$$GL(n, \mathbb{F}) \to GL(n, \mathbb{F}).$$
In the real and complex case, there is a formula for $A^{-1}$ involving the determinant and the transpose of the cofactor matrix of $A$, which shows that the entries of $A^{-1}$ are rational functions of the entries of $A$, and hence smooth. In the quaternionic case, it is easier to work with the representation of quaternionic matrices by complex matrices of twice the size.

(ii) In order to show that $O(n)$ is a smooth manifold, we can use the implicit mapping theorem. Denote by $\text{Sym}(n, \mathbb{R})$ the vector space of real symmetric matrices of order $n$, and define $f : M(n, \mathbb{R}) \to \text{Sym}(n, \mathbb{R})$ by $f(A) = AA^t$. This is a map between vector spaces whose components are quadratic polynomials. It follows that $f$ is smooth and that $df_A$ can be viewed as a map $M(n, \mathbb{R}) \to \text{Sym}(n, \mathbb{R})$ for all $A \in M(n, \mathbb{R})$. We claim that $I$ is a regular value of $f$. For the purpose of checking that, we first compute for $A \in f^{-1}(I)$ and $B \in M(n, \mathbb{R})$ that

$$df_A(B) = \lim_{h \to 0} \frac{(A + hB)(A + hB)^t - I}{h} = \lim_{h \to 0} \frac{h(AB^t + BA^t) + h^2BB^t}{h} = AB^t + BA^t.$$ 

Now given $C \in \text{Sym}(n, \mathbb{R})$, we have $df_A(\frac{1}{2}CA) = C$, and this proves that $f$ is a submersion at $A$, as desired. Hence $f^{-1}(I) = O(n)$ is an embedded submanifold of $M(n, \mathbb{R})$ of dimension

$$\dim M(n, \mathbb{R}) - \dim \text{Sym}(n, \mathbb{R}) = n^2 - \frac{n(n + 1)}{2} = \frac{n(n - 1)}{2}.$$ 

It remains to check that the group operations in $O(n)$ are smooth. Note that $O(n)$ is also an embedded submanifold of $GL(n, \mathbb{R})$, and its group operations are restrictions of the corresponding operations in $GL(n, \mathbb{R})$; from this follows that they are smooth. Hence $O(n)$ is a Lie group. The other classical matrix groups listed in Examples 1.1.2 can be similarly proven to be Lie groups.

(iii) There is a theorem of Élie Cartan asserting that any closed subgroup $G$ of a Lie group $H$ is also a Lie group with respect to the induced topology. We may want to use this theorem when necessary, without providing a proof. It immediately shows that the classical matrix groups are Lie groups.

### 2.2 Lie algebras

On one hand, the representation theoretical ideas we discussed in Chapter 1 for a (compact) topological group $G$ can be viewed as a kind of “extrinsic linearization” of $G$. Each representation is a matrix realization of $G$, so as to speak like a linear picture of $G$. The totality of linear pictures of $G$ should yield a great deal of information about the group. On the other
hand, $G$ becomes a Lie group when it is endowed with an additional, compatible smooth structure. This situation allows us to linearize the structure of $G$ and provides for an “intrinsic linearization” of the group. Then almost a miracle of nature happens, namely, the tangent space to $G$ at the identity inherits an algebraic structure that essentially captures all the curved geometry of the group; further, the infinitesimal structure completely characterizes the group locally, and indeed up to connected components and coverings in the global sense.

Whereas in representation theory the technique of integration is used, the method of intrinsic linearization builds upon differentiation and integration of vector fields. We take up such ideas next.

**Left-invariant vector fields and one-parameter subgroups**

Let $G$ be a Lie group. The simplest (non-trivial) Lie group is perhaps $\mathbb{R}$ with its additive structure, so it is natural to look for copies of $\mathbb{R}$ inside $G$. A one-parameter subgroup of $G$ is a smooth homomorphism $\varphi : \mathbb{R} \to G$, where $\mathbb{R}$ is viewed with its additive structure.

Given a one-parameter subgroup $\varphi : \mathbb{R} \to G$, one obtains an action of $\mathbb{R}$ on $G$, namely,

$$\Phi : \mathbb{R} \times G \to G, \quad \Phi(t, x) = x \cdot \varphi(t).$$

By construction, this action is **left-invariant**, in the sense that

$$\Phi(t, gx) = g \cdot \Phi(t, x).$$

(2.2.1)

The **left translation** defined by $g \in G$ is the map $L_g : G \to G, L_g(x) = gx$. It is a smooth map and indeed a diffeomorphism of $G$, its inverse being given by $L_g^{-1}$. Similarly, the **right translation** defined by $g \in G$ is the map $R_g : G \to G, R_g(x) = xg$. It is also a diffeomorphism of $G$ and its inverse is given by $R_g^{-1}$. We can now rewrite (2.2.1) as

$$\Phi(t, L_g(x)) = L_g(\Phi(t, x)),$$

(2.2.2)

for every $g, x \in G$. Differentiation (2.2.2) with respect to $t$ at $t = 0$ yields

$$X_gx = dL_g(X_x),$$

(2.2.3)

where $X$ is the vector field given by $X_x = \frac{d}{dt}|_{t=0}\Phi(t, x)$ for all $x \in G$. Equation (2.2.3) in turn is equivalent to

$$X \circ L_g = dL_g \circ X.$$

(2.2.4)

A vector field $X$ on $G$ satisfying (2.2.4) for all $g \in G$ is called **left-invariant**. We can similarly define **right-invariant** vector fields, but most often we will be considering the left-invariant variety. Note that left-invariance and right-invariance are the same property in case of an Abelian group.
The translations in $G$ define canonical identifications between the tangent spaces to $G$ at different points. For instance, $dL_g : T_h G \to T_{gh} G$ is an isomorphism for every $g, h \in G$. It is now clear that a left-invariant vector field $X$ is completely determined by its value $X_1$ at the identity of $G$.

One can invert the above process. In fact, it is not hard to see that every left-invariant vector field is automatically smooth and complete. Starting with a left-invariant vector field $X$ on $G$, we can thus integrate it to obtain its flow $\Phi : \mathbb{R} \times G \to G$. Let $\varphi : \mathbb{R} \to G$ denote the integral curve of $X$ through 1, that is, $\varphi(t) = \Phi(t, 1)$. The left-invariance of $X$ yields that $L_g \circ \varphi(t)$ must be the integral curve of $X$ through $g \in G$. Taking $g = \varphi(s)$, we obtain that

$$\varphi(s + t) = \varphi(s) \cdot \varphi(t)$$

for all $s, t \in \mathbb{R}$, that is, $\varphi$ is a one-parameter subgroup of $G$. We summarize the above discussion in the form of the following proposition.

**2.2.5 Proposition** There is a bijective correspondence between one-parameter subgroups of $G$ and left-invariant vector fields of $G$. It takes a one-parameter subgroup to the infinitesimal generator of the associated left-invariant action on $G$.

There is one more ingredient in this tale. Denote by $g$ the (real) vector space of left-invariant vector fields. As a vector space, it is isomorphic to the tangent space $T_1 G$. Recall that the Lie bracket of two vector fields $X, Y$ on $G$ (or a smooth manifold) is an infinitesimal measure of the non-commutativity of the corresponding flows. Viewing tangent vectors on a manifold as directional derivatives and vector fields as first order differential operators, the Lie bracket can be defined as

$$(2.2.6) \quad [X, Y]_x(f) = X_x(Y(f)) - Y_x(X(f))$$

for a smooth function $f$ on $G$ and $x \in G$. It turns out that the Lie bracket of two left-invariant vector fields on $G$ is also left-invariant. This makes $g$ (or, equivalently, $T_1 G$) into a Lie algebra. A vector space $V$ (finite- or infinite-dimensional) over a field, endowed with a bilinear operation $[\cdot, \cdot] : V \times V \to V$ satisfying

a. $[Y, X] = -[X, Y]$;

b. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity);

for every $X, Y \in V$ is called a **Lie algebra**. We have shown:

**2.2.7 Proposition** To every Lie group $G$ is associated a real Lie algebra $g$ of the same dimension, consisting of the left-invariant vector fields on $G$ and operation of Lie bracket of vector fields.
The Lie algebras of the classical matrix groups

The defining equations for the classical matrix groups yield, via differentiation, the equations defining the tangent spaces at the identity. For instance, we have seen in Example 1.1.2 that 
\[ O(n) = f^{-1}(I), \]
where \( f(A) = AA^t \) and then 
\[ o(n) = T_I O(n) = \ker df_I \]
where \( df_I(X) = X + X^t \). More systematically, we obtain:

\[
\begin{align*}
gl(n, \mathbb{R}) &= M(n, \mathbb{R}) \\
gl(n, \mathbb{C}) &= M(n, \mathbb{C}) \\
gl(n, \mathbb{H}) &= M(n, \mathbb{H}) \\
o(n) &= \{ X \in M(n, \mathbb{R}) \mid X + X^t = 0 \} \\
u(n) &= \{ X \in M(n, \mathbb{C}) \mid X + X^* = 0 \} \\
sp(n) &= \{ X \in M(n, \mathbb{H}) \mid X + X^* = 0 \} \\
sl(n, \mathbb{R}) &= \{ X \in M(n, \mathbb{R}) \mid \text{trace}(X) = 0 \} \\
sl(n, \mathbb{C}) &= \{ A \in M(n, \mathbb{C}) \mid \text{trace}(X) = 0 \} \\
s\circ(n) &= o(n) \\
su(n) &= u(n) \cap sl(n, \mathbb{C})
\end{align*}
\]

It remains to show that the Lie bracket in all cases is given by \([X, Y] = XY - YX\) (where \( XY \) denotes the usual product of matrices).

2.3 The exponential map

Let \( G \) be a Lie group and denote its Lie algebra by \( g \). Next we show how to organize the totality of one-parameter subgroups of \( G \) into a single map \( g \to G \). The exponential map \( \exp : g \to G \) is defined by

\[ \exp X = \varphi_X(1) \]

where \( \varphi_X : \mathbb{R} \to G \) is the integral curve of \( X \) passing through the identity at time zero. Owing to the chain rule, \( s \mapsto \varphi_X(st) \) is a one-parameter subgroup with initial velocity \( tX \). Hence \( \varphi_X(st) = \varphi_{tX}(s) \) and therefore \( \exp(tX) = \varphi_{tX}(1) = \varphi_X(t) \) for all \( t \in \mathbb{R} \).

It follows from the smooth dependence on initial conditions of solutions of ODE’s that the exponential map is smooth. Moreover

\[ d \exp_0 : T_0 g \cong g \to T_1 G \cong g \]

is given by the identity, due to

\[ d \exp_0(X) = \frac{d}{dt}|_{t=0} \exp(tX) = X. \]
It follows that $\exp$ is local diffeomorphism at $0$ and hence it can be used to introduce a local coordinate system around $1$ in $G$.\footnote{Any neighborhood of $1$ generates $G^0$.}

In case of matrix groups, it is easily seen that the exponential map coincides with the usual exponential of matrices, namely

$$\exp X = e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k. \tag{2.3.1}$$

In fact, $t \mapsto e^{tX}$ is the one-parameter subgroup with initial velocity $X$.

**Abelian groups**

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. For any $X \in \mathfrak{g}$, the flow $\Phi$ of $X$ is given by $\varphi_t = R_{\exp tX}$. Since $X, Y \in \mathfrak{g}$ commute (i.e. $[X,Y] = 0$) if and only if the corresponding flows commute, we see that $\mathfrak{g}$ is an Abelian Lie algebra (i.e. the Lie bracket is null) if and only of the elements of $G$ in $\exp[\mathfrak{g}]$ commute. Since $\exp[\mathfrak{g}]$ generates the identity component $G^0$ as a group, we deduce that $\mathfrak{g}$ is Abelian if and only of $G^0$ is an Abelian Lie group. It follows that connected Abelian Lie groups are of the form $\mathbb{R}^n/\Gamma$, where $\Gamma$ is a discrete subgroup (lattice) of $\mathbb{R}^n$. In particular, every compact connected Abelian Lie group is isomorphic to the torus $\mathbb{R}^n/\mathbb{Z}^n = T^n$.

**2.4 Lie subgroups and homomorphisms**

Let $G$ be a Lie group. A **Lie subgroup** of $G$ is a Lie group $H$ together with an injective smooth homomorphism $\varphi : H \to G$. The most common situation is when $H$ is an actual subgroup of $G$ and $\varphi$ is simply the inclusion $H \to G$.

Let $\mathfrak{g}$ be a Lie algebra. A **Lie subalgebra** of $\mathfrak{g}$ is a subspace $\mathfrak{h}$ which is closed under the bracket of $\mathfrak{g}$.

Now suppose $G$ and $H$ are Lie groups with corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, and suppose $\varphi : H \to G$ is a Lie subgroup. Since $\varphi(L_h \circ x) = L_{\varphi(h)} \circ \varphi(x)$ for all $h, x \in H$, $\varphi$ has constant rank, and since $\varphi$ is injective, it has maximal rank. It then follows that $H$ is an immersed submanifold of $G$. In particular, $d\varphi : \mathfrak{h} \to \mathfrak{g}$ is an injective homomorphism of Lie algebras, and we may and will view $\mathfrak{h}$ as a Lie subalgebra of $\mathfrak{g}$.

**2.4.1 Example** A Lie subgroup do not have to be closed, neither needs to have the relative topology, as the skew-line in the torus

$$\varphi : \mathbb{R} \to T^2, \quad \varphi(t) = (e^{it}, e^{ait})$$

shows, where $a$ is an irrational number.

As a main application of Frobenius theorem, one shows:
2.4.2 Theorem (Lie) Let $G$ be a Lie group and denote its Lie algebra by $\mathfrak{g}$. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then there is a unique connected Lie subgroup $H$ of $G$ such that the Lie algebra of $H$ is $\mathfrak{h}$.

Proof. (Sketch) Let $\mathcal{D}$ be the left-invariant distribution on $G$ defined by the values at the identity of vector fields in $\mathfrak{h}$ ($\mathcal{D}$ is a sub-bundle of the tangent bundle of $G$ which is invariant under left translations). The assumption that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ is easily seen to be equivalent to $\mathcal{D}$ being involutive (that is, closed under the Lie bracket of vector fields). By the Frobenius theorem, there is a maximal integral manifold of $\mathcal{D}$ passing through $1$, which we call $H$. It is not hard to see that a Lie subgroup of $G$ with Lie algebra $\mathfrak{h}$. □

Lie homomorphisms

Let $G$ and $H$ be Lie groups. A Lie group homomorphism is a group homomorphism $\varphi : G \to H$ which is also smooth. If, in addition, $\varphi$ is a diffeomorphism, then it is called an isomorphism. In case $G = H$, an isomorphism is called an automorphism. If $K$ is a Lie subgroup of $G$, then plainly the inclusion $K \to G$ is a Lie group homomorphism.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. A Lie algebra homomorphism is a linear $\varphi : \mathfrak{g} \to \mathfrak{h}$ that preserves brackets. If, in addition, $\varphi$ is bijective, then it is called an isomorphism. In case $\mathfrak{g} = \mathfrak{h}$, an isomorphism is called an automorphism. If $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$, then plainly the inclusion $\mathfrak{k} \to \mathfrak{g}$ is a Lie algebra homomorphism.

A Lie group homomorphism $\varphi : G \to H$ induces a linear map $d\varphi_1 : T_1 G \to T_1 H$ and hence a linear map $d\varphi : \mathfrak{g} \to \mathfrak{h}$. Indeed, for a left-invariant vector field $X$ on $G$, $X' = d\varphi(X)$ is the unique left-invariant vector field on $H$ whose value at $1 \in H$ equals $d\varphi(X_1)$. It turns out that $d\varphi \circ X = X' \circ \varphi$; one says that $X$ and $X'$ are $\varphi$-related as vector fields. Viewing $X$ as a differential operator, this means

(2.4.3) \[ X_{\varphi(x)}(f) = X_x(f \circ \varphi) \]

for a smooth function $f$ on $H$ and $x \in G$. If $Y \in \mathfrak{g}$, then $Y$ and $Y' := d\varphi(Y)$ are also $\varphi$-related, that is,

(2.4.4) \[ Y_{\varphi(x)}(f) = Y_x(f \circ \varphi). \]

A short calculation using (2.4.3) and (2.4.4) and (2.2.6) shows that

\[ [X', Y']_{\varphi(x)}(f) = [X, Y]_x(f \circ \varphi), \]

that is, $[X', Y']$ and $[X, Y]$ are $\varphi$-related. We deduce that $d\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.
Now comes the converse question: Given Lie groups $G, H$ with corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, and a Lie algebra homomorphism $\Phi : \mathfrak{g} \to \mathfrak{h}$, does there exist a Lie group homomorphism $\phi : G \to H$ such that $d\phi = \Phi$? In this generality, the result is false, and a very simple example is given by $G = \mathbb{S}^1$, $H = \mathbb{R}$ and $\Phi : \mathbb{R} \to \mathbb{R}$ any nonzero linear map, say, $\phi(t) = ct$ for some $c \in \mathbb{R} \setminus \{0\}$; indeed there are no nontrivial Lie group homomorphisms $\mathbb{S}^1 \to \mathbb{R}$ (since $\mathbb{R}$ admit no compact connected Lie subgroups other than the trivial one). Interestingly, if we interchange the roles of $\mathbb{S}^1$ and $\mathbb{R}$, namely, take $G = \mathbb{R}$ and $H = \mathbb{S}^1$, then a solution exists and is given by $\phi(t) = e^{ict}$. The underlying reason is the simple-connectedness of $\mathbb{R}$.

2.4.5 Theorem (Lie) Let $G_1$ and $G_2$ be Lie groups and assume that $G_1$ is connected and simply-connected. Then, given a homomorphism $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ between the corresponding Lie algebras, there is a unique Lie group homomorphism $\phi : G_1 \to G_2$ such that $d\phi = \Phi$.

Proof. (Sketch) The Lie algebra of $G_1 \times G_2$ is $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ and the graph of $\Phi$ is a subalgebra $\mathfrak{h}$. Owing to Theorem 2.4.2, there is a connected Lie subgroup $H$ of $G_1 \times G_2$ with Lie algebra $\mathfrak{h}$. Now $H$ should be the graph of the desired homomorphism; this gives the uniqueness, and for existence we just need to check that over a given element of $G_1$ sits exactly one element of $H$. We note that the projection $\mathfrak{h} \to \mathfrak{g}_1$ is an isomorphism and therefore the projection $H \to G_1$ is a covering (cf. Proposition 2.6.1); here we use the assumption that $G_1$ is simply-connected to deduce that $H \to G_1$ is an isomorphism. □

2.4.6 Corollary Two connected and simply connected Lie groups with isomorphic Lie algebras are isomorphic.

Homomorphisms and the exponential map

Let $\phi : G \to H$ be a Lie group homomorphism. Then $\phi$ maps one-parameter subgroups of $G$ to one-parameter subgroups of $H$, in the sense that if $\gamma : \mathbb{R} \to G$ is a one-parameter subgroup of $G$ then $\phi \circ \gamma : \mathbb{R} \to H$ is a one-parameter subgroup of $H$. It follows that

\[(2.4.7) \quad \phi \circ \exp^G = \exp^H \circ d\phi.\]

In particular, if $K$ is a Lie subgroup of $G$, then the exponential map of $K$ is the restriction of the exponential map of $G$. Moreover, the Lie algebra $K$, as a subalgebra of the Lie algebra $\mathfrak{g}$, is given by

\[(2.4.8) \quad \mathfrak{k} = \{X \in \mathfrak{g} \mid \exp^G(tX) \in \mathfrak{k}, \text{ for all } t \in \mathbb{R}\}.

2.4.9 Example One can use characterization (2.4.8) to determine the Lie algebras of the classical groups. For instance, consider $O(n)$. Let $g(t) \in O(n)$,
2.5. THE ADJOINT REPRESENTATION

Let \( G \) be a Lie group and denote its Lie algebra by \( \mathfrak{g} \). The adjoint action of \( G \) on \( G \) can be differentiated once to give the adjoint representation of \( G \) on \( \mathfrak{g} \), and again to give the adjoint representation of \( \mathfrak{g} \) on \( \mathfrak{g} \).

For \( g \in G \), the inner automorphism \( \text{Inn}_g : G \to G \) is now a smooth automorphism of \( G \), so its differential \( d(\text{Inn}_g) : \mathfrak{g} \to \mathfrak{g} \) defines an automorphism of \( \mathfrak{g} \), which we denote by \( \text{Ad}_g \). Then

\[
\text{Ad}_g X = \left. \frac{d}{dt} \right|_{t=0} g \exp tX g^{-1}.
\]

This defines a homomorphism

\[
\text{Ad} : g \in G \to \text{Ad}_g \in GL(\mathfrak{g}),
\]

which is called the adjoint representation of \( G \) on \( \mathfrak{g} \).

Recall that \( GL(\mathfrak{g}) \) is itself a Lie group isomorphic to \( GL(n, \mathbb{R}) \), where \( n = \dim \mathfrak{g} \). Its Lie algebra consists of all linear endomorphisms of \( \mathfrak{g} \) and it is denoted by \( \mathfrak{gl}(\mathfrak{g}) \) (the Lie bracket in \( \mathfrak{gl}(\mathfrak{g}) \) is \([A, B] = AB - BA\), see below). Now \( \text{Ad} : g \in G \to \text{Ad}_g \in GL(\mathfrak{g}) \) is homomorphism of Lie groups and its differential \( d(\text{Ad}) \) defines the adjoint representation of \( \mathfrak{g} \) on \( \mathfrak{g} \):

\[
\text{ad} : X \in \mathfrak{g} \to \text{ad}_X = \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\exp tX} \in \mathfrak{gl}(\mathfrak{g}).
\]

Using (2.4.7) and (2.3.1), we have

\[
\text{Ad}_{\exp X} = e^{\text{ad}_X} = I + \text{ad}_X + \frac{1}{2!}\text{ad}_X^2 + \frac{1}{3!}\text{ad}_X^3 + \cdots
\]

for all \( X \in \mathfrak{g} \).

2.5.3 Proposition \( \text{ad}_X Y = [X, Y] \) for all \( X, Y \in \mathfrak{g} \).
Proof. Let $f$ be a smooth function on $G$ and consider the smooth functions

$$F(r, s, t, u) = f(\exp(rX) \exp(sY) \exp(tX) \exp(uY))$$

and

$$G(r, s) = F(r, s, -r, -s),$$

respectively defined on $\mathbb{R}^4$ and $\mathbb{R}^2$. Then

$$\frac{\partial^2 G}{\partial r \partial s}(0, 0) = \frac{\partial^2 F}{\partial r \partial s}(0, 0) - \frac{\partial^2 F}{\partial r \partial u}(0, 0) - \frac{\partial^2 F}{\partial s \partial t}(0, 0) + \frac{\partial^2 F}{\partial t \partial u}(0, 0)$$

$$= X_1(\text{Y}f) - X_1(\text{Y}f) - Y_1(\text{X}f) + X_1(\text{Y}f)$$

$$= [X, Y]_1(f).$$

On the other hand, owing to (2.5.1)

$$\exp(s \text{Ad}_{\exp rX} Y) = \exp(rX) \exp(sY) \exp(-rX),$$

so

$$G(r, s) = f(\exp(s \text{Ad}_{\exp rX} Y) \exp(-sY))$$

and

$$\frac{\partial G}{\partial s}(r, 0) = (\text{Ad}_{\exp rX} Y)_1(f) - Y_1(f),$$

$$\frac{\partial G}{\partial r \partial s}(0, 0) = (\text{ad}_X Y)_1(f),$$

as desired. □

2.5.4 Example In case $G = GL(n, F)$, for $g \in GL(n, F)$ and $A \in M(n, F)$ we have

$$\text{Ad}_g A = \frac{d}{dt} \bigg|_{t=0} g e^{tA} g^{-1}$$

$$= g \left( \frac{d}{dt} \bigg|_{t=0} e^{tA} \right) g^{-1}$$

$$= gAg^{-1}.$$

Therefore, for $A, B \in M(n, F)$,

$$\text{ad}_A B = \frac{d}{dt} \bigg|_{t=0} \text{Ad}_{e^{tA}} B$$

$$= \frac{d}{dt} \bigg|_{t=0} e^{tA} B e^{-tA}$$

$$= (Ae^0) Be^0 + e^0 B (-Ae^0)$$

$$= AB - BA.$$

Combined with Proposition 2.5.3, this shows that the bracket in $\text{gl}(n, F) = M(n, F)$ (and hence in any matrix group) is given by $[A, B] = AB - BA$. 
2.6 Covering Lie groups

2.6.1 Proposition Let \( \varphi : G \to H \) be a homomorphism between Lie groups. Consider the induced homomorphism between the corresponding Lie algebras \( d\varphi : g \to h \). Then:

a. \( d\varphi \) is injective if and only if the kernel of \( \varphi \) is discrete.

b. \( d\varphi \) is surjective if and only if \( \varphi(G^o) = H^o \).

c. \( d\varphi \) is bijective if and only if \( \varphi \) is a smooth covering (here we assume \( G \) and \( H \) connected). In this case, the group of deck transformations is isomorphic to \( \ker \varphi \).

Proof. (a) and (b) follows from the fact that a Lie group homomorphism has constant rank as a smooth map. (c) follows from (a) and (b).

2.6.2 Theorem Every connected Lie group \( G \) has a simply-connected covering \( p : \tilde{G} \to G \) such that \( \tilde{G} \) is a connected Lie group and \( p \) is a Lie group homomorphism.

Proof. (Sketch) The topological universal covering space \( \tilde{G} \) of \( G \) can be constructed as the quotient of the space \( P(G,1) \) of continuous paths in \( G \) originating at the identity by the equivalence relation \( \sim \) that declares two paths equivalent if and only if they have the same endpoint and are homotopic with extreme points fixed. The projection \( p : \tilde{G} \to G \) maps a path to its endpoint. Since \( p \) is a local homeomorphism, we can lift the smooth structure of \( G \) to \( \tilde{G} \). Finally, given two paths \( \gamma_1, \gamma_2 \in P(G,1) \), we define \( \gamma_1 \cdot \gamma_2 \) to be path given by the pointwise product of \( \gamma_1, \gamma_2 \). It is easy to see that \( \gamma_1 \sim \gamma_1' \) and \( \gamma_2 \sim \gamma_2' \) implies \( \gamma_1 \cdot \gamma_2 \sim \gamma_1' \cdot \gamma_2' \). Hence the multiplication is well defined on \( \tilde{G} \). One checks that it is also smooth.

It follows from Proposition 2.6.1 that \( G \) and \( \tilde{G} \) in Theorem 2.6.2 have isomorphic Lie algebras. A theorem of Ado states that every real Lie algebra admits a faithful representation on \( \mathbb{R}^n \) — namely, an injective homomorphism into \( gl(n,\mathbb{R}) \) — for a sufficiently large \( n \). It follows from Theorem 2.4.2 that every real Lie algebra \( g \) can be realized as the Lie algebra of a subgroup of \( GL(n,\mathbb{R}) \). Further, owing to Theorem 2.6.2, we can find a simply-connected Lie group with Lie algebra \( g \). Hence, owing to Corollary 2.4.6, there is a bijective correspondence between simply-connected Lie groups and Lie algebras.

2.7 Problems

1. Let \( G \) be a Lie group with multiplication map \( \mu : G \times G \to G \) and inversion map \( \iota : G \to G \). Prove that \( d\mu_{(g,h)}(u,v) = (dL_g)_h(v) + (dR_h)_g(u) \) and \( d\iota_g = -(dL_{g^{-1}})_1 \circ (dR_{g^{-1}})_g \) for \( g, h \in G \) and \( u \in T_g G, v \in T_h G \).

2. Let \( G = O(n) \).
   a. Show that \( G^o \subset SO(n) \).
**b.** Prove that any element in $SO(n)$ is conjugate in $G$ to a matrix of the form
\[
\begin{pmatrix}
R_{t_1} & & \\
& \ddots & \\
& & R_{t_p}
\end{pmatrix}
\]
where $R_t$ is the $2 \times 2$ block
\[
\begin{pmatrix}
\cos t & -\sin t \\
\sin t & \cos t
\end{pmatrix}
\]
and $t_1, \ldots, t_p \in \mathbb{R}$.

**c.** Deduce from the above that $SO(n)$ is connected. Conclude that $O(n)$ has two connected components and $SO(n)$ is the identity component.

**d.** Use a similar idea to show that $U(n)$ and $SU(n)$ are connected.

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3. Show that
\[
\exp \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}
\]
for $t \in \mathbb{R}$.

4. Give examples of matrices $A, B \in gl(2, \mathbb{R})$ such that $e^{A+B} \neq e^A e^B$.

---

5. In this problem, we show that the exponential map in a Lie group does not have to be surjective.
   
   **a.** Show that every element $g$ in the image of $\exp : g \to G$ has a square root, namely, there is $h \in G$ such that $h^2 = g$.
   
   **b.** Prove that $\text{trace } A^2 \geq -2$ for any $A \in SL(2, \mathbb{R})$. (Hint: $A$ satisfies its characteristic polynomial equation $X^2 - 2(\text{trace } X)X + (\det X)I = 0$.)
   
   **c.** Deduce from the above that $\begin{pmatrix} -2 & 0 \\ 0 & -1/2 \end{pmatrix}$ does not lie in the image of $\exp : sl(2, \mathbb{R}) \to SL(2, \mathbb{R})$.

6. Let $G$ be a Lie group. Prove that $G$ does not have small subgroups, that is, there exists an open neighborhood $U$ of 1 such that $\{1\}$ is the only subgroup of $G$ entirely contained in $U$. (Hint: Use the exponential map.)

7. Let $S$ denote the vector space of $n \times n$ real symmetric matrices and let $S^+$ denote the open subset (positive cone) of $S$ consisting of positive definite matrices.
   
   **a.** Prove that the exponential map of matrices sets up a bijection from $S$ onto $S^+$. (Hint: Prove it first for diagonal matrices.)
2.7. PROBLEMS

b. Show that \( \langle X, Y \rangle = \text{trace}(XY) \) for \( X, Y \in S \) defines a positive definite symmetric bilinear form on \( S \).

c. Check that \( \text{ad}_Z X \in S \) for all \( Z \in so(n) \) and \( X \in S \), and \( \langle \text{ad}_Z X, Y \rangle + \langle X, \text{ad}_Z Y \rangle = 0 \) for all \( Z \in so(n) \) and \( X, Y \in S \).

d. For \( X \in S \), show that there is an orthogonal decomposition \( S = T_X \oplus N_X \), where \( T_X = \{ [Z, X] \in S \mid Z \in so(n) \} \) and \( N_X = \{ Y \in S \mid [Y, X] = 0 \} \).

e. For \( X \in S \), check that \( e^{-X}(d\exp)_X : T_X S \cong S \to T_I S^+ \cong S \) is given by

\[
e^{-X}(d\exp)_X = I - \frac{e^{-\text{ad}_X}}{\text{ad}_X}.
\]

(Hint: Check the formula separately on \( N_X \) and \( T_X \).)

f. Deduce from the above that \( \exp : S \to S^+ \) is a global diffeomorphism.

8  a. Determine the center of \( SU(n) \).

b. Construct a diffeomorphism \( SU(n) \times S^1 \to U(n) \). Is it an isomorphism of Lie groups?

9  Prove that the kernel of the adjoint representation of a Lie group is the center.

10 Prove that the fundamental group of a connected Lie group is Abelian. (Hint: Use Proposition 2.6.1 and Problem 9 of Chapter 1.)