Lecture Notes on Compact Lie Groups and Their Representations

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CHAPTER 1

Compact topological groups

In this introductory chapter, we essentially introduce our very basic objects of study, as well as some fundamental examples. We also establish some preliminary results that do not depend on the smooth structure, using as little as possible machinery. The idea is to paint a picture and plant the seeds for the later development of the heavier theory.

1.1 Topological groups and continuous actions

A *topological group* is a group G endowed with a Hausdorff topology such that the group operations are continuous; namely, we require that the multiplication map and the inversion map

$$\mu : G \times G \to G, \qquad \iota : G \to G$$

be continuous maps.

A *continuous action* of a topological group G on a topological space X is a continuous map

$$\Phi : G \times X \to X$$

such that

$$\begin{aligned} \Phi(1, x) &= x, \\ \Phi(g_2, \Phi(g_1, x)) &= \Phi(g_2 g_1, x), \end{aligned}$$

where g_1, $g_2 \in G$, $x \in X$ and 1 denotes the identity element of G. We shall write the above relations when there is no ambiguity as

$$1 \cdot x = x \quad \text{and} \quad g_2 \cdot (g_1 \cdot x) = (g_2 g_1) x.$$

Let Φ be a continuous action of G on X and set $\varphi_g(x) = \Phi(g, x)$. Then $\varphi_g : X \to X$ is a homeomorphism with inverse $\varphi_{g^{-1}}$ and $g \mapsto \varphi_g$ defines a homomorphism φ from G into the group $\mathrm{Homeo}(X)$ of homeomorphisms of X. If, instead, if we fix $x \in X$ and let g vary in G, we obtain the *orbit* of the action through $x \in X$, namely,

$$G(x) = \{g \cdot x \mid g \in G\}.$$

The "orbit map" $g \mapsto g \cdot x$, $G \to X$ has image $G(x)$ and sets up a bijection

$$G/G_x \cong G(x), \tag{1.1.1}$$

where

$$G_x = \{g \in G \mid g \cdot x = x\}$$

is a closed subgroup called the *isotropy group* or *stabilizer* of x. Note that $G_{g\cdot x} = gG_xg^{-1}$ for all $g \in G$. It is not difficult to see that (1.1.1) is a homeomorphism if G is compact and X is Hausdorff, where G/G_x carries the quotient topology.

If $\Psi : G \times Y \to Y$ is another cotinuous action, a continuous map $f : X \to Y$ is called *equivariant* or a *G-map* if

$$f(\Phi(g, x)) = \Psi(g, f(x))$$

for all $g \in G$, $x \in X$, or, equivalently, $f(g \cdot x) = g \cdot f(x)$. An equivariant map clearly maps orbits to orbits.

A continuous action $\varphi : G \to \mathrm{Homeo}(X)$ is called:

- *effective* if φ is injective;
- *linear* if X is a vector space V and φ_g is a linear map for all $g \in G$;
- *orhogonal* if X is an Euclidean space and φ_g is an orthogonal transformation for all $g \in G$;
- *unitary* if X is a complex vector space equipped with an Hermitian inner product and φ_g is an unitary transformation for all $g \in G$;
- *isometric* if X is a Riemannian manifold and φ_g is an isometry for all $g \in G$.

A linear action is also called a *representation*. We talk of a *real representation* in case V is a real vector space and of a *complex representation* in case V is a complex vector space.

1.1.2 Examples (i) The real line $\mathbb{R}$ with its additive group structure is an example of commutative or Abelian topological group. Similarly, $\mathbb{R}^n$ and more generally any vector space V together with addition of vectors can be considered as an Abelian topological group.

(ii) The nonzero complex numbers $\mathbb{C}^\times$ with its multiplicative structure is an Abelian topological group. The (closed) subset of $\mathbb{C}^\times$ consisting of unit complex numbers, which is homeomorphic to the circle S^1, is closed under multiplication and hence inherits the structure of (compact) Abelian topological group. More generally, a finite product

$$T^n = S^1 \times \cdots \times S^1 \quad (n \text{ factors}),$$

called a *torus*, is a compact Abelian topological group.

(iii) Any group can be considered as a topological group in a trivial way just by endowing it with the discrete topology. An important class of such examples consist of the *finite groups*. Indeed, many results about compact (Lie) groups that we will study in this book can be considered as generalizations of easier constructions for finite groups.

(iv) A fundamental class of examples of mostly non-Abelian topological groups consist of the *classical matrix groups*; they are:

$$\begin{aligned}
GL(n,\mathbb{R}) &= \{A \in M(n,\mathbb{R}) \mid \det(A) \neq 0\} \quad (\textit{real general linear group}),\\
GL(n,\mathbb{C}) &= \{A \in M(n,\mathbb{C}) \mid \det(A) \neq 0\} \quad (\textit{complex general linear group}),\\
GL(n,\mathbb{H}) &= \{A \in M(n,\mathbb{H}) \mid A \text{ is invertible}\} \quad (\textit{quaternionic general linear group}),\\
O(n) &= \{A \in GL(n,\mathbb{R}) \mid AA^t = I\} \quad (\textit{orthogonal group}),\\
U(n) &= \{A \in GL(n,\mathbb{C}) \mid AA^* = I\} \quad (\textit{unitary group}),\\
Sp(n) &= \{A \in GL(n,\mathbb{H}) \mid AA^* = I\} \quad (\textit{symplectic group}),\\
SL(n,\mathbb{R}) &= \{A \in GL(n,\mathbb{R}) \mid \det(A) = 1\} \quad (\textit{real special linear group}),\\
SL(n,\mathbb{C}) &= \{A \in GL(n,\mathbb{C}) \mid \det(A) = 1\} \quad (\textit{complex special linear group}),\\
SO(n) &= \{A \in O(n) \mid \det(A) = 1\} \quad (\textit{special orthogonal group}),\\
SU(n) &= \{A \in U(n) \mid \det(A) = 1\} \quad (\textit{special unitary group})
\end{aligned}$$

Here $M(n,\mathbb{F})$ denotes the real vector space of $n \times n$ matrices with entries in $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the reader which is unfamiliar with the quaternion algebra can now safely ignore such examples; we will come back to quaternions below); also, A^t denotes the transpose matrix of A and A^* denotes its conjugate transpose. It is easy to check that these are groups with respect to matrix multiplication. Each group G is considered with the topology induced from its inclusion in the corresponding vector space $M(n,\mathbb{F})$. Plainly, multiplication of matrices

$$M(n,\mathbb{F}) \times M(n,\mathbb{F}) \to M(n,\mathbb{F})$$

is a bilinear map and thus continuous; it follows that the induced multiplication on G is also continuous. The inversion map is only defined on $GL(n,\mathbb{F})$, and its continuity follows from the explicit formula for the inverse of a matrix given in terms of the determinant and the cofactors. Since each G is a topological subspace of an appropriate $GL(n,\mathbb{F})$, this proves that G is a topological group. We want to point out that the compact groups in the list are precisely $O(n)$, $U(n)$, $Sp(n)$, $SO(n)$ and $SU(n)$. For instance, viewing $O(n)$ as the subset of $M(n,\mathbb{R}) = \mathbb{R}^{n^2}$ consisting of matrices whose columns form an orthonormal basis of $\mathbb{R}^n$ yields that $O(n)$ is a closed and bounded (with respect to the Euclidean norm) and hence compact. The other groups are similarly shown to be compact.

(v) There are also five isolated examples

$$E_6, E_7, E_8, F_4, G_2$$

of compact simple groups that are called *exceptional*, in the sense that they do not fit into classical families like in (iv) and involve somehow exotic algebraic structures. We will say more about them in later chapters in the context of Lie theory.

Compact topological groups form a broad class that includes examples such as infinite products of circles (with the product topology) and p-adic integers (as inverse limits of finite groups). Beginning in the next chapter, we shall restrict our discussion to compact *Lie* groups, namely, those endowed with a compatible manifold structure.

1.2 Representations

Representation theory is one of the main topics in these lecture notes. If $\varphi : G \to GL(V)$ is a representation of a topological group, we will also say that V is a *linear G-space*.

Two representations $\varphi : G \to GL(V)$ and $\psi : G \to GL(W)$ of the same group G are considered *equivalent* if there exists an equivariant linear isomorphism $A : V \to W$, that is A is an isomporhism and

$$A \circ \varphi(g) = \psi(g) \circ A$$

for all $g \in G$.

Fix a representation $\varphi : G \to GL(V)$. A subspace U of V is called *invariant* if

$$\varphi(g)U \subset U$$

for all $g \in G$. In this case φ restricts to a representation of G on U, which is called a *subrepresentation* (or a *component*) of φ. It is clear that $\{0\}$ and V are always invariant subspaces of φ. The representation φ is called *irreducible* if these are the only invariant subspaces that it admits. Finally, φ is called *completely reducible* if V can be written as a non-trivial direct sum of invariant, irreducible subspaces (compare subsection 1.2).

1.2.1 Examples (i) Let G be $SO(n)$ and $V = \mathbb{R}^n$. Then left-multiplication

$$\varphi(g) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = g \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

defines a real representation. Since any unit vector can be mapped to any unit vector by an element of $SO(n)$, this representation is irreducible. For obvious reasons, this representation is called *natural* or *canonical*. Another name for it is *vector representation*. This set of names also apply to the next example.

(ii) Let G be $SU(n)$ or $U(n)$ and $V = \mathbb{C}^n$. Then left-multiplication

$$\varphi(g)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = g\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

defines a complex representation. Since any unit vector can be mapped to any unit vector by an element of G, this representation is irreducible. By ignoring the complex structure on V and viewing $\mathbb{C}^n$ as a $2n$-dimensional real vector space, we get a real representation. This process is called *realification*.

(iii) Let $G = SO(n)$ and let V consist of real symmetric $n \times n$ matrices. Then conjugation

$$\varphi(g)X = gXg^{-1} = gXg^t$$

for $g \in G$ and $X \in V$ defines a representation. This representation is not irreducible, as the subspace of scalar matrices is invariant; further, the subspace of traceless matrices V_0 is an invariant complement. It is not hard to see that $V = \mathbb{R} \cdot I \oplus V_0$ is a decomposition into irreducible components. In fact, Suppose U is a nonzero invariant subspace of V_0 and take a nonzero matrix $A \in V_0$. The conjugacy class of A contains a diagonal matrix, so we may assume A is diagonal, $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Since $\sum \lambda_i = 0$, not all λ_i are equal. Of course, the conjugacy class of A contains all diagonal matrices obtained from permutation of the diagonal entries of A, so we may assume $\lambda_1 \neq \lambda_2$. For the same reason, the $B = \mathrm{diag}(\lambda_2, \lambda_1, \lambda_3 \ldots, \lambda_n) \in U$, and then $\frac{1}{\lambda_1 - \lambda_2}(A - B) = \mathrm{diag}(1, -1, 0, \ldots, 0) \in U$. From here, one sees that U contains all diagonal matrices in V_0 and hence all matrices in V_0. (cf. Problem 7).

(iv) Let G be the additive group of the real numbers $(\mathbb{R}, +)$. Then

$$\varphi : G \to GL(\mathbb{R}^2), \qquad \varphi(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

is a representation which is not completely reducible, as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ spans an invariant subspace which does not admit an invariant complement.

(v) Let $G = SO(n)$ and let V_d be the space of real homogeneous polynomials of degree d on $x_1, \ldots, x_n$. Then

$$\varphi_d(g)p(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}) = p(g^{-1}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix})$$

defines a representation of $SO(n)$ on V_d. Consider the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

acting on V_d. Then $\Delta(V_d) \subset V_{d-2}$. Recall that Δ is the divergent of the gradient, so it is intrinsically associated to the metric structure of $\mathbb{R}^n$. Since $SO(n)$ acts on $\mathbb{R}^n$ by isometries, it commutes with Δ: $\Delta\varphi(g) = \varphi(g)\Delta$ for $g \in SO(n)$. It follows that $\mathcal{H}_d := \ker\Delta \subset V_d$ and $\mathrm{im}\Delta \subset V_{d-2}$ are invariant subspaces; the elements of $\mathcal{H}_d$ are called *(real) harmonic polynomials of degree d in n variables*.

Construction of representations

Let V and W be linear G-spaces. Then the following spaces carry naturally induced structures of linear G-spaces ($g \in G$):

a. direct sum $V \oplus W$: $g \cdot (x, y) = (g \cdot x, g \cdot y)$ ($x \in V$, $y \in W$);
b. dual space V^*: $(g \cdot \lambda)(x) = \lambda(g^{-1} \cdot x)$ ($\lambda \in V^*$, $x \in V$);
c. tensor product $V \otimes W$: $g \cdot (x \otimes y) = g \cdot x \otimes g \cdot y$ ($x \in V$, $y \in W$);
d. space of homomorphisms (linear maps) $\mathrm{Hom}(V, W)$: $(g \cdot A)x = g \cdot A(g^{-1} \cdot x)$ ($A \in \mathrm{Hom}(V, W)$, $x \in V$);
e. exterior square $\Lambda^2(V)$: $g \cdot (x \wedge y) = g \cdot x \wedge g \cdot y$ ($x, y \in V$);

to name a few of the most common instances.

Schur's lemma

Despite the simplicity of its proof, the following result has powerful applications.

1.2.2 Lemma (Schur) *Let V and W be irreducible G-spaces. If $A : V \to W$ is an equivariant linear map, then A is an isomorphism or $A = 0$.*

Proof. Since A is equivariant, its kernel $\ker A \subset V$ and its image $\mathrm{im}A \subset W$ are invariant subspaces. By the irreducibility we have that

$$\ker A = \left\{ \begin{array}{ll} \{0\} & \text{or} \\ V, & \end{array} \right. \quad \text{and} \quad \mathrm{im}\, A = \left\{ \begin{array}{ll} \{0\} & \text{or} \\ W. & \end{array} \right.$$

Suppose that $A \neq 0$. Then $\ker A = \{0\}$ and $\mathrm{im}\, A = W$. Hence A is an isomorphism. □

1.2.3 Corollary *Let V be an irreducible G-space over $\mathbb{R}$. Then the space $\mathrm{End}_G(V)$ of G-equivariant endomorphisms of V is a real division algebra.*

1.2.4 Remark Frobenius' theorem says that every (associative, finite-dimensional) real division algebra is one of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Accordingly, in view of

Corollary 1.2.3, a real irreducible representation of a topological group G on a finite-dimensional real vector space is called of *real type*, *complex type* or *quaternionic type* whether $\mathrm{End}_G(V)$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, respectively.

1.2.5 Corollary *Let V be a finite-dimensional irreducible linear G-space over $\mathbb{C}$. If $A : V \to V$ is an equivariant linear map, then $A = \lambda_0 I$ for some $\lambda_0 \in \mathbb{C}$.*

Proof. We have $A - \lambda I$ is also equivariant for all $\lambda \in \mathbb{C}$. Let λ_0 be an eigenvalue of A. Then $A - \lambda_0 I$ is not invertible, so $A - \lambda_0 I = 0$. □

1.2.6 Corollary *A finite-dimensional complex irreducible representation of an Abelian group G must be one-dimensional.*

Proof. Let $\varphi : G \to \mathbf{GL}(V)$ be a complex irreducible representation. Since G is Abelian,

$$\varphi(g') \circ \varphi(g) = \varphi(g'g) = \varphi(gg') = \varphi(g) \circ \varphi(g')$$

for all g, $g' \in G$. So for all $g \in G$, $\varphi(g) : V \to V$ is equivariant. Therefore $\varphi(g) = \lambda_g I$ where $\lambda_g \in \mathbb{C}$. Now every subspace of V is invariant, hence $\dim V = 1$ by irreducibility. □

Main problems in representation theory

We already have elements to formulate some problems in representation theory.

a. What types of groups have the property that all of their representations are completely reducible?
b. How to efficiently decide whether a given representation is irreducible?
c. How to classify irreducible representations of a given group up to equivalence?

We will solve (a) and (b) for compact topological groups in this chapter. In later chapters, we will solve (c) for compact Lie groups.

1.3 Adjoint action

Let G be a topological group. For each $g \in G$, the *inner automorphism* defined by g is the automorphism Inn_g of G given by $\mathrm{Inn}_g(x) = gxg^{-1}$. Now G acts on itself by inner automorphisms; this is the *adjoint action* of G on itself:

$$\begin{array}{rccc} \widetilde{\mathrm{Ad}}: & G & \to & \mathrm{Aut}(G) \\ & g & \mapsto & \mathrm{Inn}_g \end{array}$$

Of course, the adjoint action is trivial ($\widetilde{\mathrm{Ad}_g} = I$ for all $g \in G$) if and only if G is Abelian, so we can view the adjoint action of G on itself as a way of organizing geometrically the non-commutativity of G.

In case of matrix groups, $\widetilde{\mathrm{Ad}}$ is G-conjugation of matrices of G, and the orbits of this action are the conjugacy classes of matrices in G.

The case of S^3

Recall the (associative) real division algebra of quaternions

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}.$$

It consists of a four-dimensional vector space over $\mathbb{R}$ with basis $\{1, i, j, k\}$ and the non-commutative multiplication rules

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

If $q = a + bi + cj + dk$, its conjugate is $\bar{q} = a - bi - cj - dk$, the real and imaginary parts of q are, respectively, $\Re q = a$ and $\Im q = q - \Re q$, and the norm of q is $|q| = \sqrt{q\bar{q}} = a^2 + b^2 + c^2 + d^2$. It follows that every $q \neq 0$ has the inverse element $q^{-1} = \bar{q}/|q|^2$ (division algebra property) and thus the nonzero quaternions form a multiplicative group $\mathbb{H}^\times$.

Since $|qq'| = |q||q'|$ for all $q, q' \in \mathbb{H}$, the unit sphere

$$S^3 = \{q \in \mathbb{H} : |q| = 1\}$$

is a closed subgroup. It acts on $\mathbb{R}^4$ as follows. Identify $\mathbb{H} = \mathbb{R}^4$ and consider the standard inner product $\langle x, y\rangle = \Re(x\bar{y})$. Define

$$S^3 \times \mathbb{H} \to \mathbb{H}, \qquad q \cdot x = qxq^{-1}.$$

This action is $\mathbb{R}$-linear and orthogonal, as

$$|qxq^{-1}| = |q||x||q|^{-1} = 1 \cdot |x| \cdot 1 = |x|$$

for $q \in S^3$. This defines a homomorphism $\tilde{\psi} : S^3 \to SO(4)$.

It is clear that the real line

$$\mathbb{R} \cdot 1 = \{a : a \in \mathbb{R}\} \subset \mathbb{H}$$

is fixed by $\tilde{\psi}$. Therefore $\tilde{\psi}$ decomposes as $1 \oplus \psi$, and $\mathbb{H}$ decomposes into $\mathbb{R} \cdot 1 \oplus \Im\mathbb{H}$ where $\Im\mathbb{H}$ is the space of imaginary quaternions.

Now $\psi : S^3 \to SO(3)$ is a homomorphism. It is easy to check that $\ker \psi = \{\pm 1\}$. The induced map

$$S^3/\{\pm 1\} \to SO(3) \tag{1.3.1}$$

is continuous and injective. Since the domain is compact, it is a homeomorphism onto its image. Every element of $SO(3)$ is a product $R_i R_j R_i'$ where R_i and R_i' are rotations around the i-axis and R_j is a rotation around the j-axis (Euler decomposition). Since $\psi(e^{i\theta})$ (resp. $\psi(e^{j\theta})$, $\psi(e^{k\theta})$) is a rotation around the i- (resp. j-, k-) axis, we deduce that ψ is surjective. Now (1.3.1) is a homeomorphism. Note that $S^3/\{\pm 1\} \cong \mathbb{R}P^3$ and $\psi : S^3 \to SO(3)$ is a double covering.

It is now easy to understand geometrically the adjoint action of S^3 on itself. For $x = a + bi + cj + dk \in S^3$, there is $q \in S^3$ such that $q \cdot x = a + b'i$, where $b' = \pm\sqrt{b^2 + c^2 + d^2}$. Thus we see that each orbit intersects the circle

$$S^1 = \{\cos\theta + \sin\theta i : \theta \in \mathbb{R}\} \subset S^3 \tag{1.3.2}$$

in precisely two points. The adjoint orbit of $e^{i\theta} \in S^1$ is a round 2-sphere $S^2(\sin\theta)$. It is also clear that the circle (1.3.2) meets the adjoint orbits perpendicularly.

Incidentally, we can also view $\mathbb{H}$ as a right $\mathbb{C}$-module. In this case $\{1, j\}$ is a basis and

$$\mathbb{H} = \{\alpha + j\beta : \alpha, \beta \in \mathbb{C}\}.$$

The action of S^3 on $\mathbb{H}$ by left multiplication, namely,

$$\begin{aligned} (\alpha + j\beta) \cdot 1 &= \alpha + j\beta, \\ (\alpha + j\beta) \cdot j &= \alpha j + j\beta j = -\bar{\beta} + j\bar{\alpha}, \end{aligned}$$

defines a 2-dimensional complex representation of S^3 on $\mathbf{C}^2$,

$$\alpha + j\beta \mapsto \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix},$$

which we denote by φ_{-1}. Note that $\mathrm{im}\varphi_{-1} \subset SU(2)$, that is, φ_{-1} is unitary with respect to the standard Hermitian product on $\mathbb{C}^2$. It is easy to check that φ_{-1} in fact defines a topological isomorphism (that is, a group homomorphism and a homeomorphism) between S^3 and $SU(2)$.

1.4 Averaging method and Haar integral on compact groups

If G is a finite group and V is a linear G-space, then we can produce G-fixed points as follows. For each $x \in V$ the center of mass of the orbit $G(x)$

$$\bar{x} = \frac{1}{|G|}\sum_{g\in G} g \cdot x,$$

where $|G|$ denotes the cardinality of G, is a G-fixed point, namely

$$h \cdot \bar{x} = \frac{1}{|G|}\sum_{g\in G} h \cdot (g \cdot x) = \frac{1}{|G|}\sum_{g\in G} (hg) \cdot x = \bar{x}$$

for all $h \in G$. In terms of coordinates with respect to a fixed coordinate system, the coordinates of $\bar{x}$ turn out to be simply the average of the corresponding coordinates of the points in $G(x)$. In order to generalize this procedure to nonfinite groups, we need a notion of average value of a continuous function. We can establish such a notion in the general setting of compact topological groups as follows.

Let G be a compact topological group and let $C(G)$ denote the space of all real valued continuous functions on G. Due to compactness, every $f \in C(G)$ is uniformly continuous; it is convenient to express this condition by saying that given $\epsilon > 0$ there exists a neighborhood U of the identity $1 \in G$ such that for all $x, y \in G$,

$$xy^{-1} \in U \Rightarrow |f(x) - f(y)| < \epsilon. \tag{1.4.1}$$

Consider the continuous action of $G \times G$ on G on the left and on the right given by

$$(G \times G) \times G \to G, \qquad (g_1, g_2) \cdot g = g_1 g g_2^{-1}.$$

We consider $C(G)$ with the sup norm topology. Then there is an induced continuous action of $G \times G$ on $C(G)$ given by

$$(G \times G) \times C(G) \to C(G), \qquad [(g_1, g_2) \cdot f](x) = f((g_1, g_2)^{-1} \cdot x) = f(g_1^{-1} g g_2).$$

1.4.2 Theorem *There exists a unique $G \times G$-invariant positive continuous linear form $I : C(G) \to \mathbb{R}$ such that $I(1) = 1$; here,* positive *means we have that $I(f) \geq 0$ if $f \geq 0$, and $I(f) > 0$ if $f \geq 0$ and f is not identically zero.*

1.4.3 Remark By the Riesz-Markov-Kakutani Representation Theorem, every positive continuous linear form I on $C(G)$ arises as integration with respect to a unique regular Borel measure $d\mu$. In view of that, $I(f) = \int_G f(g)\, d\mu(g)$ is called the *Haar integral* of f.

Proof of Theorem 1.4.2. We first prove existence. For each multiset (i.e., finite subset with positive integral multiplicities) $A \subset G \times G$ set

$$\Gamma(A, f) = \frac{1}{|A|} \sum_{a \in A} m(a)(a \cdot f) \in C(G),$$

where $m(a)$ denotes the multiplicity of $a \in A$. The following multiplicative property for Γ holds:

$$\begin{aligned} \Gamma(A, \Gamma(B, f)) &= \frac{1}{|A|} \sum_{a \in A} m(a)(a \cdot \Gamma(B, f)) \\ &= \frac{1}{|A|} \frac{1}{|B|} \sum_{a \in A} \sum_{b \in B} m(a) m(b)(ab) \cdot f \\ &= \frac{1}{|AB|} \sum_{c \in AB} m(c)(c \cdot f) \\ &= \Gamma(AB, f), \end{aligned}$$

where $AB \subset G \times G$ is the multiset given by $\{ab : a \in A \text{ and } b \in B\}$ and $m(c) = \sum_{ab=c} m(a)m(b)$ for $c \in AB$.

For each $h \in C(G)$, define the variation of h to be

$$\omega(h) = \max\{h(x) : x \in G\} - \min\{h(x) : x \in G\}.$$

Set

$$\Delta_f = \{\Gamma(A, f) : A \subset G \times G \text{ is a finite multiset}\},$$

and

$$\Omega_f = \inf\{\omega(h) : h \in \Delta_f\}.$$

Note that Δ_f is an *equibounded* family of functions, that is,

$$\text{there exists } M > 0 \text{ such that } |h(x)| \leq M \text{ for all } h \in \Delta_f.$$

This follows simply because f is bounded. Moreover, we claim that Δ_f is an *equicontinuous* family of functions, that is, given $\epsilon > 0$ there exists a neighborhood V of the identity such that

$$xy^{-1} \in V \Rightarrow |h(x) - h(y)| < \epsilon$$

for all $h \in \Delta_f$. In order to show this, first consider the neighborhood U of 1 such that (1.4.1) holds and note that by compactness of G we can find another neighborhood V of 1 contained in U such that

$$g^{-1}Vg \subset U \text{ for all } g \in G. \tag{1.4.4}$$

Now if $xy^{-1} \in V$ and $a = (g_1, g_2) \in A$, then

$$|a \cdot f(x) - a \cdot f(y)| = |f(g_1^{-1}xg_2) - f(g_1^{-1}yg_2)| < \epsilon$$

because

$$g_1^{-1}xg_2(g_1^{-1}yg_2)^{-1} = g_1^{-1}(xy^{-1})g_1 \in U,$$

by (1.4.1) and (1.4.4). Therefore for $xy^{-1} \in V$ and $h = \Gamma(A, f) \in \Delta_f$ we have

$$\begin{aligned} |h(x) - h(y)| &\leq \frac{1}{|A|} \sum_{a \in A} m(a)|a \cdot f(x) - a \cdot f(y)| \\ &< \epsilon, \end{aligned}$$

which proves the claim.

Let $\{h_n\}$ be a minimizing sequence in Δ_f, namely, $\lim \omega(h_n) = \Omega_f$. Owing to the Arzelá-Ascoli theorem, there exists a convergent subsequence which we still denote with $\{h_n\}$. Put $\bar{h} = \lim h_n \in C(G)$. Then $\omega(\bar{h}) = \Omega_f$, and we claim that $\omega(\bar{h}) = 0$. Suppose, on the contrary, that $\omega(\bar{h}) > 0$. Then there exists $M < \max_G \bar{h}$ and an open subset U of G such that

$$\bar{h}(x) \leq M$$

for all $x \in U$. By compactness of G we can write $G = \cup_{i=1}^{n} g_i U$ for some $g_i \in G$. Set $A = \{(g_i, 1) : i = 1, \ldots, n\}$ (all multiplicities 1). Then, if $x \in g_i U$, we have

$$[(g_i, 1) \cdot \bar{h}](x) = \bar{h}(g_i^{-1} x) \leq M,$$

and this implies that

$$\min_G \bar{h} \leq \Gamma(A, \bar{h}) \leq \frac{M + (n-1) \max_G \bar{h}}{n} < \max_G \bar{h}$$

Now $\omega(\Gamma(A, \bar{h})) < \omega(\bar{h}) = \Omega_f$. Since

$$\lim \Gamma(A, h_n) = \lim \Gamma(A, \bar{h}) \quad \text{and} \quad \lim \omega(\Gamma(A, h_n)) = \omega(\Gamma(A, \bar{h})),$$

for n sufficiently big, we now get that

$$\omega(\Gamma(A, h_n)) < \Omega_f,$$

a contradiction to the fact that $\Gamma(A, h_n) \in \Delta_f$ (by the multiplicative property of Γ). Hence $\omega(\bar{h}) = 0$.

Now $\bar{h}$ is a constant, and we set $I(f) = \bar{h}$. Notice that I is linear because $\Gamma(A, f)$ is linear on f for fixed A; I is positive because $\Gamma(A, f) \geq 0$ for $f \geq 0$; and I is equivariant because $\Delta_{b \cdot f} = \Delta_f$ for all $b \in G \times G$. Further, I is continuous as $|I(f)| \leq ||f||_\infty$ for all $f \in C(G)$. This completes the proof of existence.

Suppose now I' is another $G \times G$-invariant positive linear form on $C(G)$ with $I'(1) = 1$. Given $f \in C(G)$ and $\epsilon > 0$, there exists a multiset $A \subset G \times G$ with

$$|\Gamma(A, f) - I(f)| < \epsilon. \tag{1.4.5}$$

By linearity and equivariance of I',

$$I'(\Gamma(A, f)) = \Gamma(A, I'(f)) = I'(f).$$

Further,

$$I'(I(f)) = I(f) I'(1) = I(f).$$

Finally, combining (1.4.5) with the positivity of I' yields

$$\begin{aligned} |I'(f) - I(f)| &= |I'(\Gamma(A, f)) - I'(I(f))| \\ &\leq I'\left(|\Gamma(A, f) - I(f)|\right) \\ &< I'(\epsilon) \\ &= \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that $I'(f) = I(f)$. □

Existence of invariant inner (Hermitian) product and complete reducibility

1.4.6 Proposition *Let G be a compact topological group and let V be a real (resp. complex) linear G-space. Then there exists a G-invariant inner (resp. Hermitian) product $\langle\cdot,\cdot\rangle$ on V, namely*

$$\langle g\cdot u, g\cdot v\rangle = \langle u,v\rangle$$

for u, $v\in V$ and $g\in G$.

Proof. Let $\langle\cdot,\cdot\rangle_0$ be an arbitrary inner (resp. Hermitian) product on V. Set

$$\langle u,v\rangle = \int_G \underbrace{\langle g\cdot u, g\cdot v\rangle_0}_{=f(g)}\,d\mu(g).$$

Then $\langle\cdot,\cdot\rangle$ is an inner (resp. Hermitian) product on V and, for $h\in G$,

$$\begin{aligned}\langle h\cdot u, h\cdot v\rangle &= \int_G \langle g\cdot(h\cdot u), g\cdot(h\cdot v)\rangle_0\,d\mu(g)\\ &= \int_G \langle (gh)\cdot u, (gh)\cdot v\rangle_0\,d\mu(g)\\ &= \int_G f(gh)\,d\mu(g)\\ &= \int_G f(g)\,d\mu(g)\\ &= \langle u,v\rangle,\end{aligned}$$

which completes the proof. □

1.4.7 Corollary *Every real (resp. complex) representation of a compact topological group is equivalent to an orthogonal (resp. unitary) one.*

1.4.8 Theorem *Every finite-dimensional representation of a compact topological group is completely reducible.*

Proof. Let V be a real (resp. complex) linear G-space. If it is not irreducible, there is a proper invariant subspace U. Equip V with an invariant inner (resp. Hermitian) product. Then the orthogonal complement $U^\perp$ is also invariant and $V = U\oplus U^\perp$. Since the dimensions of U and $U^\perp$ are strictly smaller than that of V, the result now follows from an induction on the dimension of V. □

1.5 The character theory of Frobenius-Schur

Throughout this section we assume that representations are finite-dimensional. Let G be a compact topological group. We consider the following inner product on the space $C(G,\mathbb{C})$ of complex-valued continuous functions on G:

$$(f_1, f_2) = \int_G f_1(g)\overline{f_2(g)}dg \quad (f_1, f_2 \in C(G,\mathbb{C})). \tag{1.5.1}$$

If $\varphi : G \to U(V)$ is a unitary representation and u, $v \in V$, define the *matrix coefficient*

$$\varphi_{u,v}(g) = \langle \varphi(g)v, u\rangle \quad (g \in G).$$

Then $\varphi_{u,v} \in C(G,\mathbb{C})$. If $\{e_1, \ldots, e_n\}$ is an orthonormal basis of V then

$$\varphi(g)e_j = \sum_{i=1}^{n} \varphi_{e_i,e_j}(g)e_i$$

and the $\varphi_{e_i,e_j}(g)$ are the usual matrix coefficients of $\varphi(g)$ with respect to the fixed orthonormal basis. Further, the *character* of φ is the element of $C(G,\mathbb{C})$ defined by

$$\chi_\varphi(g) = \operatorname{trace}\varphi(g) = \sum_{i=1}^{n} \varphi_{ii}(g),$$

where $\varphi_{ij} = \varphi_{e_i,e_j}$.

1.5.2 Remark (i) It follows from the invariance of the trace under conjugation that the character is constant in each conjugacy class, namely,

$$\chi_\varphi(hgh^{-1}) = \chi_\varphi(g) \quad \text{for all } g, h \in G.$$

(ii) For the same reason as in (i), we see that equivalent representations have the same character.

(iii) If $\varphi = \varphi_1 \oplus \varphi_2$ then $\chi_\varphi = \chi_{\varphi_1} + \chi_{\varphi_2}$.

1.5.3 Proposition (orthogonality relations) *(i) If $\varphi : G \to U(V)$ and $\varphi' : G \to U(V')$ are inequivalent irreducible unitary representations, and u, $v \in V$, and u', $v' \in V'$, then*

$$(\varphi_{u,v}, \varphi'_{u',v'}) = 0.$$

(ii) If $\varphi : G \to U(V)$ is an irreducible unitary representation, and u, v, u', $v' \in V$, then

$$(\varphi_{u,v}, \varphi_{u',v'}) = \frac{1}{\dim V}\overline{\langle u, u'\rangle}\langle v, v'\rangle.$$

Proof. (i) Let $A_0 \in \operatorname{Hom}(V, V')$. Define

$$A = \int_G \varphi'(g)A_0\varphi(g)^{-1}d\mu(g).$$

Then $A : V \to V'$ is a linear G-map, which we write $A \in \mathrm{Hom}_G(V, V')$. Schur's lemma implies that $A = 0$. Hence

$$\begin{aligned} 0 &= \langle Au, u' \rangle = \int_G \langle \varphi'(g) A_0 \varphi(g)^{-1} u, u' \rangle d\mu(g) \\ &= \int_G \langle A_0 \varphi(g)^{-1} u, \varphi'(g)^{-1} u' \rangle d\mu(g). \end{aligned}$$

Choose now $A_0(w) = \langle w, v \rangle v'$. Then

$$\begin{aligned} 0 &= \int_G \langle \varphi(g)^{-1} u, v \rangle \langle v', \varphi'(g)^{-1} u' \rangle d\mu(g) \\ &= \int_G \langle u, \varphi(g) v \rangle \langle \varphi'(g) v', u' \rangle d\mu(g) \\ &= \int_G \varphi'_{u',v'}(g) \overline{\varphi_{u,v}(g)} d\mu(g) \\ &= (\varphi'_{u',v'}, \varphi_{u,v}). \end{aligned}$$

(ii) Let $A_0 \in \mathrm{End}(V) = \mathrm{Hom}(V, V)$ and define

$$A = \int_G \varphi(g) A_0 \varphi(g)^{-1} d\mu(g).$$

Now $A \in \mathrm{End}_G(V)$ and Schur's lemma implies that $A = \lambda I$ for some $\lambda \in \mathbb{C}$. Then

$$\lambda \dim V = \operatorname{trace} A = \int_G (\operatorname{trace} A_0) d\mu(g) = \operatorname{trace} A_0.$$

Choose A_0 as in (i). Then $\operatorname{trace} A_0 = \langle v', v \rangle$ and a computation similar to the one in (i) shows that

$$(\varphi_{u',v'}, \varphi_{u,v}) = \langle Au, u' \rangle = \lambda \langle u, u' \rangle = \frac{1}{\dim V} \langle u, u' \rangle \overline{\langle v, v' \rangle}$$

and this completes the proof. □

1.5.4 Corollary *(i) If φ is irreducible, then $(\chi_\varphi, \chi_\varphi) = 1$.*
(ii) If φ and φ' are irreducible and inequivalent, then $(\chi_\varphi, \chi_{\varphi'}) = 0$.

Proof. (i) Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V. Then

$$(\chi_\varphi, \chi_\varphi) = \sum_{i=1}^{n} \sum_{j=1}^{n} (\varphi_{ii}, \varphi_{jj}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{n} \overline{\langle e_i, e_j \rangle} \langle e_i, e_j \rangle = 1.$$

(ii) This is immediate. □

Let $\hat{G}$ denote the set of equivalence classes of complex (unitary) irreducible representations of G. Every complex representation ρ can be expressed as the direct sum of irreducible ones, namely

$$\rho = \sum_{\varphi \in \hat{G}} m_\rho(\varphi)\varphi \qquad \text{(finite sum)}$$

where $m_\rho(\varphi)$ is the multiplicity of the equivalence class of φ in ρ. Now

$$\chi_\rho = \sum_{\varphi \in \hat{G}} m_\rho(\varphi)\chi_\varphi$$

which implies that

$$m_\rho(\varphi) = (\chi_\rho, \chi_\varphi).$$

1.5.5 Theorem *Let G be a compact topological group. Then two complex representations ρ and ρ' are equivalent if and only if their characters coincide as functions, namely $\chi_\rho = \chi_{\rho'}$. A complex representation ρ is irreducible if and only if $(\chi_\rho, \chi_\rho) = 1$.*

Proof. We have that $\chi_\rho = \chi_{\rho'}$ implies that $(\chi_\rho, \chi_\varphi) = (\chi_{\rho'}, \chi_\varphi)$ and therefore $m_\rho(\varphi) = m_{\rho'}(\varphi)$ for every $\varphi \in \hat{G}$. This shows that ρ and ρ' are equivalent. The other direction was already checked in Remark 1.5.2.

The last assertion follows from the fact that if

$$\rho = \sum_{\varphi \in \hat{G}} m_\rho(\varphi)\chi_\varphi$$

then

$$(\chi_\rho, \chi_\rho) = \sum_{\varphi \in \hat{G}} m_\rho(\varphi)^2$$

and this finishes the proof. □

1.5.6 Corollary *Let G and H denote compact topological groups.*

a. If $\varphi : G \to U(V)$ and $\psi : H \to U(W)$ are irreducible representations, then the outer tensor product $\varphi\hat{\otimes}\psi : G \times H \to U(V \otimes W)$, given by $\varphi\hat{\otimes}\psi(g,h)(v \otimes w) = \varphi(g)v \otimes \psi(h)w$, is also an irreducible representation.

b. Every irreducible representation of $G \times H$ arises as in part (i).

Proof. For part (i), we just note that $\chi_{\varphi\hat{\otimes}\psi}(g,h) = \chi_\varphi(g) \cdot \chi_\psi(h)$ and then

$$\begin{aligned}\int_{G\times H} |\chi_{\varphi\hat{\otimes}\psi}(g,h)|^2 d\mu(g,h) &= \int_G |\chi_\varphi(g)|^2 d\mu(g) \int_H |\chi_\psi(h)|^2 d\mu(h) \\ &= 1 \cdot 1 \\ &= 1.\end{aligned}$$

For part (ii), let U be an irreducible representation of $G \times H$. Restrict U to the subgroup $G \cong G \times \{1\}$ of $G \times H$. Let V be a G-irreducible component of U. Now $W := \mathrm{Hom}_G(V, U) \neq 0$ and this space carries an H-action by pos-composition, since the G- and H-actions on U commute. Define a linear map

$$\Phi : V \otimes W \to U, \qquad \Phi(u \otimes f) = f(u).$$

Then Φ is a $G \times H$-equivariant. Due to the irreducibility of U, Φ is surjective. Moreover, by Schur's lemma $\dim W$ is the number of components of the G-space U that are isomorphic to V, so $\dim W \leq \dim U / \dim V$. It follows that Φ is injective, hence defines a $G \times H$-equivariant isomorphism $V \otimes W \cong U$. Finally, W must be H-irreducible, for an H-invariant decomposition $W = W_1 \oplus W_2$ yields a $G \times H$-invariant decomposition $V \otimes W = V \otimes W_1 \oplus V \otimes W_2$. This completes the proof. □

1.5.7 Example (Representations of the circle) Consider

$$G = S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

Then for each $n \in \mathbb{Z}$ we define a one-dimensional representation

$$\varphi_n : S^1 \to U(1);\ z \mapsto z^n.$$

Notice that

$$\chi_n(z) := \chi_{\varphi_n}(z) = \varphi_n(z) = z^n.$$

Write $z = e^{i\theta}$ for $\theta \in \mathbb{R}$. The Frobenius-Schur orthogonality relations reduce to the Fourier orthogonality relations and they say that

$$\{e^{ni\theta} : n \in \mathbb{Z}\}$$

is an orthonormal set in $C(G, \mathbb{C})$. We also know from Fourier analysis that this set in fact spans a dense subset of $C(G, \mathbb{C})$ with regard to the sup-norm topology. It follows from the orthogonality relations that every complex irreducible continuous representation φ of S^1 is equivalent to φ_n for some $n \in \mathbb{Z}$. Hence $\hat{S}^1 = \mathbb{Z}$.

Representations of $SU(2)$

Recall the topological isomrphism $\varphi_{-1} : S^3 \to SU(2)$ given by

$$\varphi_{-1}(\alpha + j\beta) = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix},$$

where $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$. We think of φ_{-1} as the vector representation of $SU(2)$ and we produce a series $\{\varphi_k\}$ of unitary irreducible representations of $SU(2)$ indexed by a non-negative integer k. Consider the space of complex polynomials in two variables

$$\mathbb{C}[z_1, z_2] = \sum_{k=0}^{+\infty} V_k$$

where V_k is the subspace of homogeneous polynomials of degree k. Note that $\dim V_k = k + 1$. Define

$$\varphi_k(\alpha + j\beta) \cdot p_k(z_1, z_2) = p_k\left(\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$$

for $p_k \in V_k$. Then φ_k is a representation of S^3 on V_k.

1.5.8 Theorem *φ_k is irreducible and $\hat{S}^3 = \{\varphi_k\}_{k=0}^{\infty}$.*

Proof. Let $\chi_k = \chi_{\varphi_k}$. We will show that $(\chi_k, \chi_k) = 1$ and use Theorem 1.5.5 to deduce that φ_k is irreducible.

Notice that

$$\varphi_1(e^{i\theta})z_1 = e^{-i\theta}z_1 \quad \text{and} \quad \varphi_1(e^{i\theta})z_2 = e^{i\theta}z_2,$$

so

$$\varphi_k(e^{i\theta})z_1^j z_2^{k-j} = (e^{-i\theta}z_1)^j(e^{i\theta}z_2)^{k-j} = e^{(k-2j)i\theta}z_1^j z_2^{k-j}.$$

This implies that

$$\chi_k(e^{i\theta}) = \sum_{j=0}^{k} e^{(k-2j)i\theta} = \frac{e^{(k+1)i\theta} - e^{-(k+1)i\theta}}{e^{i\theta} - e^{-i\theta}}.$$

Note that χ_1 is also the character of φ_{-1}, so Theorem 1.5.5 says that φ_1 and φ_{-1} are equivalent representations. φ_{-1} is the representation on the dual space to φ_1, and we define φ_{-k} to be the representation on the dual space to φ_k. Then

$$\chi_{\varphi_{-k}} = \chi_{\varphi_k^*} = \overline{\chi}_{\varphi_k} = \chi_{\varphi_k}$$

(cf. Problem 6), so Theorem 1.5.5 says that φ_k and φ_{-k} are equivalent representations.

Let $d\sigma$ be the volume element in the metric induced on $S^3(1)$ from $\mathbb{R}^4$. Since left and right translations of S^3 are restrictions of orthogonal transformations of $\mathbb{R}^4$, $d\sigma$ defines a bi-invariant measure. By the uniqueness part of Theorem 1.4.2 we have that $d\mu$ and $d\sigma$ must be multiples of each other. Since the volume of $S^3(1)$ with respect to $d\sigma$ is $2\pi^2$, we deduce that

$$d\mu = \frac{1}{2\pi^2}d\sigma.$$

Recall that the circle $\{e^{i\theta} : 0 \leq \theta \leq \pi\}$ meets all adjoint orbits of S^3 in S^3, and those orbits are round spheres $S^2(\sin\theta)$, hence of volume $4\pi\sin^2\theta$. We are now ready to compute:

$$\begin{aligned}
(\chi_k, \chi_k) &= \frac{1}{2\pi^2}\int_{S^3(1)} \chi_k(g)\overline{\chi_k(g)}d\sigma \\
&= \frac{1}{2\pi^2}\int_0^{\pi} \chi_k(e^{i\theta})\overline{\chi_k(e^{i\theta})}4\pi\sin^2\theta d\theta \\
&= \frac{1}{4\pi}\int_0^{2\pi} \chi_k(e^{i\theta})\overline{\chi_k(e^{i\theta})}|e^{i\theta} - e^{-i\theta}|^2 d\theta \\
&= \frac{1}{4\pi}\int_0^{2\pi} |e^{(k+1)i\theta} - e^{-(k+1)i\theta}|^2 d\theta \\
&= \frac{1}{4\pi}\int_0^{2\pi} |2i\sin(k+1)\theta|^2 d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (1 - \cos 2(k+1)\theta) d\theta \\
&= 1.
\end{aligned}$$

Hence φ_k is irreducible.

Finally, suppose that $\varphi \in \hat{S}^3$ is a complex irreducible representation of dimension $k+1$, but φ is not equivalent to φ_k. Then

$$\begin{aligned}
0 &= (\chi_\varphi, \chi_l) \\
&= \frac{1}{4\pi}\int_0^{2\pi} \chi_\varphi(e^{i\theta})\overline{\chi_l(e^{i\theta})}|e^{i\theta} - e^{-i\theta}|^2 d\theta \\
&= \frac{1}{4\pi}\int_0^{2\pi} \chi_\varphi(e^{i\theta})(e^{i\theta} - e^{-i\theta})\overline{(e^{(l+1)i\theta} - e^{-(l+1)i\theta})} d\theta
\end{aligned}$$

for all $l = 0, 1, 2, \ldots$ Now $\chi_\varphi(e^{i\theta})(e^{i\theta} - e^{-i\theta})$ is odd in θ and orthogonal to $\sin(l+1)i\theta$ for all $l = 0, 1, 2, \ldots$, and this is a contradiction. Hence φ cannot exist. This proves that $\{\varphi_k\}_{k=0,1,2,\ldots}$ is a complete set of representatives of complex irreducible representations of S^3. □

The Peter-Weyl theorem for compact matrix groups

A representation $\varphi : G \to GL(V)$ is called *faithful* if φ is an injective homomorphism. A compact topological group admitting a faithful representation can also be called a *compact matrix group*. For an arbitrary representation $\varphi : G \to GL(V)$, it is convenient to call $\dim V$ the *degree* of φ, denoted $\deg\varphi$.

1.5.9 Theorem (Peter-Weyl, special case) *Let G be a compact matrix group. Then the matrix coefficients of all irreducible unitary representations of G with respect to given orthonormal bases*

$$\{\varphi_{ij} : \varphi \in \hat{G},\ 1 \leq i, j \leq \deg \varphi\}$$

span a dense subset of $C(G, \mathbb{C})$ in the sup-norm topology.

Proof. Let $\mathcal{A}$ be the vector subspace of $C(G, \mathbb{C})$ spanned by the matrix coefficients. Note that:

- $\mathcal{A}$ is a subalgebra of $C(G, \mathbb{C})$. Indeed, for if φ, $\varphi' \in \hat{G}$ then the matrix coefficients of $\varphi \otimes \varphi'$ are precisely the products $\varphi_{ij} \cdot \varphi'_{kl}$. Since $\varphi \otimes \varphi'$ decomposes into a direct sum of irreducible representations, $\varphi_{ij} \cdot \varphi'_{kl}$ is a linear combination of matrix coefficients of the components.
- $\mathcal{A}$ is closed under complex conjugation, for the matrix coefficients of φ^* are the $\overline{\varphi_{ij}}$.
- $\mathcal{A}$ contains the constants, as the trivial representation has one matrix coefficient function constant and equal to 1.
- $\mathcal{A}$ separates points. Here we use the existence of a faithful representation φ_0, which we can assume to be complex, unitary and irreducible. Given g, $g' \in G$ with $g \neq g'$, we have $\varphi_0(g) \neq \varphi_0(g')$ and then some matrix coefficient of φ_0 will take distinct values on g and g'.

The Stone-Weierstrass theorem now implies that $\mathcal{A}$ is dense in $C(G, \mathbb{C})$ in the sup-norm. □

Every compact matrix group is a closed subgroup of $GL(n, \mathbb{R})$ and hence a Lie group. We will extend Theorem 1.5.9 to arbitrary compact Lie groups and present applications in Chapter 7.

1.6 Problems

1 *a.* Let G be a compact subgroup of $GL(n, \mathbb{R})$ (resp. $GL(n, \mathbb{C})$). Prove that it is conjugate to a subgroup of $O(n)$ (resp. $U(n)$), namely, there exists $A \in GL(n, \mathbb{R})$ (resp. $A \in GL(n, \mathbb{C})$) such thay $AGA^{-1} \subset O(n)$ (resp. $AGA^{-1} \subset U(n)$).

b. Show that $O(n)$ (resp. $U(n)$) is a maximal compact subgroup of $GL(n, \mathbb{R})$ (resp. $GL(n, C)$) and that any two maximal compact subgroups of $GL(n, \mathbb{R})$ (resp. $GL(n, C)$) are conjugate.

2 Let G be a topological group and let H be a subgroup (resp. normal subgroup) of G. Check that the closure $\bar{H}$ is also a subgroup (resp. normal subgroup). In addition, if points are closed in the topology of G, check that the closure of an Abelian subgroup is also an Abelian subgroup.

3 Let G be a connected topological group. Prove that any neighborhood U of the identity generates G as a group. (Hint: Take $U = V \cap V^{-1}$ and show that $\cup_{n \geq 1} U^n$ is an open subgroup of G.)

4 Let V and W be linear G-spaces. Check that the canonical isomorphisms
(i) $V \otimes W \cong W \otimes V$;
(ii) $(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$;
(iii) $V^* \otimes W \cong \mathrm{Hom}(V, W)$; are G-equivariant.

5 Check that the circle (1.3.2) meets the adjoint orbits of S^3 perpendicularly.

6 Let φ, ψ be complex representations of a compact topological group G. Show that $\chi_{\varphi\otimes\psi} = \chi_\varphi \cdot \chi_\psi$ and $\chi_{\varphi^*} = \overline{\chi}_\varphi$.

7 Consider the representation given by $SO(n)$-conjugation of traceless real symmetric $n \times n$ matrices as in Examples 1.2.1(iii), and fill in the details to show that $U = V_0$.

8 a. Let V be a real G-space with an invariant inner product. Show that $V^* \cong V$ as G-spaces.
b. Let V be a complex G-space with an invariant Hermitian inner product. Show that $V^* \cong \bar{V}$ as G-spaces, where $\bar{V}$ is the complex vector space with the opposite complex structure of V.

9 Identify the Haar integral of S^1 as being given by $I(f) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta$ for $f \in C(S^1)$.

10 Prove that the Haar measure of the product of two compact topological groups $G \times H$ is the product measure of the Haar measures of of G and H.

11 Describe the irreducible representations of the n-torus. (Hint: Use Corollary 1.5.6 to extend Example 1.5.7).

12 Determine all the irreducible representations of $SO(3)$, up to equivalence. (Hint: Use the double covering $S^3 \to SO(3)$.)

13 Prove that a discrete normal subgroup of a topological group G centralizes the identity component G^0. Conclude that every discrete normal subgroup of a connected topological group is central.

14 Let V be an irreducible real (resp. complex) G-space. Prove that any two G-invariant inner products (resp. Hermitian inner products) differ by a multiplicative constant. (Hint: Diagonalize $\langle , \rangle_2$ with respect to $\langle , \rangle_1$, that is, find a $\langle , \rangle_1$-orthonormal basis of V whose Gram matrix of $\langle , \rangle_2$-inner products is diagonal, and use an argument similar to the proof of Schur's lemma.)

15 Let V be G-space over $\mathbb{F}$. A $\mathbb{F}$-bilinear form B is said to be *invariant* under G if $B(gu, gv) = B(u, v)$ for all $g \in G$ and all u, $v \in V$. Prove that if V is irreducible and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, then any nonzero invariant bilinear form B on V is nondegenerate. In case $\mathbb{F} = \mathbb{C}$, prove also that any nonzero invariant bilinear form B on V is symmetric or skew-symmetric. (The last result is not true if $\mathbb{F} = \mathbb{R}$; can you give a counterexample?)

CHAPTER 2

Review of Lie groups

This chapter is a quick review of the basic theory of Lie groups.

2.1 Basic definition

A Lie group is a topological group with an additional, compatible structure of smooth manifold. More precisely, a *Lie group* G is a group endowed with a smooth structure such that the group operations are smooth, namely, the multiplication map $\mu : G \times G \to G$ and the inversion map $\iota : G \to G$ are required to be smooth.

In this book we will only study compact Lie groups. It turns out that every compact Lie group is isomorphic to a group of matrices (this follows e.g. from the Peter-Weyl theorem). Therefore matrix groups supply (almost) all the intuition we need to understand compact Lie groups. Nevertheless, we cultivate the idea that the reader should know that the concept is larger than matrix groups, namely, indeed there exist Lie groups which are not isomorphic to a Lie group of matrices (e.g. the universal covering of $SL(2,\mathbb{R})$).

All topological groups listed in Examples 1.1.2 are in fact Lie groups with respect to the standard smooth structures. This is very easy to see in cases (i), (ii) and (iii). In the sequel we make some comments in case (iv).

2.1.1 Examples (i) Multiplication of matrices

$$M(n,\mathbb{F}) \times M(n,\mathbb{F}) \to M(n,\mathbb{F})$$

is a bilinear map, hence smooth. Further, the restriction

$$U \times U \to U, \tag{2.1.2}$$

where U is any open subset of $M(n,\mathbb{F})$ for which (2.1.2) makes sense, is also smooth. Note that we can take $U = GL(n,\mathbb{F})$. Consider now the inversion map

$$GL(n,\mathbb{F}) \to GL(n,\mathbb{F}).$$

In the real and complex case, there is a formula for A^{-1} involving the determinant and the transpose of the cofactor matrix of A, which shows that the entries of A^{-1} are rational functions of the entries of A, and hence smooth. In the quaternionic case, it is easier to work with the representation of quaternionic matrices by complex matrices of twice the size.

(ii) In order to show that $O(n)$ is a smooth manifold, we can use the implicit mapping theorem. Denote by $Sym(n,\mathbb{R})$ the vector space of real symmetric matrices of order n, and define $f : M(n,\mathbb{R}) \to Sym(n,\mathbb{R})$ by $f(A) = AA^t$. This is a map between vector spaces whose components are quadratic polynomials. It follows that f is smooth and that df_A can be viewed as a map $M(n,\mathbb{R}) \to Sym(n,\mathbb{R})$ for all $A \in M(n,\mathbb{R})$. We claim that I is a regular value of f. For the purpose of checking that, we first compute for $A \in f^{-1}(I)$ and $B \in M(n,\mathbb{R})$ that

$$\begin{aligned} df_A(B) &= \lim_{h\to 0} \frac{(A+hB)(A+hB)^t - I}{h} \\ &= \lim_{h\to 0} \frac{h(AB^t + BA^t) + h^2 BB^t}{h} \\ &= AB^t + BA^t. \end{aligned}$$

Now given $C \in Sym(n,\mathbb{R})$, we have $df_A(\frac{1}{2}CA) = C$, and this proves that f is a submersion at A, as desired. Hence $f^{-1}(I) = O(n)$ is an embedded submanifold of $M(n,\mathbb{R})$ of dimension

$$\dim M(n,\mathbb{R}) - \dim Sym(n,\mathbb{R}) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

It remains to check that the group operations in $O(n)$ are smooth. Note that $O(n)$ is also an embedded submanifold of $GL(n,\mathbb{R})$, and its group operations are restrictions of the corresponding operations in $GL(n,\mathbb{R})$; from this follows that they are smooth. Hence $O(n)$ is a Lie group. The other classical matrix groups listed in Examples 1.1.2 can be similarly proven to be Lie groups.

(iii) There is a theorem of Élie Cartan asserting that any closed subgroup G of a Lie group H is also a Lie group with respect to the induced topology. We may want to use this theorem when necessary, without providing a proof. It immediately shows that the classical matrix groups are Lie groups.

(iv) It is often convenient to view $GL(n,\mathbb{H})$ as a group of complex matrices of twice the size. Namely, we extend the ideas in Chapter 1 proving that $Sp(1) = S^3 \cong SU(2)$. Consider the $\mathbb{C}$-linear isomorphism $\mathbb{H}^n \to \mathbb{C}^{2n}$, where $\mathbb{H}^n$ is regarded as a right $\mathbb{C}$-module, given by

$$q = \alpha + j\beta \mapsto \vartheta(q) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

where $q \in \mathbb{H}^n$ and $\alpha, \beta \in \mathbb{C}^{2n}$. Then, a matrix $X \in M(n, \mathbb{H})$, viewed as an endomorphism of $\mathbb{H}^n$, is conjugated to an endomorphism $\Phi(X)$ of $\mathbb{C}^{2n}$:

$$\begin{array}{ccc} \mathbb{H}^n & \xrightarrow{\vartheta} & \mathbb{C}^{2n} \\ X\downarrow & & \downarrow\Phi(X) \\ \mathbb{H}^n & \xrightarrow[\vartheta]{} & \mathbb{C}^{2n} \end{array}$$

Since $(A + jB)(\alpha + j\beta) = (A\alpha - \bar{B}\beta) + j(A\beta + B\alpha)$, we see that

$$\Phi(X) = \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}. \tag{2.1.3}$$

It follows that Φ defines, by restriction, an injective homomorphism $\varphi : GL(n, \mathbb{H}) \to GL(2n, \mathbb{C})$. Thus the image of φ is a closed subgroup of $GL(2n, \mathbb{C})$, often denoted by $U^*(2n)$, which is isomorphic to $GL(n, \mathbb{H})$.

By the way, the complex $2n \times 2n$-matrices of the form (2.1.3) are exactly those $Z \in GL(2n, \mathbb{C})$ that satisfy

$$ZJ_n = J_n\bar{Z},$$

where

$$J_n = \begin{pmatrix} 0 & -I_n \\ I_n & \bar{0} \end{pmatrix}.$$

Indeed the quaternionic structure ϵ on $\mathbb{C}^{2n}$ corresponds to left right multiplication by j on $\mathbb{H}^n$, so it is given by

$$\begin{aligned} \epsilon \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \vartheta((\alpha + j\beta)j) \\ &= \vartheta((-\bar{\beta} + j\bar{\alpha}) \\ &= \vartheta(jI_n)(\bar{\alpha} + j\bar{\beta})) \\ &= \Phi(jI_n)\vartheta(\bar{\alpha} + j\bar{\beta}) \\ &= J_n \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix} \\ &= \begin{pmatrix} -\bar{\beta} \\ \bar{\alpha} \end{pmatrix}. \end{aligned}$$

In order to avoid talking about quaternionic determinants, which exist but whose theory is nonstandard, we define

$$SU^*(2n) := U^*(2n) \cap SL(2n, \mathbb{C}),$$

and define $SL(n, \mathbb{H})$ to be the corresponding subgroup of $GL(n, \mathbb{H})$.

2.2 Lie algebras

On one hand, the representation theoretical ideas we discussed in Chapter 1 for a (compact) topological group G can be viewed as a kind of "extrinsic linearization" of G. Each representation is a matrix realization of G, so as to speak like a linear picture of G. The totality of linear pictures of G should yield a great deal of information about the group. On the other hand, G becomes a Lie group when it is endowed with an additional, compatible smooth structure. This situation allows us to linearize the structure of G and provides for an "intrinsic linearization" of the group. Then almost a miracle of nature happens, namely, the tangent space to G at the identity inherits an algebraic structure that essentially captures all the curved geometry of the group; further, the infinitesimal structure completely characterizes the group locally, and indeed up to connected components and coverings in the global sense.

Whereas in representation theory the technique of integration is used, the method of intrinsic linearization builds upon differentiation and integration of vector fields. We take up such ideas next.

Left-invariant vector fields and one-parameter subgroups

Let G be a Lie group. The simplest (non-trivial) Lie group is perhaps $\mathbb{R}$ with its additive structure, so it is natural to look for copies of $\mathbb{R}$ inside G. A *one-parameter subgroup* of G is a smooth homomorhism $\varphi : \mathbb{R} \to G$, where $\mathbb{R}$ is viewed with its additive structure.

Given a one-parameter subgroup $\varphi : \mathbb{R} \to G$, one obtains a action of $\mathbb{R}$ on G, namely,

$$\Phi : \mathbb{R} \times G \to G, \qquad \Phi(t, x) = x \cdot \varphi(t).$$

By construction, this action is *left-invariant*, in the sense that

$$\Phi(t, gx) = g \cdot \Phi(t, x). \tag{2.2.1}$$

The *left translation* defined by $g \in G$ is the map $L_g : G \to G$, $L_g(x) = gx$. It is a smooth map and indeed a diffeomorphism of G, its inverse being given by $L_{g^{-1}}$. Similarly, the *right translation* defined by $g \in G$ is the map $R_g : G \to G$, $R_g(x) = xg$. It is also a diffeomorphism of G and its inverse is given by $R_{g^{-1}}$. We can now rewrite (2.2.1) as

$$\Phi(t, L_g(x)) = L_g(\Phi(t, x)), \tag{2.2.2}$$

for every g, $x \in G$. Differentiation (2.2.2) with respect to t at $t = 0$ yields

$$X_{gx} = dL_g(X_x), \tag{2.2.3}$$

where X is the vector field given by $X_x = \frac{d}{dt}|_{t=0}\Phi(t, x)$ for all $x \in G$. Equation (2.2.3) in turn is equivalent to

$$X \circ L_g = dL_g \circ X. \tag{2.2.4}$$

A vector field X on G satisfying (2.2.4) for all $g \in G$ is called *left-invariant*. We can similarly define *right-invariant* vector fields, but most often we will be considering the left-invariant variety. Note that left-invariance and right-invariance are the same property in case of an Abelian group.

The translations in G define canonical identifications between the tangent spaces to G at different points. For instance, $dL_g : T_hG \to T_{gh}G$ is an isomorphism for every g, $h \in G$. It is now clear that a left-invariant vector field X is completely determined by its value X_1 at the identity of G.

One can invert the above process. In fact, it is not hard to see that every left-invariant vector field is automatically smooth and complete. Starting with a left-invariant vector field X on G, we can thus integrate it to obtain its flow $\Phi : \mathbb{R} \times G \to G$. Let $\varphi : \mathbb{R} \to G$ denote the integral curve of X through 1, that is, $\varphi(t) = \Phi(t, 1)$. The left-invariance of X yields that $L_g \circ \varphi(t)$ must be the integral curve of X through $g \in G$. Taking $g = \varphi(s)$, we obtain that

$$\varphi(s + t) = \varphi(s) \cdot \varphi(t)$$

for all s, $t \in \mathbb{R}$, that is, φ is a one-parameter subgroup of G. We summarize the above discussion in the form of the following proposition.

2.2.5 Proposition *There is a bijective correspondence between one-parameter subgroups of G and left-invariant vector fields of G. It takes a one-parameter subgroup to the infinitesimal generator of the associated left-invariant action on G.*

There is one more ingredient in this tale. Denote by $\mathfrak{g}$ the (real) vector space of left-invariant vector fields. As a vector space, it is isomorphic to the tangent space T_1G. Recall that the Lie bracket of two vector fields X, Y on G (or a smooth manifold) is an infinitesimal measure of the non-commutativity of the corresponding flows. Viewing tangent vectors on a manifold as directional derivatives and vector fields as first order differential operators, the Lie bracket can be defined as

$$[X,Y]_x(f) = X_x(Y(f)) - Y_x(X(f)) \tag{2.2.6}$$

for a smooth funcion f on G and $x \in G$. It turns out that the Lie bracket of two left-invariant vector fields on G is also left-invariant (this follows from (2.4.3) where we take φ to be a left-translation). This makes $\mathfrak{g}$ (or, equivalently, T_1G) into a Lie algebra. A vector space V (finite- or infinite-dimensional) over a field, endowed with a bilinear operation $[\cdot,\cdot] : V \times V \to V$ satisfying

a. $[Y,X] = -[X,Y]$;

b. $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ (Jacobi identity);

for every X, $Y \in V$ is called a *Lie algebra*. We have shown:

2.2.7 Proposition *To every Lie group G is associated a real Lie algebra $\mathfrak{g}$ of the same dimension, consisting of the left-invariant vector fields on G and operation of Lie bracket of vector fields.*

The Lie algebras of the classical matrix groups

The defining equations for the classical matrix groups yield, via differentiation, the equations defining the tangent spaces at the identity. For instance, we have seen in Example 1.1.2 that $O(n) = f^{-1}(I)$, where $f(A) = AA^t$ and then $\mathfrak{o}(n) = T_I O(n) = \ker df_I$ where $df_I(X) = X + X^t$. More systematically, we obtain:

$$
\begin{aligned}
\mathfrak{gl}(n,\mathbb{R}) &= M(n,\mathbb{R}) \\
\mathfrak{gl}(n,\mathbb{C}) &= M(n,\mathbb{C}) \\
\mathfrak{gl}(n,\mathbb{H}) &= M(n,\mathbb{H}) \\
\mathfrak{o}(n) &= \{X \in M(n,\mathbb{R}) \mid X + X^t = 0\} \\
\mathfrak{u}(n) &= \{X \in M(n,\mathbb{C}) \mid X + X^* = 0\} \\
\mathfrak{sp}(n) &= \{X \in M(n,\mathbb{H}) \mid X + X^* = 0\} \\
\mathfrak{sl}(n,\mathbb{R}) &= \{X \in M(n,\mathbb{R}) \mid \operatorname{trace}(X) = 0\} \\
\mathfrak{sl}(n,\mathbb{C}) &= \{A \in M(n,\mathbb{C}) \mid \operatorname{trace}(X) = 0\} \\
\mathfrak{so}(n) &= \mathfrak{o}(n) \\
\mathfrak{su}(n) &= \mathfrak{u}(n) \cap \mathfrak{sl}(n,\mathbb{C})
\end{aligned}
$$

It remains to show that the Lie bracket in all cases is given by $[X, Y] = XY - YX$ (where XY denotes the usual product of matrices).

2.3 The exponential map

Let G be a Lie group and denote its Lie algebra by $\mathfrak{g}$. Next we show how to organize the totality of one-parameter subgroups of G into a single map $\mathfrak{g} \to G$. The *exponential map* $\exp : \mathfrak{g} \to G$ is defined by

$$\exp X = \varphi_X(1)$$

where $\varphi_X : \mathbb{R} \to G$ is the integral curve of X passing through the identity at time zero. Owing to the chain rule, $s \mapsto \varphi_X(st)$ is a one-parameter subgroup with initial velocity tX. Hence $\varphi_X(st) = \varphi_{tX}(s)$ and therefore $\exp(tX) = \varphi_{tX}(1) = \varphi_X(t)$ for all $t \in \mathbb{R}$.

It follows from the smooth dependence on initial conditions of solutions of ODE's that the exponential map is smooth. Moreover

$$d\exp_0 : T_0\mathfrak{g} \cong \mathfrak{g} \to T_1G \cong \mathfrak{g}$$

is given by the identity, due to

$$d\exp_0(X) = \frac{d}{dt}\Big|_{t=0} \exp(0 + tX) = X.$$

It follows that exp is local diffeomorphism at 0 and hence it can be used to introduce a local coordinate system around 1 in G.[1]

In case of matrix groups, it is easily seen that the exponential map coincides with the usual exponential of matrices, namely

$$\exp X = e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k. \tag{2.3.1}$$

In fact, $t \mapsto e^{tX}$ is the one-parameter subgroup with initial velocity X.

2.4 Lie homomorphisms and Lie subgroups

Let G and H be Lie groups. A *Lie group homomorphism* is a group homomorphism $\varphi : G \to H$ which is also smooth. If, in addition, φ is a diffeomorphism, then it is called an *isomorphism*. In case $G = H$, an isomorphism is called an *automorphism*. In the cases $H = GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$, or $H = GL(V)$ for some vector space V, a homomorphism $\varphi : G \to H$ is called a *representation* of G.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. A *Lie algebra homomorphism* is a linear $\varphi : G \to H$ that preserves brackets. If, in addition, φ is bijective, then it is called an *isomorphism*. In case $\mathfrak{g} = \mathfrak{h}$, an isomorphism is called an *automorphism*. In the cases $\mathfrak{h} = \mathfrak{gl}(n, \mathbb{R})$ or $\mathfrak{gl}(n, \mathbb{C})$, or $\mathfrak{h} = \mathfrak{gl}(V)$ for some vector space, a homomorphism $\Phi : \mathfrak{g} \to \mathfrak{h}$ is called a *representation* of $\mathfrak{g}$.

A Lie group homomorphism $\varphi : G \to H$ induces a linear map $d\varphi_1 : T_1 G \to T_1 H$ and hence a linear map $d\varphi : \mathfrak{g} \to \mathfrak{h}$. Indeed, for a left-invariant vector field X on G, $X' = d\varphi(X)$ is the unique left-invariant vector field on H whose value at $1 \in H$ equals $d\varphi(X_1)$. It turns out that $d\varphi \circ X = X' \circ \varphi$; one says that X and X' are *φ-related* as vector fields. Viewing X as a differential operator, this means

$$X'_{\varphi(x)}(f) = X_x(f \circ \varphi) \tag{2.4.1}$$

for a smooth function f on H and $x \in G$. If $Y \in \mathfrak{g}$, then Y and $Y' := d\varphi(Y)$ are also φ-related, that is,

$$Y'_{\varphi(x)}(f) = Y_x(f \circ \varphi). \tag{2.4.2}$$

A short calculation using (2.4.1) and (2.4.2) and (2.2.6) shows that

$$[X', Y']_{\varphi(x)}(f) = [X, Y]_x(f \circ \varphi), \tag{2.4.3}$$

that is, $[X', Y']$ and $[X, Y]$ are φ-related. We deduce that $d\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

[1] Any neighborhood of 1 generates G^0.

Lie subgroups

Let G be a Lie group. Loosely speaking, a Lie subgroup of G is a Lie group which is both a submanifold and a subgroup. Formally, a *Lie subgroup* of G is a Lie group H together with a map $\varphi : H \to G$ which is a homomorphism and a injective immersion (of course the topology of H can be finer than the relative topology; cf. Example 2.4.4). The most common situation is when H is an actual subgroup of G and φ is simply the inclusion $H \to G$; in most cases, we can replace H by $\varphi(H)$ and assume we are in this stuation.

Let $\mathfrak{g}$ be a Lie algebra. A *Lie subalgebra* of $\mathfrak{g}$ is a subspace $\mathfrak{h}$ which is closed under the bracket of $\mathfrak{g}$.

Now suppose G and H are Lie groups with corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, and suppose $\iota : H \to G$ is a Lie subgroup, where ι is the inclusion. Then $d\iota : \mathfrak{h} \to \mathfrak{g}$ is an injective homomorphism of Lie algebras, and we may and will view $\mathfrak{h}$ as a Lie subalgebra of $\mathfrak{g}$.

2.4.4 Example A Lie subgroup do not have to be closed, neither needs to have the relative topology, as the skew-line in the torus

$$\varphi : \mathbb{R} \to T^2, \qquad \varphi(t) = (e^{it}, e^{ait})$$

shows, where a is an irrational number.

As a main application of Frobenius theorem, one shows:

2.4.5 Theorem (Lie) *Let G be a Lie group and denote its Lie algebra by $\mathfrak{g}$. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then there is a unique connected Lie subgroup H of G such that the Lie algebra of H is $\mathfrak{h}$.*

Proof. (Sketch) Let $\mathcal{D}$ be the left-invariant distribution on G defined by the values at the identity of vector fields in $\mathfrak{h}$ ($\mathcal{D}$ is a sub-bundle of the tangent bundle of G which is invariant under left translations). The assumption that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ is easily seen to be equivalent to $\mathcal{D}$ being involutive (that is, closed under the Lie bracket of vector fields). By the Frobenius theorem, there is a maximal integral manifold of $\mathcal{D}$ passing through 1, which we call H. It is not hard to see that an abstract subgroup of G. The fact that the restriction of the group operations of G to H are smooth depends on the fact that H is an integral manifold of an involutive distribution. □

2.4.6 Corollary *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then there is a bijective correspondence between connected Lie subgroups of G and Lie subalgebras of $\mathfrak{g}$.*

We have seen that a Lie group homomorphism induces a homomorphism between the corresponding Lie algebras. Now comes the converse question: Given Lie groups G, H with corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, and a Lie algebra homomorphism $\Phi : \mathfrak{g} \to \mathfrak{h}$, does there exist a Lie group homomorphism $\varphi : G \to H$ such that $d\varphi = \Phi$? In this generality, the result is false, and a very simple example is given by $G = S^1$, $H = \mathbb{R}$ and $\Phi : \mathbb{R} \to \mathbb{R}$ any nonzero linear map, say, $\Phi(t) = ct$ for some $c \in \mathbb{R} \setminus \{0\}$; indeed there are no nontrivial Lie group homomorphisms $S^1 \to \mathbb{R}$ (since $\mathbb{R}$ admit no compact connected Lie subgroups other than the trivial one). Interestingly, if we interchange the roles of S^1 and $\mathbb{R}$, namely, take $G = \mathbb{R}$ and $H = S^1$, then a solution exists and is given by $\varphi(t) = e^{ict}$. The underlying reason is the simple-connectedness of $\mathbb{R}$.

2.4.7 Theorem (Lie) *Let G_1 and G_2 be Lie groups and assume that G_1 is connected and simply-connected. Then, given a homomorphism $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ between the corresponding Lie algebras, there is a unique Lie group homomorphism $\varphi : G_1 \to G_2$ such that $d\varphi = \Phi$.*

Proof. (Sketch) The Lie algebra of $G_1 \times G_2$ is $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ and the graph of Φ is a subalgebra $\mathfrak{h}$. Owing to Theorem 2.4.5, there is a connected Lie subgroup H of $G_1 \times G_2$ with Lie algebra $\mathfrak{h}$. Now H should be the graph of the desired homomorphism; this gives the uniqueness, and for existence we just need to check that over a given element of G_1 sits exactly one element of H. We note that the projection $\mathfrak{h} \to \mathfrak{g}_1$ is an isomorphism and therefore the projection $H \to G_1$ is a covering (cf. Proposition 2.6.4); here we use the assumption that G_1 is simply-connected to deduce that $H \to G_1$ is an isomorphism. □

2.4.8 Corollary *Two connected and simply connected Lie groups with isomorphic Lie algebras are isomorphic.*

Homomorphisms and the exponential map

Let $\varphi : G \to H$ be a Lie group homomorphism. Then φ maps one-parameter subgroups of G to one-parameter subgroups of H, in the sense that if $\gamma : \mathbb{R} \to G$ is a one-parameter subgroup of G then $\varphi \circ \gamma : \mathbb{R} \to H$ is a one-parameter subgroup of H. It follows that

$$\varphi \circ \exp^G = \exp^H \circ d\varphi. \tag{2.4.9}$$

In particular, if K is a Lie subgroup of G, then the exponential map of K is the restriction of the exponential map of G. Moreover, the Lie algebra K, as a subalgebra of the Lie algebra $\mathfrak{g}$, is given by

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \exp^G(tX) \in \mathfrak{k}, \text{ for all } t \in \mathbb{R}\}. \tag{2.4.10}$$

2.4.11 Example One can use characterization (2.4.10) to determine the Lie algebras of the classical groups. For instance, consider $O(n)$. Let $g(t) \in O(n)$, $-\epsilon < t < \epsilon$ be a smooth curve, with $g(0) = 1$. Then differentiating the identity $g(t)g(t)^t = I$ at $t = 0$ yields $g(0)'g(0) + g(0)g(0)' = 0$, or $g(0) + g(0)' = 0$, that is, $T_I O(n)$ consists of skew-symmetric matrices. Conversely, if $A + A^t = 0$, then $g = \exp(tA)$ satisfies

$$gg^t = e^{tA}(e^{tA})^t = e^{tA}e^{tA^t} = e^{tA}e^{-tA} = e^0 = 1,$$

that is, $\exp(tA) \in O(n)$ for all $t \in \mathbb{R}$.

2.5 The adjoint representation

Let G be a Lie group and denote its Lie algebra by $\mathfrak{g}$. The adjoint action of G on G can be differentiated once to give the adjoint representation of G on $\mathfrak{g}$, and again to give the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$.

For $g \in G$, the inner automorphism $\mathrm{Inn}_g : G \to G$ is now a smooth automorphism of G, so its differential $d(\mathrm{Inn}_g) : \mathfrak{g} \to \mathfrak{g}$ defines an automorphism of $\mathfrak{g}$, which we denote by Ad_g. Then

$$\mathrm{Ad}_g X = \frac{d}{dt}\Big|_{t=0} g \exp tX g^{-1}. \tag{2.5.1}$$

In terms of (2.4.9), his equation implies

$$\exp \mathrm{Ad}_g X = g \exp X g^{-1}$$

for all $g \in G$ and $X \in \mathfrak{g}$.

We have defined a homomorphism

$$\mathrm{Ad} : g \in G \to \mathrm{Ad}_g \in GL(\mathfrak{g}).$$

It is called the *adjoint representation* of G on $\mathfrak{g}$.

Recall that $GL(\mathfrak{g})$ is itself a Lie group isomorphic to $GL(n, \mathbb{R})$, where $n = \dim \mathfrak{g}$. Its Lie algebra consists of all linear endomorphisms of $\mathfrak{g}$ and it is denoted by $\mathfrak{gl}(\mathfrak{g})$ (the Lie bracket in $\mathfrak{gl}(\mathfrak{g})$ is $[A, B] = AB - BA$, see below). Now $\mathrm{Ad} : g \in G \to \mathrm{Ad}_g \in GL(\mathfrak{g})$ is homomorphism of Lie groups and its differential $d(\mathrm{Ad})$ defines the *adjoint representation* of $\mathfrak{g}$ on $\mathfrak{g}$:

$$\mathrm{ad} : X \in \mathfrak{g} \to \mathrm{ad}_X = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp tX} \in \mathfrak{gl}(\mathfrak{g}). \tag{2.5.2}$$

Using (2.4.9) and (2.3.1), we have

$$\begin{aligned} \mathrm{Ad}_{\exp X} &= e^{\mathrm{ad}_X} \\ &= I + \mathrm{ad}_X + \frac{1}{2}\mathrm{ad}_X^2 + \frac{1}{3!}\mathrm{ad}_X^3 + \cdots \end{aligned} \tag{2.5.3}$$

for all $X \in \mathfrak{g}$.

2.5.4 Proposition $\mathrm{ad}_X Y = [X, Y]$ *for all* $X, Y \in \mathfrak{g}$.

Proof. Let f be a smooth function on G and consider the smooth functions

$$F(r,s,t,u) = f(\exp(rX)\exp(sY)\exp(tX)\exp(uY))$$

and

$$G(r,s) = F(r,s,-r,-s),$$

respectively defined on $\mathbb{R}^4$ and $\mathbb{R}^2$. Then

$$\begin{aligned}
\frac{\partial^2 G}{\partial r \partial s}(0,0) &= \frac{\partial^2 F}{\partial r \partial s}(0,0) - \frac{\partial^2 F}{\partial r \partial u}(0,0) - \frac{\partial^2 F}{\partial s \partial t}(0,0) + \frac{\partial^2 F}{\partial t \partial u}(0,0) \\
&= X_1(Yf) - X_1(Yf) - Y_1(Xf) + X_1(Yf) \\
&= [X,Y]_1(f).
\end{aligned}$$

On the other hand, owing to (2.5.1)

$$\exp(s\mathrm{Ad}_{\exp rX}Y) = \exp(rX)\exp(sY)\exp(-rX),$$

so

$$G(r,s) = f(\exp(s\mathrm{Ad}_{\exp rX}Y)\exp(-sY))$$

and

$$\frac{\partial G}{\partial s}(r,0) = (\mathrm{Ad}_{\exp rX}Y)_1(f) - Y_1(f),$$

$$\frac{\partial G}{\partial r \partial s}(0,0) = (\mathrm{ad}_X Y)_1(f),$$

as desired. □

2.5.5 Example In case $G = GL(n,\mathbb{F})$, for $g \in GL(n,\mathbb{F})$ and $A \in M(n,\mathbb{F})$ we have

$$\begin{aligned}
\mathrm{Ad}_g A &= \frac{d}{dt}\Big|_{t=0} g e^{tA} g^{-1} \\
&= g\left(\frac{d}{dt}\Big|_{t=0} e^{tA}\right) g^{-1} \\
&= gAg^{-1}.
\end{aligned}$$

Therefore, for A, $B \in M(n,\mathbb{F})$,

$$\begin{aligned}
\mathrm{ad}_A B &= \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{e^tA} B \\
&= \frac{d}{dt}\Big|_{t=0} e^{tA} B e^{-tA} \\
&= (Ae^0)Be^0 + e^0 B(-Ae^0) \\
&= AB - BA.
\end{aligned}$$

Combined with Proposition 2.5.4, this shows that the bracket in $\mathfrak{gl}(n,\mathbb{F}) = M(n,\mathbb{F})$ (and hence in any matrix group) is given by $[A,B] = AB - BA$.

2.6 Quotients and coverings of Lie groups

2.6.1 Theorem *Let G be a Lie group and let H be a closed subgroup of G. Then there is a unique smooth structure on the topological quotient G/H such that the left multiplication $\lambda : G \times G/H \to G/H$, $\lambda(g, xH) = (gx)H$ is smooth. Moreover, $\pi : G \to G/H$ is a surjective submersion and* $\dim G/H = \dim G - \dim H$.

Proof. (Sketch) Recall that, being closed, H is a Lie subgroup of G with respect to the induced topology. π is an open map and hence maps a countable basis of G to a countable basis of G/H. The closedness of H and the openness of π also imply that G/H is Hausdorff. To construct a local chart of G/H around $p_0 = \pi(1) = 1H$, denote the Lie algebras of G and H by $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and choose a complementary subspace $\mathfrak{m}$ to $\mathfrak{h}$ in $\mathfrak{g}$. One uses the Inverse Mapping Theorem and the fact that H has the relative topology to show that there is a neighborhood V of 0 in $\mathfrak{m}$ such that $\pi \circ \exp$ defines a homeomorphism from V onto an open neighborhood of $p \in G/H$, whose inverse is a then a local chart ψ arount p. The local charts around other points in G/H are taken to be of the form $\psi^g = \psi \circ \lambda_{g^{-1}}$. One needs to show that the changes of local charts are smooth, but then the smoothness of λ is automatic. π is a submersion since its local representation is the linear projection $\mathfrak{g} \to \mathfrak{m}$ along $\mathfrak{h}$, and $\dim G/H = \dim \mathfrak{m} = \dim \mathfrak{g} - \dim \mathfrak{h}$. To see that the uniqueness part, denote by $(G/H)_1$ the topological quotient endowed with a different smooth structure such that λ is smooth. Consider the map $f : G/H \to (G/H)_1$ given by $f(gH) = \lambda(g, p)$. It is clearly well-defined, smooth and bijective. One checks by direct computation that df_{1H} is injective. Therefore f is an immersion at $1H$ and hence, by G-equivariance, an immersion everywhere. Now $\dim G/H \leq \dim(G/H)_1$ and we cannot have strict inequality (since f is bijective). It follows f is a diffeomorphism. □

2.6.2 Corollary *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let H be a closed normal subgroup of G with Lie algebra $\mathfrak{h}$. Then G/H is a Lie group, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and the Lie algebra of G/H can be identified with the quotient Lie algebra $\mathfrak{g}/\mathfrak{h}$.*

Proof. We already know that G/H is a smooth manifold and an abstract group. We need to show that the group operations are smooth. Denote by $\bar{\mu}$ the group multiplication on G/H. We have a commutative diagram:

$$\begin{array}{ccc} G \times G/H & \xrightarrow{\ \lambda\ } & G/H \\ & \searrow^{\pi\times\mathrm{id}} & \uparrow{\scriptstyle \bar{\mu}} \\ & & G/H \times G/H \end{array}$$

Since π is a submersion and λ is smooth, also $\bar{\mu}$ is smooth. Similarly, the inversion map in G/H is smooth. To see that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, see Problem 14. Finally, the differential $d\pi_1 : T_1G \cong \mathfrak{g} \to T_{1\cdot H}(G/H)$ has kernel $\mathfrak{h}$ and induces a Lie algebra isomorphism of $\mathfrak{g}/\mathfrak{h}$ with the Lie algebra of G/H. □

2.6.3 Proposition *Let $\varphi : G \to H$ be a Lie group homomorphism. Then:*
(a) $\ker\varphi$ *is a closed, normal Lie subgroup of G and its Lie algebra is* $\ker d\varphi$*.*
(b) The image $\varphi(G)$ is a Lie subgroup of H, its Lie algebra is $d\varphi[\mathfrak{g}]$, and it is isomorphic to $G/\ker\varphi$.

Proof. (a) $\ker\varphi$ is a known to be a closed normal subgroup of G; being closed, it is a Lie subgroup. It follows from (2.4.9) and (2.4.10) that it has $\ker d\varphi$ as its Lie algebra.

(b) Since $\varphi \circ L_g = L_{\varphi(g)} \circ \varphi$ for all $g \in G$, the homomorphism φ has constant rank. It follows from the Rank Theorem that its image $\varphi(G)$ is an immersed submanifold of H of dimension equal to the rank of φ. Now the inclusion of $\varphi(G)$ into H is an injective immersion and a homomorphism, and hence $\varphi(G)$ is a Lie subgroup of H. From (2.4.10) we see that $d\varphi(\mathfrak{g})$ is contained in the Lie algebra of $\varphi(G)$, and then we have equality by dimensional reasons. Regarding the last assertion, φ induces a smooth and bijective map $\bar{\varphi} : G/\ker\varphi \to \varphi(G)$. Since $d\bar{\varphi} : \mathfrak{g}/\ker d\varphi \to d\varphi[\mathfrak{g}]$ is an isomorphism, $\bar{\varphi}$ has maximal rank at $1 \cdot (\ker\varphi)$ and hence everywhere; we deduce that it is a diffeomorphism. □

2.6.4 Proposition *Let $\varphi : G \to H$ be a homomorphism between Lie groups. Consider the induced homomorphism between the corresponding Lie algebras $d\varphi : \mathfrak{g} \to \mathfrak{h}$. Then:*
a. $d\varphi$ is injective if and only if the kernel of φ is discrete.
b. $d\varphi$ is surjective if and only if $\varphi(G^\circ) = H^\circ$.
c. $d\varphi$ is bijective if and only if φ is a smooth covering (here we assume G and H connected). In this case, the group of deck transformations is isomorphic to $\ker\varphi$*.*

Proof. (a) and (b) follows from Proposition 2.6.3. (c) follows from (a) and (b). □

2.6.5 Theorem *Every connected Lie group G has a simply-connected covering $p : \tilde{G} \to G$ such that $\tilde{G}$ is a connected Lie group and p is a Lie group homomorphism.*

Proof. (Sketch) The topological universal covering space $\tilde{G}$ of G can be constructed as the quotient of the space $P(G,1)$ of continuous paths in G originating at the identity by the equivalence relation $\sim$ that declares two paths equivalent if and only if they have the same endpoint and are homotopic with extreme points fixed. The projection $p : \tilde{G} \to G$ maps a path to its endpoint. Since p is a local homeomorphism, we can lift the smooth sttructure of G to $\tilde{G}$. Finally, given two paths γ_1, $\gamma_2 \in P(G,1)$, we define $\gamma_1 \cdot \gamma_2$ to be path given by the pointwise product of γ_1, γ_2. It is easy to see that $\gamma_1 \sim \gamma_1'$ and $\gamma_2 \sim \gamma_2'$ implies $\gamma_1 \cdot \gamma_2 \sim \gamma_1' \cdot \gamma_2'$. Hence the multiplication is well defined on $\tilde{G}$. One checks that it is also smooth. □

It follows from Proposition 2.6.4 that G and $\tilde{G}$ in Theorem 2.6.5 have isomorphic Lie algebras. A theorem of Ado states that every real Lie algebra admits a faithful representation on $\mathbb{R}^n$ — namely, an injective homomorphism into $\mathfrak{gl}(n,\mathbb{R})$ — for a sufficiently large n. It follows from Theorem 2.4.5 that every real Lie algebra $\mathfrak{g}$ can be realized as the Lie algebra of a subgroup of $GL(n,\mathbb{R})$. Further, owing to Theorem 2.6.5, we can find a simply-connected Lie group with Lie algebra $\mathfrak{g}$. Hence, owing to Corollary 2.4.8, there is a bijective correspondence between simply-connected Lie groups and Lie algebras. Taken together with Theorem 2.4.7, this yields:

2.6.6 Theorem *There is an equivalence between the category of simply-connected Lie groups and their morphisms and the category of real Lie algebras and their morphisms.*

Abelian groups

Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any $X \in \mathfrak{g}$, the flow Φ of X is given by $\Phi(t,\cdot) = R_{\exp tX}$. Since X, $Y \in \mathfrak{g}$ commute (i.e. $[X,Y]=0$) if and only if the corresponding flows commute, we see that $\mathfrak{g}$ is an Abelian Lie algebra (i.e. the Lie bracket is null) if and only of the elements of G in $\exp[\mathfrak{g}]$ commute. Since $\exp[\mathfrak{g}]$ generates the identity component G° as a group, we deduce that $\mathfrak{g}$ is Abelian if and only of G° is an Abelian Lie group.

In case G is Abelian, we note that $\exp : \mathfrak{g} \to G$ is a Lie group homomorphism. In fact, given X, $Y \in \mathfrak{g}$, both $t \mapsto \exp t(X+Y)$ and $t \mapsto \exp tX \exp tY$ are one-parameter groups with initial velocity $X+Y$, so they coincide. Now $\exp : \mathfrak{g} \to G$ is the universal covering.

It follows that connected Abelian Lie groups are of the form $\mathbb{R}^n/\Gamma$, where Γ is a discrete subgroup of $\mathbb{R}^n$, that is isomorphic to $T^k \times \mathbb{R}^{n-k}$. In particular, every compact connected Abelian Lie group is isomorphic to the torus $\mathbb{R}^n/\mathbb{Z}^n = T^n$.

2.7 Problems

1 Let G be a Lie group with multiplication map $\mu : G \times G \to G$ and inversion map $\iota : G \to G$. Prove that $d\mu_{(g,h)}(u,v) = (dL_g)_h(v) + (dR_h)_g(u)$ and $d\iota_g = -(dL_{g^{-1}})_1 \circ (dR_{g^{-1}})_g$ for g, $h \in G$ and $u \in T_gG$, $v \in T_hG$.

2 Let G be a connected Lie group and let X be a smooth vector field on G. Show that X is left-invariant of and only if $[X,Y]=0$ for all right-invariant vector fields Y on $\mathfrak{g}$.

3 Let $G = O(n)$.

a. Show that $G^\circ \subset SO(n)$.
b. Prove that any element in $SO(n)$ is conjugate in G to a matrix of the form

$$\begin{pmatrix} R_{t_1} & & & & & \\ & \ddots & & & & \\ & & R_{t_p} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

where R_t is the 2×2 block

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

and $t_1, \ldots, t_p \in \mathbb{R}$.
c. Deduce from the above that $SO(n)$ is connected. Conclude that $O(n)$ has two connected components and $SO(n)$ is the identity component.
d. Use a similar idea to show that $U(n)$ and $SU(n)$ are connected.

4 Show that

$$\exp\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

for $t \in \mathbb{R}$.

5 Give examples of matrices A, $B \in \mathfrak{gl}(2, \mathbb{R})$ such that $e^{A+B} \neq e^A e^B$.

6 In this problem, we show that the exponential map in a Lie group does not have to be surjective.
a. Show that every element g in the image of $\exp : \mathfrak{g} \to G$ has a square root, namely, there is $h \in G$ such that $h^2 = g$.
b. Prove that $\operatorname{trace} A^2 \geq -2$ for any $A \in SL(2, \mathbb{R})$. (Hint: A satisfies its characteristic polynomial equation $X^2 - 2(\operatorname{trace} X)X + (\det X)I = 0$.)
c. Deduce from the above that $\begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ does not lie in the image of $\exp : \mathfrak{sl}(2, \mathbb{R}) \to SL(2, \mathbb{R})$.

7 Consider the smooth map $\varphi : S^3 \times S^3 \to SO(4)$ defined by $\varphi(q_1, q_2)x = q_1 x q_2^{-1}$, where we view S^3 as the unit quaternions we identify $\mathbb{R}^4$ with the quaternions.
a. Check that φ is a Lie group homomorphism and $\ker \varphi = \{\pm(1, 1)\}$.
b. Compute $d\varphi$ and show that this is an injective map. (Hint: Identify the Lie algebra of S^3 with $\Im \mathbb{H}$.)
c. Deduce that φ is a covering of Lie groups and $\pi_1(SO(4)) \cong \mathbb{Z}_2$.

8 Show that the isomorphism $GL(n, \mathbb{H}) \to U^*(2n)$ constructed in Examples 2.1.1(iv) restricts to an isomorphism $Sp(n) \cong SU^*(2n) \cap U(2n)$.

9 Use the Gram-Schmidt orthogonalization process to show that $GL(n, \mathbb{R})$ is diffeomorphic to $O(n) \times V_+$, where V_+ is the the cone of upper triangular matrices with positive diagonal entries.

10 Let G be a Lie group. Prove that G does not have *small* subgroups, that is, there exists an open neighborhood U of 1 such that $\{1\}$ is the only sbgroup of G entirely contained in U. (Hint: Use the exponential map.)

11 Let S denote the vector space of $n \times n$ real symmetric matrices and let S^+ denote the open subset (positive cone) of S consisting of positive definite matrices.

$a.$ Prove that the exponential map of matrices sets up a bijection from S onto S^+. (Hint: Prove it first for diagonal matrices.)

$b.$ Show that $\langle X, Y \rangle = \mathrm{trace}(XY)$ for X, $Y \in S$ defines a positive definite symmetric bilinear form on S.

$c.$ Check that $\mathrm{ad}_Z X \in S$ for all $Z \in \mathfrak{so}(n)$ and $X \in S$, and $\langle \mathrm{ad}_Z X, Y \rangle + \langle X, \mathrm{ad}_Z Y \rangle = 0$ for all $Z \in \mathfrak{so}(n)$ and $X, Y \in S$.

$d.$ For $X \in S$, show that there is an orthogonal decomposition $S = T_X \oplus N_X$, where $T_X = \{[Z, X] \in S \mid Z \in \mathfrak{so}(n)\}$ and $N_X = \{Y \in S \mid [Y, X] = 0\}$.

$e.$ For $X \in S$, check that $e^{-X}(d\exp)_X : T_X S \cong S \to T_I S^+ \cong S$ is given by

$$e^{-X}(d\exp)_X = \frac{I - e^{-\mathrm{ad}_X}}{\mathrm{ad}_X}.$$

(Hint: Check the formula separately on N_X and T_X.)

$f.$ Deduce from the above that $\exp : S \to S^+$ is a global diffeomorphism.

12 $a.$ Determine the center of $SU(n)$.

$b.$ Construct a diffeomorphism $SU(n) \times S^1 \to U(n)$. Is it an isomorphism of Lie groups?

13 Prove that the kernel of the adjoint representation of a connected Lie group is the center.

14 Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and consider a connected Lie subgroup A of G with Lie algebra $\mathfrak{a}$. Prove that A is a normal subgroup of G if and only if $\mathfrak{a}$ is an ideal of $\mathfrak{g}$.

15 Prove that the fundamental group of a connected Lie group is Abelian. (Hint: Use Proposition 2.6.4 and Problem 13 of Chapter 1.)

CHAPTER 3

Structure of compact Lie groups

The main goal of this chapter is to reduce the classification of compact connected Lie groups to the classification of simply-connected compact connected simple Lie groups and the determination of their centers. The main technique is linearization, that is, passage to the Lie algebra. Along the way, we shall see the relation of compacity of the group with semisimplicity of its Lie algebra.

Throughout this chapter, unless explicitly mentioned otherwise, G denotes a compact connected Lie group and $\mathfrak{g}$ denotes its Lie algebra.

3.1 Invariant inner product on the Lie algebra

Many important results for compact Lie groups can be derived from the existence of a bi-invariant measure. In 1897 Adolf Hurwitz introduced this idea under the name of "invariant integration" and in 1933 Alfred Haar considered the more general idea of a "left invariant Haar measure" on locally compact topological groups.

A left invariant Haar measure on a locally compact topological group is a nonzero regular Borel measure that is invariant under left-translations. Its existence can be proved using techniques of functional analysis. In the case of compact Lie groups, an argument based on differential forms quickly yieds the existence of a left invariant Haar measure that is also right-invariant and we talk of a bi-invariant Haar measure. In our case, we have already constructed a bi-invariant Haar measure on a compact topological group (Remark 1.4.3).

Let μ be a bi-invariant Haar measure on G. Then

$$\int_G f(gxh^{-1})\, d\mu(x) = \int_G f(x)\, d\mu(x)$$

for every continuous function f on G and all $g, h \in G$. The following result is a special case of Proposition 1.4.6.

3.1.1 Proposition *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Then there exists a positive definite inner product $\langle , \rangle$ on $\mathfrak{g}$ with respect to which Ad_g is an orthogonal transformation; we say that $\langle , \rangle$ is* Ad-invariant.

3.1.2 Proposition *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Then: an inner product $\langle , \rangle$ on $\mathfrak{g}$ is Ad-invariant if and only if*

$$\langle \mathrm{ad}_X Y, Z\rangle + \langle Y, \mathrm{ad}_X Z\rangle = 0$$

for all X, Y, $Z \in \mathfrak{g}$; we say it is ad-invariant.

Proof. Let $O(\mathfrak{g})$ denote the orthogonal group of a given inner product $\langle , \rangle$ on $\mathfrak{g}$, and let $SO(\mathfrak{g})$ denote its identity component; this is a Lie group and we let $\mathfrak{so}(\mathfrak{g})$ denote its Lie algebra. For each $X \in \mathfrak{g}$, owing to (2.5.3) and (2.4.10), $\mathrm{ad}_X \in \mathfrak{so}(\mathfrak{g})$ if and only if $\mathrm{Ad}_{\exp(tX)} \in SO(\mathfrak{g})$ for all $t \in \mathbb{R}$, and this is equivalent to the statement of the proposition, if we use the connectedness of G to have that $\exp[\mathfrak{g}]$ generates G. □

3.2 Compact Lie algebras

We call a Lie algebra $\mathfrak{g}$ *compact* if there exists a compact Lie group G whose Lie algebra is isomorphic to $\mathfrak{g}$. The main goal of this section is to prove the following result.

3.2.1 Theorem *Every compact Lie algebra $\mathfrak{g}$ admits a decomposition into a direct sum of ideals,*

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r,$$

where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and the $\mathfrak{g}_i$ are compact simple ideals.

We call a Lie algebra *simple* if it is not Abelian and it does not admit non-trivial ideals. The center of a Lie algebra is the subspace of elements that commute with every other element; namely, for a Lie algebra $\mathfrak{g}$, the *center* of $\mathfrak{g}$ is

$$Z(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X,Y] = 0 \text{ for all } Y \in \mathfrak{g}\} = \ker \mathrm{ad}.$$

If G is a Lie group with Lie algebra $\mathfrak{g}$, then of course the center of G is

$$Z(G) = \{g \in G \mid gh = hg \text{ for all } h \in G\}.$$

Being a closed subgroup, $Z(G)$ is a Lie subgroup of G and it follows from Proposition 2.6.3(a) and Exercise 13 of Chapter 2 that the Lie algebra of $Z(G)$ is $Z(\mathfrak{g})$.

3.2.2 Examples (i) Plainly $\mathfrak{u}(n)$ is a compact Lie algebra. Let $X \in \mathfrak{u}(n)$. Writing

$$X = \frac{\operatorname{trace} X}{n} I + \left(X - \frac{\operatorname{trace} X}{n} I \right)$$

shows that $\mathfrak{u}(n) = \sqrt{-1}\mathbb{R} \cdot I \oplus \mathfrak{su}(n)$ is a direct sum of ideals. It follows from Problem 12 of Chapter 2 that $\sqrt{-1}\mathbb{R} \cdot I$ is the center of $\mathfrak{u}(n)$. Plainly $\mathfrak{su}(n)$ is a compact Lie algebra, and we shall soon see that it is a simple Lie algebra.

(ii) Since $SU(2)$ is the universal covering Lie group of $SO(3)$, the Lie algebras $\mathfrak{su}(2) = \mathfrak{so}(3)$. Now it follows from Problem 7 in Chapter 2 that $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. Plainly $\mathfrak{so}(n)$ is a compact Lie algebra, and we shall see that it is simple if $n = 3$ or $n \geq 5$ (of course, $\mathfrak{so}(2)$ is one-dimensional and hence Abelian).

Adjoint group

Let $\mathfrak{g}$ be a Lie algebra. Then $GL(\mathfrak{g})$ is a Lie group with Lie algebra $\mathfrak{gl}(\mathfrak{g})$ consisting of all endomorphisms of the vector space underlying $\mathfrak{g}$. The group of automorphisms of $\mathfrak{g}$, denoted by $\operatorname{Aut}(\mathfrak{g})$, is clearly a closed subgroup of $GL(\mathfrak{g})$. Recall that a closed subgroup of a Lie group is a Lie subgroup with the subspace topology. Hence $\operatorname{Aut}(\mathfrak{g})$ is a Lie subgroup of $GL(\mathfrak{g})$. Its Lie algebra consists of the endomorphisms $D \in \mathfrak{gl}(\mathfrak{g})$ such that

$$\exp(tD) \cdot [X, Y] = [\exp(tD) \cdot X, \exp(tD) \cdot Y]$$

for X, $Y \in \mathfrak{g}$, $t \in \mathbb{R}$. Differentiating this equation at $t = 0$, we obtain

$$D[X, Y] = [DX, Y] + [X, DY] \tag{3.2.3}$$

for X, $Y \in \mathfrak{g}$. The endomorphisms D satisfying equation (3.2.3) are called *derivations* of $\mathfrak{g}$. The space $\operatorname{Der}(\mathfrak{g})$ of all derivations of $\mathfrak{g}$ is closed under the Lie bracket of $\mathfrak{gl}(\mathfrak{g})$, as is easily seen. Conversely, if D is a derivation of $\mathfrak{g}$, then one checks by induction that

$$D^m[X, Y] = \sum_{i+j=m} \frac{m!}{i!j!} [D^i X, D^j X]$$

for all $m \geq 1$. It follows that

$$e^D[X, Y] = \sum_{m \geq 0} \frac{1}{m!} D^m[X, Y] = [e^D X, e^D Y],$$

that is, e^D is an automorphism. Therefore the Lie algebra of $\operatorname{Aut}(\mathfrak{g})$ is $\operatorname{Der}(\mathfrak{g})$.

In particular, the Jacobi identity shows that $\operatorname{ad}_X \in \operatorname{Der}(\mathfrak{g})$ for all $X \in \mathfrak{g}$, so that the image of the adjoint representation $\operatorname{ad}[\mathfrak{g}]$ is a subalgebra of $\operatorname{Der}(\mathfrak{g})$, again by Jacobi. The derivation property says that $[D, \operatorname{ad}_X] = \operatorname{ad}_{DX}$ for $D \in \operatorname{Der}(\mathfrak{g})$ and $X \in \mathfrak{g}$, so $\operatorname{ad}[\mathfrak{g}]$ is indeed an *ideal* of $\operatorname{Der}(\mathfrak{g})$.

The elements of $\mathrm{ad}[\mathfrak{g}]$ are called *inner derivations*. Let $\mathrm{Inn}(\mathfrak{g})$ be the connected subgroup of $\mathrm{Aut}(\mathfrak{g})$ defined by $\mathrm{ad}[\mathfrak{g}]$. In view of Problem 14 of Chapter 2, $\mathrm{Inn}(\mathfrak{g})$ is a normal subgroup of $\mathrm{Aut}(\mathfrak{g})$, and in accordance with the next proposition, that group is called the *adjoint group* of $\mathfrak{g}$ and its elements are called *inner automorphisms* of $\mathfrak{g}$.

Compactness is not needed for the following result.

3.2.4 Proposition *The adjoint group* $\mathrm{Inn}(\mathfrak{g})$ *is canonically isomorphic to* $G/Z(G)$*, where* G *is any connected Lie group with Lie algebra* $\mathfrak{g}$ *and* $Z(G)$ *denotes the center of* G*.*

Proof. The image of the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is contained in $\mathrm{Inn}(\mathfrak{g})$, because $\mathrm{Ad}_{\exp X} = e^{\mathrm{ad}_X}$ for $X \in \mathfrak{g}$, and the image of $\exp$ generates G. Since $d(\mathrm{Ad}) = \mathrm{ad}$, Proposition 2.6.3 says that the Lie algebra of the image of Ad is $\mathrm{ad}[\mathfrak{g}]$, thus we get equality $\mathrm{Ad}(G) = \mathrm{Inn}(\mathfrak{g})$. Finally, note that the kernel of Ad is $Z(G)$ (cf. Problem 13 in Chapter 2). □

Fix a real Lie algebra $\mathfrak{g}$. Regarding *all* connected Lie groups that have Lie algebras isomorphic to $\mathfrak{g}$, now the following picture emerges. If $G_1 \to G_2$ is a covering homomorphism, then G_1 and G_2 have isomorphic Lie algebras, but the converse statement does not hold, namely, there exist Lie groups with isomorphic Lie algebras such that neither one of them covers the other one[1]. However, there exists a simply-connected Lie group $\tilde{G}$ with Lie algebra $\mathfrak{g}$, and $\tilde{G}$ does cover any other Lie group with Lie algebra $\mathfrak{g}$. Moreover, in the case in which $\mathfrak{g}$ is centerless, the adjoint group $\bar{G} := \mathrm{Inn}(\mathfrak{g})$ has Lie algebra $\mathfrak{g}$ and it is covered by any connected Lie group G with Lie algebra $\mathfrak{g}$, since $\bar{G} \cong G/Z(G)$. Hence $\tilde{G}$ sits at the top of the hierarchy and $\bar{G}$ sits at its bottom.

Cartan-Killing form

Let $\mathfrak{g}$ be a Lie algebra. The *Cartan-Killing form* of $\mathfrak{g}$ is the symmetric bilinear form

$$\beta(X, Y) = \mathrm{trace}\,\mathrm{ad}X\mathrm{ad}Y \qquad \text{(trace)}$$

where $X, Y \in \mathfrak{g}$.

3.2.5 Proposition *a. If* $\mathfrak{a} \subset \mathfrak{g}$ *is an ideal, then the Cartan-Killing form of* $\mathfrak{a}$ *is the restriction of* β *to* $\mathfrak{a} \times \mathfrak{a}$*.*

b. If $s \in \mathrm{Aut}(\mathfrak{g})$*, then* $\beta(sX, sY) = \beta(X, Y)$ *for* $X, Y \in \mathfrak{g}$*.*

c. $\beta(\mathrm{ad}_X Y, Z) + \beta(Y, \mathrm{ad}_X Z) = 0$ *for* $X, Y, Z \in \mathfrak{g}$ *(*β *is* ad*-invariant).*

Proof. (a) If $X, Y \in \mathfrak{a}$ then $\mathrm{ad}_X \mathrm{ad}_Y$ maps $\mathfrak{g}$ into $\mathfrak{a}$. (b) If $s \in \mathrm{Aut}(\mathfrak{g})$ then $\mathrm{ad}_{sX} = s \circ \mathrm{ad}_X \circ s^{-1}$. (c) It follows from Jacobi. □

[1] Example?

Semisimplicity

In order to avoid unnecessary technicalities, we adopt the following non-standard, but completely equivalent definition. We call a Lie algebra $\mathfrak{g}$ *semisimple* if β is nondegenerate.[2] Note that by the ad-invariance of β, its kernel is always an ideal of the underlying Lie algebra.

A Lie group G is called *semisimple* (resp. simple) if its Lie algebra is semisimple (resp. simple). Note that the definition of simple Lie group is different from the definition of simple abstract group, in that a simple Lie group is allowed to contain non-trivial discrete normal subgroups; for instance, $SU(n)$ is considered a simple Lie group but it has a finite center (cf. Problem 12 in Chapter 2).

3.2.6 Proposition *Let $\mathfrak{g}$ be a semisimple Lie algebra, let $\mathfrak{a} \subset \mathfrak{g}$ be an ideal and*

$$\mathfrak{a}^\perp = \{X \in \mathfrak{g} : \beta(X, \mathfrak{a}) = 0\}.$$

Then $\mathfrak{a}^\perp$ is an ideal, $\mathfrak{a}$ and $\mathfrak{a}^\perp$ are semisimple and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ (direct sum of ideals).

Proof. The ad-invariance of β implies that $\mathfrak{a}^\perp$ is an ideal of $\mathfrak{g}$. Then $\mathfrak{a} \cap \mathfrak{a}^\perp$ is an ideal of $\mathfrak{g}$, and again by ad-invariance of β, $\mathfrak{a} \cap \mathfrak{a}^\perp$ is Abelian; in fact, for every $Z \in \mathfrak{g}$ and $X, Y \in \mathfrak{a} \cap \mathfrak{a}^\perp$,

$$\beta(Z, [X, Y]) = \beta([Z, X], Y) = 0,$$

so $[X, Y] = 0$ by nondegeneracy of β. Fix now a complementary subspace $\mathfrak{b}$ of $\mathfrak{a} \cap \mathfrak{a}^\perp$ in $\mathfrak{g}$. Then, for $X \in \mathfrak{a} \cap \mathfrak{a}^\perp$ and $Y \in \mathfrak{g}$, the linear map $\mathrm{ad}_X \mathrm{ad}_Y$ maps $\mathfrak{a} \cap \mathfrak{a}^\perp$ to zero and $\mathfrak{b}$ to $\mathfrak{a} \cap \mathfrak{a}^\perp$, so it has no diagonal elements and thus $\beta(X, Y) = 0$, yielding $X = 0$ by nondegeneracy of β. We have shown that $\mathfrak{a} \cap \mathfrak{a}^\perp = 0$. The nondegeneracy of β also implies that $\dim \mathfrak{a} + \dim \mathfrak{a}^\perp = \dim \mathfrak{g}$, whence $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$. That $\mathfrak{a}$ and $\mathfrak{a}^\perp$ are semisimple is a consequence of Proposition 3.2.5(a) and $\beta(\mathfrak{a}, \mathfrak{a}^\perp) = 0$. □

3.2.7 Corollary *A semisimple Lie algebra $\mathfrak{g}$ is centerless and decomposes into a direct sum of simple ideals $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$. In particular, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.*

Proof. Clearly the center of $\mathfrak{g}$ is contained in the kernel of the Cartan-Killing form, hence it is zero. If $\mathfrak{a}$ is a proper ideal of $\mathfrak{g}$, then the proposition says that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ and the decomposition result follows by induction on the dimension of $\mathfrak{g}$. Finally,

$$[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}_1, \mathfrak{g}_1] \oplus \cdots \oplus [\mathfrak{g}_r, \mathfrak{g}_r]$$

and each $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$ since $[\mathfrak{g}_i, \mathfrak{g}_i]$ is a nonzero ideal of $\mathfrak{g}_i$. □

[2] The more usual definition are that a Lie algebra is called *semisimple* if it does not admit nontrivial solvable ideals; in particular, a simple Lie algebra is semisimple. Cartan's criterium for semisimplicity is that the Killing form be nondegenerate.

3.2.8 Proposition *If* $\mathfrak{g}$ *is semisimple then* $\mathrm{ad}[\mathfrak{g}] = \mathrm{Der}(\mathfrak{g})$, *i.e. every derivation is inner.*

Proof. Since $\mathfrak{g}$ is centerless, $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is injective, so $\mathrm{ad}[\mathfrak{g}]$ is semisimple. Denote by β and β' the Cartan-Killing forms of $\mathrm{ad}[\mathfrak{g}]$ and $\mathrm{Der}(\mathfrak{g})$, respectively. $\mathrm{ad}[\mathfrak{g}]$ is an ideal of $\mathrm{Der}(\mathfrak{g})$ and β is the restriction of β' to $\mathrm{ad}[\mathfrak{g}]$. Consider $\mathfrak{a} = \mathrm{Der}(\mathfrak{g})^\perp$ with respect to β' and note that $\mathfrak{a}$ is a complementary subspace to $\mathrm{ad}[\mathfrak{g}]$ in $\mathrm{Der}(\mathfrak{g})$, which is moreover an ideal of $\mathrm{Der}(\mathfrak{g})$. Now $\mathrm{ad}_{DX} = [D, \mathrm{ad}_X] \in \mathrm{ad}[\mathfrak{g}] \cap \mathfrak{a} = 0$ for $D \in \mathfrak{a}$ and $X \in \mathfrak{g}$. Since $\ker \mathrm{ad} = 0$, this implies $\mathfrak{a} = 0$, as desired. □

3.2.9 Corollary *If* $\mathfrak{g}$ *is semisimple then* $\mathrm{Inn}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})^0$.

Proof. $\mathrm{Inn}(\mathfrak{g})$ is connected and both hand sides have the same Lie algebra. □

Main result

3.2.10 Theorem *Let* $\mathfrak{g}$ *be a Lie algebra. The following assertions are equivalent:*

a. $\mathfrak{g}$ *is a compact Lie algebra.*

b. $\mathrm{Inn}(\mathfrak{g})$ *is compact.*

c. $\mathfrak{g}$ *admits an* ad*-invariant positive definite inner product.*

d. $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$ *where* $\mathfrak{z}$ *is the center of* $\mathfrak{g}$ *and* $[\mathfrak{g}, \mathfrak{g}]$ *is semisimple with negative definite Cartan-Killing form.*

Proof. (c) implies (d). Let $\langle , \rangle$ be an ad-invariant positive definite inner product on $\mathfrak{g}$,

$$\langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle = 0$$

for $X, Y, Z \in \mathfrak{g}$. The center $\mathfrak{z}$ is ad-invariant, so also its $\langle , \rangle$-orthogonal complement $\mathfrak{z}^\perp$ is ad-invariant. Now $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{z}^\perp$, direct sum of ideals. The Cartan-Killing form of $\mathfrak{z}^\perp$ is the restriction of the Cartan-Killing form β of $\mathfrak{g}$. Owing to the ad-invariance of $\langle , \rangle$, ad_X is skew-symmetric with respect to $\langle , \rangle$ for $X \in \mathfrak{g}$, thus it has purely imaginary eigenvalues. Therefore $\beta(X, X) = \mathrm{trace}\, \mathrm{ad}_X^2 \leq 0$ and equality holds if and only if $\mathrm{ad}_X = 0$ if and only if $X \in \mathfrak{z}$. This proves that $B|_{\mathfrak{z}^\perp \times \mathfrak{z}^\perp}$ is negative definite and hence $\mathfrak{z}^\perp$ is semisimple and $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{z}^\perp, \mathfrak{z}^\perp] = \mathfrak{z}^\perp$.

(d) implies (b). We have $\mathrm{Inn}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{z}) \times \mathrm{Inn}([\mathfrak{g}, \mathfrak{g}]) = \mathrm{Inn}([\mathfrak{g}, \mathfrak{g}])$ since $\mathfrak{z}$ is Abelian thus, without loss of generality, we may assume $\mathfrak{g}$ is semisimple with negative definite Cartan-Killing form. Let $O(\mathfrak{g}) \subset GL(\mathfrak{g})$ the compact subgroup of β-preserving transformations. Clearly $\mathrm{Aut}(\mathfrak{g})$ is contained in $O(\mathfrak{g})$ as a closed, thus compact subgroup. Now Corollary 3.2.9 yields the result.

(b) implies (c). Since the group of inner automorphisms $\mathrm{Inn}(\mathfrak{g})$ is compact, by Proposition 3.1.1 there exists an Ad-invariant positive definite inner product on $\mathfrak{g}$. It is also ad-invariant.

Now (b), (c) and (d) are equivalent. We next show that (b) and (d) imply (a). $\mathrm{Inn}([\mathfrak{g},\mathfrak{g}]) = \mathrm{Inn}(\mathfrak{g})$ is compact and and $\mathrm{Inn}([\mathfrak{g},\mathfrak{g}])$ has Lie algebra $\mathrm{ad}([\mathfrak{g},\mathfrak{g}]) \cong [\mathfrak{g},\mathfrak{g}]$ since $[\mathfrak{g},\mathfrak{g}]$ is semisimple. Now $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g},\mathfrak{g}]$ is the Lie algebra of $S^1 \times \cdots \times S^1 \times \mathrm{Inn}([\mathfrak{g},\mathfrak{g}])$.

Finally (a) implies (b). Since $\mathfrak{g}$ is the Lie algebra of a compact connected Lie group G, $\mathrm{Inn}(\mathfrak{g}) \cong G/Z(G)$ is compact. □

3.2.11 Corollary *A semisimple Lie algebra is compact if and only if its Cartan-Killing form is negative definite.*

Proof of Theorem 3.2.1. It follows from Theorem 3.2.10 and Corollaries 3.2.7 and 3.2.11.

3.2.12 Remark The kernel of the Cartan-Killing form β of a Lie algebra $\mathfrak{g}$ is an ideal. It follows that in case of simple $\mathfrak{g}$, either β is nondegenerate and then $\mathfrak{g}$ is semisimple, or $\beta = 0$. In case $\mathfrak{g}$ is simple and compact, Theorem 3.2.10 says that the second possibility cannot occur. We deduce that a compact simple Lie algebra is semisimple (according to our definitions). We shall see later that indeed the compactness assumption is unnecessary, that is, every simple Lie algebra is semisimple.

Geometry of compact Lie groups with bi-invariant metrics

Recall that a Riemannian metric on a smooth manifold M is simply a smoothly varying assignment of an inner product $\langle,\rangle_p$ on the tangent space T_pM for each $p \in M$; here the smoothness of $\langle,\rangle$ refers to the fact that $p \mapsto \langle X_p, Y_p\rangle_p$ defines a smooth function on M for all smooth vector fields X, Y on M.

As an important application of Proposition 3.1.1, we show that a compact Lie group G admits a *bi-invariant Riemannian metric*, that is, a Riemannian metric with respect to which left translations and right translations are isometries.

Indeed there is a bijective correspondence between postive-definite inner products on $\mathfrak{g}$ and left-invariant Riemannian metrics on G: every inner product on $\mathfrak{g} = T_1G$ gives rise to a left-invariant metric on G by declaring the left-translations to be isometries, namely, $dL_g : T_1G \to T_gG$ is a linear isometry for all $g \in G$; conversely, every left-invariant metric on G is completely determined by its value at 1. Now, when is a left-invariant metric $\langle,\rangle$ on G also right-invariant? Note that differentiation of the obvious formula $R_g = L_g \circ \mathrm{Inn}_{g^{-1}}$ at 1 yields

$$d(R_g)_1 = (dL_g)_1 \circ \mathrm{Ad}_{g^{-1}},$$

where $g \in G$. We deduce that g is right-invariant if and only if $\langle,\rangle_1$ is Ad-invariant. Thus the existence of a bi-invariant metric on G follows from Proposition 3.1.1.

Let G be a compact connected Lie group endowed with a bi-invariant Riemannian metric and denote its Lie algebra by $\mathfrak{g}$. The Koszul formula for the Levi-Cività connection ∇ on G is

$$\begin{aligned}\langle\nabla_X Y, Z\rangle = \frac{1}{2}\big(&X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ &+\langle[X,Y],Z\rangle + \langle[Z,X],Y\rangle + \langle[Z,Y],X\rangle\big),\end{aligned} \tag{3.2.13}$$

where X, Y, $Z \in \mathfrak{g}$. The inner product of left-invariant vector fields is constant, so the three terms in the first line of the right hand-side of (3.2.13) vanish; we apply ad-invariance of the inner product on $\mathfrak{g}$ to manipulate the remaining three terms and we arrive at

$$\nabla_X Y = \frac{1}{2}[X,Y].$$

In particular, $\nabla_X X = 0$ so every one-parameter subgroup of G is a geodesic thorugh 1. Since there one-parameter groups going in all directons, they comprise all geodesics of G through 1 (of course, the geodesics through other points in G differ from one-parameter groups by a left translation).

3.2.14 Remark The statement about one-parameter groups coinciding with Riemannian geodesics through 1 is equivalent to saying that the exponential map of the Lie group coincides with the Riemannian exponential map $\mathrm{Exp}_1 : T_1G \to G$ that maps each $X \in T_1G$ to the value at time 1 of the geodesic through 1 with initial velocity X. Now geodesics through 1 are defined for all values of the parameter; in view of the Hopf-Rinow theorem, this means that G is complete as a Riemannian manifold, and any point in G can be joined by a geodesic to 1, or Exp_1 is surjective. We deduce the the exponential map of a compact Lie group is a surjective map (compare Problem 6 in Chapter 2).

Now we compute the Riemannian sectional curvature of G. Let $X, Y \in \mathfrak{g}$ be an orthonormal pair. Then

$$\begin{aligned} K(X,Y) &= -\langle R(X,Y)X,Y\rangle \\ &= -\langle \nabla_X\nabla_Y X + \nabla_Y\nabla_X X - \nabla_{[X,Y]}X, Y\rangle \\ &= \frac{1}{4}||[X,Y]||^2, \end{aligned}$$

using again ad-invariance of the inner product.

Finally, let $\{X_i\}_{i=1}^n$ be an orthonormal basis of $\mathfrak{g}$. Then the Ricci curvature

$$\begin{aligned} Ric(X,X) &= \sum_{i=1}^n K(X,X_i) \\ &= \frac{1}{4}\sum_{i=1}^n ||[X,X_i]||^2. \end{aligned}$$

It follows that $Ric(X,X) \geq 0$ and $Ric(X,X) = 0$ if and only if X lies in the center of $\mathfrak{g}$.

The universal covering of compact semisimple Lie groups

There are several proofs of the following theorem. Here we use basic Riemannian geometry. Later we will see an algebraic proof based on lattices.

3.2.15 Theorem (Weyl) *Let G be a compact connected semisimple Lie group. Then the universal covering Lie group $\tilde{G}$ is also compact (equivalently, the fundamental group of G is finite).*

Proof. The universal covering $\tilde{G}$ has a structure of Lie group so that the projection $\tilde{G} \to G$ is a smooth homomorphism. Equip G with a bi-invariant Riemannian metric. Since $\mathfrak{g}$ is centerless, $\mathrm{Ric}(X,X) > 0$ for $X \neq 0$. By compactness of the unit sphere, $\mathrm{Ric}(X,X) \geq a\langle X, X\rangle$ for some $a > 0$. The Bonnet-Myers theorem yields that $\tilde{G}$ is compact. □

The structure of compact connected Lie groups

We close this section with a description of the structure of compact connected Lie groups toward a classification, by putting together previous results.

Let G be a compact connected Lie group. Then its Lie algebra $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$, where $\mathfrak{z}$ is the center and $\mathfrak{g}'$ is compact semisimple (Theorem 3.2.10). The kernel of the adjoint representation of G is $Z(G)$ (Problem 13 of Chapter 2), so its Lie algebra is the kernel of the adjoint representation of $\mathfrak{g}$, namely, $\mathfrak{z}$. Now the connected subgroup of G associated to $\mathfrak{z}$ is the identity component $Z(G)^0$ of the center $Z(G)$. Since $Z(G)$ is a closed subgroup of G, so is $Z(G)^0$. Let G_{ss} be the connected subgroup of G associated to $\mathfrak{g}'$. The Cartan-Killing form is negative-definite, so $\mathrm{Inn}(\mathfrak{g}') = G_{ss}/Z(G_{ss})$ is compact and thus its universal covering $\tilde{G}_{ss}$ is compact by Weyl's theorem 3.2.15. In particular, $Z(G_{ss})$ is finite.

Note that the universal covering $\widetilde{Z(G)^0} = \mathbb{R}^k$, where $k = \dim \mathfrak{z}$, and $\mathbb{R}^k \times \tilde{G}_{ss}$ is a simply connected Lie group with Lie algebra $\mathfrak{z} \oplus \mathfrak{g}'$, and hence it is the universal covering $\tilde{G}$ of G. The covering projection maps $\mathbb{R}^k$ to $Z(G)^0$ and $\tilde{G}_{ss}$ to G_{ss}. It follows that G_{ss} is a closed subgroup of G. Since $\tilde{G} = \mathbb{R}^k \cdot \tilde{G}_{ss}$, we also deduce that $G = Z(G)^0 \cdot G_{ss}$.[3] The intersection $Z(G)^0 \cap G_{ss}$ is a finite group D, because $\mathfrak{z} \cap \mathfrak{g}' = 0$. It is clear that $D = Z(G_{ss})$, and this is another way to see that $Z(G_{ss})$ is finite, Now $Z(G)^0 \cdot G_{ss} = (Z(G)^0 \times G_{ss})/Z(G_{ss})$. We have proved:

[3]Given a group G and subsets A, $B \subset G$, we write $A \cdot B = \{ab \in G \mid a \in A,\ b \in B\}$. If A and B are subgroups and, say, the elements of A normalize B, then $(a'b')(ab) = (a'a)((a^{-1}b'a)b) \in A \cdot B$ and $(ab)^{-1} = b^{-1}a^{-1} = a^{-1}(ab^{-1}a^{-1}) \in A \cdot B$ for a, $b \in B$, so $A \cdot B$ is a subgroup of G.

3.2.16 Theorem *A compact connected Lie group G is a finite quotient of the direct product of a simply-connected compact connected semisimple Lie group $\tilde{G}_{ss}$ and a torus $Z(G)^0$.*

Continuing with the above, the Lie algebra $\mathfrak{g}'$ decomposes into a direct sum of simple ideals $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ (Corollary 3.2.7). Let G_i (resp. $\tilde{G}_i$) be the connected subgroup of G_{ss} (resp. $\tilde{G}_{ss}$) associated to $\mathfrak{g}_i$. Then G_i (resp. $\tilde{G}_i$) is a closed normal subgroup of G_{ss} (resp. $\tilde{G}_{ss}$) and a simple Lie group. We have $\tilde{G}_{ss} = \tilde{G}_1 \times \cdots \times \tilde{G}_r$ and G_{ss} is the quotient of $G_1 \times \cdots \times G_r$ by a finite central subgroup.

3.2.17 Examples (i) $U(n) = (S^1 \times SU(n))/\mathbb{Z}_n$ where $\mathbb{Z}_n$ is embedded into $S^1 \times SU(n)$ as $z \mapsto (z, z^{-1}I)$.

(ii) $SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by $(-I, -I) \in SU(2) \times SU(2)$.

All in all, we have reduced the classification of compact connected Lie groups to the classification of compact simple Lie algebras and the determination of the centers of the corresponding simply-connected Lie groups. The result that we shall prove, accredited to Killing and Cartan, is that the simply-connected compact connected simple Lie groups are the three families of classical groups $SU(n)$, $Sp(n)$, $Spin(n)$ (the universal covering of $SO(n)$) and the five exceptional groups G_2, F_4, E_6, E_7, E_8, of dimensions 14, 52, 78, 133, 248, respectively.

3.3 Complex semisimple Lie algebras

Il apparut que, entre deux vérités du domaine réel, le chemin le plus facile et le plus court passe bien souvent par le domaine complexe. (Attributed to Jacques Hadamard.)

Although it is not strictly necessary, we find it convenient to complexify all Lie algebras and work in the complex domain. The notion of semisimplicity introduced for real Lie algebras makes sense over any field of characteristic zero. It is almost immediate to see that the complexification of a real semisimple Lie algebra is semisimple.

On the other hand, it is a non-trivial result that every complex semisimple Lie algebra is the complexification of a *compact* Lie algebra. We give a proof of this theorem, although the result is not necessary for readers interested only on compact Lie algebras. Another advantage of making one familiar with complex semisimple Lie algebras is that it will be easier to later understand noncompact semisimple Lie algebras (although we do not discuss them in this book).

Complexification and realification

Let $\mathfrak{g}$ be a real Lie algebra. The *complexification* of $\mathfrak{g}$ is the complex Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$, where the Lie bracket is extended complex bilinearly. It is clear that this makes $\mathfrak{g}^{\mathbb{C}}$ into a complex Lie algebra. Let β be the Cartan-Killing form of $\mathfrak{g}$. Let $\{X_1, \ldots, X_n\}$ be a basis of $\mathfrak{g}$ over $\mathbb{R}$. Then β is non-degenerate if and only if the (symmetric) matrix $(\beta(X_i, X_j))$ is non-singular. Note that $\{X_1, \ldots, X_n\}$ is a also a basis of $\mathfrak{g}^{\mathbb{C}}$ over $\mathbb{C}$. It follows that $\dim_{\mathbb{C}} \mathfrak{g}^{\mathbb{C}} = \dim_{\mathbb{R}} \mathfrak{g}$ and that $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}^{\mathbb{C}}$ is semisimple.

Now let $\mathfrak{g}$ be a complex Lie algebra. The *realification* of $\mathfrak{g}$ is the underlying real Lie algebra $\mathfrak{g}^{\mathbb{R}}$ obtained by restriction of the scalars of $\mathfrak{g}$ to $\mathbb{R}$. (Note that the inverse process of complexification is not realification, but it is taking a real form, see below.) If $\{X_1, \ldots, X_n\}$ is a basis of $\mathfrak{g}$ over $\mathbb{C}$, then $\{X_1, \ldots, X_n, \sqrt{-1}X_1, \ldots, \sqrt{-1}X_n\}$ is a basis of $\mathfrak{g}^{\mathbb{R}}$ over $\mathbb{R}$, so $\dim_{\mathbb{R}} \mathfrak{g}^{\mathbb{R}} = 2 \dim_{\mathbb{C}} \mathfrak{g}$. For $X \in \mathfrak{g}^{\mathbb{R}}$, the matrix of $\mathrm{ad}_X : \mathfrak{g}^{\mathbb{R}} \to \mathfrak{g}^{\mathbb{R}}$ with respect to this basis has the block form

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

where $A + \sqrt{-1}B$ is the matrix of $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$, X viewed as an element of $\mathfrak{g}$. It follows that the Cartan-Killing forms $\beta_{\mathfrak{g}}$ of $\mathfrak{g}$ and $\beta_{\mathfrak{g}^{\mathbb{R}}}$ of $\mathfrak{g}^{\mathbb{R}}$ are related by

$$\beta_{\mathfrak{g}^{\mathbb{R}}} = 2\Re\{\beta_{\mathfrak{g}}\}.$$

Since $\Im\{\beta_{\mathfrak{g}}\}(X, Y) = \Re\{\beta_{\mathfrak{g}}\}(X, -\sqrt{-1}Y)$, we see that $X \in \ker \beta_{\mathfrak{g}^{\mathbb{R}}}$ if and only if $X \in \ker \beta_{\mathfrak{g}}$. It follows that $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}^{\mathbb{R}}$ is semisimple.

3.3.1 Lemma *Let $\mathfrak{g}$ be a real simple Lie algebra. Then the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is not simple if and only if $\mathfrak{g}$ is the realification of a complex simple Lie algebra.*

Proof. Assume that $\mathfrak{g}$ is the realification of a complex Lie algebra $\mathfrak{h}$ (note that $\mathfrak{h}$ must be simple, for the realification of an ideal of $\mathfrak{h}$ would be an ideal of $\mathfrak{g}$). Then there is complex structure J on $\mathfrak{g}$ such that $J[X, Y] = [JX, Y] = [X, JY]$ for $X, Y \in \mathfrak{g}$. The **C**-linear extension of J to $\mathfrak{g}^{\mathbb{C}}$ admits eigenvalues $\pm\sqrt{-1}$ and corresponding eigenspace decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{a}_{\sqrt{-1}} + \mathfrak{a}_{-\sqrt{-1}}$ where $\mathfrak{a}_{\pm\sqrt{-1}} = \{Z \mp \sqrt{-1}JZ : Z \in \mathfrak{g}\}$. It is easy to see that $\mathfrak{g}^{\mathbb{C}} = \mathfrak{a}_{\sqrt{-1}} + \mathfrak{a}_{-\sqrt{-1}}$ is a direct sum of ideals. In particular we see that $\mathfrak{g}^{\mathbb{C}}$ is not simple which proves half the lemma. Note that $\mathfrak{a}_{\sqrt{-1}}$ is isomorphic as a complex Lie algebra via $\frac{1}{2}(Z - \sqrt{-1}JZ) \mapsto Z$ to $\mathfrak{h}$, and $\mathfrak{a}_{-\sqrt{-1}}$ is isomorphic as a complex Lie algebra via $\frac{1}{2}(Z + \sqrt{-1}JZ) \mapsto Z$ to $\mathfrak{h}$ endowed with the conjugate complex structure.

Conversely assume that the complexification $\mathfrak{g}^{\mathbb{C}}$ can be written as a direct sum of simple ideals $\mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r$ for some $r > 1$ where the $\mathfrak{h}_i$ are complex simple Lie algebras. Let $\pi_i : \mathfrak{g} \to \mathfrak{h}_i$ be the composition of the inclusion map $\mathfrak{g} \to \mathfrak{g}^{\mathbb{C}}$ followed by the projection $\mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r \to \mathfrak{h}_i$. We claim that $\pi_i(\mathfrak{g}) \neq 0$ for all i. In fact, if Z is a nonzero element of $\mathfrak{h}_i$, we write $Z = X + \sqrt{-1}Y$ for some $X, Y \in \mathfrak{g}$ and then $Z = \pi_i(X) + \sqrt{-1}\pi_i(Y)$ which implies that either $\pi_i(X) \neq 0$ or $\pi_i(Y) \neq 0$, and this proves the claim. Since $\mathfrak{g}$ is simple, we have that π_i is injective and then the real dimension of $\mathfrak{h}_i$ cannot be less than the real dimension of $\mathfrak{g}$, namely

$$\dim_{\mathbf{R}} \mathfrak{h}_i \geq \dim_{\mathbf{R}} \mathfrak{g} = \frac{1}{2} \dim_{\mathbf{R}} \mathfrak{g}^{\mathbb{C}}.$$

This implies that $r = 2$ and that $\dim_{\mathbf{R}} \mathfrak{h}_i = \dim_{\mathbf{R}} \mathfrak{g}$. Now $\pi : \mathfrak{g} \to \mathfrak{h}_i$ is an isomorphism and we can transfer the complex structure from $\mathfrak{h}_i$ to $\mathfrak{g}$ which completes the proof. □

3.3.2 Example Let $\mathfrak{g} = \mathfrak{u}(n)$. The elements of $\mathfrak{g}$ are skew-Hermitian matrices, and the elements of $\sqrt{-1}\mathfrak{g}$ are Hermitian matrices. Since every complex matrix uniquely decomposes into a sum of a skew-Hermitian and a Hermitian matrix, we deduce that

$$\mathfrak{u}(n)^{\mathbb{C}} = \mathfrak{u}(n) + \sqrt{-1}\mathfrak{u}(n) = \mathfrak{gl}(n, \mathbb{C}).$$

If we impose the trace zero condition, we get

$$\mathfrak{su}(n)^{\mathbb{C}} = \mathfrak{su}(n) + \sqrt{-1}\mathfrak{su}(n) = \mathfrak{sl}(n, \mathbb{C}).$$

3.3.3 Example Let $\mathfrak{g} = \mathfrak{so}(n)$. Then the complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}(n, \mathbb{C}) = \{A \in \mathfrak{gl}(n, \mathbb{C}) \mid A + A^t = 0\}$.

3.3.4 Example Due to Examples 2.1.1 and Problem 8 in Chapter 2, the Lie algebra $\mathfrak{g} = \mathfrak{sp}(n)$ can be viewed as the Lie algebra of complex matrices

$$\left\{ \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix} : A \in \mathfrak{u}(n),\ B \in \mathrm{Sym}(n, \mathbb{C}) \right\}.$$

Recall that $\mathfrak{u}(n)^{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$; for $A \in \mathfrak{u}(n)$, we can rewrite $\bar{A}$ as $-A^t$, and we note that the latter expression is $\mathbb{C}$-linear in A. Moreover, the linear system

$$\begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\bar{B}_1 \\ B_1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\bar{B}_2 \\ B_2 & 0 \end{pmatrix}$$

has a unique solution in $B_1, B_2 \in \mathrm{Sym}(n, \mathbb{C})$ for given $B, C \in \mathrm{Sym}(n, \mathbb{C})$. It follows that

$$\mathfrak{g}^{\mathbb{C}} =: \mathfrak{sp}(n, \mathbb{C}) = \left\{ \begin{pmatrix} A & C \\ B & -A^t \end{pmatrix} : A \in \mathfrak{gl}(n, \mathbb{C}),\ B,\ C \in \mathrm{Sym}(n, \mathbb{C}) \right\}.$$

Real forms

Let $\mathfrak{g}$ be a complex Lie algebra. A *real form* of $\mathfrak{g}$ is a (real) Lie subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}^{\mathbb{R}}$ such that $\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}}$. A real form $\mathfrak{g}_0$ of $\mathfrak{g}$ is called *compact* if it is a compact Lie algebra.

3.3.5 Example $\mathfrak{su}(n)$ and $\mathfrak{sl}(n,\mathbb{R})$ are real forms of $\mathfrak{sl}(n,\mathbb{C})$. Only the first one is a compact real form.

3.3.6 Example The complex Lie algebra $\mathfrak{so}(n,\mathbb{C})$ has a compact real form $\mathfrak{so}(n)$ and, in case $n \geq 3$, noncompact real forms

$$\mathfrak{so}(p,q) = \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid A^t I_{p,q} + I_{p,q}A = 0\},$$

where $p+q=n$ with p, $q>0$, and

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

The following theorem is a consequence of Cartan's classification of real simple Lie algebras (1914). Hermann Weyl published an intrinsic proof in 1925-6 based on the detailed structure of semisimple Lie algebras. It allows to prove many results for complex semisimple Lie algebras by reducing them to a similar result for compact semisimple Lie algebras, where we have a bi-invariant measure. This process is dubbed "Weyl's unitary trick". We avoid the more elaborate algebraic machinery by using an argument based on a proof given by R. Richardson (1968), following a suggestion of Cartan himself, but delegate the proof to an appendix.

3.3.7 Theorem *Every complex semisimple Lie algebra admits a compact real form. Any two compact real forms of a complex semisimple Lie algebra are conjugate under an inner automorphism.*

3.4 Problems

1 Consider the adjoint representation ad of a Lie algebra $\mathfrak{g}$ on itself. Show that the invariant subspaces of ad are precisely the ideals of $\mathfrak{g}$. Conclude that ad is completely reducible (resp. irreducible) for a semisimple (resp. simple) Lie algebra.

2 Let $\mathfrak{g}$ be a semisimple Lie algebra. Show that the only ideals in $\mathfrak{g}$ are the sums of some of its simple ideals.

3 Let $\mathfrak{g}$ be a compact simple Lie algebra. Prove that the ad-invariant inner product on $\mathfrak{g}$ is unique, up to a multiplicative constant. (Hint: Problem 14 in Chapter 1).

4 Let G be a Lie group equal to one of $SO(n)$ ($n \geq 3$) or $SU(n)$ ($n \geq 2$), and denote its Lie algebra by $\mathfrak{g}$. Prove that for any $c > 0$

$$\langle X, Y \rangle = -c \operatorname{trace}(XY),$$

where $X, Y \in \mathfrak{g}$, defines a Ad-invariant positive definite inner product on $\mathfrak{g}$. Conclude that the Cartan-Killing form of $\mathfrak{g}$ is of this form for some $c < 0$.

5 Explain why $\mathfrak{sl}(2, \mathbb{R})$ is not a compact Lie algebra.

6 Prove that a complex Lie algebra whose realification is a compact Lie algebra must be Abelian.

7 Prove that a real Lie algebra with a positive-definite Cartan-Killing form must be zero-dimensional.

8 A Lie algebra $\mathfrak{g}$ is called *nilpotent* if the *lower central series* of ideals of $\mathfrak{g}$

$$\mathcal{C}^0\mathfrak{g} \supset \mathcal{C}^1\mathfrak{g} \supset \cdots \supset C^q\mathfrak{g} \supset \cdots,$$

defined by $\mathcal{C}^0\mathfrak{g} = \mathfrak{g}$ and $\mathcal{C}^q\mathfrak{g} = [\mathfrak{g}, \mathcal{C}^{q-1}\mathfrak{g}]$ for $q \geq 1$, terminates at zero, that is, $\mathcal{C}^p\mathfrak{g} = 0$ for some $p \geq 1$.

a. Show that the space of strictly upper triangular matrices in $\mathfrak{gl}(n, \mathbb{R})$ is a nilpotent Lie algebra.

b. Check that if $\mathfrak{g}$ is nilpotent then, for all $X \in \mathfrak{g}$, ad_X is nilpotent as an endomorphism of $\mathfrak{g}$ (that is, $\mathrm{ad}_x^m = 0$ for some $m > 0$); we say that $\mathfrak{g}$ is ad-*nilpotent* (Engel's Theorem is the statement that every ad-nilpotent Lie algebra is nilpotent).

c. Prove that the Cartan-Killing form of a nilpotent Lie algebra is null.

9 Compute the Killing form of $\mathfrak{gl}(n, \mathbb{F})$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ directly from the formula $\mathrm{ad}_X Y = XY - YX$.

10 Obtain the following expressions for the Cartan-Killing forms of the classical complex Lie algebras:

$$\mathfrak{sl}(n, \mathbb{C}) : \beta(X, Y) = 2n \operatorname{trace}(XY);$$
$$\mathfrak{so}(n, \mathbb{C}) : \beta(X, Y) = (n-2) \operatorname{trace}(XY);$$
$$\mathfrak{sp}(n, \mathbb{C}) : \beta(X, Y) = 2(n+1) \operatorname{trace}(XY).$$

11 Let G be a compact connected Lie group of dimension at least 3 with Lie algebra $\mathfrak{g}$, and denote the Cartan-Killing form of $\mathfrak{g}$ by β.

a. Let ω be the left-invariant 3-form on G whose value at the identity is

$$\omega_1(X, Y, Z) = \beta([X, Y], Z)$$

for $X, Y, Z \in \mathfrak{g}$. Prove that ω is skew-symmetric and right-invariant, so it defines a bi-invariant differential form of degree 3 on G.

b. In case $G = S^3$, show that $\frac{1}{16}\omega$ coincides with the volume form, with respect to some orientation. Deduce that the bi-invariant integral on S^3 is given by $\int_{S^3} f(g)\,d\mu(g) = \pm\frac{1}{32\pi^2}\int_{S^3} f\omega$ for $f \in C(S^3)$.

3.A Existence of compact real forms

Proof of Theorem 3.3.7. (Sketch) A complex Lie algebra of dimension n can be thought of $\mathbb{C}^n$ with a skew-symmetric multiplication satisfying the Jacobi identity. In other words, the Lie bracket belongs to the space $V_n = \Lambda^2(\mathbb{C}^{n*}) \otimes_{\mathbb{C}} \mathbb{C}^n$ and its coordinates satisfy quadratic polynomial equations corresponding to the Jacobi equation. Fix a basis $(e_1, \ldots, e_n)$ of $\mathbb{C}^n$. Then

$$\mu(e_i, e_j) = \sum_{k=1}^{n} \mu_{ij}^k e_k$$

for some $\mu_{ij}^k \in \mathbb{C}$. The Jacobi condition is

$$\sum_{m=1}^{n} (\mu_{ij}^m \mu_{mk}^\ell + \mu_{jk}^m \mu_{mi}^\ell + \mu_{ki}^m \mu_{mj}^\ell) = 0$$

for all i, j, k, $\ell = 1, \ldots, n$. Now there is a closed subvariety $\mathcal{L}_n$ of V_n parametrizing all complex n-dimensional Lie algebras.

Consider the natural action of $G := GL(n, \mathbb{C})$ on V_n, namely,

$$g \cdot \mu(x, y) := g(\mu(g^{-1}x, g^{-1}y))$$

for $g \in G$, $\mu \in V_n$, x, $y \in \mathbb{C}^n$. It amounts to "change of basis" in the Lie algebra. The idea of this proof is to find a suitable basis whose real span will be a compact real form. Thus we identify the given complex semisimple Lie algebra with $(\mathbb{C}^n, \mu)$ and need to find $g \in G$ such that $\nu := g \cdot \mu$ has coordinates ν_{ij}^k all real and a negative definite Cartan-Killing form on the real span of $(e_1, \ldots, e_n)$.

Denote the Cartan-Killing form of $\nu \in \mathcal{L}_n$ by B_ν. By semisimplicity, may assume the basis of $\mathbb{C}^n$ has been chosen so that $B_\mu(e_i, e_j) = -\delta_{ij}$ for all i, j. We shall restrict to changes of basis that preserve B_μ. This will ensure that a potential real form is compact. So let

$$H = \{g \in G \mid B_{g\mu} = B_\mu\}$$

and consider the orbit $H(\mu) =: Y$.

CLAIM. Y is a closed subvariety of $\mathcal{L}_n$. In order to prove the claim, let $X := G(\mu)$ and $Z = \{\nu \in \mathcal{L}_n \mid B_\nu = B_\mu\}$. Note that Z is a closed subvariety of $\mathcal{L}_n$. Plainly, $Y = X \cap Z$. For all $\nu \in Z$, $(\mathbb{C}^n, \nu)$ is semisimple and

$$\dim H(\nu) = \dim H - \dim H_\nu = \dim H - n$$

is independent of ν, as $H_\nu = \mathrm{Aut}(\mathbb{C}^n, \nu) = \mathrm{Der}(\mathbb{C}^n, \nu)^0 \cong (\mathbb{C}^n, \nu)$ has dimension n. Now all orbits of H in Z have the same dimension. It is an elementary result of algebraic actions that the lowest dimensional orbit of H in Z is closed in Z. It follows that all H-orbits in Z are closed in Z, and hence in $\mathcal{L}_n$, including Y, as we wished.

Endow $\mathbb{C}^n$ with the Hermitian inner product such that $(e_1, \ldots, e_n)$ is a unitary basis. This specifies a subgroup K of G isomorphic to $U(n)$. Recall that its Lie algebra $\mathfrak{k}$ is a real form of $\mathfrak{g}$.

Consider $\rho_\mu : H \to \mathbb{R}$ given by $\rho_\mu(h) = ||h\mu||^2$. Since $H(\mu)$ is closed in $\mathcal{L}_n$, there exists a point of minimum of ρ_μ, which we may assume to be $1 \in H$. Therefore

$$0 = (d\rho_\mu)_1(A) = 2\Re(A\mu, \mu) \tag{3.A.1}$$

for all $A \in \mathfrak{h}$. Note that

$$\mathfrak{h} \cong \mathfrak{so}(n, \mathbb{C}) = \mathfrak{so}(n) + \sqrt{-1}\mathfrak{so}(n) \subset \mathfrak{u}(n) + \sqrt{-1}\mathfrak{u}(n) = \mathfrak{g},$$

so that $\mathfrak{h}$ is invariant under taking the transpose conjugate matrix. Now we can apply (3.A.1) to $[A, A^*]$ and obtain

$$||A^*\mu||^2 - ||A\mu||^2 = \Re([A, A^*]\mu, \mu) = 0.$$

This equation shows that $A\mu = 0$ if and only if $A^*\mu = 0$, that is, also $\mathfrak{h}_\mu$ is invariant under taking the transpose conjugate matrix. This means

$$\begin{aligned} \mathfrak{h}_\mu &= \mathfrak{h}_\mu \cap \mathfrak{so}(n) + \mathfrak{h}_\mu \cap (\sqrt{-1}\mathfrak{so}(n)) \\ &= \mathfrak{h}_\mu \cap \mathfrak{so}(n) + \sqrt{-1}(\mathfrak{h}_\mu \cap \mathfrak{so}(n)). \end{aligned}$$

We have shown that $\mathfrak{h}_\mu \cap \mathfrak{so}(n)$ is a compact real form of $\mathfrak{h}_\mu$. But $\mathfrak{h}_\mu = \mathrm{Der}(\mathbb{C}^n, \mu)$ is isomorphic to $(\mathbb{C}^n, \mu)$.

It remains to prove the uniqueness, up to conjugacy. We first observe that a real form of $(\mathbb{C}^n, \mu)$ is equivalent to a critical point of ρ_μ. Indeed, we first compute:

$$\mu_{ij}^k = -B_\mu(\mu(e_i, e_j), e_k) = -B_\mu(e_i, \mu(e_j, e_k)) = \mu_{jk}^i$$

and

$$\delta_{ij} = -B_\mu(e_i, e_j) = -\sum_{k\ell} \mu_{jk}^\ell \mu_{i\ell}^k = -\sum_{k\ell} \mu_{i\ell}^k \mu_{\ell j}^k = \sum_{k\ell} \mu_{i\ell}^k \mu_{j\ell}^k.$$

In particular, $\sum_{k\ell}(\mu_{k\ell}^i)^2 = \sum_{k\ell}(\mu_{i\ell}^k)^2 = 1$ for all i, and therefore

$$\sum_{ijk} (\nu_{ij}^k)^2 = n \text{ for all } \nu \in H(\mu).$$

Hence

$$||\nu||^2 = \sum_{ijk} |\nu_{ij}^k|^2 \geq \sum_{ijk} \Re(\nu_{ij}^k)^2 = n,$$

and equality holds if and only if all ν_{ij}^k are real.

Now suppose $(e_1', \ldots, e_n')$ is another basis of $\mathbb{C}^n$ whose real span yields a compact real form of μ. We may choose this basis so that $B_\mu(e_i', e_j') = -\delta_{ij}$ for al i, j. Let $h \in G$ be such that $he_i = e_i'$ for all i. Then $\nu := h^{-1} \cdot \mu$ has real coefficients in the basis $(e_1, \ldots, e_k)$ and $B_{h^{-1}\mu}(e_i, e_j) = -1$ for all i, j. It follows that $h \in H$. We have shown that a real form of $(\mathbb{C}^n, \mu)$ indeed corresponds to a point of minimum of ρ_μ on H.

Next, suppose 1 and h^{-1} are two points of minimum ρ_μ in $H = SO(n, \mathbb{C})$. Write $h^{-1} = k \exp A$ for $k \in SO(n)$ and $A \in \sqrt{-1}\mathfrak{so}(n)$.[4] Put

$$f(t) = ||h^{-1} \cdot \mu||^2 = ||\exp(tA) \cdot \mu||^2 = \sum_i e^{2c_i t} ||\mu_i||^2,$$

where μ_i are the eigenvectors of A and c_i the corresponding eigenvalues. Then f is a strictly convex function unless, for each i, we have $\mu_i = 0$ or $c_i = 0$, that is, unless $h^{-1}\mu = k\mu$. Since $f(0)$ and $f(1)$ are points of minimum, we must have $h^{-1}\mu = k\mu \in (K \cap H)(\mu)$, where $K \cap H = SO(n)$. We have shown that the set of minima is the $K \cap H$-orbit of μ.

Finally, $hk\mu = \mu$, so $hk \in H_\mu = \mathrm{Aut}(\mu)$; moreover, $k \cdot \mathrm{span}_{\mathbb{R}}(e_1, \ldots, e_n) = \mathrm{span}_{\mathbb{R}}(e_1, \ldots, e_n)$, so

$$hk \cdot \mathrm{span}_{\mathbb{R}}(e_1, \ldots, e_n) = h \cdot \mathrm{span}_{\mathbb{R}}(e_1, \ldots, e_n) = \mathrm{span}_{\mathbb{R}}(e_1', \ldots, e_n'),$$

as wished. □

[4]Need to explain this Cartan decomposition.

Root theory

One says an endomorphism A of a real or complex vector space V is *semisimple* if every invariant subspace admits an invariant complement. It is easy to see that, over $\mathbb{C}$, this is equivalent to A being diagonalizable. In this case, we can write $A = A_1 \oplus \cdots \oplus A_r$ and $V = V_1 \oplus \cdots \oplus V_r$, where A_i is a scalar operator on V_i for each i.

In our case, for a given complex semisimple Lie algebra $\mathfrak{g}$ (say the complexification of a compact Lie algebra $\mathfrak{u}$), we want to understand its fine structure, namely, describe its multiplication table in conceptual terms. It is very natural to look at $\mathrm{ad}[\mathfrak{g}]$, the algebra of endomorphisms of $\mathfrak{g}$ generated by ad_X, for all $X \in \mathfrak{g}$. We first observe that $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is semisimple (not for all $X \in \mathfrak{g}$, but) for all $X \in \mathfrak{u}$, hence diagonalizable. This is already very good, but to be really useful we need a notion of simultaneous diagonalization. A commuting family of semisimple endomorphisms can be diagonalized in the same basis. Since $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$ for all $X, Y \in \mathfrak{g}$, we need to consider operators $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ where X belongs to an Abelian subalgebra $\mathfrak{h}$. Of course, the bigger the $\mathfrak{h}$, the better. This brings us to the notion of a Cartan subalgebra (CSA) of a semisimple Lie algebra.

The counterpart of a CSA on the group level is the notion of a maximal torus. If U is a compact Lie group, there is an inner product on its Lie algebra $\mathfrak{u}$ with respect to which the transformations $\mathrm{Ad}_u : \mathfrak{u} \to \mathfrak{u}$ for $u \in U$ are all orthogonal and hence diagonalizable over $\mathbb{C}$. In this chapter, we introduce maximal tori and CSA, but we will focus on the adjoint representation of a CSA. In later chapters, when we talk about the Weyl formulae and the Peter-Weyl theorem, we will come back to maximal tori.

4.1 Maximal tori

Let U be a compact connected Lie group, let $\mathfrak{u}$ denote its Lie algebra and assume U is not Abelian. We claim U contains proper subgroups isomorphic to a torus $T^n = S^1 \times \cdots \times S^1$, with $n \geq 1$ factors. In fact, for any nonzero $X \in \mathfrak{u}$, the closure of the image of the one-parameter subgroup defined by X,

$$\{\exp tX \mid t \in \mathbb{R}\},$$

is a compact connected Abelian subgroup, hence isomorphic to a torus. A *maximal torus* of G is a torus subgroup which is not properly contained in a bigger torus. By dimensional reasons, maximal tori exist.

4.1.1 Examples

$$\left\{ \begin{pmatrix} e^{it_1} & & \\ & \ddots & \\ & & e^{it^n} \end{pmatrix} \Bigg| \, t_1, \ldots, t_n \in \mathbb{R} \right\}$$

is a torus T in $U(n)$. This is a maximal torus of $U(n)$ because an element $g \in U(n)$ that commutes with all elements of T must lie in T. Indeed g commutes with a diagonal matrix with all entries distinct, which implies that g preserves its eigenspaces in $\mathbb{C}^n$. Hence g is diagonal.

Similarly,

$$\left\{ \begin{pmatrix} \cos t_1 & -\sin t_1 & & & \\ \sin t_1 & \cos t_1 & & & \\ & & \ddots & & \\ & & & \cos t_n & -\sin t_n \\ & & & \sin t_n & \cos t_n \end{pmatrix} : \, t_1, \ldots, t_n \in \mathbb{R} \right\}$$

is a torus in $SO(2n)$,

$$\left\{ \begin{pmatrix} \cos t_1 & -\sin t_1 & & & & \\ \sin t_1 & \cos t_1 & & & & \\ & & \ddots & & & \\ & & & \cos t_n & -\sin t_n & \\ & & & \sin t_n & \cos t_n & \\ & & & & & 1 \end{pmatrix} : \, t_1, \ldots, t_n \in \mathbb{R} \right\}$$

is a torus in $SO(2n+1)$, and one checks these are also maximal tori.

4.1.2 Lemma *Let $T^n = \mathbb{R}^n/\mathbb{Z}^n$ be an n-torus. Then T^n is* monogenic, *that is, there is $g \in T^n$ such that the cyclic group $\langle g \rangle$ generated by g is dense in T^n.*

Proof. Fix a countable basis $\{U_i\}_{i \in \mathbb{N}}$ of open sets of T^n. Given any cube C_0 in T^n (i.e. projection of a product of closed intervals in $\mathbb{R}^n$), we shall construct a descending chain of cubes $C_0 \supset C_1 \supset C_2 \supset \cdots$ whose intersection contains a generator g as desired.

We proceed by induction. Suppose we have already defined $C_0 \supset C_1 \supset \cdots \supset C_{i-1}$ and C_{i-1} has side ϵ. Take an integer $N_i > 1/\epsilon$. Then $C_{i-1}^{N_i} = T^n$.

By continuity, we can find a cube C_i contained in C_{i-1} such that $C_i^{N_i} \subset U_i$. Let $g \in \cap_{i=0}^{\infty} C_i$. Then $g^{N_i} \in U_i$ for all i, so $\langle g \rangle$ is dense in T^n. □

Let T be a maximal torus of U, and denote its Lie algebra by $\mathfrak{t}$. Then $\mathfrak{t}$ is an Abelian Lie subalgebra of $\mathfrak{u}$. Indeed it is a *maximal Abelian subalgebra*, for if $\mathfrak{s}$ is an Abelian subalgebra containing $\mathfrak{t}$, then the associated connected Lie subgroup S of U is Abelian, so its closure $\bar{S}$ is a torus containing T. By maximality of T, $\bar{S} = T$ and thus $\mathfrak{s} \subset \mathfrak{t}$.

For later use, we introduce the following terminology. Let $\mathfrak{g}$ be a Lie algebra and let Σ be a subset of $\mathfrak{g}$. The *centralizer* of Σ in $\mathfrak{g}$ is

$$Z_{\mathfrak{g}}(\Sigma) = \{X \in \mathfrak{g} \ : \ [X, Y] = 0 \text{ for all } Y \in \Sigma\}.$$

The *normalizer* of Σ in $\mathfrak{g}$ is

$$N_{\mathfrak{g}}(\Sigma) = \{X \in \mathfrak{g} \ : \ [X, Y] \in \Sigma \text{ for all } Y \in \Sigma\}.$$

Note that $Z_{\mathfrak{g}}(\Sigma)$ and $N_{\mathfrak{g}}(\Sigma)$ are subalgebras of $\mathfrak{g}$, by Jacobi.

Similarly, if G is a Lie group with Lie algebra $\mathfrak{g}$, we can define the *centralizer* and the *normalizer* of Σ in G respectively as

$$Z_G(\Sigma) = \{g \in G \mid \mathrm{Ad}_g Y = Y \text{ for all } Y \in \Sigma\}$$

and

$$N_G(\Sigma) = \{g \in G \mid \mathrm{Ad}_g Y \in \Sigma \text{ for all } Y \in \Sigma\}.$$

These are subgroups of G (closed, if Σ is a closed subset of $\mathfrak{g}$).

4.1.3 Theorem (É. Cartan's Maximal Torus Theorem) *Any two maximal tori in a compact connected Lie group U are conjugate under an inner automorphism. In particular, given a maximal torus T of U, every element of U is conjugate under an inner automorphism to an element in T; equivalently, T intersects every conjugacy class of U.*

Proof. Let T and T' be maximal tori in U. It suffices to show that their Lie algebras $\mathfrak{t}$ and $\mathfrak{t}'$ are conjugate under the adjoint representation. Choose $H \in \mathfrak{t}$ that generates a dense one-parameter subgroup of T. Then $\mathfrak{t}$ is the centralizer of H in $\mathfrak{u}$.

Let $H' \in \mathfrak{u}$ be arbitrary and use the compactness of U to choose a critical point $u_0 \in U$ of the smooth function $f(u) = \langle H, \mathrm{Ad}_u H' \rangle$, where $\langle \cdot, \cdot \rangle$ denotes an ad invariant positive definite bilinear form on $\mathfrak{u}$. Then, for $X \in \mathfrak{u}$,

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \langle H, \mathrm{Ad}_{(\exp tX)u_0} H' \rangle \\ &= \langle H, [X, \mathrm{Ad}_{u_0} H'] \rangle \\ &= \langle X, [\mathrm{Ad}_{u_0} H', H] \rangle. \end{aligned}$$

Since X is arbitrary and $\langle\cdot,\cdot\rangle$ is non-degenerate, $[\mathrm{Ad}_{k_0}H', H] = 0$. Now $\mathrm{Ad}_{u_0}H' \in \mathfrak{t}$, and this proves the last assertion in the statement of the Theorem.

In particular, we apply the above reasoning to the case H' generates a dense one-parameter subgroup of $\mathfrak{t}'$. Then $\mathrm{Ad}_{u_0}H' \in \mathfrak{t}$ implies that $\mathfrak{t} \subset Z_{\mathfrak{u}}(\mathrm{Ad}_{u_0}H') = \mathrm{Ad}_{u_0}Z_{\mathfrak{u}}(H') = \mathrm{Ad}_{u_0}\mathfrak{t}'$. Now $\dim\mathfrak{t} \leq \dim\mathfrak{t}'$ and, by symmetry, we obtain equality. Hence $\mathfrak{t} = \mathrm{Ad}_{u_0}\mathfrak{t}'$. □

We define the *rank* of a compact Lie group to be the dimension of a maximal torus. Since any two maximal tori are conjugate, they all have the same dimension.

4.1.4 Remark It follows from Theorem 4.1.3 that the exponential map of a compact connected Lie group G is surjective. Indeed, it is explicitly surjective in the case of a torus; any element of G sits inside a maximal torus T; since $\exp^T$ is the restriction of $\exp^G$, we are done (compare Remark 3.2.14).

4.2 Cartan subalgebras

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A *Cartan subalgebra* (CSA) of $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{h}$ such that:

(a) $\mathfrak{h}$ is a maximal Abelian subalgebra of $\mathfrak{g}$;

(b) ad_H is a semisimple endomorphism of $\mathfrak{g}$.

4.2.1 Proposition *Every complex semisimple Lie algebra $\mathfrak{g}$ contains a CSA. Further, two CSA's of $\mathfrak{g}$ are conjugate under an inner automorphism.*

Proof. Consider the case in which $\mathfrak{g}$ is the complexification of a compact Lie algebra $\mathfrak{u}$; by Theorem 3.3.7, this is the general case. We prove the existence of CSA and relegate the proof of uniqueness to the appendix. Any nonzero vector of $\mathfrak{u}$ spans over $\mathbb{R}$ a one-dimensional Abelian subalgebra of $\mathfrak{u}$. Now a maximal Abelian subalgebra $\mathfrak{t}$ of $\mathfrak{u}$ exists by dimensional reasons. We will show that $\mathfrak{h} := \mathfrak{t}^{\mathbb{C}}$ is a CSA of $\mathfrak{g}$.

It is clear that $\mathfrak{h}$ is Abelian. Suppose $Z \in \mathfrak{g}$ centralizes $\mathfrak{h}$. Write $Z = X + iY$ where $X, Y \in \mathfrak{u}$. Then X and Y centralize $\mathfrak{t}$, which implies $X, Y \in \mathfrak{t}$ by maximality of $\mathfrak{t}$, and therefore $Z \in \mathfrak{h}$. This proves that $\mathfrak{h}$ is maximal Abelian.

Since $\mathfrak{u}$ is a compact Lie algebra, there is an ad-invariant inner product on $\mathfrak{u}$ (Theorem 3.2.10). Therefore $\mathrm{ad}_X : \mathfrak{u} \to \mathfrak{u}$ is diagonalizable over $\mathbb{C}$ for all $X \in \mathfrak{u}$. It follows that $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is semisimple for all $X \in \mathfrak{u}$. Now let $Z \in \mathfrak{h}$ and write $Z = X + iY$ for $X, Y \in \mathfrak{t}$. Since $\mathfrak{t}$ is Abelian, ad_X and $\mathrm{ad}_{iY} = i\mathrm{ad}_Y$ are commuting semisimple endomorphisms of $\mathfrak{g}$. Hence also $\mathrm{ad}_Z : \mathfrak{g} \to \mathfrak{g}$ is semisimple. This completes the proof that $\mathfrak{h}$ is a CSA of $\mathfrak{g}$. □

We define the *rank* of a complex semisimple Lie algebra to be the dimension of a CSA. Since any two CSA are conjugate, they all have the same dimension.

4.3 Case study: representations of $SU(2)$

In section 1.5, we classified all irreducible unitary representations of $SU(2)$. Now we take a closer look at their structure with a view toward generalization.

Recall that $SU(2)$ has, up to equivalence, exactly one irreducible representation φ_k on in dimension $k+1$ for each non-negative integer k. Here the representation space V_k is the space of homogeneous complex polynomials of degree k in two variables z_1, z_2. A basis is thus given by

$$z_1^k,\ z_1^{k-1}z_2, \ldots,\ z_2^k. \tag{4.3.1}$$

If we view $SU(2)$ as the unit quaternions S^3 and consider the circle $S^1 = \{e^{it} \mid t \in \mathbb{R}\}$, then (4.3.1) is a basis of eigenvectors of $\varphi_k(e^{it})$, for

$$\varphi_k(e^{it}) z_1^{k-j} z_2^j = e^{(2j-k)it} z_1^{k-j} z_2^j.$$

In terms of $SU(2)$, we have

$$\varphi_k \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} = \begin{pmatrix} e^{-kit} & & & \\ & e^{-(k-2)it} & & \\ & & \ddots & \\ & & & e^{kit} \end{pmatrix}.$$

If we linearize the representation, we have a better chance of understading how it acts on V_k. Consider the induced representation $d\varphi_k : \mathfrak{su}(2) \to \mathfrak{su}(k+1)$. Now

$$iH = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

is a basis of $\mathfrak{su}(2)$ and we clearly have

$$d\varphi_k(iH) = \begin{pmatrix} -ki & & & \\ & -(k-2)i & & \\ & & \ddots & \\ & & & ki \end{pmatrix}.$$

We also compute, for $v_j = z_1^{k-j} z_2^j$, that

$$\begin{aligned} d\varphi_k(X)v_j &= \frac{d}{dt}\Big|_{t=0} \varphi_k \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} v_j \\ &= \frac{d}{dt}\Big|_{t=0} (\cos t z_1 - \sin t z_2)^{k-j} (\sin t z_1 + \cos t z_2)^j \\ &= -(k-j) v_{j+1} + j v_{j-1} \end{aligned}$$

and

$$\begin{aligned} d\varphi_k(Y)v_j &= \frac{d}{dt}\Big|_{t=0}\varphi_k\begin{pmatrix}\cos t & i\sin t\\ i\sin t & \cos t\end{pmatrix}v_j \\ &= \frac{d}{dt}\Big|_{t=0}(\cos tz_1 - i\sin tz_2)^{k-j}(-i\sin tz_1 + \cos tz_2)^j \\ &= -i(k-j)v_{j+1} - ijv_{j-1}. \end{aligned}$$

With respect to the dual basis $\{v_j^*\}_{j=0}^k$, the dual representation has complex conjugate matrix. Thefore

$$\begin{aligned} d\varphi_{-k}(iH)v_j^* &= i(k-2j)v_j^*, \\ d\varphi_{-k}(X)v_j^* &= -(k-j)v_{j+1}^* + jv_{j-1}^*, \\ d\varphi_{-k}(Y)v_j^* &= i(k-j)v_{j+1}^* + ijv_{j-1}^*. \end{aligned}$$

Further, if we complexify everything, the structure of the representation reveals itself transparent. Consider $\pi_k := (d\varphi_{-k})^c : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{sl}(k+1,\mathbb{C})$. Note that $\mathfrak{sl}(2,\mathbb{C})$ admits a natural basis

(4.3.2) $$H = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad E = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad F = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},$$

with brackets

$$[E,F] = H, \quad [H,E] = 2E, \quad [H,F] = -2F.$$

Since $E = \frac{1}{2}(X - iY)$ and $F = -\frac{1}{2}(X + iY)$, we easily see that

$$\pi_k(E)v_j^* = jv_{j-1}^* \quad \text{and} \quad \pi_k(F)v_j^* = (k-j)v_{j+1}^*.$$

Finally, in the renormalized basis $\{v_j'\}$ with $v_j' = \alpha_j v_j^*$, where $\alpha_{j+1} = (k-j)\alpha_j$ and $\alpha_0 = 1$, we finally deduce that

$$\pi_k(H) \text{ has matrix } \begin{pmatrix} k & & & & \\ & k-2 & & & \\ & & \ddots & & \\ & & & -(k-2) & \\ & & & & -k \end{pmatrix};$$

$$\pi_k(E) \text{ has matrix } \begin{pmatrix} 0 & k & & & & \\ & 0 & 2(k-1) & & & \\ & & 0 & 3(k-2) & & \\ & & & & \ddots & \\ & & & & & k \\ & & & & & 0 \end{pmatrix};$$

$$\pi_k(F) \text{ has matrix } \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & 1 & 0 \end{pmatrix}.$$

In this form, the behavior of π_k is easy to describe. There is an basis consisting of eigenvectors of $\pi_k(H)$, ordered by their eigenvalues k, $k-2, \ldots, -k$, such that $\pi_k(E)$ maps a $(k-2j)$-eigenvector to a $(k-2j)+2$-eigenvector, whereas $\pi_k(F)$ maps a $(k-2j)$-eigenvector to a $(k-2j)-2$-eigenvector.

Note that $\{\pi_k\}_{k=0}^{\infty}$ is a complete set of representatives of equivalence classes of irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$. We shall use this description of π_k to analyse the structure of arbitrary complex semisimple Lie algebras in the next section.

4.4 Root space decomposition

Henceforth we fix a complex semisimple Lie algebra $\mathfrak{g}$ and a CSA $\mathfrak{h}$. We shall analyse the structure in light of the spectral decomposition of the adjoint representation of $\mathfrak{g}$ restricted to $\mathfrak{h}$.

Since $[\mathrm{ad}_H, \mathrm{ad}'_H] = \mathrm{ad}_{[H,H']}$,

$$\{\mathrm{ad}_H \mid H \in \mathfrak{h}\}$$

is a commuting family of semisimple endomorphisms of $\mathfrak{g}$, so we can find a common eigenspace decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \sum_\alpha \mathfrak{g}_\alpha. \tag{4.4.1}$$

Here

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{ad}_H X = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$$

is an eigenspace and the eigenvectors $\alpha(H)$ depend linearly on $H \in \mathfrak{h}$, that is, they are linear functionals $\alpha : \mathfrak{h} \to \mathbb{C}$. Note that $\mathfrak{h} \subset \mathfrak{g}_0$, since $\mathfrak{h}$ is Abelian, and indeed the equality holds, since $\mathfrak{h}$ is maximal Abelian:

$$\boxed{\mathfrak{g}_0 = \mathfrak{h}} \tag{4.4.2}$$

By finite-dimensionality, the sum in (4.4.1) is finite in the sense that $\mathfrak{g}_\alpha = 0$ but for finitely many $\alpha \in \mathfrak{h}^*$, and we put

$$\Delta := \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0 \text{ and } \alpha \neq 0\}.$$

The elements of $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ are called the *roots* of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and Δ is called the *root system*. For a root $\alpha \in \Delta$, $\mathfrak{g}_\alpha$ is the associated *root space*, and its nonzero elements are the associated *root vectors*. The following is called the *root space decomposition* of $\mathfrak{g}$ with respect to $\mathfrak{h}$:

(4.4.3) $$\boxed{\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha}$$

4.4.4 Lemma (Fundamental calculation) *It holds*

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$$

for all $\alpha, \beta \in \mathfrak{h}^*$.

Proof. Let $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$ and $H \in \mathfrak{h}$. Then $[H, [X, Y]] = [[H, X], Y] + [X, [H, Y]] = [\alpha(H)X, Y] + [X, \beta(H)Y] = (\alpha(H) + \beta(H))[X, Y]$. ☐

It follows from Lemma 4.4.4 that for $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$ with $\alpha + \beta \neq 0$,

$$\mathrm{ad}_X \mathrm{ad}_Y : \mathfrak{g}_\gamma \to \mathfrak{g}_{\gamma+(\alpha+\beta)}$$

has null trace. We deduce:

(4.4.5) $$\boxed{B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \text{ if } \alpha, \beta \in \Delta, \alpha + \beta \neq 0.}$$

In particular, owing to (4.4.2),

(4.4.6) $$\boxed{B(\mathfrak{h}, \mathfrak{g}_\alpha) = 0 \text{ for all } \alpha \in \Delta.}$$

Since B is nondegenerate:

(4.4.7) $$\boxed{B : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C} \text{ is nondegenerate.}}$$

and

(4.4.8) $$\boxed{B : \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C} \text{ is nondegenerate for all } \alpha \in \Delta.}$$

In particular,

(4.4.9) $$\boxed{\alpha \in \Delta \text{ if and only if } -\alpha \in \Delta.}$$

We also have

(4.4.10) $$\boxed{\mathfrak{h}^* = \sum_{\alpha \in \Delta} \mathbb{C}\alpha.}$$

Indeed $\cap_{\alpha \in \Delta} \ker \alpha$ equals the center of $\mathfrak{g}$, which is zero.

Next, for each $\alpha \in \Delta$, use (4.4.7) to define $H_\alpha \in \mathfrak{h}$ satisfying $B(H_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{h}$. We also put $B(\alpha, \beta) := B(H_\alpha, H_\beta)$ for all $\alpha, \beta \in \Delta$. Now a simple calculation, for $H \in \mathfrak{h}$, $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$,

$$\begin{aligned} B(H, [X, Y]) &= B([H, X], Y) = \alpha(H)B(X, Y) = B(H, H_\alpha)B(X, Y) \\ &= B(H, B(X, Y)H_\alpha) \end{aligned}$$

yields

(4.4.11) $\boxed{[X, Y] = B(X, Y)H_\alpha \text{ for all } X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_{-\alpha} \text{ and } \alpha \in \Delta.}$

In particular, due to (4.4.8),

(4.4.12) $\boxed{[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}H_\alpha.}$

We may suppose that $\mathfrak{g}$ admits a compact real form $\mathfrak{u}$ and $\mathfrak{h}$ is the complexification of a Lie algebra $\mathfrak{t}$ of $\mathfrak{u}$ (either by assumption, or by Theorem 3.3.7). Since $\mathrm{ad}_H : \mathfrak{u} \to \mathfrak{u}$ is skew-symmetric with respect to the real inner product $-B : \mathfrak{u} \times \mathfrak{u} \to \mathbb{R}$ for all $H \in \mathfrak{u}$, the root α takes imaginary values on $\mathfrak{t}$. Since B is negative definite on $\mathfrak{t}$ and $B(H_\alpha, H) = \alpha(H) \in i\mathbb{R}$ for all $H \in \mathfrak{t}$, we deduce that $iH_\alpha \in \mathfrak{t}$. Set $\mathfrak{h}_\mathbb{R} := \sum_{\alpha \in \Delta} \mathbb{R}H_\alpha$. Owing to (4.4.10),

(4.4.13) $\boxed{\mathfrak{h}_\mathbb{R} = i\mathfrak{t}.}$

Moreover,

(4.4.14) $\boxed{\text{The roots take real values on } \mathfrak{h}_\mathbb{R} \text{ and } B : \mathfrak{h}_\mathbb{R} \times \mathfrak{h}_\mathbb{R} \to \mathbb{R} \text{ is a real inner product.}}$

Denote the restriction of B to $\mathfrak{h}_\mathbb{R}$ or $\mathfrak{h}_\mathbb{R}^*$ by $\langle , \rangle$ and write $\langle \alpha, \alpha \rangle = ||\alpha||^2$ for $\alpha \in \Delta$; also, put $\bar{H}_\alpha = \frac{2}{||\alpha||^2} H_\alpha$ ($\bar{H}_\alpha$ is sometimes called the *coroot* associated to α) and use (4.4.8) to find $E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha \in \mathfrak{g}_{-\alpha}$ ($\alpha \in \Delta$) such that $B(E_\alpha, F_\alpha) = \frac{2}{||\alpha^2||}$. Now, owing to (4.4.11),

$$[E_\alpha, F_\alpha] = \bar{H}_\alpha, \; [\bar{H}_\alpha, E_\alpha] = 2E_\alpha, \; [\bar{H}_\alpha, F_\alpha] = -2F_\alpha,$$

that is,

(4.4.15) $\boxed{\mathfrak{g}[\alpha] := \mathbb{C}\bar{H}_\alpha + \mathbb{C}E_\alpha + \mathbb{C}F_\alpha \text{ is a subalgebra isomorphic to } \mathfrak{sl}(2, \mathbb{C}).}$

Also, $i\bar{H}_\alpha$, $E_\alpha - F_\alpha$, $i(E_\alpha + F_\alpha)$ span over $\mathbb{R}$ a subalgebra isomorphic to $\mathfrak{su}(2)$.

4.4.16 Remark With the notation above,

$$\mathfrak{u} := \mathfrak{t} + \sum_{\alpha\in\Delta^+} \mathbb{R}(E_\alpha - F_\alpha) + \mathbb{R}i(E_\alpha + F_\alpha)$$

is a compact real form of $\mathfrak{g}$. Indeed, if

$$X = iH + \sum_{\alpha\in\Delta^+} a_\alpha(E_\alpha - F_\alpha) + ib_\alpha(E_\alpha + F_\alpha)$$

with $H \in \mathfrak{h}_\mathbb{R}$ and a_α, $b_\alpha \in \mathbb{R}$, then the Cartan-Killing form β of $\mathfrak{g}$ has

$$\beta(X,X) = -\beta(H,H) - \frac{4}{||\alpha||^2} \sum_{\alpha\in\Delta^+} a_\alpha^2 + b_\alpha^2 < 0.$$

In the next section, we shall use our knowledge about representations of $\mathfrak{g}[\alpha]$ to investigate the structure of root systems.

4.5 Root systems

Throughout this section, we fix a complex semisimple Lie algebra and a CSA $\mathfrak{h}$, with associated root system Δ.

Fix $\alpha \in \Delta$ and put $V := \mathfrak{h} + \sum_{c\in\mathbb{C}\setminus\{0\}} \mathfrak{g}_{c\alpha}$. In view of the Fundamental Calculation (4.4.4), the adjoint action of $\mathfrak{g}$ restricted to $\mathfrak{g}[\alpha] \cong \mathfrak{sl}(2,\mathbb{C})$ defines a representation ρ on V. Recall that the set of eigenvalues of $\rho(\bar{H}_\alpha)$, as a multiset, is a union of sets each of which has the form

(4.5.1) $$\{-m, -m-2, \ldots, m\}$$

for some nonnegative integer m.

If $0 \neq X \in \mathfrak{g}_{c\alpha}$, then $\rho(\bar{H}_\alpha)X = [\bar{H}_\alpha, X] = c\alpha(\bar{H}_\alpha)X = 2cX$, implying that $2c \in \mathbb{Z}$.

We also know some components of V. In fact, ρ is trivial on $\ker\alpha \subset \mathfrak{h}$ (so $\rho(\bar{H}_\alpha)$ has eigenvalues 0 there) and ρ is irreducible on $\mathfrak{g}[\alpha] \subset \mathbb{C}\bar{H}_\alpha + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$ (so $\rho(\bar{H}_\alpha)$ has eigenvalues 0, 2, -2 there). In particular, $W := \ker\alpha + \mathfrak{g}[\alpha]$ contains the zero eigenspace of $\rho(\bar{H}_\alpha)$ (that is, $\mathfrak{h}$). We claim that W equals V. Indeed, if this is not the case, take an irreducible component U in an invariant complement of W in V. The subrepresentation U contains $\mathfrak{g}_{c\alpha}$ for some c, where the eigenvalue of $\rho(\bar{H}_\alpha)$ is the nonzero integer $n = 2c$; in case n is even, U contains also a zero eigenvector, by the form (4.5.1), which is forbidden since $W \supset \mathfrak{h}$; in case n is odd, U contains an eigenvector with eigenvalue 1, by the form (4.5.1), and this implies that $\beta := \frac{1}{2}\alpha \in \Delta$; we now replace α by β in the definition of V and deduce that $2\beta = \alpha \in \Delta$, which says that $\rho(\bar{H}_\beta)$ has an eigenvalue 4, a contradiction to the above.

Now

(4.5.2) $$\boxed{\dim \mathfrak{g}_\alpha = 1.}$$

and

(4.5.3) $\boxed{\text{The only multiples of } \alpha \in \Delta \text{ which are roots are } \pm\alpha.}$

Next, let α, $\beta \in \Delta$, $\beta \neq \pm\alpha$, and consider $\rho = \mathrm{ad}|_{\mathfrak{g}[\alpha]}$ acting on $V = \sum_{k\in\mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$. For $X \in \mathfrak{g}_{\beta+k\alpha}$, we have that

$$\begin{aligned}\rho(\bar{H}_\alpha)X &= [\bar{H}_\alpha, X] = (\beta+k\alpha)(\bar{H}_\alpha)X \\ &= \left(2\frac{\langle\beta,\alpha\rangle}{||\alpha||^2} + 2k\right)X.\end{aligned}$$

Since $\dim \mathfrak{g}_{\beta+k\alpha} = 1$ for $\beta + k\alpha \in \Delta$, we see that V is irreducible and the sequence of eigenvalues of $\rho(\bar{H}_\alpha)$ has the form $2\frac{\langle\beta,\alpha\rangle}{||\alpha||^2} + 2k$, where $-p \leq k \leq q$, for integers p, $q \geq 0$. Owing to (4.5.1), this set is symmetric with respect to 0. In particular, $-\left(2\frac{\langle\beta,\alpha\rangle}{||\alpha||^2} - 2p\right) = 2\frac{\langle\beta,\alpha\rangle}{||\alpha||^2} + 2q$. It follows that

(4.5.4) $\boxed{\begin{array}{l}\text{For } \alpha,\ \beta \in \Delta,\ \beta \neq \pm\alpha, \text{ there exist integers } p,\ q \geq 0 \text{ such that} \\ 2\frac{\langle\beta,\alpha\rangle}{||\alpha||^2} = p - q \text{ where } \beta + k\alpha \in \Delta \text{ if and only if } -p \leq k \leq q.\end{array}}$

$\{\beta + k\alpha \in \Delta \mid -p \leq k \leq q\}$ is called the *α-string of roots through β*.

Further,

(4.5.5) $\boxed{[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta} \text{ if } \alpha,\ \beta,\ \alpha+\beta \in \Delta.}$

In fact $\alpha + \beta \in \Delta$ implies $q \geq 1$. If it were $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$, then $\sum_{-p\leq k\leq 0} \mathfrak{g}_{\beta+k\alpha}$ would be a proper invariant subspace of $V = \sum_{-p\leq k\leq q} \mathfrak{g}_{\beta+k\alpha}$, but V $\mathrm{ad}|_{\mathfrak{g}[\alpha]}$-irreducible in view of the above discussion.

We further refine the information about the length of strings of roots.

(4.5.6) $\boxed{\text{If } \alpha,\ \beta \in \Delta,\ \beta \neq \pm\alpha, \text{ then } 2\frac{\langle\beta,\alpha\rangle}{||\alpha||^2} \in \{0, \pm1, \pm2, \pm3\}.}$

Indeed we already know $2\frac{\langle\beta,\alpha\rangle}{||\alpha||^2}$ is an integer; also α, β are linearly independent by (4.5.3), so the Cauchy-Schwarz inequality says that

$$0 \leq \left|2\frac{\langle\beta,\alpha\rangle}{||\alpha||^2}\right| \cdot \left|2\frac{\langle\alpha,\beta\rangle}{||\beta||^2}\right| = 4\frac{\langle\beta,\alpha\rangle^2}{||\alpha||^2||\beta||^2} < 4.$$

Further:

(4.5.7) $\boxed{\text{If } \alpha,\ \beta \in \Delta,\ \beta \neq \pm\alpha, \text{ then the } \alpha\text{-string through } \beta \text{ has length} \leq 4.}$

In fact, that string coincides with the α-string through $\gamma = \beta - p\alpha$, that is, $\{\gamma + k\alpha \mid 0 \leq k \leq p+q\}$, and due to (4.5.4) and (4.5.6),

$$-2\frac{\langle\gamma,\alpha\rangle}{||\alpha||^2} = p + q \leq 3.$$

Weyl group

Consider next $\mathfrak{h}_{\mathbb{R}}^* = \sum_{\alpha\in\Delta} \mathbb{R}\alpha$ (resp. $\mathfrak{h}_{\mathbb{R}}$) equipped with the inner product $\langle , \rangle$ induced from the Cartan-Killing form. The *reflection* in the hyperplane $\alpha^\perp$ (resp. $\ker\alpha$) is

$$(4.5.8) \qquad s_\alpha(\lambda) := \lambda - 2\frac{\langle\lambda,\alpha\rangle}{||\alpha||^2}\alpha \qquad \big(\text{resp. } s_\alpha(H_\lambda) := H_\lambda - 2\frac{\langle\lambda,\alpha\rangle}{||\alpha||^2}H_\alpha\big).$$

The *Weyl group* W of $\mathfrak{g}$ relative to $\mathfrak{h}$ is the group generated by the orthogonal transformations $\{s_\alpha \mid \alpha \in \Delta\}$.

(4.5.9) W permutes the roots; in particular, W is finite.

In fact, using (4.5.4), for each $\beta \in \Delta$ we have $s_\alpha(\beta) = \beta + (q-p)\alpha \in \Delta$, since $-p \le q - p \le q$.

For each $\alpha \in \Delta$, $\ker\alpha \subset \mathfrak{h}_{\mathbb{R}}$ is called a *singular hyperplane*; $\cup_{\alpha\in\Delta}\ker\alpha$ is called the set of *singular elements* of $\mathfrak{h}_{\mathbb{R}}$, and its complement $\mathfrak{h}'_{\mathbb{R}}$ in $\mathfrak{h}_{\mathbb{R}}$ is called the set of *regular elements* of $\mathfrak{h}_{\mathbb{R}}$. The connected components of of $\mathfrak{h}'_{\mathbb{R}}$ are called *Weyl chambers*.

(4.5.10) W acts simply transitively on the set of Weyl chambers.

Indeed it follows from (4.5.9) that W permutes the chambers. Given two chambers $\mathcal{C}$ and $\mathcal{C}'$, we choose $H \in \mathcal{C}$, $H' \in \mathcal{C}'$ and join them by an arc in $\mathfrak{h}_{\mathbb{R}}$ that at each time meets at most one singular hyperplane; the product in order of the reflections in the hyperplanes met by the arc is an element of W that sends $\mathcal{C}$ to $\mathcal{C}'$.[1]

Positive roots and simple roots

Fix a Weyl chamber $\mathcal{C}$. It is clear that each root takes a constant sign on $\mathcal{C}$. We define

$$\Delta^+ := \{\alpha \in \Delta \mid \alpha(\mathcal{C}) > 0\}.$$

In view of (4.4.9), this determines a partition of the root systems into two halves:

(4.5.11) $\Delta = \Delta^+\dot\cup(-\Delta^+)$; and $\alpha, \beta \in \Delta^+$, $\alpha+\beta \in \Delta$ implies $\alpha+\beta \in \Delta^+$.

Δ^+ is called a *positive root system* and its elements are called *positive roots*. A positive root is called *simple* if it cannot be written as a sum of two positive roots. Denote by Π the set of simple roots.

(4.5.12) If $\alpha, \beta \in \Pi$, $\alpha \neq \beta$ then $\alpha - \beta \notin \Delta$ and $\langle\alpha,\beta\rangle \le 0$.

[1]Missing argument for simple transitivity. Humphreys, use repr theory.

Indeed $\alpha = \beta + (\alpha - \beta)$ and $\beta = \alpha + (\beta - \alpha)$; if $\alpha - \beta$ is a root, then either it is positive or its negative is positive; in either case we reach a contradiction to the fact that α and β are simple. Now $p = 0$ in (4.5.4), so $2\frac{\langle \beta, \alpha \rangle}{||\alpha||^2} = -q \leq 0$.

Since all simple roots lie in the same open half-space (as they are all positive), with all mutual angles non-acute by (4.5.12), an elementary argument in linear algebra shows that they are linearly independent. Therefore:

(4.5.13) Every positive root can be written uniquely as a non-negative integral linear combination of simple roots; and Π is a basis of $\mathfrak{h}_{\mathbb{R}}^*$ over $\mathbb{R}$.

The fact that every positive root is a positive sum of simple roots follows immediately from the definition of simple root; the uniqueness part follows from the linear independence of Π; and the last statement now follows from (4.4.10).

Now we enumerate $\Pi = \{\alpha_1, \ldots, \alpha_r\}$, where $r = \dim_{\mathbb{C}} \mathfrak{h}$, and define

$$a_{ij} = 2\frac{\langle \alpha_j, \alpha_i \rangle}{||\alpha_i||^2}.$$

The a_{ij} are called *Cartan integers* and the $r \times r$ matrix (a_{ij}) is called the *Cartan matrix*.

A positive root is built up through roots by starting with a simple root and adding one simple root at a time. Indeed, define the *level* of $\beta = \sum_{i=1}^r m_i \alpha_i$ to be $\sum_{i=1}^r m_i$.

(4.5.14) Every positive root of level $m + 1$ can be written in at least one way as a the sum of a positive root of level m and a simple root.

To check this, suppose $\beta = \sum_{i=1}^r m_i \alpha_i$ has level $m + 1$. Then

$$0 < \langle \beta, \beta \rangle = \sum_{i-1}^{r} m_{i_0} \langle \beta, \alpha_i \rangle$$

shows that $\langle \beta, \alpha_{i_0} \rangle$ and $m_{i_0} > 0$ for some i_0. By (4.5.4), $\beta - \alpha_{i_0} \in \Delta$ is a root, as wished.

Now (4.5.11) and (4.5.14) imply that Π and (a_{ij}) determine Δ.

Isomorphism theorem

Consider a standard triple H_i, E_i, F_i spanning $\mathfrak{g}[\alpha_i]$ for each $i = 1, \ldots, r$, as in (4.4.15). Then we have the relations

$$[H_i, H_j] = 0,\ [E_i, F_i] = \delta_{ij} H_i,\ [H_i, E_j] = a_{ij} E_j,\ [H_i, F_j] = -a_{ij} F_j,$$

for all i, j. For each $\alpha \in \Delta^+$, use (4.5.13) to write

$$\alpha = \alpha_{i_1} + \cdots + \alpha_{i_s}, \tag{4.5.15}$$

sum of simple roots, where each partial sum is a root; this decomposition is unique, up to reordering. Now define

$$\begin{aligned} E_\alpha &= [E_{i_s}, [E_{i_{s-1}}, \cdots, [E_{i_2}, E_{i_1}] \cdots]] \\ F_\alpha &= [F_{i_s}, [F_{i_{s-1}}, \cdots, [F_{i_2}, F_{i_1}] \cdots]]. \end{aligned} \tag{4.5.16}$$

In view of (4.5.5),

$$\{H_i\}_{1 \le i \le r} \cup \{E_\alpha, F_\alpha\}_{\alpha \in \Delta^+} \tag{4.5.17}$$

is a basis of $\mathfrak{g}$. Now a delicate argument of induction on s shows that the multiplication table of this basis has rational entries determined by the Cartan matrix. Here the difficulty lies in the fact that a different order in (4.5.15) entails different vectors E'_α, F'_α in (4.5.16), and the main step is to show that E'_α (resp. F'_α) is a rational multiple of E_α (resp. F_α), where the multiplicative constant is determined by the permutation and the Cartan matrix. This proves:

4.5.18 Theorem (Isomorphism Theorem) *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be complex semisimple Lie algebras with corresponding CSA $\mathfrak{h}$ and $\mathfrak{h}'$. Denote by Δ and Δ' the associated root systems. If $\Phi : \mathfrak{h} \to \mathfrak{h}'$ is a linear isometry with respect to the Cartan-Killing forms such that $\Phi^*\Delta' = \Delta$, then Φ extends to an isomorphism $\mathfrak{g} \to \mathfrak{g}'$.*

4.5.19 Corollary *Every element of the Weyl group W of a complex semisimple Lie algebra $\mathfrak{g}$ is induced by an automorphism of $\mathfrak{g}$.*

Proof. Each generator s_α of W as in (4.5.8) is a linear isometry of the CSA that preserves the root system. □

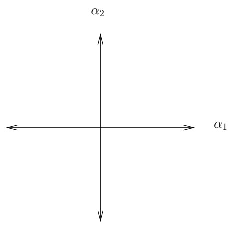
In particular, a complex semisimple Lie algebra is determined, up to isomorphism, by Π and (a_{ij}). However, three arbitrary choices have been made in the construction of these invariants:

(a) A CSA $\mathfrak{h}$ of $\mathfrak{g}$.
(b) A Weyl chamber $\mathcal{C}$ of $\mathfrak{g}$ relative to $\mathfrak{h}$.
(c) A labeling of the simple roots in Π.

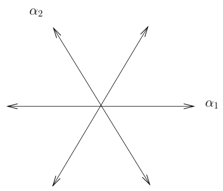
In case of (a), any two CSA's of $\mathfrak{g}$ are conjugate under an inner automorphism (Proposition 4.2.1). In case of (b), any two Weyl chambers are related by the action of an element of the Weyl group, and the latter is induced by an automorphism of $\mathfrak{g}$ (Corollary 4.5.19). We avoid the choice in (c) by introducing the *Dynkin diagram* of $\mathfrak{g}$: take r nodes, one for each simple root in Π; if $i \neq j$, join α_i to α_j by $a_{ij}a_{ji} = 4\cos^2 \angle(\alpha_i, \alpha_j) \in \{0, 1, 2, 3\}$ lines; if $||\alpha_i|| > ||\alpha_j||$, we further orient the lines in the direction of the shorter root. Note that $a_{ij} \neq 0$ if and only $a_{ji} \neq 0$, and in this case $\frac{a_{ij}}{a_{ji}} = \frac{||\alpha_j||^2}{||\alpha_i||^2}$.

Root systems of rank two

The conditions on the angles and lengths of the simple roots strongly limit the possibilities for Dynkin diagrams. As a warm up, we list all the possibilities in case of rank $r = 2$.

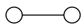


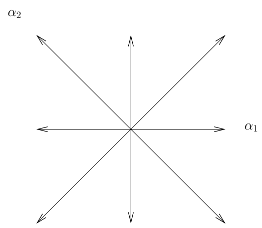
$a_{12} = a_{21} = 0 \qquad \Delta^+ = \{\alpha_1, \alpha_2\}$



$a_{12} = a_{21} = -1$

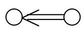
$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$

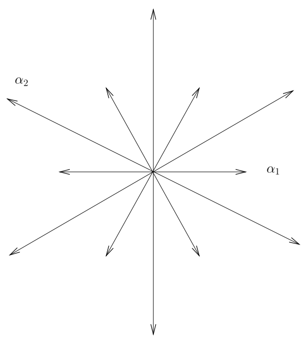




$a_{12} = -2 \;\; a_{21} = -1$

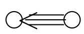
$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$





$a_{12} = -3 \;\; a_{21} = -1$

$$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$$



4.6 Classification of root systems

The Isomorphism Theorem 4.5.18 reduces the classification of complex semisimple Lie algebras to the classification of Dynkin diagrams and the construction of a complex semisimple Lie algebra for each diagram in the list. On can then use Theorem 3.3.7 or simply construct explicitly the compact real form to obtain the classification of compact semisimple Lie algebras, and hence, owing to Theorem 2.4.7, the classification of simply-connected compact connected Lie groups.

Further, if the Dynkin diagram is disconnected, then there is a partition of the simple roots $\Pi = \Pi_1\dot{\cup}\Pi_2$ with $\langle\alpha,\beta\rangle = 0$ for all $\alpha \in \Pi_1$, $\beta \in \Pi_2$. By (4.5.4), $\Delta = \Delta_1\dot{\cup}\Delta_2$, where each $\alpha \in \Delta_i$ is a linear combination of roots in Π_i. It follows that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where Δ_i is the root system of $\mathfrak{g}_i$ with respect to its CSA $\mathfrak{h} \cap \mathfrak{g}_i$. Hence we may restrict the discussion below to connected Dyinkin diagrams. In the connected case, the ratios between lengths of simples roots are determined by the angles; indeed, two simple roots connected by one line (resp. two, three lines) have the same length (resp. have length ratio $\sqrt{2}$, $\sqrt{3}$). Therefore we may in principle study configurations of unit vectors and later analyse the possibilities for lengths; that is, for now we consider diagrams as above, but without arrows in the multiple lines. Such diagrams, without the arrows, are sometimes called *Coxeter diagrams* or *Coxeter graphs*.

We are now faced with a problem in Euclidean geometry, namely, the determination of all configurations of n linearly independent unit vectors $(e_1,\ldots,e_n)$ with preassigned angles in the set $\{\pi/2, 2\pi/3, 3\pi/4, 5\pi/6\}$ among them. We call a diagram *admissible* if it corresponds to such a configuration. The determination of admissible diagrams is based on the simple observation that a $n\times n$ real symmetric matrix is the matrix of scalar products of n linearly independent vectors in a Euclidean space if and only if it is positive definite. Indeed in the sequel we will only need the necessary condition, that is,

$$\left|\sum_{i=1}^n x_i e_i\right|^2 = \sum_{i,j=1}^n x_i x_j \langle e_i, e_j\rangle \geq 0$$

for all $x_1,\ldots,x_n \in \mathbb{R}$ and equality holds if and only if all $x_i = 0$.

4.6.1 Lemma *Subdiagrams of admissible diagrams are admissible.*

We call a pair of nodes a *link* if they are connected by at least one line.

4.6.2 Lemma *An admissible diagram with n nodes contains at most $n-1$ links.*

Proof.

$$0 < \left|\sum_{i=1}^n e_i\right|^2 = n + 2\sum_{i<j}\langle e_i, e_j\rangle.$$

and note that $\langle e_i, e_j\rangle \leq -\frac{1}{2}$ for all i, j. □

It follows from Lemmas 4.6.1 and 4.6.2 that

4.6.3 Lemma *An admissible diagram contains no cycles (closed paths).*

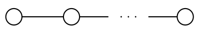
4.6.4 Lemma *Not more than three lines can be joined to a given node.*

Proof. Suppose $e_1, \dots, e_m$ are joined to e_{m+1}. Owing to Lemma 4.6.3, $e_1, \dots, e_m$ is an orthonormal set. Since e_1 does not belog to their span,

$$1 = |e_1|^2 > \sum_{i=1}^{m} \langle e_1, e_i \rangle^2.$$

Now just recall that the number of lines joined to e_1 is $\sum_{i=1}^{m} 4\langle e_1, e_i \rangle^2$. □

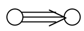
4.6.5 Lemma *Suppose an admissible diagram contains a subdiagram of type*



such that only the end nodes are connected to other nodes. Then the diagram obtained by contracting this subdiagram to a single node is admissible.

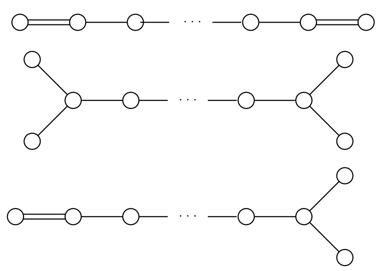
Proof. Suppose that $e_1, \dots, e_m$ correspond to the nodes of the subdiagram. Then $e_1 + \cdots + e_m$ is a unit vector and $(e_1 + \cdots + e_m, e_{m+1}, \dots, e_n)$ is a configuration corresponding to the contracted diagram. □

4.6.6 Lemma *The only admissible diagram with a triple link is*



Moreover, the following arrangements are forbidden for an admissible diagram: two double links; two triple nodes; a double link and a triple node.

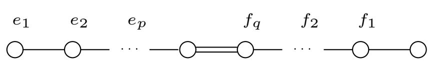
Proof. In view of Lemma 4.6.5, the diagrams



contract to a diagram with a quadruple node and hence are forbidden by Lemma 4.6.4. □

It only remains to determine the admissible diagrams with one double link and diagrams with one triple node.

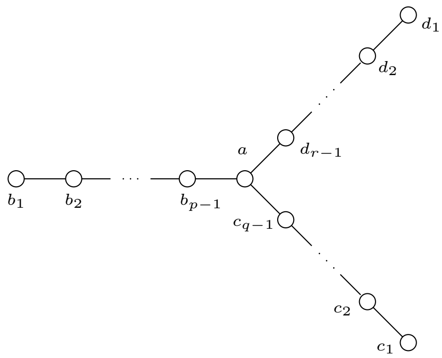
4.6.7 Lemma *Suppose*



is an admissible diagram with a double link, where we indicate the unit vectors corresponding to the nodes and $p \geq q \geq 1$. Then $q = 1$ or $p = q = 2$.

Proof. Let $e = \sum_{i=1}^{p} ie_i$ and $f = \sum_{j=1}^{q} jf_j$. Then the Cauchy-Schwarz inequality $\langle e, f\rangle^2 \leq ||e||^2 \cdot ||f||^2$ yields $(p-1)(q-1) < 2$, which implies the desired result. □

4.6.8 Lemma *Suppose*



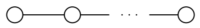
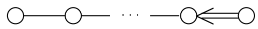
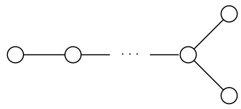
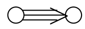
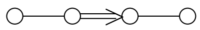
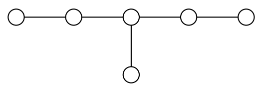
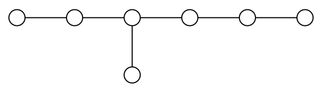
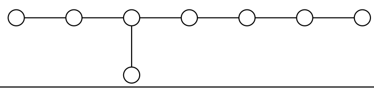
is an admissible diagram with one triple node, where we indicate the unit vectors corresponding to the nodes and $p \geq q \geq r \geq 2$. Then $q = r = 2$ or $r = 2$, $q = 3$ and $p \leq 5$.

Proof. Let $b = \sum_{i=1}^{p-1} ib_i$, $c = \sum_{j=1}^{q-1} jc_j$ and $d = \sum_{k=1}^{r-1} kd_k$. Then a, b, c, d are linearly independent vectors, so

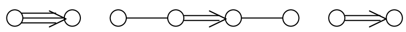
$$\begin{aligned} 1 &= ||a||^2 \\ &> \frac{\langle a, b\rangle^2}{||b||^2} + \frac{\langle a, c\rangle^2}{||c||^2} + \frac{\langle a, d\rangle^2}{||d||^2} \\ &= \frac{1}{2}\left(1 - \frac{1}{p}\right) + \frac{1}{2}\left(1 - \frac{1}{q}\right) + \frac{1}{2}\left(1 - \frac{1}{r}\right), \end{aligned}$$

which gives $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. This implies the desired result. □

4.6.9 Theorem *The following is a complete list of connected Dynkin diagrams:*

Cartan type	*Diagram*	*Condition*
A_n		–
B_n		$n \geq 2$
C_n		$n \geq 3$
D_n		$n \geq 4$
G_2		–
F_4		–
E_6		–
E_7		–
E_8		–

Proof. The list of admissible Coxeter graphs is obtained from Lemmas 4.6.4, 4.6.6, 4.6.7 and 4.6.8. Regarding the lengths of roots, the diagrams



are symmetric, so it does not matter which root is shorter. Therefore the only essential difference is in the Coxeter graph

for $n \geq 3$, which gives rise to two different Dynkin diagrams, namely,

 and .

This yields the table. □

The Cartan types A_n, B_n, C_n, D_n correspond to the classical series of compact connected simple Lie groups $SU(n)$, $SO(2n+1)$, $Sp(n)$, $SO(2n)$, respectively. In fact, in the next chapter we will compute explicitly the root systems of the classical groups. The remaining types are called *exceptional*, and the construction of the corresponding Lie algebras is more involved.

4.6.10 Example We will determine the root system of type A_n from the simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. We need to determine which nonnegative integral linear combinations $\sum_{i=1}^{n} m_i\alpha_i$ are roots. The roots of level one are the simple roots, by definition. For level two, there are no roots of the form $2\alpha_j$, due to (4.5.3), and we need to find out which sums $\alpha_i + \alpha_j$ with $i \neq j$ are roots. By (xix) $\alpha_i - \alpha_j \notin \Delta$, so in view of (4.5.4) $\alpha_i + \alpha_j$ is a root if and only $\langle \alpha_i, \alpha_j \rangle < 0$ if and only if $|i - j| < 1$ by the form of A_n. Next, suppose inductively that a root of level m has the form $\beta = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+m-1}$. No root is a linear combination of simple roots with coefficients of mixed sign, so $\beta - \alpha_j$ is never a root. In view of (4.5.4), $\beta + \alpha_j$ is a root if and only $\langle \beta, \alpha_j \rangle < 0$ if and only if $j = i - 1$ or $j = i + m$ by the form of A_n. We have shown that

$$\Delta^+ = \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j \mid 1 \leq i \leq j \leq n\}.$$

In particular, there are $n^2 + n$ roots in A_n.

4.7 Problems

1 This is a direct construction of the complex irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$, so it thus provides an alternative to section 4.3. Let π be a complex irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ on V, and recall the basis (4.3.2).

a. Let $v \in V$ be an eigenvector of $\pi(H)$ with eigenvalue λ. Use the commutation relations to show that $\pi(E)v$ and $\pi(F)v$ are an eigenvectors of $\pi(H)$ with eigenvalues $\lambda + 2$, $\lambda - 2$, respectively.

b. By finite-dimensionality, there is an eigenvalue λ of $\pi(H)$ such that $\pi(E)v = 0$. Use irreducibility to show that the vectors v, $\pi(F)v$, $\pi(F)^2(v), \ldots$ span V. Deduce that all the eigenspaces of $\pi(H)$ are one-dimensional.

c. By finite-dimensionality there is a smallest non-negative integer k such that $\pi(F)^{k+1}v = 0$. Note that $\dim V = k + 1$, and use again the commutation relations to prove that $\lambda = k$ is a non-negative integer.

d. Deduce from the above that there is at most one complex irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ of dimension $k + 1$; denote it by π_k. Identify π_0, π_1 and π_2 with the trivial representation on $\mathbb{C}$, the vector representation on $\mathbb{C}^2$, and the adjoint representation on $\mathbb{C}^3$, respectively.

e. Show that the symmetric power $S^k(\pi_1)$ is irreducible and hence that $S^k(\pi_1) = \pi_k$. (Hint: Compute the eigenvalues of $S^k(\pi_1)(H)$.)

2 Use root systems to check the following special isomorphisms in low dimensions:

$$\begin{gathered}\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{C}) \cong \mathfrak{sp}(1,\mathbb{C}) \\ \mathfrak{so}(5,\mathbb{C}) \cong \mathfrak{sp}(2,\mathbb{C}) \\ \mathfrak{sl}(4,\mathbb{C}) \cong \mathfrak{so}(6,\mathbb{C}) \\ \mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})\end{gathered}$$

The universal covering Lie group pof $SO(n)$ for $n \geq 3$ is denoted by $Spin(n)$. Deduce from the above that[2]

$$\begin{aligned} Spin(3) &= SU(2) \\ Spin(4) &= SU(2) \times SU(2) \\ Spin(5) &= Sp(2) \\ Spin(6) &= SU(4) \end{aligned}$$

3 Let Δ be a root system and assume $\alpha, \beta \in \Delta$ with $\alpha + \beta \in \Delta$. Using only basic properties of root systems (no classification), prove that

$$(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Delta$$

is a root system of type A_2, B_2 or G_2.

4 Consider a root system Δ with an ordering and denote by $\alpha_1, \ldots, \alpha_n$ the simple roots, where $\alpha_1 > \cdots > \alpha_n$. Prove that, in the following cases for the type of Δ, the indicated expression is a root and it belongs to the closed positive Weyl chamber $\bar{\mathcal{C}}$:

$$\begin{aligned} &\mathsf{B_n} : \alpha_1 + \cdots + \alpha_n \\ &\mathsf{C_n} : \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n \\ &\mathsf{F_4} : \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_n \\ &\mathsf{G_2} : 2\alpha_1 + \alpha_2 \end{aligned}$$

5 Use the root space decomposition to:

$a.$ Classify compact connected Lie groups of dimension 3.

$b.$ Prove that there exist no compact semisimple Lie groups in dimensoions 4, 5 or 7.

6 Let $\mathfrak{g}$ be a complex simple Lie algebra with root system Δ and Weyl group W.

$a.$ Prove that two simple roots whose nodes in the Dinkin diagram are connected by a single edge are in the same orbit under W.

[2]It is known that $SU(n)$ and $Sp(n)$ are connected and simply-connected.

b. Deduce that all roots in Δ of a particular length form a single orbit under W.

7 Consider a root system Δ with an ordering. Let α and β be two roots whose nodes in the Dynkin diagram are joined by n edges, where $0 \leq k \leq 3$. Let s_α and s_β be the associated reflections in the Weyl group. Show that

$$(s_\alpha s_\beta)^m = 1, \text{ where } m = \begin{cases} 2 & \text{if } k = 0; \\ 3 & \text{if } k = 1; \\ 4 & \text{if } k = 2; \\ 6 & \text{if } k = 3. \end{cases}$$

8 Prove that any element of order 2 in the Weyl group of a root system is the product of two commuting reflections.

9 Let Δ be a root system of type B_3, namely:

$$\Delta = \{\pm(\theta_1 \pm \theta_2), \pm(\theta_2 \pm \theta_3), \pm(\theta_1 \pm \theta_3)\} \cup \{\pm\theta_1, \pm\theta_2, \pm\theta_3\}.$$

Prove that the orthogonal projection of Δ on the hyperplane orthogonal to $\theta_1 + \theta_2 + \theta_3$ is congruent to a root system of type G_2.

10 Let e_1, e_2, e_3, e_4 be the canonical basis of Euclidean space $\mathbb{R}^4$. Consider the configuration Δ of 48 vectors

$$\pm e_i \ (1 \leq i \leq 4), \ \pm e_i \pm e_j \ (1 \leq i < j \leq 4), \ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4).$$

a. Let α, $\beta \in \Delta$. Show that $a_{\alpha,\beta} := 2\frac{\langle\beta,\alpha\rangle}{||\alpha||^2}$ is an integer and $s_\alpha(\beta) := \beta - a_{\alpha,\beta}\alpha \in \Delta$.
b. Check that Δ is congruent to the root system of type F_4.
c. Deduce that the dimension of F_4 is 52 and compute the order of its Weyl group to be 1152.

4.A Compact reals forms of CSA's

4.A.1 Lemma *Every CSA of $\mathfrak{g}$ is the complexification of a maximal Abelian subalgebra of a compact real form $\mathfrak{u}$ of $\mathfrak{g}$.*

Proof. Let $\mathfrak{h}$ be a CSA of $\mathfrak{g}$. By (4.4.7), the Cartan-Killing form of $\mathfrak{g}$ is non-degenerate on $\mathfrak{h}$. Therefore, in the proof of Theorem 3.3.7, we can choose the basis $(e_1, \ldots, e_n)$ satisfying $B(e_i, e_j) = -\delta_{ij}$ so that $(e_1, \ldots, e_k)$ spans $\mathfrak{h}$ over $\mathbb{C}$. Then the real span of $(e_1, \ldots, e_k)$ will be a maximal Abelian subalgebra of the compact real form of $\mathfrak{g}$ given by the real span of $(e_1, \ldots, e_n)$. □

Proof of uniqueness part of Theorem 4.2.1. We continue with the same notation and let $\mathfrak{h}'$ be another CSA of $\mathfrak{g}$. We want to show that $\mathfrak{h}$ and $\mathfrak{h}'$ are conjugate under an inner automorphism. First we observe that $\mathfrak{h}'$ is the complexification of $\mathfrak{t}' = \mathfrak{h}' \cap \mathfrak{u}'$ for some compact real form $\mathfrak{u}'$ of $\mathfrak{g}$, where $\mathfrak{t}'$ is a maximal Abelian subalgebra of $\mathfrak{u}'$, owing to Lemma 4.A.1. By the uniqueness part of Theorem 3.3.7, there is $g \in G := \mathrm{Int}(\mathfrak{g})$ such that $\mathrm{Ad}_g\mathfrak{u}' = \mathfrak{u}$. Now $\mathrm{Ad}_g\mathfrak{t}'$ and $\mathfrak{t}$ are two maximal Abelian subalgebras of $\mathfrak{u}$. By Theorem 4.1.3, there is $u \in U$, where U is the connected subgroup of G with Lie algebra $\mathfrak{u}$, such that $\mathrm{Ad}_{ug}\mathfrak{t}' = \mathfrak{t}$. By taking complexifications, we finally obtain that $\mathrm{Ad}_{ug}\mathfrak{h}' = \mathfrak{h}$. □

CHAPTER 5

Weight theory

In this chapter, we classify the representations of compact connected Lie groups. In view of Theorem 1.4.8, it is enough to classify irreducible representations. By Schur's lemma, the center acts as a scalar, so we may restrict to semisimple Lie groups. Our line of reasoning is as follows. First we consider the case of simply-connected Lie groups. Their irreducible representations are in bijective correspondence with the irreducible representations of their Lie algebras.

Next, a compact semisimple Lie algebra $\mathfrak{u}$ can be complexified to a complex semisimple Lie algebra $\mathfrak{g}$ and, conversely, every complex semisimple Lie algebra $\mathfrak{g}$ admits a compact real form $\mathfrak{u}$, which is semisimple and unique up to inner automorphism. So we need to understand the relation between the irreducible representations of $\mathfrak{u}$ and $\mathfrak{g}$. This is given by complexification, in one direction, and realification and taking real forms, in the other direction.

The main step is the classification of complex irreducible representations of a complex semisimple Lie algebra. This is where weight theory enters the picture.

5.1 Linearization of representations

A representation of a Lie group gives rise to a representation of its Lie algebra. A *representation* of a Lie algebra $\mathfrak{g}$ is a homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$; here $\mathfrak{g}$ and V can real or complex, but of course either $\mathfrak{g}$ and $\mathfrak{gl}(V)$ are both real or they are both complex in order that the definition make sense; hence if $\mathfrak{g}$ is complex, then V must be also complex and $\mathfrak{gl}(V)$ must be considered with its structure of complex Lie algebra. Suppose now G is a Lie group with Lie algebra $\mathfrak{g}$; since a representation $\rho : G \to GL(V)$ is a homomorphism, it makes sense to consider its differential $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, which gives rise to a representation of $\mathfrak{g}$. Conversely, given a Lie algebra $\mathfrak{g}$ and a representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$, due to Theorem 2.4.7 there is a unique representation $\rho : \tilde{G} \to GL(V)$, where $\tilde{G}$ is the connected simply-connected Lie group with Lie algebra $\mathfrak{g}$, such that $d\rho = \pi$.

Assume G is a connected Lie group with Lie algebra $\mathfrak{g}$. If $\rho : G \to GL(V)$ is a representation, then it follows from the identity $\rho(\exp X) = e^{d\rho(X)}$ for $X \in \mathfrak{g}$ that ρ and $d\rho$ have the same invariant subspaces. In particular, ρ is irreducible (completely reducible) if and only if $d\rho$ has the same property. It also follows from the same identity that if $\rho_1 : G \to GL(V_1)$, $\rho_2 : G \to GL(V_2)$ are representations, then ρ_1 and ρ_2 are equivalent if and only if $d\rho_1$ and $d\rho_2$ are equivalent.

5.1.1 Proposition *a. Let $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ be representations and consider their tensor product $\rho := \rho_1 \otimes \rho_2 : G \to GL(V_1 \otimes V_2)$. Then the representation $\pi := d\rho : \mathfrak{g} \to \mathfrak{gl}(V_1 \otimes V_2)$ is given by*

$$\pi(X)(v_1 \otimes v_2) = \pi_1(X)v_1 \otimes v_2 + v_1 \otimes \pi_2(X)v_2,$$

where $\pi_1 = d\rho_1$ and $\pi_2 = d\rho_2$.

b. Let $\rho : G \to GL(V)$ be a representation and consider the dual representation $\rho^* : G \to GL(V^*)$. Then the representation $\pi^* = d(\rho^*) : \mathfrak{g} \to \mathfrak{gl}(V^*)$ is given by

$$\pi^*(X)f = -f \circ \pi(X),$$

where $\pi = d\rho$.

Proof. Given $X \in \mathfrak{g}$, for part (i) we differentiate throughout the equation $\rho(\exp tX)(v_1 \otimes v_2) = \rho_1(\exp tX)v_1 \otimes \rho_2(\exp tX)v_2$ at $t = 0$ and use the bilinearity of tensor product to obtain the desired formula. For part (ii), we differentiate $\rho^*(\exp tX)f = f \circ \rho(\exp tX)^{-1} = f \circ \rho(\exp(-tX))$. □

5.2 Complexification of real representations

If $\rho : G \to GL(V)$ is a real representation of an arbitrary Lie group, then the *complexification* of ρ is the complex representation $\rho^c : G \to GL(V)$, where $V^c = V \otimes_{\mathbb{R}} \mathbb{C}$ and $\rho^c(g)$ is the $\mathbb{C}$-linear extension of $\rho(g)$ to a complex linear transformation of V^c. Note that ρ^c is indeed a Lie group homomorphism $G \to GL(V^c)$. In case we start with an orthogonal representation $\rho : G \to O(V)$, the complexification is naturally a unitary representation $\rho^c : G \to U(V)$ with respect to the extension of the Euclidean inner product $\langle , \rangle$ on V to a Hermitian inner product $(,)$on V^c, namely,

$$(v_1 + iv_2, v_1' + iv_2') = \langle v_1, v_1' \rangle + \langle v_2, v_2' \rangle + i(\langle v_2, v_1' \rangle - \langle v_1, v_2' \rangle)$$

for $v_1, v_2 \in V$.

Next we put together the techniques of linearization and complexification. Start with a real representation $\rho : G \to GL(V)$ of a (real) Lie group G. Then we have the complex representation $d(\rho^c) : \mathfrak{g} \to \mathfrak{gl}(V^c)$ of the Lie algebra $\mathfrak{g}$ of G. A moment's thought shows that also $d\rho$ can be complexified, and then $(d\rho)^c = d(\rho^c)$.

Finally, if $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a complex representation of a real Lie algebra $\mathfrak{g}$ (e.g. π is of the form $d(\rho^c)$ as in the previous paragraph), then $\mathfrak{gl}(V)$ has the structure of a complex Lie algebra, so π extends $\mathbb{C}$-linearly to a complex Lie algebra homomorphism $\mathfrak{g}^c \to \mathfrak{gl}(V)$, which we denote by the same letter π. In fact, in this case in some sense it does not matter very much whether we view the domain of π as being defined on $\mathfrak{g}$ or $\mathfrak{g}^c$.

The following result shows that it is enough to study complex representations.

5.2.1 Proposition *Let $\rho_1 : G \to GL(V_1)$, $\rho_2 : G \to GL(V_2)$ be real representations. Then ρ_1, ρ_2 are equivalent if and only if ρ_1^c, ρ_2^c are equivalent.*

Proof. An equivalence $A : V_1 \to V_2$, $A\rho_1(g) = \rho_2(g)A$ for $g \in G$, yields via complexification an equivalence $A^c : V_1^c \to V_2^c$, $A^c\rho_1^c(g) = \rho_2^c(g)A^c$ for $g \in G$. Conversely, suppose there is an equivalence $B : V_1^c \to V_2^c$, $B\rho_1^c(g) = \rho_2^c(g)B$ for $g \in G$. Restricting this identity to V_1 and using the fact that $\rho_1^c(g)$, $\rho_2^c(g)$ map V_1 to V_1, one shows that the real and imaginary parts of $B|V_1$ satisfy $\Re(B|V_1)\rho_1(g) = \rho_2(g)\Re(B|V_1)$, $\Im(B|V_1)\rho_1(g) = \rho_2(g)\Im(B|V_1)$. Then the map

$$A(x) = \Re(B|V_1) + x\,\Im(B|V_1) : V_1 \to V_2$$

for $x \in \mathbb{R}$ also satisfies $A(x)\rho_1(g) = \rho_2(g)A(x)$. Since $\det A(i) \neq 0$, the polynomial $\det A(x)$ is not identically zero. Hence there exists $x \in \mathbb{R}$ such that $A(x)$ is invertible. □

As regards the relation of irreducibility to complexification, the situation is slightly subtle.

5.2.2 Proposition *Let $\rho : G \to GL(V)$ be a real representation.*

a. If ρ^c is irreducible, then ρ is irreducible.

b. Suppose ρ is irreducible. Then ρ^c is not irreducible if and only if ρ is a complex representation π viewed as a real one (namely, there exists a structure of complex vector space on V with respect to which $\rho(g)$ is complex linear for $g \in G$); in this case $\rho^c = \pi \oplus \bar{\pi}$.

Proof. (a) If W is an invariant (real) subspace of V, then W^c is an invariant complex subspace of V^c.

(b) Imitate the argument in the proof of Lemma 3.3.1. □

5.3 Real forms and realifications of complex representations

Let $\pi : G \to GL(V)$ be a complex representation of a Lie group G on the complex vector space V. The *realification* of π is the real representation $\pi^r : G \to GL(V^r)$, where we forget the complex structure of V and view V as real vector space of twice the dimension. A *real form* of π (or an *invariant* *real form* of V) is a real vector subspace W of V, invariant under $\pi(g)$ for all $g \in G$, such that $W^c = V$; if W is a real form of V, then there is an induced representation $\rho : G \to GL(W)$ such that $\rho^c = \pi$. Whereas the realification of π always exists and is unique, real forms neither have to exist nor are necessarily unique.

Now let $\rho : G \to GL(W)$ be a real representation of a Lie group G on a real vector space W. An *invariant complex structure* on W is a complex structure J (that is, an endomorphism $J : W \to W$ with $J^2 = -I$) such that $\rho(g) \circ J = J \circ \rho(g)$ for all $g \in G$. If W admits an invariant complex structure, then ρ is the realification of a complex representation of G on the complex vector space $V = (W, J)$.

5.3.1 Proposition *Let $\pi : G \to GL(V)$ be a complex irreducible representation. Then the realification $\pi^r : G \to GL(V^r)$ is irreducible if and only if π admits no invariant real form.*

Proof. If W is an invariant real form of π and $\rho : G \to GL(W)$ is the induced representation, $\rho^c = \pi$, then $\pi^r = \rho^{cr}$ decomposes as $\rho \oplus \rho$ with respect to the real invariant (irreducible) decomposition $V = W \oplus iW$, so π^r is reducible.

Conversely, assume π^r is reducible and let W be an invariant (real) subspace of V^r. Denote the invariant complex structure on V^r by J. Then $W \cap JW$ and $W + JW$ are invariant complex subspaces of $(W, J) = V$. By irreducibility of π, we must have $W \cap JW = \{0\}$ and $W \oplus JW = V^r$, that is, $W^c = V$ and π admits a real form, namely, the representaion induced by π^r on W. □

5.3.2 Proposition *Let $\pi_1 : G \to GL(V_1)$ and $\pi_2 : G \to GL(V_2)$ be a complex representations.*

a. If π_1 and π_2 are equivalent, then the realifications π_1^r and π_2^r are also equivalent.

b. If π_1 and π_2 are irreducible and π_1^r and π_2^r are equivalent then π_1 is equivalent to π_2 or $\bar\pi_2$.

Proof. Part (i) is obvious. Suppose π_1^r and π_2^r are equivalent. Thanks to Propositions 5.2.2 and 5.2.1,

$$\pi_1 \oplus \bar\pi_1 \sim \pi_1^{rc} \sim \pi_2^{rc} \sim \pi_2 \oplus \bar\pi_2,$$

so π_1 is equivalent to π_2 or $\bar\pi_2$. □

5.3.3 Examples (i) Let ρ denote the vector representation of $SO(n)$ on $\mathbb{R}^n$. Since $SO(n)$ maps any unit vector in $\mathbb{R}^n$ to any unit vector, ρ is irreducible. Let us check there is no invariant complex structure if $n \neq 2$. Of course, it suffices to consider n even, say $n = 2m$. Suppose $m \geq 2$ and J is a complex structure on $\mathbb{R}^{2m}$ that commutes with $\rho(g)$ for all $g \in SO(n)$. By replacing the inner product $\langle , \rangle$ by

$$\langle u, v\rangle_1 = \frac{1}{2}(\langle u, v\rangle + \langle Ju, Jv\rangle)$$

we may assume J is orthogonal (indeed here $\langle , \rangle_1 = \langle , \rangle$, but we do not need to know that). Take a complex basis $\{e_1, \ldots, e_m\}$ consisting of unit vectors, so that $\mathcal{B} = \{e_1, Je_1, \ldots, e_m, Je_m\}$ is a real, orthonormal basis of $\mathbb{R}^n$. Since $SO(n)$ maps any orthonormal basis of $\mathbb{R}^n$ to any orthonormal basis, there is an element in $SO(n)$ that maps e_1 to e_2 and fixes all the other elements of $\mathcal{B}$. This contradicts the fact that ρ commutes with J. It follows that the complexification ρ^c acts irreducibly on $\mathbb{C}^n$. Note however that the realification (see below) $(\rho^c)^r = \rho \oplus \rho$ is reducible, namely, $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$ is an invariant decomposition into real subspaces. The case of $SO(2)$ is the special case $U(1)$ of the next example.

(ii) Let ρ denote the vector representation of $U(n)$ on $\mathbb{C}^n$. Since $SO(n)$ maps any unit vector in $\mathbb{C}^n = \mathbb{R}^{2n}$ to any unit vector, ρ is irreducible, even as a real representation, that is, ρ^r is irreducible. Of course, ρ^r admits an invariant complex structure, so $(\rho^r)^c = \rho \oplus \bar{\rho}$.

Reduction of the classification problem

Let G be a compact connected semisimple Lie group. Every representation of G is completely reducible, so we may restrict to irreducible representations. In view of to Propositions 5.2.1, 5.2.2, 5.3.1 and 5.3.2, a real irreducible representation ρ of G fall into one of two classes: (I)ρ^c is irreducible and then ρ is a real form of a complex irreducible representation of G; (II) ρ^c is reducible and then ρ is the realification of a complex irreducible representation π of G. In class I ρ is a real form of ρ^c, whereas in class II π admits no real forms and $\rho = \pi^r$. Hence it is enough to classify complex irreducible representations and to determine their real forms (in case they exist).

Any representation of G can be lifted to a representation of its universal covering Lie group $\tilde{G}$, which is also compact by Theorem 3.2.15 (but not all representations of $\tilde{G}$ induce representations of G), so we will assume G is simply-connected. Owing to Theorem 2.4.7, there is a bijective correspondence between (irreducible) representations of G and (irreducible) representations of its Lie algebra $\mathfrak{g}$. And complex (irreducible) representations of $\mathfrak{g}$ are the same as complex (irreducible) representations of $\mathfrak{g}^c$. Hence we are led to the classification of complex irreducible representations of complex semisimple Lie algebras.

5.4 Weight space decomposition

Let $\mathfrak{g}$ denote a complex semisimple Lie algebra and let $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ denote a complex representation. We will now construct a decomposition of V that mimics the root space decomposition of $\mathfrak{g}$; in fact, it will be essentially the same thing in case $\pi = \mathrm{ad}$.

Fix a CSA $\mathfrak{h}$ of $\mathfrak{g}$. We may assume $\mathfrak{g}$ admits a compact real form $\mathfrak{u}$ and $\mathfrak{h}$ is the complexification of a Lie algebra $\mathfrak{t}$ of $\mathfrak{u}$ (either by assumption, or by Theorem 3.3.7). By Theorem 1.4.6, there exists an inner product on V such that π maps the elements of $\mathfrak{u}$ to skew-Hermitian endomorphisms of V; in particular, these are semisimple (diagonalizable over $\mathbb{C}$). Since $\{\rho(H) \mid H \in \mathfrak{h}\}$ is a commuting family of semisimple endomorphisms of V, we can find a common eigenspace decomposition of V:

$$V = \sum_{\lambda} V_\lambda. \tag{5.4.1}$$

Here

$$V_\lambda = \{v \in V \mid \pi(H)v = \lambda(H)v\}$$

is a common eigenspace and $\lambda : \mathfrak{h} \to \mathbb{C}$ is a linear functional. By finite-dimensionality, the sum in (5.4.1) is finite in the sense that $V_\lambda = 0$ but for finitely many $\lambda \in \mathfrak{h}^*$, and we put

$$\Lambda_\pi := \{\lambda \in \mathfrak{h}^* \mid V_\lambda \neq 0\}.$$

The elements of Λ_π are called the *weights* of V, and Λ_π is called the *weight system*; note that zero can be a weight. For a weight $\lambda \in \Lambda_\pi$, V_λ is the associated *weight space*, and its nonzero elements are the associated *weight vectors*. The dimension $m(\lambda)$ of V_λ is called the *multiplicity* of λ, so that Λ_π is a multiset. The following is called the *weight space decomposition* of $\mathfrak{g}$ with respect to $\mathfrak{h}$:

$$V = \sum_{\lambda \in \Lambda_\pi} V_\lambda. \tag{5.4.2}$$

As a special case, $\Lambda_{\mathrm{ad}} = \Delta \cup \{0\}$, $m(\alpha) = 1$ for $\alpha \in \Delta$ and $m(0) = \dim \mathfrak{h}$.

5.4.3 Example Let π denote the vector representation of $\mathfrak{u} = \mathfrak{su}(n)$ on $\mathbb{C}^n$. This is an irreducible representation, which we can also view as a representation of the complexified Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. Let $\mathfrak{t}$ be the subspace of $\mathfrak{u}$ consisting of diagonal matrices $H = \mathrm{diag}(it_1, \ldots, it_n)$ ($t_j \in \mathbb{R}$) with $\sum_{j=1}^n t_j = 0$ and let $\mathfrak{h}$ denote its complexification, namely the subspace of $\mathfrak{g}$ consisting of diagonal matrices $H = \mathrm{diag}(a_1, \ldots, a_n)$ ($a_j \in \mathbb{C}$) with $\sum_{j=1}^n a_j = 0$. We already know that $\mathfrak{h}$ is a CSA of $\mathfrak{g}$. Denote by $\theta_1, \ldots, \theta_n \in \mathfrak{h}^*$ the linear functionals such that $\theta_j(H) = a_j$. Then the weight system

$$\Lambda_\pi = \{\theta_1, \ldots, \theta_n\}.$$

5.4.4 Proposition *Let $\mathfrak{g}$ be a complex semisimple Lie algebra.*

a. Let π_1 and π_2 be representations of $\mathfrak{g}$ on V_1, V_2 with weight systems Λ_{π_1}, Λ_{π_2}, resp. Then the weight system of the tensor product representation $\pi_1 \otimes \pi_2 : \mathfrak{g} \to \mathfrak{gl}(V_1 \otimes V_2)$ is

$$\begin{aligned} \Lambda_{\pi_1 \otimes \pi_2} &= \Lambda_{\pi_1} + \Lambda_{\pi_2} \\ &= \{\lambda_1 + \lambda_2 \mid \lambda_1 \in \Lambda_{\pi_1},\ \lambda_2 \in \Lambda_{\pi_2}\} \end{aligned}$$

where $m(\lambda) = \sum_{\lambda_1+\lambda_2=\lambda} m(\lambda_1)m(\lambda_2)$ for $\lambda \in \Lambda_{\pi_1 \otimes \pi_2}$.

b. Let π be a representation of $\mathfrak{g}$ on V. Then $\Lambda_{\pi^*} = -\Lambda_\pi$ as a multiset.

Proof. Let $\mathfrak{h}$ be a CSA of $\mathfrak{g}$. For part (i), If $v_1 \in V_1$ is a λ_1-weight vector π_1 and $v_2 \in V_2$ is a λ_2-weight vector of π_2, then, in view of Proposition 5.1.1(i), $v_1 \otimes v_2$ is a $\lambda_1 + \lambda_2$-weight vector of $\pi_1 \otimes \pi_2$. For part (ii), using Proposition 5.1.1(ii) we see that, given any basis of V consisting of weight vectors of π, the dual basis of V^* consists of weight vectors of π^*, with exactly the opposite weights. □

5.4.5 Example (Adjoint representation of type A_n) Let $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$ be the Lie algebra of traceless $n+1 \times n+1$ complex matrices and choose the subspace $\mathfrak{h}$ of diagonal matrices as CSA. Let $\mathbb{C}^{n+1}$ denote the vector representation of $\mathfrak{g}$ and $G = SL(n+1, \mathbb{C})$.

On one hand, the G-action on $\mathrm{End}(\mathbb{C}^{n+1})$ is given by $g \cdot A = g \circ A \circ g^{-1}$. We identify $\mathrm{End}(\mathbb{C}^{n+1})$ with $M(n+1, \mathbb{C})$ and now we have $SL(n+1, \mathbb{C})$-conjugation of matrices in $M(n+1, \mathbb{C})$, which decomposes into G-invariant subspaces as $\mathfrak{sl}(n+1, \mathbb{C}) \oplus \mathbb{C} \cdot I$; the first component is the adjoint representation of $SL(n+1, \mathbb{C})$ and the second component is the trivial representation.

On the other hand, $\mathrm{End}(\mathbb{C}^{n+1}) \cong \mathbb{C}^{n+1*} \otimes \mathbb{C}^{n+1}$. In view of Proposition 5.4.4 and Example 5.4.3, this representation has weight system $\{\theta_i - \theta_j \mid 1 \le i,\ j \le n+1\}$, which decomposes as

$$\{\pm(\theta_i - \theta_j) \mid 1 \le i < j \le n+1\} \cup \{\underbrace{0, \ldots, 0}_{n \text{ times}}\} \cup \{0\}.$$

We deduce that

$$\Delta(\mathfrak{g}, \mathfrak{h}) = \{\pm(\theta_i - \theta_j) \mid 1 \le i < j \le n+1\}.$$

Note that the rank of $\mathfrak{sl}(n+1, \mathbb{C})$ is n, which is the number of zero weights in its adjoint representation, whereas the remaining zero weight comes from the trivial component.

We impose an ordering on the roots such that

$$\Delta^+(\mathfrak{g}, \mathfrak{h}) = \{\theta_i - \theta_j \mid 1 \le i < j \le n+1\},$$

by choosing the Weyl chamber to be defined by the inequalities

$$\theta_i - \theta_j > 0 \text{ for } i < j.$$

Then the associated basis of simple roots is

$$\Pi = \{\alpha_1 := \theta_1 - \theta_2, \dots, \alpha_n := \theta_n - \theta_{n+1}\}.$$

Since the other positive roots

$$\theta_i - \theta_j = \alpha_i + \cdots + \alpha_{j-1}$$

for $i < j$, we deduce that Δ is of type A_n (cf. Example 4.6.10).

In particular, this shows that $\mathfrak{sl}(n+1, \mathbb{C})$ is a complex simple Lie algebra and $\mathfrak{su}(n)$ is a compact simple Lie algebra.

Properties of weight systems

5.4.6 Lemma (Fundamental calculation) *It holds*

$$\pi(\mathfrak{g}_\alpha)V_\lambda \subset V_{\lambda+\alpha}$$

for all $\alpha \in \Delta$, $\lambda \in \mathfrak{h}^*$.

Proof. Let $X \in \mathfrak{g}_\alpha$, $v \in V_\lambda$ and $H \in \mathfrak{h}$. Then $\pi(H)\pi(X)v = \pi(X)\pi(H)v + \pi([H, X])v = \pi(X)\lambda(H)v + \pi(\alpha(H)X)v = (\lambda(H) + \alpha(H))\pi(X)v$. □

Choose a positive Weyl chamber in $\mathfrak{h}_\mathbb{R}$. This specifies an ordered basis of simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$. Now we introduce a total ordering on $\mathfrak{h}_\mathbb{R}^*$, called *lexicographic order*:

$$\sum_{i=1}^r m_i\alpha_i \succ \sum_{i=1}^r n_i\alpha_i$$

if and only if there is i_0 such that $m_i = n_i$ for $1 \leq i \leq i_0 - 1$ and $m_{i_0} > n_{i_0}$.

Each weight $\lambda \in \Lambda_\pi$ takes real values on $\mathfrak{h}_\mathbb{R}$, because it must take pure imaginary values on the subalgebra $i\mathfrak{h}_\mathbb{R} = \mathfrak{h} \cap \mathfrak{u}$, so we may view $\lambda \in \mathfrak{h}_\mathbb{R}^*$. Now the weights are totally ordered by the total ordering in $\mathfrak{h}_\mathbb{R}^*$. We denote by μ_π the *greatest* or *highest weight* of π.

Let $\lambda \in \Lambda_\pi$ and $\alpha \in \Delta$. Similar to the adjoint case, we examine the restriction of π to $\mathfrak{g}[\alpha]$ to conclude that

$$2\frac{\langle \lambda, \alpha \rangle}{||\alpha||^2} = p - q \text{ where } \lambda + k\alpha \in \Lambda \text{ if and only if } -p \leq k \leq q.$$

$\{\lambda + k\alpha \in \Delta \mid -p \leq k \leq q\}$ is called the α-string of weights through λ. Here $\langle \lambda, \nu \rangle = \langle H_\lambda, H_\nu \rangle$, where H_λ is defined by $\langle H_\lambda, H \rangle = \lambda(H)$ for all $H \in \mathfrak{h}_\mathbb{R}$. We deduce, as in (4.5.5), that

(5.4.7) If $\lambda \in \Lambda_\pi, \alpha \in \Delta, \lambda + \alpha \in \Lambda_\pi$ then $\pi[\mathfrak{g}_\alpha]V_\lambda \neq 0$.

We also see that that the Weyl group W preserves Λ_π; indeed for a generator $s_\alpha \in W$ and $\lambda \in \Lambda_\pi$ we have

$$s_\alpha(\lambda) = \lambda - 2\frac{\langle \lambda, \alpha \rangle}{||\alpha||^2}\alpha = \lambda + (q-p)\alpha \in \Lambda_\pi$$

as $-p \leq q - p \leq q$. Since the weights of a representation of $\mathfrak{g}[\alpha] \cong \mathfrak{sl}(2, \mathbb{C})$ are symmetric about 0, by this analysis we further see that the multiplicities of the weights are invariant under the Weyl group.

Now assume V is irreducible and $\mu_\pi \neq 0$, so that V is not the trivial one-dimensional representation, and choose a highest weight $v \in V_{\mu_\pi}$. By the Fundamental Calculation 5.4.6, $\pi[\mathfrak{g}_\alpha]v \in V_{\mu_\pi} + \alpha$ if $\mu_\pi + \alpha$ is a weight, and is zero otherwise. Note $\mu_\pi + \alpha$ is not a weight for $\alpha > 0$ because in that case $\mu_\pi + \alpha > \mu_\pi$; and $\pi[\mathfrak{h}]v = \mathbb{C}v$.

Plainly, $\sum_{n \geq 0} \pi[\mathfrak{g}]^n \cdot v$ is an invariant subspace of V. By the irreducibility assumption, it must coincide with V. Recall the basis (4.5.17) of $\mathfrak{g}$. Using the commutation rules, we see that every element of $\pi[\mathfrak{g}]^n$ is a linear combination of elements of the form

$$\pi[F_{\beta_m}] \cdots \pi[F_{\beta_1}]\pi[E_{\alpha_\ell}] \cdots \pi[F_{\alpha_1}]\pi[H_{i_k}] \cdots \pi[H_{i_1}].$$

It follows that

$$\sum_{\lambda \in \Lambda_\pi} V_\lambda = V = \mathbb{C}v + \sum_{m \geq 1} \sum_{\beta_1, \ldots, \beta_m \in \Delta} \pi[F_{\beta_m}] \cdots \pi[F_{\beta_1}]v.$$

Further, any root vector F_{β_j} is obtained from the root vectors F_{α_i} associated to the simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ as in (4.5.16), so that

$$\sum_{\lambda \in \Lambda_\pi} V_\lambda = \mathbb{C}v + \sum_{s \geq 1} \sum_{\alpha_{i_1}, \ldots, \alpha_{i_s} \in \Pi} \pi[F_{\alpha_{i_s}}] \cdots \pi[F_{\alpha_{i_1}}]v.$$

This equation shows that every weight has the form $\mu_\pi - \alpha_{i_1} - \cdots - \alpha_{i_s}$ for some simple roots $\alpha_{i_1}, \cdots, \alpha_{i_s}$. Further, we deduce:

The multiplicity of the highest weight μ_π is one.

The highest weight μ_π is depends only on the simple system Π but not on the ordering used to define Π.

Every weight ν is obtained from μ_π by successive subtraction of simple roots, yielding a finite decreasing sequence of weights from μ_π to ν.

Now we have a precise description of the sets of weights of π in terms of its highest weight. Recall that Π is a basis of $\mathfrak{h}^*_\mathbb{R}$. We next introduce a two lattices in $\mathfrak{h}^*_\mathbb{R}$. The first one is the so called *(integral) weight lattice* of $\mathfrak{g}$:

$$L_{wt} = \{\lambda \in \mathfrak{h}^*_\mathbb{R} \mid 2\frac{\langle \lambda, \alpha_i \rangle}{||\alpha_i||^2} \in \mathbb{Z} \text{ for all } \alpha_i \in \Pi\}.$$

The second lattice is the *root latice* of $\mathfrak{g}$:

$$L_{rt} = \sum_{i=1}^{r} \mathbb{Z} \cdot \alpha_i.$$

Note that L_{rt} is a sublattice of L_{wt} and every weight of π lies in L_{wt}.[1]

5.4.8 Lemma *Λ_π precisely consists of the elements of L_{wt} that are congruent to μ_π modulo L_{rt} and lie in the convex hull $\mathcal{P}_\pi$ of the images of μ_π under the Weyl group W.*

Proof. First we note that Λ_π is contained in $\mu_\pi - C_\pi$, where C_π is the positive real cone spanned by the simple roots $\alpha_i \in \Pi$ sich that $\langle \mu_\pi, \alpha_i \rangle \neq 0$ (or, equivalently, $\pi(F_{\alpha_i})v \neq 0$). Moreover, Λ_π contains the α-string of weights

$$\mu_\pi, \mu_\pi - \alpha_i, \ldots, \mu_\pi - 2\frac{\langle \mu_\pi, \alpha_i \rangle}{||\alpha_i||^2} = s_{\alpha_i}(\mu_\pi),$$

so that $s_{\alpha_i}(\mu_\pi)$ is a vertex of $\mathcal{P}_\pi$, and indeed every vertex adjacent to μ_π must be of the form $s_{\alpha_i}(\mu_\pi)$ for some j. Applying the same reasoning to each successive vertex, shows that every vertex of $\mathcal{P}_\pi$ is W-conjugate to μ_π.

Now $\Lambda_\pi \subset \mathcal{P}_\pi \cap (\mu_\pi + L_{rt})$. Since the sets of weights of the form $\{\lambda + k\beta\}$ for $\lambda \in \Gamma_\pi$ and $\beta \in \Delta$ are connected strings, proceeding by induction on the dimension of the faces one sees that $\Lambda_\pi = \mathcal{P}_\pi \cap (\mu_\pi + L_{rt})$. $\square$

Lemma 5.4.8 shows that μ_π determines Λ_π, up to the multiplicities. We can extract a basis of V from the set of weight vectors

$$v, \pi[F_{\alpha_{i_s}}] \cdots \pi[F_{\alpha_{i_1}}]v$$

for $\alpha_{i_1}, \cdots, \alpha_{i_s} \in \Pi$. A delicate argument shows that the action of $\pi(E_\alpha)$ and $\pi(F_\beta)$ on the basis elements is determined by the Cartan matrix, similar to the adjoint representation. Thus one can reconstruct π, up to equivalence, from μ_π.

We say an element of $\mathfrak{h}_{\mathbb{R}}^*$ is *dominant* if $\langle \lambda, \alpha \rangle \geq 0$ (of course it suffices to require the inequality for simple roots), and we denote the set of dominant weights by L_{wt}^+. It follows from (5.4.7) that $\mu_\pi \in L_{wt}^+$.

5.4.9 Theorem (Theorem of the Highest Weight (Cartan 1913)) *Let $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ be an irreducible representation of a complex semisimple Lie algebra. Fix a CSA $\mathfrak{h}$ and a positive Weyl chamber. Then every irreducible representation π of $\mathfrak{g}$ is determined, up to equivalence, by its highest weight μ_π . Conversely, given a dominant integral weight μ, there exists a unique irreducible representation of $\mathfrak{g}$ with μ as its highest weight vector.*

[1] These are also W-invariant.

Proof. (Sketch) For the uniqueness result, we have sketched above the idea that μ_π determines π. This result is also a consequence of the Weyl character formula, to be proved in Chapter 6. We give in Remark 5.4.11 below a simple, independent argument.

For the existence result, a sleek proof comes out of the Peter-Weyl theorem. We will also take the invariant theoretic approach and construct some of the irreducible representations as subrepresentations of tensor powers of standard represention, in addition to the spin representations. Note that in account of Theorem 3.2.1 and Corollary 1.5.6, it suffices to construct irreducible representations of complex simple Lie algebras.

Other two approaches for the existence result respectively rely on Verma modules and the Borel-Weil theorem, but we do not discuss them here. □

Reducible representations

We extend the definition of a highest weight vector to the case of a reducible representation as follows. Consider a non-necessarily irreducible representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$. A *highest weight vector* of π is a nonzero vector $v \in V$ which is an eigenvector of $\pi(H)$ for all $H \in \mathfrak{h}$ and lies in the kernel of $\pi(E_\alpha)$ for all $\alpha \in \Delta^+$. The *weight* of v is the element $\lambda \in \mathfrak{h}^*$ such that $\pi(H)v = \lambda(H)v$ for all $H \in \mathfrak{h}$.

5.4.10 Proposition *Let $v \in V$ be a highest weight vector of π Then the subspace*

$$W = \sum_{n \geq 0} \pi[\mathfrak{g}]^n v = \mathbb{C}v + \sum_{m \geq 1} \sum_{\beta_1,\ldots,\beta_m \in \Delta} \pi[F_{\beta_m}] \cdots \pi[F_{\beta_1}]v$$

is an irreducible subrepresentation with highest weight λ.

Proof. It is clear that W is an invariant subspace. Suppose $W = W_1 \oplus W_2$ is an invariant decomposition. Then each W_i is a sum of weight spaces of $\mathfrak{h}$. Since the weight space of λ in W is $\mathbb{C}v$ and thus one-dimensional, it must be contained in either W_1 or W_2. It follows that $W = W_1$ or $W = W_2$. □

5.4.11 Remark We can now give another argument for the uniqueness result of Theorem 5.4.9. Suppose $\pi_1 : \mathfrak{g} \to \mathfrak{gl}(V_1)$ and $\pi_2 : \mathfrak{g} \to \mathfrak{gl}(V_2)$ are two irreducible representations with the same highest weight μ. Take highest weight vectors v_1 and v_2 of π_1 and π_2, resp. Then $(v_1, v_2) \in V_1 \oplus V_2$ is also a λ-weight vector and a highest weight vector of $\pi_1 \oplus \pi_2$. Let W be the irreducible subrepresentation generated by (v_1, v_2). The projections $W \to V_1$ and $W \to V_2$ are equivariant maps, so by Schur lemma they are isomorphisms. Hence $V_1 \cong V_2$.

5.4.12 Example (Cartan composition) Let V_{λ_1} and V_{λ_2} be complex irreducible representations of $\mathfrak{g}$ with highest weights λ_1 and λ_2, respectively. Then

$V_{\lambda_1} \otimes V_{\lambda_2}$ in general does not have to be irreducicle, but there is an irreducible subrepresentation $V_{\lambda_1+\lambda_2} \subset V_{\lambda_1} \otimes V_{\lambda_2}$ with highest weight vector $\lambda_1 + \lambda_2$, called the *Cartan composition* of V_{λ_1} and V_{λ_2}, which is generated by $v_{\lambda_1} \otimes v_{\lambda_2}$, where v_{λ_i} is a highest weight vector of V_{λ_i}. For instance, in Example 5.4.5 we showed that in case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ the Cartan composition of the vector representation $\mathbb{C}^n$ and and its dual $\mathbb{C}^{n*}$ is the adjoint representation.

The next example is a generalization of Example 5.4.5.

5.4.13 Example Let $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a faithful representation. Then we claim that $\mathrm{ad} \subset \pi \otimes \pi^*$.

Indeed, by faithfulness $\mathfrak{g} \cong \pi[\mathfrak{g}] \subset \mathfrak{gl}(V)$. On the other hand, as $\mathfrak{g}$-representations,

$$W := \mathfrak{gl}(V) = \mathrm{End(V)} = V \otimes V^*.$$

Define $\rho : \mathfrak{g} \to \mathfrak{gl}(W)$ by $\rho(X)A = \pi(X)A - A\pi(X) = [\pi(X), A]$. It follows from Jacobi that ρ is a representation. Moreover,

$$\rho(X)\pi(Y) = [\pi(X), \pi(Y)] = \pi[X, Y] = \pi(\mathrm{ad}_X Y)$$

for all X, $Y \in \mathfrak{g}$, that is, the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\pi} & \pi[\mathfrak{g}] \\ {\scriptstyle \mathrm{ad}_X}\downarrow & & \downarrow{\scriptstyle \rho(X)} \\ \mathfrak{g} & \xrightarrow[\pi]{} & \pi[\mathfrak{g}] \end{array}$$

is commutative, proving that $\pi[\mathfrak{g}]$ is an invariant subspace of W, and that ad and ρ are equivalent representations.

The next example is a specialization of the previous example.

5.4.14 Example Suppose $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a faithful representation and B is a nondegenerate bilinear form on V. We claim that if B is symmetric (resp. skew-symmetric) then $\mathrm{ad} \subset \Lambda^2\pi$ (resp. $\mathrm{ad} \subset S^2\pi$).

Indeed B defines an equivariant map $V \to V^*$ and $\pi = \pi^*$. Now we have, as $\mathfrak{g}$-representations,

$$\mathfrak{gl}(V) = \mathrm{End(V)} = V \otimes V^* = \otimes^2 V = S^2V \oplus \Lambda^2 V.$$

In view of Example 5.4.13, we need only check that $\pi[\mathfrak{g}] \subset \Lambda^2 V$ (resp. $\pi[\mathfrak{g}] \subset S^2V$) in case B is symmetric (resp. skew-symmetric).

The subspace S^2V (resp. $\Lambda^2 V$) of $\otimes^2 V$ is spanned by vectors of the form $u \otimes v + v \otimes u$ for u, $v \in V$ (resp. $u \otimes v - v \otimes u$) for u, $v \in V$. These correspond to elements of $\mathfrak{gl}(V)$ of the form $B(\cdot,u)v+B(\cdot,v)u$ (resp. $B(\cdot,u)v-B(\cdot,v)u$). Now $\mathfrak{gl}(V)=\mathrm{sym}(V)\oplus\mathrm{skew}(V)$ where

$$\begin{aligned}\mathrm{sym}(V)&=\{A\in\mathfrak{gl}(V)\mid B(Ax,y)-B(x,Ay)=0\text{ for all }x,\,y\in V\}\\&\cong S^2V\end{aligned}$$

and

$$\begin{aligned}\mathrm{skew}(V)&=\{A\in\mathfrak{gl}(V)\mid B(Ax,y)+B(x,Ay)=0\text{ for all }x,\,y\in V\}\\&\cong \Lambda^2V.\end{aligned}$$

The invariance of B under π means that

$$B(\pi(X)u,v)+B(u,\pi(X)v)=0$$

for all $X\in\mathfrak{g}$ and all u, $v\in V$. If B is symmetric (resp. skew-symmetric), this precisely says that $\pi[\mathfrak{g}]\subset\mathrm{skew}(V)$ (resp. $\pi[\mathfrak{g}]\subset\mathrm{sym}(V)$), as we wished.

5.4.15 Example (Adjoint representation of type C_n) Let $\mathfrak{g}=\mathfrak{sp}(n,\mathbb{C})$ be the complexification of $\mathfrak{u}=\mathfrak{sp}(n)$, viewed as a Lie algebra of complex matrices as in Example 3.3.4:

$$\mathfrak{sp}(n,\mathbb{C})=\left\{\begin{pmatrix}A & C\\ B & -A^t\end{pmatrix}\;:\;A\in\mathfrak{gl}(n,\mathbb{C}),\;B,\;C\in\mathrm{Sym}(n,\mathbb{C})\right\}.$$

Let π denote the vector representation of $\mathfrak{g}$ on $V=\mathbb{C}^{2n}$. This is a faithful representation. Note that if $X\in\mathfrak{g}$ then $X^tJ_n+J_nX=0$. This exactly means that $\mathfrak{g}$ preserves the skew-symmetric bilinear form B on $\mathbb{C}^{2n}$ given by $B(u,v)=u^tJv$. Therefore we can apply the result of Example 5.4.14 to obtain $\mathrm{ad}\subset S^2\pi$. Since $\dim\mathfrak{g}=2n^2+n=\dim S^2(\mathbb{C}^{2n})$, in this case we get equality, namely, the adjoint representation of $\mathfrak{sp}(n,\mathbb{C})$ coincides with $S^2(\mathbb{C}^{2n})$.

It is easy to see that a CSA $\mathfrak{h}$ of $\mathfrak{g}$ consists of the diagonal matrices in $\mathfrak{g}$, that is, matrices of the form $H=\mathrm{diag}(a_1,\ldots,a_n,-a_1,\ldots,-a_n)$ ($a_j\in\mathbb{C}$). Denote by $\theta_1,\ldots,\theta_n\in\mathfrak{h}^*$ the linear functionals such that $\theta_j(H)=a_j$. Then the weight system

$$\Lambda_\pi=\{\theta_1,\ldots,\theta_n,-\theta_1,\ldots,-\theta_n\}.$$

The weight system of $S^2\pi$ consists of all sums of two weights of π, where the order is unimportant. We get the zero weight n times, as $\theta_i+(-\theta_i)=0$ for $i=1,\ldots,n$, the number n corresponding to the rank of $\mathfrak{g}$. The nonzero weights comprise the root system of $\mathfrak{g}$:

$$\Delta(\mathfrak{sp}(n,\mathbb{C}))=\{\pm2\theta_i\mid 1\leq i\leq n\}\cup\{\pm(\theta_i\pm\theta_j)\mid 1\leq i<j\leq n\}.$$

Note that it is of type C_n.

In particular, this example shows that $\mathfrak{sp}(n,\mathbb{C})$ is a complex simple Lie algebra and $\mathfrak{sp}(n)$ is a compact simple Lie algebra.

5.4.16 Example (Adjoint representation of types B_n and D_n) Let $\mathfrak{g} = \mathfrak{so}(m,\mathbb{C})$ be the complexification of $\mathfrak{u} = \mathfrak{so}(m)$. Then

$$\mathfrak{so}(m,\mathbb{C}) = \{X \in \mathfrak{gl}(m,\mathbb{C}) \mid X + X^t = 0\}.$$

Let π denote the vector representation of $\mathfrak{g}$ on $V = \mathbb{C}^m$. This is a faithful representation. Note that the condition $X^t + X = 0$ exactly means that $\mathfrak{g}$ preserves the symmetric bilinear form B on $\mathbb{C}^m$ given by $B(u,v) = u^t v$. Therefore we can apply the result of Example 5.4.14 to obtain $\mathrm{ad} \subset \Lambda^2\pi$. Since $\dim\mathfrak{g} = \frac{1}{2}m(m-1) = \dim\Lambda^2(\mathbb{C}^m)$, in this case we get equality, namely, the adjoint representation of $\mathfrak{so}(m,\mathbb{C})$ coincides with $\Lambda^2(\mathbb{C}^m)$. To continue, we will need to distinguish between the cases of m even and m order.

(i) Suppose $m = 2n$. A CSA $\mathfrak{h}$ of $\mathfrak{g}$, coming from the complexification of the Lie algebra of a maximal torus of $SO(2n)$ is given by matrices of the form ($a_j \in \mathbb{C}$)

$$H = \begin{pmatrix} 0 & -ia_1 & & & \\ ia_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -ia_n \\ & & & ia_n & 0 \end{pmatrix}.$$

Denote by $\theta_1, \ldots, \theta_n \in \mathfrak{h}^*$ the linear functionals such that $\theta_j(H) = a_j$. Then the weight system

$$\Lambda_\pi = \{\theta_1, \ldots, \theta_n, -\theta_1, \ldots, -\theta_n\}.$$

The weight system of $\Lambda^2\pi$ consists of all sums of two different weights of π, where the order is unimportant. We get the zero weight n times, as $\theta_i + (-\theta_i) = 0$ for $i = 1, \ldots, n$, the number n corresponding to the rank of $\mathfrak{g}$. The nonzero weights comprise the root system of $\mathfrak{g}$:

$$\Delta(\mathfrak{so}(2n,\mathbb{C})) = \{\pm(\theta_i \pm \theta_j) \mid 1 \le i < j \le n\}.$$

Note that it is of type D_n.

(ii) Suppose $m = 2n+1$. A CSA $\mathfrak{h}$ of $\mathfrak{g}$ now consists of matrices of the form ($a_j \in \mathbb{C}$)

$$H = \begin{pmatrix} 0 & -ia_1 & & & & \\ ia_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & -ia_n & \\ & & & ia_n & 0 & \\ & & & & & 0 \end{pmatrix}.$$

Denote by $\theta_1, \ldots, \theta_n \in \mathfrak{h}^*$ the linear functionals such that $\theta_j(H) = a_j$. Then the weight system

$$\Lambda_\pi = \{\theta_1, \ldots, \theta_n, -\theta_1, \ldots, -\theta_n, 0\}.$$

The weight system of $\Lambda^2\pi$ again consists of all sums of two different weights of π, where the order is unimportant. We get the zero weight n times, as $\theta_i + (-\theta_i) = 0$ for $i = 1, \ldots, n$, the number n corresponding to the rank of $\mathfrak{g}$. The nonzero weights comprise the root system of $\mathfrak{g}$:

$$\Delta(\mathfrak{so}(2n+1, \mathbb{C})) = \{\pm(\theta_i \pm \theta_j) \mid 1 \leq i < j \leq n\} \cup \{\pm\theta_i \mid 1 \leq i \leq n\}.$$

Note that it is of type B_n.

In particular, cases (i) and (ii) in this example show that $\mathfrak{so}(m, \mathbb{C})$ is a complex simple Lie algebra and $\mathfrak{so}(m)$ is a compact simple Lie algebra.

5.4.17 Example Let $\mathfrak{u} = \mathfrak{so}(3)$ and $\mathfrak{g} = \mathfrak{so}(3, \mathbb{C})$. From Example 5.4.16(ii), we see that $\Delta = \{\pm\theta_1\}$ and $\mathfrak{g}$ has rank 1, that is,

$$\Lambda_{ad} = \{\pm\theta_1, 0\}.$$

Since the vector representation π of $\mathfrak{g}$ on $\mathbb{C}^3$ has the same weight system, we deduce $\pi = \mathrm{ad}$, that is, $\mathfrak{so}(3, \mathbb{C}) = \mathbb{C}^3$ as $\mathfrak{so}(3, \mathbb{C})$-modules.

5.4.18 Example Due to Problem 7 in Chapter 2, $\mathfrak{so}(4, \mathbb{C}) = \mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C})$. In terms of root systems,

$$\begin{aligned} \Delta(\mathsf{D}_2) &= \{\pm(\theta_1 \pm \theta_2)\} \\ &= \{\pm(\theta_1 + \theta_2)\} \cup \{\pm(\theta_1 - \theta_2)\} \\ &= \Delta(\mathsf{A}_1) \cup \Delta(\mathsf{A}_1), \end{aligned}$$

since $\theta_1 + \theta_2 \perp \theta_1 - \theta_2$.

Fundamental weights

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, fix a CSA $\mathfrak{h}$ and a chamber Weyl. As usual, we denote the root system by Δ, the system of positive roots by Δ^+ and the simple roots by $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. We have discussed how complex irreducible representations of $\mathfrak{g}$ are determined by their highest weight vectors, which can be any dominant integral weight. We shall now show how to parameterize the dominat integral weights in a consistent way.

Recall the Euclidean space $\mathfrak{h}^*_{\mathbb{R}}$ and its Euclidean inner product $\langle \cdot, \cdot \rangle$ coming from the restriction of the Cartan-Killing form. Define elements

$$\varpi_1, \ldots, \varpi_n \in \mathfrak{h}^*_{\mathbb{R}}$$

by the rule

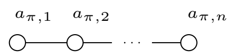
$$2\frac{\langle \varpi_i, \alpha_j \rangle}{||\alpha_j||^2} = \delta_{ij},$$

where δ_{ij} denotes the delta Kronecker. We call ϖ_i the *fundamental weight* (or *basic weight*) associated to the simple root α_i, and the corresponding irreducible representation the *fundamental representation* (or *basic representation*).

Note $\varpi_1, \dots, \varpi_n$ form an integral basis of L_{wt}^+. For each irreducible representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$, if μ_π denotes its highest weight, then

$$\mu_\pi = \sum_{i=1}^{n} a_{\pi,i}\varpi_i$$

for some nonnegative integers $a_{\pi,i}$. The *Schläfli-Dynkin* diagram of π is the Dynkin diagram of $\mathfrak{g}$ with $a_{\pi,i}$ written over the node corresponding to α_i. For instance, in case $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$, we get:



5.4.19 Example Let $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$. Its root system $\Delta = \{\pm(\theta_i - \theta_j) \mid 1 \leq i < j \leq n\}$ has basis of simple roots given by $\Pi = \{\theta_1 - \theta_2, \dots, \theta_{n-1} - \theta_n\}$ (cf. Example 4.6.10). Recall that the Cartan-Killing form is a multiple of the trace form $(X, Y) \mapsto \mathrm{trace}(XY)$; the exact multiple is unimportant for the construction of the Dynkin diagram, so we may assume it is 1. Now it is easy to see that $H_{\theta_i - \theta_j} = E_{ii} - E_{jj}$, where E_{ab} is the $n \times n$ matrix with all entries zero, but the (a, b)-entry, which is equal to 1. It follows that $\langle \theta_a - \theta_b, \theta_c - \theta_d \rangle = \delta_{ac} + \delta_{bd} - \delta_{ad} - \delta_{bc}$.

Therefore the number of lines in the Dynkin diagram joining $\theta_i - \theta_{i+1}$ and $\theta_j - \theta_{j+1}$ is

$$4\frac{\langle \theta_i - \theta_{i+1}, \theta_j - \theta_{j+1} \rangle^2}{||\theta_i - \theta_{i+1}||^2 ||\theta_j - \theta_{j+1}||^2} = \begin{cases} 1 & \text{if } i+1 = j; \\ 0 & \text{if } i+1 < j. \end{cases}$$

We see again that Δ is of type A_n.

Note that

$$\langle \theta_1 + \cdots + \theta_i, \theta_j - \theta_{j+1} \rangle = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

It follows that the fundamental weight

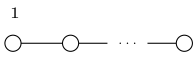
$$\varpi_i = \theta_1 + \cdots + \theta_i$$

for $i = 1, \dots, n$.

The vector representation π_1 has highest weight vector $\theta_1 = \varpi_1$ (Example 5.4.3), its dual has highest weight $-\theta_{n+1} = \varpi_n$, and the adjoint representation has highest weight equal to the highest root

$$\begin{aligned} \theta_1 - \theta_{n+1} &= \alpha_1 + \cdots + \alpha_n \\ &= 2\theta_1 + \theta_2 + \cdots + \theta_n \\ &= \varpi_1 + \varpi_n. \end{aligned}$$

It follows that the diagrams of π_1, π_1^* and ad are respectively



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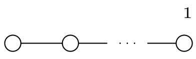
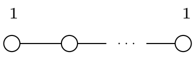


and



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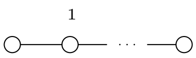
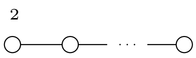
Further, the representations $\Lambda^2\pi_1$ and $S^2\pi_1$, with highest weights $\theta_1 + \theta_2$ and $2\theta_1$ are



and


.

Note that the fundamental representation π_i with highest weight ϖ_i is $\Lambda^i\pi_1$. Since

$$\theta_1 + \cdots + \theta_i = -\theta_{i+1} + \cdots - \theta_{n+1},$$

we see that $\Lambda^i\pi_1 = \Lambda^{n+1-i}\pi_1^*$.

5.5 Problems

1 Let $G = SU(n)$ and $V = \Lambda^k\mathbb{C}^n$.

a. List the weight vectors of this representation in terms of the canonical basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$.

b. Show there is only one highest weight vector (up to multiples) and deduce that V is irreducible.

2 Consider the Lie algebra of type B_2 ($\mathfrak{g} = \mathfrak{sp}(2,\mathbb{C}) = \mathfrak{so}(5,\mathbb{C})$; the corresponding simply-connected compact Lie group is $Sp(2)$).

a. Check that the fundamental weights $\varpi_1 = \alpha_1 + \alpha_2$ and $\varpi_2 = \frac{1}{2}\alpha_1 + \alpha_2$, where α_1 is the long simple root and α_2 is the short simple root.

b. For the representation π_λ with highest weight $\lambda = \varpi_1 + \varpi_2$, use Lemma 5.4.8 to verify that π_λ has precisely 12 weights as indicated in Fig. 5.1. Also, write down those weights explicitly in terms of the simple roots and the fundamental weights.

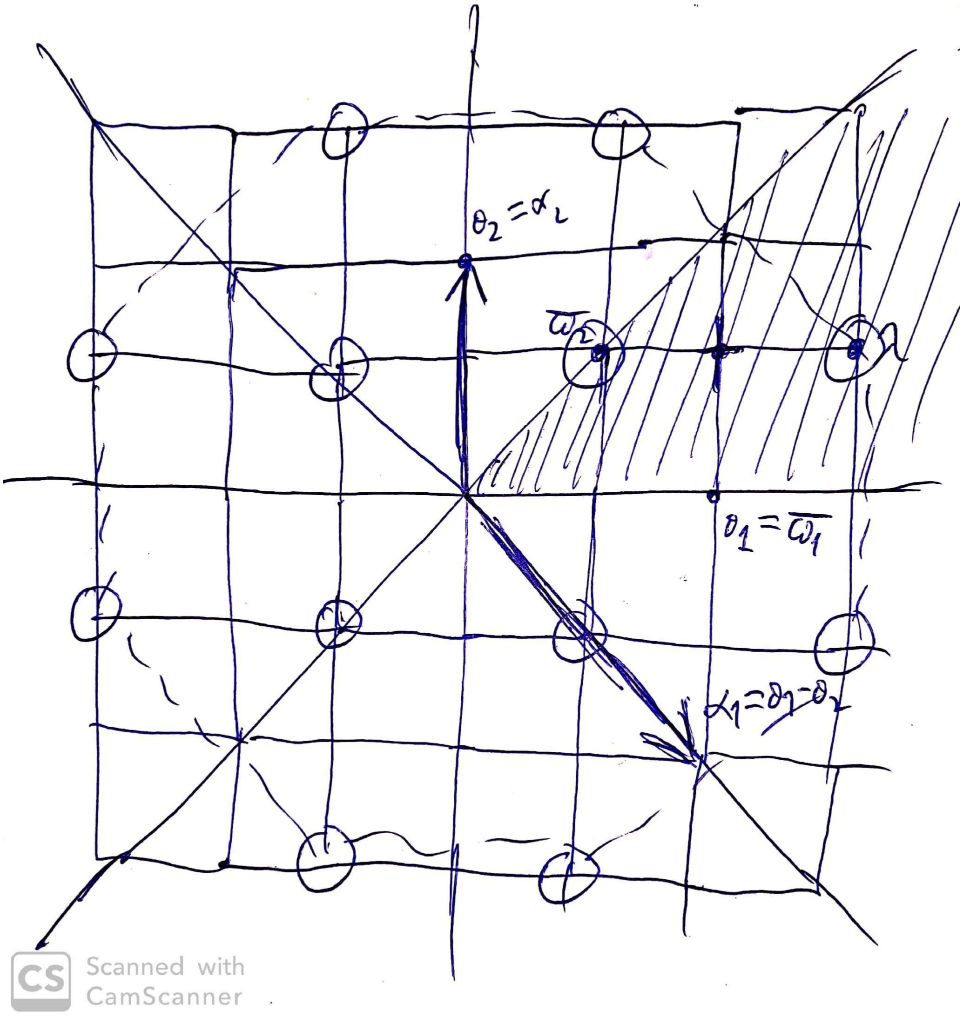


Figure 5.1: Weights of $\pi_{1,1}$.

The multiplicity of the 8 weights in the outer layer is 1, since these are in W-orbit of the highest weight. The multiplicity of the 4 weights in the inner layer could be 1 or higher; we need more work to determine it (cf. Problem 10 in Chapter 6).

3 Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ and $V = \Lambda^2(S^2\mathbb{C}^3)$. In this problem, we show that V is irreducible.

a. Check that the weights of V and their multiplicities are as follows:

$$\begin{array}{rc} 3\theta_i + \theta_j & 1 \\ 2\theta_i + 2\theta_j & 1 \\ 2\theta_i + \theta_j + \theta_k = \theta_i & 2 \end{array}$$

where i, j, k denote pairwise different indices.

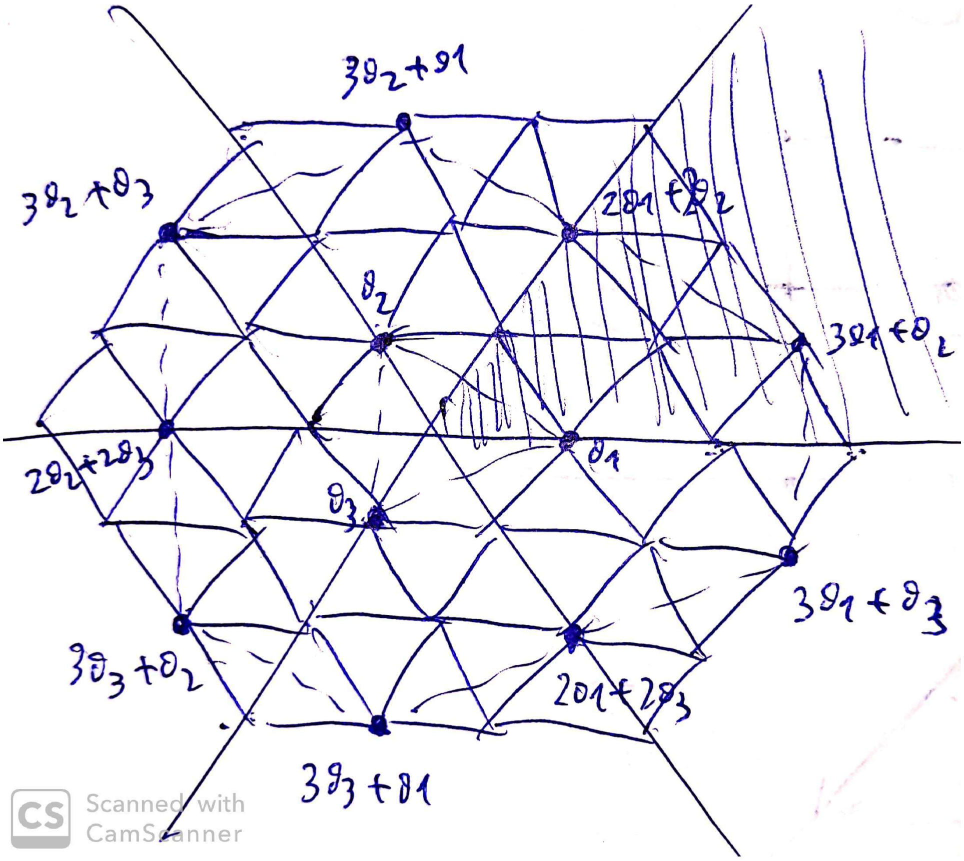


Figure 5.2: Weights of $\Lambda^2(S^2\mathbb{C}^3)$.

- b. Check that the highest weight is $3\theta_1 + \theta_2$ and that $e_1^2 \wedge (e_1e_2)$ is a highest weight vector.
- c. Denote by W the irreducible subrepresentation of V generated by $e_1^2 \wedge (e_1e_2)$ and deduce from Lemma 5.4.8 that W has the same weights as V.

We want to see that $W = V$. For that purpose, we need to show that the weight spaces of V of dimension 2 are contained in W.

- d. Check that $e_1^2 \wedge (e_2e_3)$ and $(e_1e_2) \wedge (e_1e_3)$ are linearly independent weight vectors of weight $2\theta_1 + \theta_2 + \theta_3$.
- e. Recall that $e_1^2 \wedge (e_1e_2) \in W$, and explain why $e_1^2 \wedge e_2^2 \in W$.
- f. Compute $E_{31}(e_1^2 \wedge (e_1e_2))$ and $E_{32}(e_1^2 \wedge e_2^2)$ (E_{ij} is the matrix with 1 in the (i,j)-entry and zero in the other entries).
- g. Use part (f) and the fact that the Weyl group acts on the weights preserving the multiplicities to conclude that $W = V$ and hence V is irreducible.

4 Let λ be a dominant algebraically integral weight and denote by π_λ the irreducible representation with highest weight λ. Show that 0 occurs as a weight of π_λ if and only if λ is a linear combination of roots.

5 Check that the fundamental weights are as indicated:
a. $\mathsf{B_n}$: $\varpi_1 = \theta_1, \ldots, \varpi_{n-1} = \theta_1 + \cdots + \theta_{n-1}$, $\varpi_n = \frac{1}{2}(\theta_1 + \cdots + \theta_n)$.
b. $\mathsf{C_n}$: $\varpi_1 = \theta_1, \ldots, \varpi_n = \theta_1 + \cdots + \theta_n$.
c. $\mathsf{D_n}$: $\varpi_1 = \theta_1, \ldots, \varpi_{n-2} = \theta_1 + \cdots + \theta_{n-2}$, $\varpi_{n-1} = \frac{1}{2}(\theta_1 + \cdots + \theta_{n-1} - \theta_n)$, $\varpi_n = \frac{1}{2}(\theta_1 + \cdots + \theta_{n-1} + \theta_n)$.

6 Consider the Lie algebra of type $\mathsf{G_2}$. Let α_1 denote the long simple root and let α_2 denote the short simple root. Check that $\varpi_1 = 2\alpha_1 + 3\alpha_2$ and $\varpi_2 = \alpha_1 + 2\alpha_2$.

7 Let π_n be $n+1$-dimensional irreducible unitary representation of $SU(n)$. Prove the following *Clebsch-Gordan* decompositions:

$$\pi_m \otimes \pi_n = \sum_{k=0}^{\min\{m,n\}} \pi_{m+n-2k},$$

$$S^2(\pi_n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \pi_{2n-4k},$$

$$\Lambda^2(\pi_n) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \pi_{2n-4k-2}.$$

The projections $\pi_m \otimes \pi_n \to \pi_{m+n}$ and $S^2(\pi_n) \to \pi_{2n}$ can be identified with "multiplication of polynomials".

8 Let $\mathfrak{g}$ be a complex semisimple Lie algebra. For each dominant integral weight λ, denote by π_λ the complex irreducible representation of $\mathfrak{g}$ with highest weight vector.
a. Prove that if a pure tensor is a highest weight vector of $\pi_\lambda \otimes \pi_\mu$ then it must be the highest weight vector of the Cartan component $\pi_{\lambda+\mu}$.
b. Show that the Cartan component $\pi_{\lambda+\mu}$ indeed occurs with multiplicity one in $\pi_\lambda \otimes \pi_\mu$.
c. Prove that if $\pi_\nu \subset \pi_\lambda \otimes \pi_\mu$, then $\nu = \lambda + \mu'$, where μ' is a weight of π_μ.

9 Let π be a complex representation of a Lie group G on a finite-dimensional vector space V. The representation π is said to be of *real type* if it comes from a representation of G on a real vector space by extension of scalars, namely, it admits a real form; and π is said to be of *quaternionic type* if it comes from a representation on a quaternionic vector space by restriction of scalars, that is, V is a quaternionic vector space viewed as complex and $\pi(g)$ is a quaternionic linear transformation for all $g \in G$. If π is neither of real type nor of quaternionic type, one can say it is of *complex type*.

a. Prove that π is of real type (resp. quaternionic type) if and only there exists a conjugate linear map $\epsilon : V \to V$ that commutes with the action of G and such that $\epsilon^2 = I$ (resp. $\epsilon^2 = -I$). (Hint: ϵ is complex conjugation, resp. multiplication by j).

b. Assume G is compact. Prove that π is of real type (resp. quaternionic type) if and only there exists a G-invariant nondegenerate symmetric (resp. skew-symmetric) bilinear form on V. (Hint: Consider $B(u, v) = \langle u, \epsilon v \rangle$, where $\langle , \rangle$ is a G-invariant Hermitian product on V and ϵ is an in part (a). Conversely, if B exists and one defines ϵ by this formula, one can check ϵ^2 is a definite, Hermitian operator on V, which can be then renormalized.)

c. Check that π is equivalent to its contragredient representation on V^* if and only if π is of real or quaternionic type.

d. In case V is irreducible, show that π must be exactly of one of those three types: real, quaternionic or complex.

CHAPTER 6

The Weyl formulae

In this chapter and in the following, our approach will be more in an analytic vein and focused more on the group level. We will recover and/or complete some results from earlier chapters.

We know from Chapter 1 that, for a complex (unitary) representation π of a compact Lie group G, the character function

$$\chi_\pi : G \to \mathbb{C},\ \chi_\pi(g) = \mathrm{trace}\,\pi(g)$$

completely determines the equivalence class of the representation. The character function, in its turn, is Ad-invariant, so it is determined by its values on a maximal torus T of G. Now

$$\chi_\pi(\exp H) = \sum_\mu m_\mu e^{\mu(H)}$$

where H belongs the Lie algebra $\mathfrak{t}$ of T, μ runs through the weights of π, and m_μ denotes the multiplicity of μ.

The character function is thus a complete invariant of the representation, but how to compute it? How to determine the mutiplicities of the weights? Even for a given π, this in general is labourious and indeed a daunting task. Well, of course we can restrict to irreducible representations, and from Chapter 5, we know that an irreducible π is completely determined by its highest weight μ_π, which (say, in case G is simply-connected) can be any dominant weight; and the multiplicity of μ_π is always 1; and the multiplicities of the weights are invariant under the action of Weyl group[1]. The other weights of π can be obtained from π by taking the integral weights in the convex hull of the W-orbit of μ_π, which are congruent to μ_π modulo the root lattice. Still, this does not tell much about the actual values of the multiplicities.

The main result in this chapter is the Weyl Character Formula. The WCF gives not only the multiplicities, but the full character of an irreducible representation π in terms of its highest weight μ_π. There are a few

[1] Ref?

other formulae for the character and for the multiplicities, as well generalizations and extensions of WCF (the most notable perhaps being Freudenthal's, Kostant's, Steinberg's, Harishchandras's, Demazure's, Weyl-Kac's, etc.); however, WCF remains the most "classical" one, and a very beautiful and most useful theoretical statement, although perhaps not the most suitable formula for practical computations.

In the case G is simply-connected, the formula can be stated as

$$\chi_\lambda(\exp H) = \frac{\mathrm{Alt}_{\mathrm{W}}\, e^{(\lambda+\delta)(H)}}{\mathrm{Alt}_W\, e^{\delta(H)}},$$

where $\lambda = \mu_\pi$ is the highest weight, δ is half the sum of the positive roots,

$$\mathrm{Alt}_W = \sum_{w \in W} \mathrm{sgn}(W) w$$

is the antisymmetrization operator, based on W, acting on characters of the maximal torus. The difficulty in using this formula is of course in computing the quotient between the linear combinations of exponentials. It is rather surprising that the numerator and the denominator of the formula involve a much smaller number of terms than its quotient (the character function itself).[2]

The basic idea of Hermann Weyl's proof relies on an integration formula, bearing his own name, which (together with the fact that the L^2-norm over G of the character of an irreducible representation is 1) implies that the L^2-norm over T of the function given by χ_λ multiplied by the the denominator of WCF is equal to $|W|$; this function is a linear combination of torus characters, which are themselves orthonormal, and thus it follows that the function involves exactly $|W|$ of such torus characters; on the other hand, we know that $e^{\lambda+\delta}$ must be a torus character appearing in the function, only once, and cannot cancel out; since its W-translates must also appear, each with multiplicity one, and there are exactly $|W|$ of them, it follows that these are all torus characters that contribute to the function, and hence we get the numerator of WCF.

Throughout this chapter we fix a compact connected Lie group G of dimension n and rank k and a maximal torus T of G. Also, let Δ be the root system of $\mathfrak{g}^c$ with respect to $\mathfrak{t}^c$ and fix an ordering of the roots. Then we have the (complex root) decomposition

$$\mathfrak{g}^c = \mathfrak{t}^c + \sum_{\alpha \in \Delta} \mathfrak{g}^c_\alpha$$

[2]Note that this formula can also be expressed as $\chi_\lambda \cdot \mathcal{A}_\delta = \mathcal{A}_{\lambda+\delta}$, where $\mathcal{A}_\mu = \mathrm{Alt}_W\, e^\mu$; this identity can be interpreted in the group algebra of the algebraically integral weights, written multiplicatively as formal linear combinations of e^μ.

and the real root decomposition

$$\mathfrak{g} = \mathfrak{t} + \sum_{\alpha\in\Delta^+} (\mathfrak{g}^c_\alpha + \mathfrak{g}^c_{-\alpha}) \cap \mathfrak{g}.$$

For brevity, write $\mathfrak{g}_{(\pm\alpha)} := (\mathfrak{g}^c_\alpha + \mathfrak{g}^c_{-\alpha}) \cap \mathfrak{g}$.

6.1 The analytic Weyl group

The Weyl group W was defined in Chapter 4 as the group generated by the reflections $\{s_\alpha\}$ associated to the roots $\alpha \in \Delta$. We can realize W on the group level as follows.

The normalizer of T in G is the closed subgroup

$$N_G(T) = \{g \in G \mid gTg^{-1} = g\}.$$

Of course, $N_G(T)$ acts by inner automorphisms on T. The kernel of this action is the centralizer

$$Z_G(T) = \{g \in G \mid gtg^{-1} = t \text{ for all } t \in T\}.$$

6.1.1 Lemma *The centralizer of any torus S in G is the union of all maximal tori containing S; in particular, $Z_G(S)$ is connected.*

It follows from the Lemma 6.1.1 that $Z_G(T) = T$. The analytically defined Weyl group is defined as

$$W_G = N_G(T)/T.$$

6.1.2 Proposition *Let G be a compact connected semisimple Lie group with Lie algebra $\mathfrak{g}$, and fix a maximal torus T with Lie algebra $\mathfrak{t}$, so that $\mathfrak{h} = \mathfrak{t}^c$ is a CSA of $\mathfrak{g}^c$. Then the analytic Weyl group $W_G = N_G(T)/T$ is canonically isomorphic to the algebraic Weyl group $W_\mathfrak{g}$ defined in Chapter 4 (as the group generated by the reflections on the singular hyperplanes).*

Proof. Let $\alpha \in \Delta^+$ and consider the reflection s_α. We will exhibit an element $n_\alpha \in N_G(T)$ such that $\mathrm{Ad}_{n_\alpha} = s_\alpha$ on $\mathfrak{h}$.

Recall the Lie algebra $\mathfrak{g}[\alpha] = \mathbb{C}\bar{H}_\alpha + \mathbb{C}E_\alpha + \mathbb{C}F_\alpha \cong \mathfrak{sl}(2,\mathbb{C})$. The element $X_\alpha := \frac{1}{2}(E_\alpha - F_\alpha)$ belongs to a compact real form $\mathfrak{su}(2)$. Note that $\mathrm{ad}_{X_\alpha} H = 0$ if $H \in \ker\alpha$ and $\mathrm{ad}^2_{X_\alpha} \bar{H}_\alpha = -\bar{H}_\alpha$. For $H \in \mathfrak{h}_\mathbb{R} = i\mathfrak{t}$, we compute

$$\begin{aligned}\mathrm{Ad}_{\exp(tX_\alpha)} H &= e^{t\,\mathrm{ad}_{X_\alpha}} \\ &= \sum_{k=0}^\infty \frac{t^k}{k!} \mathrm{ad}^k_{X_\alpha} H,\end{aligned}$$

which yields H, if $H \in \ker\alpha$, and

$$\begin{aligned}
\mathrm{Ad}_{\exp(tX_\alpha)}\bar{H}_\alpha &= \sum_{m=0}^{\infty}\frac{t^{2m}}{(2m)!}(-1)^m\bar{H}_\alpha + \sum_{m=0}^{\infty}\frac{t^{2m+1}}{(2m+1)!}(-1)^m\mathrm{ad}_{X_\alpha}\bar{H}_\alpha \\
&= \cos t\,\bar{H}_\alpha + \sin t\,\mathrm{ad}_{X_\alpha}\bar{H}_\alpha \\
&= -\bar{H}_\alpha
\end{aligned}$$

if $t=\pi$. Hence we can take $n_\alpha := \exp(\pi X_\alpha)$.

Note that W_G permutes the roots. Indeed, for $n \in N_G(T) = N_G(\mathfrak{t})$, $\alpha \in \Delta$ and $E_\alpha \in (\mathfrak{g}^c)_\alpha$, we have

$$\begin{aligned}
[H, \mathrm{Ad}_n E_\alpha] &= \mathrm{Ad}_n[\mathrm{Ad}_{n^{-1}}H, E_\alpha] \\
&= \alpha(\mathrm{Ad}_{n^{-1}}H)\mathrm{Ad}_n E_\alpha \\
&= (\mathrm{Ad}_n^t\alpha)(H)\,\mathrm{Ad}_n E_\alpha,
\end{aligned}$$

showing that $\mathrm{Ad}_n E_\alpha$ is a root vector for the root $\mathrm{Ad}_n^t\alpha$. It follows that W_G acts on the set of Weyl chambers. Since $W_\mathfrak{g}$ acts *simply* transitively on the set of Weyl chambers, we deduce that $W_G \cong W_\mathfrak{g}$. □

6.1.3 Corollary *The multiplicities of the weights of a representation of a complex semisimple Lie algebra* $\mathfrak{g}$ *are invariant under the Weyl group.*

6.1.4 Proposition *The inclusion* $T \to G$ *induces a homeomorphism between orbit spaces* $h : T/W \to G/\mathrm{Ad}$.

Proof. h is well-defined since W is a subquotient of G. It is surjective by the Maximal Torus Theorem 4.1.3. Let us prove injectivity. Suppose s, $t \in T$ are G-conjugate, $s = gtg^{-1}$ for some $g \in G$. Then $Z_G(s)^0$ is a compact connected Lie group and T, gTg^{-1} are two maximal tori in it so, again by 4.1.3, there is $z \in Z_G(s)^0$ such that $T = z(gTg^{-1})z^{-1} = (zg)T(zg)^{-1}$. Now $n := zg \in N_G(T)$ and $ntn^{-1} = s$. Finally, h is continuous and T/W is compact, so h is a homeomorphism. □

6.2 Invariant integration on a (compact) Lie group

On a compact Lie group, the bi-invariant Haar integral constructed in section 1.4 has a description in terms of volume forms. Let G be a Lie group. A differential form ω on G is called *left-invariant* if $L_g^*\omega = \omega$ for all $g \in G$. Similarly, and defines right-invariant differential forms.

Since a left-invariant form is determined by its value at 1, the space of left-invariant n-forms on G is one-dimensional for $n = \dim G$. For a nonzero left invariant n-form ω, consider the associated orientation on G. For each compactly supported continuous function f on G,

(6.2.1) $$f \mapsto \int_G f\omega$$

a positive continuous linear functional and hence yields a regular Borel measure on G. Since $L_g : G \to G$ is a diffeomorphism that preserves the orientation of G, we have $\int_G f\omega = \int_G L_g^*(f\omega) = \int_G (f \circ L_g)\omega$ for all $g \in G$, and then (6.2.1) is called a *left Haar integral* on G. In case G is compact, there is a unique left invariant n-form ω with $\int_G \omega = 1$, up to sign. Henceforth we will identify this n-form with the associated measure on G and denote them by dg.

What about right-invariance of (6.2.1)? For each $h \in G$, $R_h^* dg$ is a left-invariant form on G and thus we can write $R_h^* dg = \tilde{\lambda}(h)dg$ for a homomorphism $\tilde{\lambda} : G \to \mathbb{R}^\times$. Now $R_h : G \to G$ is a diffeomorphism which preserves (resp. reverses) the orientation if $\tilde{\lambda}(h) > 0$ (resp. if $\tilde{\lambda}(h) < 0$), so

$$\begin{aligned}\int_G f\,dg &= (\operatorname{sgn}\tilde{\lambda}(h)) \int_G R_h^*(f\,dg) \\ &= (\operatorname{sgn}\tilde{\lambda}(h)) \int_G (f \circ R_h)\tilde{\lambda}(h)dg \\ &= \int_G (f \circ R_h)\lambda(h)dg,\end{aligned}$$

where the homomorphism $\lambda = |\tilde{\lambda}| : G \to (0, +\infty)$ is called the *modular function* on G. In case G is a compact Lie group, $\lambda(G)$ is a compact subgroup of $(0, +\infty)$ and thus trivial. This show that the left Haar integral is also right-invariant, and hence we have a two-sided Haar integral on G.

Denote the Lie algebra of G by $\mathfrak{g}$. Fix a basis $X_1 \ldots, X_n$ of $\mathfrak{g}$. Let $\theta_1, \ldots, \theta_n$ be the dual basis of $\mathfrak{g}^*$. Then an explicit left-invariant n-form on G is given by $\omega = \theta_1 \wedge \ldots \wedge \theta_n$. In case G is compact and endowed a bi-invariant Riemannian metric, and we take $X_1 \ldots, X_n$ to be orthonormal, dg coincides with the *Riemannian volume form*, and the Haar measure coincides with the Riemannian measure.

Integration on compact homogeneous spaces

Assume now G is a compact connected Lie group and H is a connected closed subgroup and consider the homogeneous space G/H and the submersion $\pi : G \to G/H$. Recall the left multiplication $\lambda : G \times G/H \to G/H$ given by $\lambda(g, xH) = (gx)H$.

6.2.2 Proposition *There is a volume form $d(gH)$ on G/H, unique up to sign, which is left-invariant in the sense that $\lambda^* d(gH) = d(gH)$, and which satisfies $\int_{G/H} d(gH) = 1$.*

Proof. Let $m = \dim G/H$ and choose a nonzero $\eta_1 \in \Lambda^m(T_1(G/H))$. The isotropy action of H at the basepoint $1H$ is the homomorphism $H \to GL(T_1(G/H))$ given by $h \mapsto d\lambda_h$. Now η_1 can be extended by left-translation to a differential m-form η on G/H if and only η_1 is invariant under the isotropy action at $1H$, namely, invariant under

$$T_h := \Lambda^m(d(\lambda_h)_{1H}) : \Lambda^m(T_{1H}(G/H)) \to \Lambda^m(T_{1H}(G/H))$$

for all $h \in H$. Since $\Lambda^m(T_{1H}(G/H))$ is one-dimensional, there is a homomorphism $c : H \to \mathbb{R}^\times$ such that $T_h(\eta_1) = c(h)\eta_1$. By assumption H is compact and connected, so its image under c is a compact and connected subgroup of $\mathbb{R}^\times$. It follows that $c \equiv 1$ and thus η_1 is invariant under the isotropy action.

Since G/H is compact, $\int_{G/H} \eta$ is finite. We put $d(gH) = \frac{1}{\int_{G/H}\eta}\eta$. □

Let dg, dh and $d(gH)$ be the normalized volume forms on G, H and G/H, respectively. Consider the orthogonal projection $T_1(G) \to T_1(H)$; pull-back $dh_1 \in \Lambda^k(T_1H)$ ($k = \dim H$) to an element $d\tilde{h}_1 \in \Lambda^k(T_1G)$ and extend this to a left-invariant k-form $d\tilde{h}$ on G. Then $d\tilde{h}|_H = dh$ and $d\tilde{h} \wedge \pi^* d(gH)$ is a left-invariant n-form ($n = \dim G$) on G, and hence $c \cdot d\tilde{h} \wedge \pi^* d(gH) = dg$ for some nonzero constant c, which we may assume positive.

6.2.3 Lemma $c = 1$.

Proof. Consider $\pi : G \to G/H$. The level sets of π ("fibers") are the left cosets of H. Locally, G splits as $H \times G/H$, and dg splits accordingly. For a continuous function $f : G \to \mathbb{R}$, we use a partition of unity to decompose $\int_G f$ and apply Fubini's Theorem to each piece. We get

$$\int_G f(g)\, dg = c \int_{G/H} \left(\int_H f(gh)\, dh \right) d(gH). \tag{6.2.4}$$

In particular, taking $f \equiv 1$ and using that G, G/H and H all have volume 1 (by assumption) yields $c = 1$. □

Formula (6.2.4) is a kind of Fubini's theorem for the Lie group G. In particular, by applying it to a function of the form $f = F \circ \pi$ for some $F : G/H \to \mathbb{R}$, we get that

$$\int_G F \circ \pi = \int_{G/H} F;$$

in other words, the measure on G/H is the push-forward measure of the measure on G.

Riemannian interpretation

We continue considering the compact homogeneous manifold G/H and the projection $\pi : G \to G/H$, which is a submersion. Denote the Lie algebras of G and H by $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Equip $\mathfrak{g}$ with an Ad-invariant inner product and let $\mathfrak{h}^\perp$ denote the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{h} + \mathfrak{h}^\perp$ is an $\mathrm{Ad}|_H$-invariant decomposition. The differential $d\pi_1 : T_1G \to T_{1H}G/H$ has kernel $\mathfrak{h} \subset \mathfrak{g} \cong T_1G$, and hence induces an identification $\mathfrak{h}^\perp \cong T_{1H}(G/H)$. Now G is endowed with a bi-invariant Riemannian metric and G/H is endowed with a G-invariant Riemannian metric. Also H inherits a bi-invariant metric.

Let $H_1, \ldots, H_k$ be an orthonormal basis of $\mathfrak{h}$ and let $X_1, \ldots, X_m$ be an orthonormal basis of $\mathfrak{h}^\perp$. Let $\bar{\phi}_1, \ldots, \bar{\phi}_k$ be the dual basis of $\mathfrak{h}^*$, and let $\phi_1, \ldots, \phi_k, \theta_1, \ldots, \theta_m$ be the dual basis of $\mathfrak{g}^*$. Then $\phi_i|_{\mathfrak{h}} = \bar{\phi}_i$ for $i = 1, \ldots, k$. Now the Riemannian volume forms of G, H and G/H satisfy $d\mathrm{vol}_G = \phi_1 \wedge \ldots, \phi_k \wedge \theta_1 \wedge \ldots \wedge \theta_m$, $d\mathrm{vol}_H = \bar{\phi}_1 \wedge \ldots \wedge \bar{\phi}_k$ and $\theta_1 \wedge \ldots \wedge \theta_m = \pi^* d\mathrm{vol}_{G/H}$. For a continuous function $f : G \to \mathbb{R}$, formula (6.2.4) takes the form

$$\int_G f(g) d\mathrm{vol}_G = \int_{G/H} \left(\int_H f(gh) d\mathrm{vol}_H \right) d\mathrm{vol}_{G/H}.$$

In particular,

$$\mathrm{vol}(G) = \mathrm{vol}(H) \cdot \mathrm{vol}(G/H).$$

6.3 Weyl integration formula

For each $\alpha \in \Delta$, $X_\alpha \in \mathfrak{g}^c_\alpha$ and $\exp H \in T$ with $H \in \mathfrak{t}$, we have

$$\begin{aligned} \mathrm{Ad}_{\exp H} X_\alpha &= e^{\mathrm{ad}_H} X_\alpha \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} (\mathrm{ad}_H)^j X_\alpha \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} \alpha(H)^j X_\alpha \\ &= e^{\alpha(H)} X_\alpha. \end{aligned}$$

In particular, if $\exp H = \exp H'$ then $e^\alpha(H) = e^\alpha(H')$ (equivalently, $\alpha(H) - \alpha(H') \in 2\pi\sqrt{-1}\mathbb{Z}$). It follows there is a homomorphism $\xi_\alpha : T \to S^1$, which we call the *character* associated to α, such that $d\xi_\alpha = \alpha$:

$$\begin{array}{ccc} \mathfrak{t} & \xrightarrow{\alpha} & \sqrt{-1}\mathbb{R} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle e} \\ T & \xrightarrow[\xi_\alpha]{} & S^1 \end{array}$$

An element $g \in G$ is called *regular* if it is contained in exactly one maximal torus (equivalently, $\dim Z_G(g) = k$) and *singular* otherwise. Denote by G_{reg} (resp. G_{sing}) the set of regular elements of G and put $T_{reg} = G_{reg} \cap T$

($T_{sing} = G_{sing} \cap T$). Note that G_{reg} and G_{sing} are Ad-invariant subsets of G, and

$$T_{sing} = \cup_{\alpha \in \Delta^+} \ker \xi_\alpha.$$

The identity component of $\ker \xi_\alpha$ is a codimension one torus of T, but in general $\ker \xi_\alpha$ does not need to be connected.

The first lemma is the qualitative assertion of the fact that "almost all" elements of G are regular.

6.3.1 Lemma *$\dim G_{sing} \leq n-3$ in the sense that $G \setminus G_{sing}$ is contained in a finite union of sets, each of which is the image of a smooth map $M^{n-3} \to G$.*

Proof. If $g \in G_{sing}$, then g is Ad-conjugate to an element in T_{sing}. But T_{sing} is a finite union of codimension one subtori, so $\dim T_{sing} \leq k-1$. Since $\dim Z_G(g) \geq k+2$, the adjoint orbit of g has dimension $\dim G/Z_G(g) \leq n-k-2$. Now $\dim G_{sing} = \dim G \cdot T_{sing} \leq (n-k-2)+(k-1) = n-3$. □

Next we consider the homogeneous manifold G/T. Identify $T_{1T}(G/T) = \mathfrak{g}/\mathfrak{t} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_{(\pm\alpha)}$; this is also $\mathfrak{t}^\perp$, with respect to an Ad-invariant inner product on $\mathfrak{g}$.

6.3.2 Lemma *The map $\psi : G/T \times T_{reg} \to G_{reg}$ given by $\psi(gT, t) = gtg^{-1}$ is a smooth covering of cardinality $|W|$.*

Proof. Note that ψ is well-defined (use T is Abelian) and G-equivariant. We compute for $X \in \mathfrak{t}^\perp$ and $H \in \mathfrak{t}$:

$$\begin{aligned} dL_{t^{-1}} d\psi_{(1T,t)}(X, dL_t(H)) &= \frac{d}{ds}|_{s=0} L_{t^{-1}} \psi(e^{sH}, te^{sH}) \\ &= \frac{d}{ds}|_{s=0} t^{-1} e^{sX} te^{sH} e^{-sX} \\ &= \mathrm{Ad}_{t^{-1}} X + H - X. \end{aligned} \tag{6.3.3}$$

Now $dL_{t^{-1}} d\psi_{(1T,t)}(\mathrm{id} \times dL_t) : \mathfrak{g}/\mathfrak{t} \times \mathfrak{t} \to \mathfrak{g}$ has block form

$$\begin{array}{c} \\ \mathfrak{t}^\perp \\ \mathfrak{t} \end{array} \begin{array}{c} \begin{array}{cc} \mathfrak{t}^\perp & \mathfrak{t} \end{array} \\ \left(\begin{array}{c|c} \mathrm{Ad}_{t^{-1}} - I & 0 \\ \hline 0 & I \end{array} \right) \end{array} \tag{6.3.4}$$

where

$$(\mathrm{Ad}_{t^{-1}} - I) \otimes \mathbb{C}|_{\mathfrak{g}_{(\pm\alpha)}} = \begin{pmatrix} e^{-\alpha(H)} - 1 & 0 \\ 0 & e^{\alpha(H)} - 1 \end{pmatrix}. \tag{6.3.5}$$

Here $t = \exp H \in T_{reg}$, so $\alpha(H) \notin 2\pi\sqrt{-1}\mathbb{Z}$, and this implies ψ is a local diffeomorphism at $(1T, t)$, and hence everywhere by equivariance.

Note that ψ is the restriction of a proper map $G/T \times T \to G$, so it is also proper. Therefore ψ is a covering. Given $t \in T_{reg} \subset G_{reg}$, we have $\psi(gT, s) = t$ if and only if $s = g^{-1}tg$, that is, $g \in N_G(T)$. Hence $\psi^{-1}(t)$ has exactly $|W|$ elements, and so does the preimage of any element in G', by equivariance. □

We equip G and T with normalized Haar measures (of total mass 1) and consider the induced invariant measure on G/T, as discussed in section 6.2 (also of total mass 1). The following integration formula shows that integration on G can be done first along the adjoint orbits, then along the maximal torus, and finally we divide by the order of the Weyl group since every adjoint orbit meets the maximal torus in that number of points.

6.3.6 Theorem (Weyl Integration Formula) *Let G be a compact connected Lie group, and let T be a maximal torus of G. Let dg and dt be Haar measures on G and T, respectively, of total mass 1, and let f be an* Ad*-invariant continuous function on G. Then*

$$\int_G f(g)\,dg = \frac{1}{|W|}\int_T f(t)v(t)\,dt,$$

where

$$\begin{aligned} v(t) &= |\Delta(t)|^2 \\ &= \prod_{\alpha\in\Delta^+} |e^{\frac{\alpha(H)}{2}} - e^{-\frac{\alpha(H)}{2}}|^2 \\ &= 2^{|\Delta|}\prod_{\alpha\in\Delta^+} \sin^2\frac{\alpha(H)}{2i} \end{aligned}$$

and

$$\begin{aligned} \Delta(t) &= \prod_{\alpha\in\Delta^+} (1-\xi_{-\alpha}(t)) \\ &= \prod_{\alpha\in\Delta^+} (1-e^{-\alpha(H)}) \end{aligned}$$

for $H \in \mathfrak{t}$, $\exp H = t \in T$. Further, $v(t)$ is the volume of the adjoint orbit in G through $t \in T$ with respect to a bi-invariant metric on G that induces the normalized Haar measure.

Proof. Proof. Using Lemmata 6.3.1 and 6.3.2, and Fubini's Theorem, we can write

$$\begin{aligned}
\int_G f(g)\,dg &= \int_{G_{reg}} f(g)\,dg \\
&= \frac{1}{|W|}\int_{G/T\times T_{reg}} f(\psi(gT,t))\psi^* dg \\
&= \frac{1}{|W|}\int_{G/T\times T_{reg}} f(t)(\det d\psi_{(gT,t)})\pi_1^* d(gT)\wedge \pi_2^* dt \\
&= \frac{1}{|W|}\int_T f(t)\left[\int_{G/T} \det d\psi_{(gT,t)} d(gT)\right] dt, \qquad (6.3.7)
\end{aligned}$$

where we have used $\pi_1^* d(gT)\wedge\pi_2^* dt$ is the volume form of $G/T\times T$, for $\pi_1 : G/T\times T\to G/T$ and $\pi_2 : G/T\to T$ the projections.

For fixed $t\in T_{reg}$, the term in brackets records the m-dimensional volume of the adjoint orbit through t ($m = \dim G - \operatorname{rank} G$). Moreover, the map $G/T\to \mathrm{Ad}_G(t)$, $gT\mapsto gtg^{-1}$ is an equivariant map and hence its Jacobian determinant $\det d\psi_{(gT,t)}$ is constant along G/T. Indeed, $\psi(gT,t) = \mathrm{Inn}_g\psi(1T,t)$ so

$$\begin{aligned}
d\psi_{(gT,t)} &= d(\mathrm{Inn}_g)_t\circ d\psi_{(1T,t)} \\
&= d(L_{gtg^{-1}})\circ \mathrm{Ad}_g\circ d(L_{t^{-1}})_t\circ d\psi_{(1T,t)}
\end{aligned}$$

Recall that left translations preserve the volume forms and $\det \mathrm{Ad}_g = 1$ since G is connected. Therefore

$$\det d\psi_{(gT,t)} = \det d\psi_{(1T,t)} =: v(t).$$

Finally, by identifying $\mathfrak{t}^\perp\times\mathfrak{t}\cong\mathfrak{t}^\perp\oplus\mathfrak{t}=\mathfrak{g}$ and noting that $\pi_1^* d(gT)\wedge \pi_2^* dt|_{(1T,t)}$ coincides with dg_1, it follows from (6.3.3), (6.3.4) and (6.3.5) that (recall $\alpha(H)\in i\mathbb{R}$)

$$v(t) = \prod_{\alpha\in\Delta^+} |e^{-\alpha(H)}-1|^2$$

for $t=\exp H$. □

6.4 Lattices

Let G be a compact connected Lie group and let T be a maximal torus in G. Then we have the system of roots Δ. We choose a Weyl chamber, so this specifies the positive roots Δ^+ and the simple roots $\Pi = \{\alpha_1,\dots,\alpha_n\}$. Recall that the roots take real values on $\mathfrak{t}_\mathbb{R} := i\mathfrak{t}$ and the Cartan-Killing form of $\mathfrak{g}$ restricts to a positive definite inner product on that space.

We also recall the weight lattice L_{wt} and the root lattice L_{rt} that were defined in section 5.4. These are W-invariant lattices in $\mathfrak{t}_{\mathbb{R}}^*$ with $L_{rt} \subset L_{wt}$. There are dual lattices

$$L_{wt}^* := \{H \in \mathfrak{t}_{\mathbb{R}} \ : \ \lambda(H) \in \mathbb{Z} \text{ for all } \lambda \in L_{wt}\}$$

(the *fundamental lattice*) and

$$L_{rt}^* := \{H \in \mathfrak{t}_{\mathbb{R}} \ : \ \lambda(H) \in \mathbb{Z} \text{ for all } \lambda \in L_{rt}\}$$

(the *central lattice*) in $\mathfrak{t}_{\mathbb{R}}$, which are also W-invariant, with $L_{wt}^* \subset L_{rt}^*$. The fundamental lattice has an integral basis given by $(\bar{H}_{\alpha_1}, \ldots, \bar{H}_{\alpha_n})$, where $\bar{H}_{\alpha_i}$ is the coroot associated to the simple root α_i (see Chapter 4); for this reason, it is also called the *coroot lattice*.

The lattices L_{wt}^* and L_{rt}^* do not depend on the group G, but only on its Lie algebra. On the other hand, the *unit lattice*

$$I_G = \exp_G^{-1}(1) \subset \mathfrak{t}$$

depends on G. In fact, if $p : G' \to G$ is covering of Lie groups, then there is a commutative diagram

$$\begin{array}{ccc} & \mathfrak{g} & \\ {\scriptstyle \exp_{G'}} \swarrow & & \searrow {\scriptstyle \exp_G} \\ G' & \xrightarrow{\quad p \quad} & G \end{array}$$

so that $I_{G'}$ is a sublattice of I_G and

$$I_G/I_{G'} = \exp_{G'}^{-1}(\ker p)/\exp_{G'}^{-1}(1) \xrightarrow[\exp_{G'}]{\cong} \ker p. \tag{6.4.1}$$

6.4.2 Example (i) Let $G = SU(2)$ and let T be the subgroup consisting of diagonal matrices. Then $\mathfrak{t}_{\mathbb{R}} = \{H_a = \operatorname{diag}(a, -a) \ : \ a \in \mathbb{R}\}$, and $\Delta = \{\pm\alpha\}$, where $\alpha = 2\theta$ and $\theta(H_a) = a$. Now $\lambda = n\theta \in L_{wt}$ if and only if $2\frac{\langle \lambda, \alpha \rangle}{||\alpha||^2} = n \in \mathbb{Z}$, so $L_{wt} = \{n\theta \mid n \in \mathbb{Z}\}$ and $L_{rt} = \{2n\theta \mid n \in \mathbb{Z}\}$. Moreover, $H_a \in L_{wt}^*$ if and only if $\theta(H_a) \in \mathbb{Z}$, so $L_{wt}^* = \mathbb{Z} \cdot H_1$ and the coroot is H_1; also, $L_{rt}^* = \mathbb{Z} \cdot H_{\frac{1}{2}}$. Finally, $I_G = \{\operatorname{diag}(2\pi\sqrt{-1}a, -2\pi\sqrt{-1}a) \ : \ a \in \mathbb{Z}\} = 2\pi\sqrt{-1}L_{wt}^*$. The Cartan-Killing form of $\mathfrak{g} = \mathfrak{su}(2)$ is $\beta(X, Y) = -4 \operatorname{trace}(X^*Y)$, so $||H_a||^2 = -8a^2$, but we do not need this information in the calculations above.

(ii) Let $G = SO(3)$ and

$$\mathfrak{t} = \left\{ \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ : \ a \in \mathbb{R} \right\}, \quad \theta\left(\begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = ia.$$

Then $\Delta = \{\pm\alpha\}$ where $\alpha = \theta$. Now $L_{wt} = \{\frac{n}{2}\theta \mid n \in \mathbb{Z}\}$ and $L_{rt} = \{n\theta \mid n \in \mathbb{Z}\}$. Therefore the coroot of α is $\left(\begin{smallmatrix} 0 & 2i & 0 \\ -2i & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, which spans L_{wt}^* over $\mathbb{Z}$. Also $\left(\begin{smallmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ spans L_{rt}^* over $\mathbb{Z}$, and

$$I_G = \left\{ \begin{pmatrix} 0 & -2\pi n & 0 \\ 2\pi n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : n \in \mathbb{Z} \right\} = 2\pi\sqrt{-1}L_{rt}^*.$$

Below it is convenient to use the following terminology: the *character group* of the torus T is the multiplicative group $\mathrm{Hom}(T, S^1)$; its elements are called *characters* of T.

The linear functionals in

$$\mathcal{A}_G := 2\pi\sqrt{-1}I_G^* = \left(\frac{1}{2\pi\sqrt{-1}}I_G\right)^* \subset \sqrt{-1}\mathfrak{t}_{\mathbb{R}}^* = \mathfrak{t}^*$$

are sometimes called *analytically integral*. The reason is, if $\lambda \in \mathcal{A}_G$, then $\lambda(H) \in 2\pi\sqrt{-1}\mathbb{Z}$ for all $H \in I_G$, that is, $e^{\lambda(H)} = 1$ for all $H \in \ker \exp_G$. Therefore there is a character ξ_λ associated to λ, namely $\lambda = d\xi_\lambda$ and

$$\begin{array}{ccc} \mathfrak{t} & \xrightarrow{\lambda} & \sqrt{-1}\mathbb{R} \\ {\scriptstyle \exp_G}\downarrow & & \downarrow{\scriptstyle e} \\ T & \xrightarrow[\xi_\lambda]{} & S^1 \end{array}$$

is commutative. Conversely, if $\lambda \in \mathfrak{t}^*$ is a linear functional for which there exists a character $\xi_\lambda : T \to S^1$ making the diagram above to commute, namely $\lambda = d\xi_\lambda$, then $\lambda \in \mathcal{A}_G$.

Note that also $\mathcal{A}_G$ is W-invariant. We deduce from the above that there is a W-equivariant homomorphism $\lambda \mapsto \xi_\lambda$ from the additive group $\mathcal{A}_G$ to the character group of T.

By comparison, the linear functionals in L_{wt} are sometimes called *algebraically integral*. Recall that L_{wt} is spanned over $\mathbb{Z}$ by the fundamental weights.

6.4.3 Proposition *In general, it is true that:*

(6.4.4) $$L_{rt} \subset \mathcal{A}_G \subset L_{wt}.$$

Equivalently, by dualization,

(6.4.5) $$2\pi\sqrt{-1}L_{wt}^* \subset I_G \subset 2\pi\sqrt{-1}L_{rt}^*.$$

Moreover, if φ is a unitary representation of G on V and λ is a weight of the complexification π of $d\varphi$, then $\lambda \in \mathcal{A}_G$.

Proof. We first prove the last assertion. Let λ be a weight of π, as in the statement, with weight space V_λ. Then $\varphi|_T$ acts on V_λ by a character ξ_λ, with $d\xi_\lambda = \lambda$ on $\mathfrak{t}$. Therefore $\lambda \in \mathcal{A}_G$.

We apply the preceding result to the adjoint representation: the roots are weights of ad, and $\mathrm{ad} = d(\mathrm{Ad})$; therefore every root is analytically integral, proving the first inclusion in (6.4.4).

The second inclusion in (6.4.4) is equivalent to the first one in (6.4.5). Consider a coroot $\bar{H}_\alpha$ for some $\alpha \in \Delta$, which is a generator of L_{wt}^*. Then there is a Lie subalgebra $\mathfrak{g}[\alpha] \cong \mathfrak{sl}(2,\mathbb{C})$ of $\mathfrak{g}$ and an associated homomorphism $\psi : SU(2) \to G$ (since $SU(2)$ is simply-connected) such that $(d\psi)^c(H_1) = \bar{H}_\alpha$, where $H_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Now

$$\exp(2\pi\sqrt{-1}\bar{H}_\alpha) = \exp(d\psi(2\pi\sqrt{-1}H_1)) = \psi(\exp 2\pi\sqrt{-1}H_1) = \psi(1) = 1,$$

proving that $2\pi\sqrt{-1}\bar{H}_\alpha \in I_G$. □

6.4.6 Theorem *The fundamental group and the center of G are respectively given by the finite Abelian groups:*

$$\pi_1(G) = L_{wt}/\mathcal{A}_G \quad \textit{and} \quad Z(G) = \mathcal{A}_G/L_{rt}.$$

Further, for the simply-connected (compact) Lie group $\tilde{G}$ with Lie algebra $\mathfrak{g}$, we have:

$$\mathcal{A}_{\tilde{G}} = L_{wt} \quad \textit{and} \quad Z(\tilde{G}) = L_{wt}/L_{rt}.$$

Proof. (Sketch) The hard part is to show that $\mathcal{A}_{\tilde{G}} = L_{wt}$, that is, every algebraically integral form is analytically integral on $\tilde{G}$. We can prove this up to the existence part of the Theorem of the Highest Weight 5.4.9.[3]

Recall that L_{wt} is spanned over $\mathbb{Z}$ by the fundamental weights $\varpi_1, \ldots, \varpi_n$; it suffices to check that these are analytically integral on $\tilde{G}$. By 5.4.9, ϖ_i is the highest weight of a fundamental representation π_i of $\mathfrak{g}$; one can also construct π_i directly. Since $\tilde{G}$ is simply-connected, π_i comes from a representation of $\tilde{G}$. It follows from the last assertion of Proposition 6.4.3 that ϖ_i is analyically integral.

The result about $\pi_1(G)$ follows from this. Apply (6.4.1) to the universal covering $\tilde{G} \to G$ to get

$$\pi_1(\tilde{G}) \cong I_G/I_{\tilde{G}} \cong I_{\tilde{G}}^*/I_G^* = \mathcal{A}_{\tilde{G}}/\mathcal{A}_G = L_{wt}/\mathcal{A}_G.$$

To prove the result about $Z(G)$, note that $\exp : 2\pi\sqrt{-1}L_{rt}^* \to \ker \mathrm{Ad} = Z(G)$ is a surjective homomorphism with kernel I_G. Therefore

$$Z(G) \cong 2\pi\sqrt{-1}L_{rt}^*/I_G \cong \mathcal{A}_G/L_{rt}.$$

The rest follows. □

[3]One can construct the fundamental representations. Or prove the Borel-Weil Theorem (from Peter-Weyl).

6.4.7 Remark A geometric arguement to show that $\mathcal{A}_{\tilde{G}} = L_{wt}$ goes as follows. Let $\tilde{T}$ be maximal torus of $\tilde{G}$ (which covers the maximal torus T of G).

The first claim is that the map $\iota_\# : \pi_1(\tilde{T}) \to \pi_1(\tilde{G})$ induced by the inclusion $\iota : \tilde{T} \to \tilde{G}$ is surjective. In fact, equip $\tilde{G}$ with a bi-invariant Riemannian metric. Using a result of Cartan in Riemannian geometry, every loop in $\tilde{G}$ at the identity can be deformed to a shortest curve in the same homotopy class of loops, by a shortening process, which will then be a geodesic. Now this geodesic is a closed one-parameter group $\gamma(t) = \exp(tX)$ where $X \in \mathfrak{g}$ and $\gamma(0) = \gamma(1) = 1$. By the Theorem of the Maximal Torus, we can find $g \in \tilde{G}$ such that $\mathrm{Ad}_g X \in \mathfrak{t}$ and then $\gamma_1(t) = g\exp(tX)g^{-1} = \exp(t\mathrm{Ad}_g X)$ is a loop in $\tilde{T}$. Since $\tilde{G}$ is connected, there is a path $g(s)$ joining 1 to g. Therefore $\gamma_s(t) = g(s)\exp(tX)g(s)^{-1}$ is a homotopy between $\gamma_0 = \gamma$ and γ_1.

Now $\pi_1(\tilde{G}) \cong \pi_1(\tilde{T})/\ker\iota_\#$, and we need to identify $\pi_1(\tilde{T})$ and $\ker\iota_\#$.

Since $\exp^{\tilde{T}} : \mathfrak{t} \to \tilde{T}$ is a covering homomorphism and $\mathfrak{t}$ is simply-connected, $\pi(\tilde{T}) \cong \ker\exp^{\tilde{T}} = \ker\exp^{\tilde{G}} = I_{\tilde{G}}$.

Next, note that $2\pi\sqrt{-1}L_{wt}^* \subset \ker\iota_\#$. Indeed, if $\bar{H}_\alpha$ is a coroot, then the loop $t \mapsto \exp(2\pi\sqrt{-1}t\bar{H}_\alpha)$ is homotopically trivial in $\tilde{G}$. The reason is because it is the continuous image of the homotopically trivial loop $t \mapsto \exp(2\pi\sqrt{-1}t\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right))$ in $SU(2)$ (recall $SU(2)$ is simply-connected) under the homomorphism $SU(2) \to \tilde{G}$ associated to α.

It turns out that we have equality $2\pi\sqrt{-1}L_{wt}^* = \ker\iota_\#$, but proving that requires more effort and we skip the details. Putting all together yields

$$\begin{aligned}\pi_1(\tilde{G}) \cong \pi_1(\tilde{T})/\ker\iota_\# = I_{\tilde{G}}/2\pi\sqrt{-1}L_{wt}^* &\cong \frac{1}{2\pi\sqrt{-1}}I_{\tilde{G}}/L_{wt}^* \\ &\cong L_{wt}^{**}/\left(\frac{1}{2\pi\sqrt{-1}}I_{\tilde{G}}\right)^* \cong L_{wt}/\mathcal{A}_G.\end{aligned}$$

6.5 Weyl character formula

Let $\delta = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha \in L_{wt}$.

6.5.1 Lemma *For all $w \in W$, we have $w \cdot \delta - \delta \in L_{rt}$.*

Proof. We first claim that if α is a simple root, then s_α sends α to $-\alpha$ (clear) and permutes all of the other positive roots. To check the second assertion, let $\beta \in \Delta^+$, $\beta \neq \alpha$, and write $\beta = m_1\alpha_1 + \cdots + m_n\alpha_n$, where $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, $m_1, \ldots, m_n \geq 0$, and, say, $\alpha = \alpha_1$. Since $\beta \in \Delta^+ \setminus \{\alpha_1\}$, there is a coefficient $m_i > 0$ with $i \geq 2$. Now

$$s_\alpha(\beta) = \left(m_1 - 2\frac{\langle\beta,\alpha_1\rangle}{||\alpha_1||^2}\right)\alpha_1 + m_2\alpha_2 + \cdots + m_n\alpha_n.$$

and $m_i > 0$, so $s_\alpha(\beta) \in \Delta^+$, as wished.

It follows from the above that, for a simple root α, $s_\alpha(\delta) = \delta - \alpha \in \delta + L_{rt}$. Since any element $w \in W$ can be written as a product of reflections on simple roots, we are done. $\square$

For λ analytically integral and dominant, put

$$\chi_\lambda(t) = \frac{\sum_{w\in W}(\operatorname{sgn} w)\xi_{w\cdot(\lambda+\delta)-\delta}(t)}{\Delta(t)} \tag{6.5.2}$$

where

$$\Delta(t) = \prod_{\alpha\in\Delta^+}(1 - \xi_{-\alpha}(t))$$

and $t \in T_{reg}$. It follows from Lemma 6.5.1 that $w \cdot (\lambda + \delta) - \delta$ is analytically integral, so χ_λ is well-defined.

6.5.3 Lemma *χ_λ is W-invariant and hence extends to a continuous* Ad*-invariant function on G.*

Proof. For $H \in \mathfrak{t}$, note that

$$\begin{aligned}\Delta(\exp H) &= \prod_{\alpha\in\Delta^+}(1 - e^{-\alpha(H)}) \\ &= e^{-\delta(H)} \prod_{\alpha\in\Delta^+}(e^{\frac{\alpha(H)}{2}} - e^{-\frac{\alpha(H)}{2}}).\end{aligned}$$

Therefore we can write

$$\chi_\lambda(\exp H) = \frac{N(H)}{D(H)}, \tag{6.5.4}$$

where

$$N(H) = \sum_{w\in W}(\operatorname{sgn} w)e^{w\cdot(\lambda+\delta)(H)}$$

and

$$\begin{aligned}D(H) &= e^{\delta}(H)\Delta(\exp H) \\ &= \prod_{\alpha\in\Delta^+}(e^{\frac{\alpha(H)}{2}} - e^{-\frac{\alpha(H)}{2}}).\end{aligned} \tag{6.5.5}$$

For $w' \in W$, we compute that

$$\begin{aligned}N(w'H) &= \sum_{w\in W}(\operatorname{sgn} w)e^{w\cdot(\lambda+\delta)(w'H)} \\ &= \sum_{w\in W}(\operatorname{sgn} w)e^{w'^{-1}w\cdot(\lambda+\delta)(H)} \\ &= \sum_{w\in W}(\operatorname{sgn} w'w)e^{w\cdot(\lambda+\delta)(w'H)} \\ &= (\operatorname{sgn} w')N(H),\end{aligned}$$

which means that N is W*-alternating*. Next we show that also D is W-alternating, and this will imply that χ is W-invariant. For that purpose, it suffices to check that $D(\exp s_\alpha H) = -D(\exp H)$ for a reflection s_α on a positive root α. But this follows immediately from formula (6.5.5) and the fact that s_α sends α to $-\alpha$ and permutes the other roots. □

6.5.6 Lemma *$\{\chi_\lambda \mid \lambda$ is dominant and analytically integral$\}$ is an orthonormal set in $L^2(G)$.*

Proof. Apply the Weyl Integration Formula to $f = |\chi_\lambda|^2$

$$\begin{aligned}
\int_G |\chi_\lambda(g)|^2\, dg &= \frac{1}{|W|}\int_T |\chi_\lambda(t)|^2 |\Delta(t)|^2\, dt \\
&= \frac{1}{|W|}\int_T \left|\sum_{w\in W} \operatorname{sgn}(w)\xi_{w\cdot(\lambda+\delta)-\delta}(t)\right|^2 dt \\
&= \frac{1}{|W|}\sum_{w,w'\in W} \operatorname{sgn}(ww')\int_T \xi_{w\cdot(\lambda+\delta)-\delta}(t)\xi_{w'\cdot(\lambda+\delta)-\delta}(t)\, dt \\
&= 1.
\end{aligned}$$

In the last equality, we have used the fact that $\xi_{w\cdot(\lambda+\delta)-\delta}$ for $w \in W$ are distinct characters of T, as $\lambda + \delta$ sits in the open Weyl chamber, and W acts simply transitively on the Weyl chambers. Further, $\xi_\nu : T \to \mathbb{C}$ is an irreducible representation of T, so the character theory of Schur says that

$$\int_T \xi_\nu(t)\bar{\xi}_{\nu'}(t)\, dt = \begin{cases} 1 & \text{if } \nu = \nu', \\ 0 & \text{if } \nu \neq \nu'. \end{cases}$$

Similarly, one checks that $\int_G \chi_\lambda \bar{\chi}_{\lambda'} = 0$ for $\lambda \neq \lambda'$. □

Now let χ be the character of an irreducible representation π of G:

$$\chi(t) = \operatorname{trace}\pi(t) = \sum_{\mu\in\Lambda_\pi} m_\mu \xi_\mu(t)$$

where Λ_π is the weight system and m_μ is the multiplicity of $\mu \in \Lambda_\pi$. Then, for $H \in \mathfrak{t}$,

$$\chi(\exp H) = \sum_{\mu\in\Lambda_\pi} m_\mu e^{\mu(H)}$$

and (recall (6.5.5))

$$D(H)\chi(\exp H) = \sum_{\nu\in\mathcal{A}_G} n_\nu e^{(\nu+\delta)(H)}, \tag{6.5.7}$$

where $n_\nu \in \mathbb{Z}$ and $n_\nu = 0$ but for finitely many ν. Since the left hand-side of (6.5.7) is W-alternating, we have

$$n_{w\cdot\nu} = \operatorname{sgn}(w)n_\nu$$

for all $w \in W$. In particular, $n_\nu = 0$ if $\nu + \delta$ lies in a wall of Weyl chamber. Denote by $(L_{wt}^{++})_{reg}$ the *strictly* dominant algebraically integral weights, i. e. those algebraically integral weights that sit in the open Weyl chamber. Since W acts simply transitively on the Weyl chambers,

$$\begin{aligned}\chi(\exp H)D(H) &= \sum_{w\in W} \sum_{\substack{\nu\in\mathcal{A}_G \\ \nu+\delta\in(L_{wt}^{++})_{reg}}} n_{w\cdot\nu} e^{w\cdot(\nu+\delta)(H)} \\ &= \sum_{\substack{\nu\in\mathcal{A}_G \\ \nu+\delta\in(L_{wt}^{++})_{reg}}} n_\nu \sum_{w\in W} \operatorname{sgn}(w) e^{w\cdot(\nu+\delta)(H)}.\end{aligned}$$

Multiplying through by $e^{-\delta(H)}$ and noting that $\nu + \delta$ is strictly dominant if and only if ν is dominant, we obtain

$$\chi(\exp H)\Delta(\exp H) = \sum_{\nu\in\mathcal{A}_G^{++}} n_\nu \chi_\nu(\exp H)\Delta(\exp H).$$

or

$$\chi(t)\Delta(t) = \sum_{\nu\in\mathcal{A}_G^{++}} n_\nu \chi_\nu(t)\Delta(t)$$

for $t \in T$.

Since χ is the character of a complex *irreducible* representation, $\int_G |\chi|^2 = 1$. Now the Weyl Integration Formula and Lemma 6.5.6 give

$$\begin{aligned}1 &= \frac{1}{|W|}\int_T |\chi|^2|\Delta|^2 \\ &= \frac{1}{|W|}\sum_{\nu,\,\nu'\in\mathcal{A}_G^{++}} n_\nu n_{\nu'} \int_T \chi_\nu\bar{\chi}_{\nu'}|\Delta|^2 \\ &= \sum_{\nu,\,\nu'\in\mathcal{A}_G^{++}} n_\nu n_{\nu'} \int_G \chi_\nu\bar{\chi}_{\nu'} \\ &= \sum_{\nu\in\mathcal{A}_G^{++}} n_\nu^2.\end{aligned}$$

We deduce that there exists $\lambda \in \mathcal{A}_G^{++}$ such that $n_\lambda = \pm 1$ and $n_\nu = 0$ for $\nu \neq \lambda$. It follows that

$$\chi \cdot \Delta = \pm\chi_\lambda \cdot \Delta.$$

In order to decide the sign, note that

$$e^\delta(\Delta \circ \exp) = e^\delta + \sum_{\nu\prec\delta} a_\nu e^\nu$$

and

$$\chi \circ \exp = m_{\lambda_\pi} e^{\lambda_\pi} + \sum_{\mu\prec\lambda_\pi} m_\mu e^\mu,$$

where λ_π is the highest weight of π, so

$$e^\delta(\chi \circ \exp)(\Delta \circ \exp) = m_{\lambda_\pi} e^{\lambda_\pi + \delta} + \text{lower order terms.} \tag{6.5.8}$$

On the other hand,

$$\begin{aligned} e^\delta(\chi_\lambda \circ \exp)(\Delta \circ \exp) &= (e^{\lambda+\delta} + \text{lower order terms}) \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}) \\ &= e^{\lambda+\delta} + \text{lower order terms.} \end{aligned} \tag{6.5.9}$$

Comparing (6.5.8) and (6.5.9), we finally obtain $\lambda = \lambda_\pi$ and $n_\lambda = m_{\lambda_\pi} = 1$. We summarize our findings in the celebrated

6.5.10 Theorem (Weyl Character Formula) *The character of a complex irreducible representation π of a compact connected semisimple Lie G is given by χ_λ, the* Ad*-invariant extension of (6.5.2), where λ is the highest weight of π.*

6.5.11 Corollary *The multiplicity of the highest weight vector of a complex irreducible representation of G is* 1*.*

6.5.12 Corollary *Two complex irreducible representations of G are equivalent if and only if they have the same highest weight.*

Proof. Due to WCF, the highest weight determines the character function of the representation which, in turn, determines the representation, owing to the character theory of Frobenius-Schur. □

6.5.13 Corollary (Weyl Denominator Formula)

$$D = \sum_{w \in W} \operatorname{sgn}(w) e^{w \cdot \delta}.$$

Proof. Just take the trivial representation $\pi = I$, corresponding to $\lambda = 0$, in the WCF. Since $\chi_0 = 1$, the desired formula is immediately obtained from (6.5.4). □

It follows from the Weyl denominator formula 6.5.13 that the character χ_λ of the complex irreducible representation of G with highest weight λ satisfies

$$\chi_\lambda(\exp H) = \frac{\sum_{w \in W} \operatorname{sgn}(w) e^{w \cdot (\lambda+\delta)(H)}}{\sum_{w \in W} \operatorname{sgn}(w) e^{w \cdot \delta(H)}} \tag{6.5.14}$$

for all $H \in \mathfrak{t}$.

6.5.15 Corollary (Weyl dimension formula) *The degree of a complex irreducible representation π with highest weight λ is*

$$\deg \pi = \prod_{\alpha \in \Delta^+} \frac{\langle \lambda + \delta, \alpha \rangle}{\langle \delta, \alpha \rangle}.$$

Proof.

$$\begin{aligned}\deg\pi &= \chi_\lambda(1)\\ &= \lim_{t\to 0}\chi_\lambda(\exp t\delta).\end{aligned}$$

We compute the numerator of (6.5.14) as

$$\begin{aligned}\sum_{w\in W}\operatorname{sgn}(w)e^{\langle w\cdot(\lambda+\delta),t\delta\rangle} &= \sum_{w\in W}\operatorname{sgn}(w)e^{\langle t(\lambda+\delta),w\cdot\delta\rangle}\\ &= \prod_{\alpha\in\Delta^+} e^{\langle t(\lambda+\delta),\frac{\alpha}{2}\rangle} - e^{\langle t(\lambda+\delta),\frac{\alpha}{2}\rangle},\end{aligned}$$

where we have used (6.5.5) and Corollary 6.5.13. Therefore

$$\begin{aligned}\deg\pi &= \lim_{t\to 0}\prod_{\alpha\in\Delta^+}\frac{e^{\frac{t}{2}\langle\lambda+\delta,\alpha\rangle} - e^{\frac{t}{2}\langle\lambda+\delta,\alpha\rangle}}{e^{\frac{t}{2}\langle t\delta,\alpha\rangle} - e^{\frac{t}{2}\langle\delta,\alpha\rangle}}\\ &= \prod_{\alpha\in\Delta^+}\lim_{t\to 0}\frac{\sin\frac{t}{2i}\langle\lambda+\delta,\alpha\rangle}{\sin\frac{t}{2i}\langle\delta,\alpha\rangle}\\ &= \prod_{\alpha\in\Delta^+}\frac{\langle\lambda+\delta,\alpha\rangle}{\langle\delta,\alpha\rangle},\end{aligned}$$

as wished. □

6.6 Problems

1 Identify the calculation of the character χ_k and and the computation of $\int_{S^3}|\chi_k|^2$ in Theorem 1.5.8 as instances of the Weyl character formula and the Weyl integration formula.

2 Consider $G = SU(2)$ and identify the fundamental weight $\varpi_1 = \theta$. Apply the Weyl dimension formula to check that $\dim\pi_{n\varpi_1} = n+1$.

3 *a.* Consider left-multiplication in $GL(n,\mathbb{R})$ and compute that $\det L_x = (\det x)^n$ for $x \in GL(n,\mathbb{R})$.
b. Check that $|\det x|^{-n}\,dx$ is a two-sided Haar measure on $GL(n,\mathbb{R})$, where dx denotes Lebesgue measure on $M(n,\mathbb{R})$.

4 Find a root α for $Sp(2)$ such that $\ker\xi_\alpha$ has two connected components.

5 Let $\mathfrak{g}$ be a complex semisimple Lie algebra of rank n, $\mathfrak{h}$ a CSA and Δ the root system, with an ordering of the roots. Define $\delta := \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ and show that $\delta = \sum_{i=1}^n \varpi_i \in L_{wt}^+$.

6 Let $\mathfrak{g}$ be a complex semisimple Lie algebra fix a CSA and an ordering of the roots. Show that if λ is a dominant algebraically integral weight and ϖ is a fundamental weight, then $\deg \pi_{\lambda+\varpi} > \deg \pi_\lambda$ (where π_μ denotes the irreducible representation with highest weight μ). Deduce that the non-trivial representations of $\mathfrak{g}$ of smallest dimension are fundamental representations.

7 Let G be a compact connected semisimple Lie group.

a. Show that G admits representations of arbitrarily high dimension.

b. Show that G admits (if at all) at most finitely many nonequivalent representations in any given dimension.

8 Let G be a compact connected semisimple Lie group, fix a maximal torus and an ordering of the roots. Assume that $\delta = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ is analytically integral and denote by π_δ the irreducible representation of G with highest weight δ. Prove that $\deg \pi_\delta = 2^{|\Delta^+|}$.

9 Check that $\delta = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ is analytically integral for $SU(n)$ and $Sp(n)$ and $SO(2n)$, but not for $SO(2n+1)$.

10 Use the Weyl dimension formula to show that the multiplicity of the weights of the representation $\pi_{1,1}$ of $Sp(2)$, lying in the inner polygon of the diagram of weights, as given in Problem 2 of Chapter 5, is 2.

11 Let G be a simply-conncted compact connected simple Lie group of rank two and denote by ϖ_1, ϖ_2 the fundamental weights. Let $\pi_{a,b}$ denote the irreducible representation of G of highest weight $a\varpi_1 + b\varpi_2$ (a, $b \in \mathbb{Z}$). Prove that $\deg \pi_{a,b}$ is as indicated, in the following cases:

a. $SU(3)$: $\frac{1}{2}(a+1)(b+1)(a+b+2)$;

b. $Sp(2)$: $\frac{1}{6}(a+1)(b+1)(a+b+2)(2a+b+3)$;

c. G_2: $\frac{1}{120}(a+1)(b+1)(a+b+2)(2a+b+3)(a+3b+4)(2a+3b+5)$.

12 Consider $G = SU(3)$ and $V = S^q(\mathbb{C}^3)$. Identify the highest weight of V and use the formula from Problem 11 to prove that V is irreducible for all $q \geq 1$.

13 Consider the contraction map

$$c : S^2(\mathbb{C}^3) \otimes \mathbb{C}^{3*} \to \mathbb{C}^3$$

given by $uv \otimes w^* \mapsto w^*(u)v + w^*(v)u$.

a. Show that this map is well defined and $SU(3)$-equivariant. Deduce that $\ker c$ is G-invariant.

b. Check that $e_1^2 \otimes e_3^* \in \ker c$, where $\{e_i\}$ is the canonical basis of $\mathbb{C}^3$ and $\{e_i^*\}$ is the dual basis. Deduce that $\pi_{2\theta_1-\theta_3}$ is a subrepresentation of $\ker c$ (in the notation of Problem 11, $\pi_{2\theta_1-\theta_3} = \pi_{2,1}$).

c. Use Problem 11 to prove that

$$S^2(\mathbb{C}^3) \otimes \mathbb{C}^{3*} = \pi_{2,1} \oplus \mathbb{C}^3.$$

CHAPTER 7

The Peter-Weyl Theorem

The Peter-Weyl Theorem is a basic result in Harmonic Analysis that essentially describes the regular representation of a compact topological group G by giving an orthonormal basis of the space of square integrable functions on G, in terms of the matrix coefficients of all irreducible representations of G. It was proved by Hermann Weyl and his student Fritz Peter in 1927, building on previous work of Frobenius and Schur for finite groups.

In this chapter, all compact topological groups are assumed Hausdorff.

7.1 Statement and proof of the theorem

Let G be a compact topological group. Fix a bi-invariant Haar measure on G of total volume 1. We consider the convolution algebra $(C(G), *)$ of complex valued continuous functions on G where

$$f_1 * f_2(x) = \int_G f_1(xg^{-1}) f_2(g)\, dg = \int_G f_1(g) f_2(g^{-1}x)\, dg.$$

We also consider the Banach spaces $L^1(G)$, $L^2(G)$ and $L^\infty(G)$, note the inequalities

$$||f||_1 \leq ||f||_2 \leq ||f||_\infty,$$

(use Cauchy-Schwarz for the first one) and recall that $L^2(G)$ is a Hilbert space whose associated inner product with be denoted by $(,)$. Let $\hat{G}$ be the set of equivalence classes of irreducible finite-dimensional complex representations of G. For $\pi \in \hat{G}$ and u, v vectors in the underlying representation space V_π, we recall the *matrix coefficient*

$$f_{u,v}(x) = \langle u, \pi(x)v \rangle \qquad (x \in G).$$

Note that $f_{u,v}$ is really a matrix coefficient of π^*. We have $f_{u,v} \in L^2(G)$.

7.1.1 Proposition (Schur orthogonality relations) *We have:*

a. If $\pi \in \hat{G}$ and u, v, u', $v' \in V_\pi$, then

$$(f_{u,v}, f_{u',v'}) = \frac{1}{\dim V_\pi} \langle u, u' \rangle \overline{\langle v, v' \rangle}.$$

b. If π, $\pi' \in \hat{G}$ are inequivalent, and u, $v \in V_\pi$, u', $v' \in V_{\pi'}$, then

$$(f_{u,v}, f_{u',v'}) = 0.$$

The *character* of $\pi \in \hat{G}$ is the smooth function

$$\chi_\pi : G \to \mathbb{C}, \qquad \chi_\pi(x) = \operatorname{trace} \pi(x).$$

We recall that equivalent representations have the same character and that $\chi_\pi(gxg^{-1}) = \chi_\pi(x)$. We also have $\chi_{\pi \oplus \pi'} = \chi_\pi + \chi_{\pi'}$ and $\chi_{\pi \otimes \pi'} = \chi_\pi \chi_{\pi'}$. A consequence of the Schur orthogonality relations is:

7.1.2 Corollary *For π, $\pi' \in \hat{G}$, $(\chi_\pi, \chi_{\pi'})$ is 1 if π and π' are equivalent and 0 otherwise. It follows that two representations of G are equivalent if and only if they have the same character.*

Let $\{\pi^k\}$ be a maximal set of mutually nonequivalent irreducible unitary finite dimensional representations of G acting on spaces V_k of dimension d_k. For each index k, we choose an orthonormal basis $\{e_1^k, \ldots, e_{d_k}^k\}$ of V_k and consider the matrix coefficients $\pi_{ij}^k(x) = \overline{f_{e_i^k, e_j^k}(x)}$. The Schur orthogonality relations say that

$$\{d_k^{1/2} \pi_{ij}^k\}_{i,j,k}$$

is an orthonormal set in $L^2(G)$. Since $C(G)$ is dense in $L^2(G)$, the following theorem implies that this is an orthonormal basis of $L^2(G)$; in particular, $\hat{G}$ is at most countable (note that the theorem says that the *set* is dense, not its span).

7.1.3 Theorem (Peter-Weyl, first version) *Let G be a compact topological group. Then the set of matrix coefficients of representations in $\hat{G}$ is dense in $C(G)$ in the L^∞-norm.*

If $\varphi \in C(G)$ and $f \in L^1(G)$, set $T_\varphi f = \varphi * f$. Plainly, we have:

7.1.4 Lemma *If $\varphi \in C(G)$ and $f \in L^1(G)$, then $||T_\varphi f||_\infty \leq ||\varphi||_\infty ||f||_1$. In particular, T_φ is a bounded operator on $L^1(G)$.*

7.1.5 Lemma *If $\varphi \in C(G)$ and $f \in L^2(G)$, then T_φ is a compact bounded operator on $L^2(G)$ with norm bounded by $||\varphi||_\infty$. If, in addition, $\varphi(x^{-1}) = \overline{\varphi(x)}$ for $x \in G$, then T_φ is self-adjoint.*

Proof. We have

$$||T_\varphi f||_2 \leq ||T_\varphi f||_\infty \leq ||\varphi||_\infty ||f||_1 \leq ||\varphi||_\infty ||f||_2.$$

To prove compactness of T_φ, we need to show that it takes bounded sequences to sequences with convergent subsequences. Let (f_n) be a sequence bounded in $L^2(G)$. Then it is bounded in $L^1(G)$, say $||f_n|| \leq M$ for some $M > 0$ and all n, and it follows from Lemma 7.1.4 that $(T_\varphi f_n)$ is (equi)bounded in $L^\infty(G)$. Since convergence in L^2-norm is implied by convergence in L^∞-norm, the result will follow from the Arzelà-Ascoli theorem if we can show that this sequence is also equicontinuous. Compactness of G implies that φ is uniformly continuous; this means that given $\epsilon > 0$ there exists a neighborhood U of 1 such that $|\varphi(gx) - \varphi(x)| < \epsilon/M$ for $g \in U$. Now

$$\begin{aligned} |T_\varphi f_n(gx) - T_\varphi f_n(x)| &= \left| \int_G \left(\varphi(gxh^{-1}) - \varphi(xh^{-1}) \right) f_n(h)\, dh \right| \\ &\leq \int_G \left| \varphi(gxh^{-1}) - \varphi(xh^{-1}) \right| |f_n(h)|\, dh \\ &< \frac{\epsilon}{M} ||f_n||_1 \\ &\leq \epsilon, \end{aligned}$$

proving equicontinuity of $(T_\varphi f_n)$. The last assertion is easy. □

Introduce the *right-regular representation* R of G on $L^2(G)$ by the rule $(R_g f)(x) = f(xg)$. Similarly, the *left-regular representation* L of G on $L^2(G)$ is given by $(L_g f)(x) = f(g^{-1}x)$. Both actions of G on $L^2(G)$ are continuous, in the sense that $G \times L^2(G) \to L^2(G)$ is continuous,[1] and preserve the L^2-norm.

7.1.6 Lemma *If $\varphi \in C(G)$ and $\lambda \in \mathbb{C}$, the λ-eigenspace $V_\lambda = \{f \in L^2(G) \mid T_\varphi f = \lambda f\}$ is invariant under the right-regular representation of G.*

Proof. In fact T_φ commutes with the right-regular representation, as is easily checked. □

Proof of Theorem 7.1.3. Let $f \in C(G)$. We shall prove that for any $\epsilon > 0$ there exists a matrix coefficient f' of an irreducible representation such that $||f - f'||_\infty < \epsilon$. Compactness of G implies that f is uniformly continuous, so $||R_g f - f||_\infty < \epsilon/2$ if $g \in U$ for some neighborhood U of 1. Choose a continuous $\varphi : G \to [0, +\infty)$ with support contained in U, total integral 1 and satisfying $\varphi(g^{-1}) = \varphi(g)$ for all g. One easily checks that

$$||T_\varphi f - f||_\infty < \frac{\epsilon}{2}. \tag{7.1.7}$$

[1] Need a check!

Owing to Lemma 7.1.5, T_φ is a compact self-adjoint operator on $L^2(G)$. By the spectral theorem, there is Hilbert direct sum decomposition of $L^2(G) = \overline{\oplus_\lambda V_\lambda}$, where V_λ is the λ-eigenspace of T_φ, and $\dim V_\lambda < \infty$ if $\lambda \neq 0$. Let f_λ be the component of f in V_λ, and choose $q > 0$ such that $\sum_{0<|\lambda|<q} |f_\lambda|^2 < \epsilon^2/(4||\varphi||_\infty^2)$. Then

$$||\sum_{0<|\lambda|<q} f_\lambda||_1 < \epsilon/(2||\varphi||_\infty). \tag{7.1.8}$$

Finally, put $f'' = \sum_{|\lambda|\geq q} f_\lambda$ and $f' = T_\varphi(f'')$. By using (7.1.7), (7.1.8) and Lemma 7.1.4, we can write

$$\begin{aligned} ||f - f'||_\infty &\leq ||f - T_\varphi f||_\infty + ||T_\varphi(f - f'')||_\infty \\ &< \frac{\epsilon}{2} + ||\varphi||_\infty \frac{\epsilon}{2||\varphi||_\infty} \\ &= \epsilon. \end{aligned}$$

There remains only to identify f' as a matrix coefficient of some irreducible finite-dimensional representation of G. Note that f', f'' lie in $\sum_{|\lambda|\geq q} V_\lambda$, and the latter is a finite-dimensional (since the eigenvalues of a compact self-adjoint operator on a Hilbert space can only accumulate at 0), R-invariant subspace of $L^2(G)$, so the result follows from the following lemma. □

7.1.9 Lemma *If $f \in L^2(G)$ lies in a finite-dimensional, R-invariant subspace of $L^2(G)$, then f is a matrix coefficient of some irreducible finite-dimensional representation of G.*

Proof. Let $V \subset L^2(G)$ be a R-irreducible subspace containing f. The linear functional $u \in V \mapsto u(1) \in \mathbb{C}$ can be represented as $u(1) = \overline{\langle u, v \rangle}$ for some $v \in V$. In particular, $f(x) = R_x f(1) = \overline{\langle R_x f, v \rangle} = \langle v, R_x f \rangle$ is a matrix coefficient. □

7.1.10 Theorem (Peter-Weyl, second version) *Let G be a compact topological group. Then there is a $G \times G$-equivariant isometric identification (Hilbert space direct sum)*

$$L^2(G) = \overline{\bigoplus_{\pi \in \hat{G}} V_\pi \otimes V_{\pi^*}},$$

induced by $u \otimes v^* \mapsto \sqrt{\dim V_\pi}\, f_{u,v}$ for u, $v \in V_\pi$, where $G \times G$ acts on $L^2(G)$ by the left- and right-regular representation.

Proof. Note that

$$f_{\pi(g)u,v}(x) = \langle \pi(g)u, \pi(x)v \rangle = \langle u, \pi(g^{-1}x)v \rangle = (L_g f_{u,v})(x)$$

for all $\pi \in \hat{G}$, u, $v \in V_\pi$ and g, $x \in G$. Similarly, $f_{u,\pi(g)v} = R_g f_{u,v}$. This proves the equivariance of the map. Its isometric property is derived from the Schur orthogonality relations. □

7.1.11 Corollary *For each $x \in G \setminus \{1\}$, there exists an irreducible representation π of G such that $\pi(x) \neq I$.*

Proof. Let $f \in C(G)$ be such that $f(x) = 1$ and $f(1) = 0$. By the Peter-Weyl Theorem, there exists a matrix coefficient f' of an irreducible representation $\pi \in \hat{G}$ such that $f'(x) \neq f'(1)$. It follows that $\pi(x) \neq \pi(1) = I$. □

7.1.12 Corollary *The set $\{\chi_\pi \mid \pi \in \hat{G}\}$ of character functions of irreducible representations of G is an orthonormal basis of the space $L^2(G)^{\mathrm{Ad}}$ of* Ad*-invariant L^2-functions on G.*

Proof. We already know that the set in the statement is orthonormal, so it is enough to prove that it spans a dense subspace of $L^2(G)^{\mathrm{Ad}}$. Indeed, given $f \in L^2(G)^{\mathrm{Ad}}$ and $\epsilon > 0$, owing to Peter-Weyl we can write

$$||f - \sum_{i \in I} f_i|| < \epsilon,$$

where f_i is a matrix coefficient of an irreducible representation π_i and I is a finite set. Since f is Ad-invariant, we get

$$||f - \sum_{i \in I} \int_G f_i|| < \epsilon.$$

Next we remark that, for a matrix coefficient of the form $f_{u,v}$ for u, $v \in V_\pi$ and $\pi \in \hat{G}$, we have that

$$\int_G f_{u,v}(gxg^{-1})\, dg = \left\langle \int_G \pi(gxg^{-1})u\, dg, v \right\rangle = \frac{\langle u, v\rangle}{\dim V_\pi} \chi_\pi(x),$$

in view of the calculations in Proposition 1.5.3. Hence each $\int_G f_i$ is a multiple of χ_{π_i}, and we are done. □

7.2 Highest weight classification

7.2.1 Theorem *Let G be a compact connected semisimple Lie group. Then there is a bijective correspondence between complex irreducible representations and dominant analytically integral weights, given by taking the highest weight.*

Proof. Let π be a complex irreducible representation of G. Since G is connected, π is completely characterized by the induced representation of its Lie algebra $\mathfrak{g}$. Take a CSA of the complexification and choose a positive Weyl chamber. The highest weight μ_π of π is a dominant weight by the results in Chapter 5 (see Theorem 5.4.9), and it is analytically integral by Proposition 6.4.4; also, it completely determines the representation, see

Remark 5.4.11. To prove the theorem, it only remains to construct a complex irreducible representation for a given dominant analytically integral weight $\lambda \in \mathcal{A}_G^{++}$.

Consider the function χ_λ is in (6.5.2). We know from Lemma 6.5.6 that $\int_G |\chi_\lambda|^2 = 1$, so χ_λ is a non-zero Ad-invariant function in $C(G)$, and also that $\int_G \chi_\lambda \bar{\chi}_{\lambda'} = 0$ for $\lambda \neq \lambda'$.

Now the Weyl character formula says that the character of any complex irreducible representation π of G is of the form χ_{μ_π} for some dominant analytically integral weigth μ_π, namely, the highest weight of π. If χ_λ is not the character of an irreducible representation, then $\lambda \neq \mu_\pi$ for all $\pi \in \hat{G}$ and therefore $\int_G \chi_\lambda \bar{\chi}_{\mu_\pi} = 0$ for all $\pi \in \hat{G}$, by the above. However, χ_λ cannot be orthogonal to all χ_{μ_π}, owing to Corollary 7.1.12. □

7.3 Other applications of Peter-Weyl

For our first application, compare Corollary 1.5.6.

7.3.1 Proposition *Let G_1 and G_2 be compact topological groups. Then the complex irreducible representations of $G_1 \times G_2$ are precisely those of the form $\pi(g_1, g_2) = \pi_1(g_1) \otimes \pi_2(g_2)$, where $g = (g_1, g_2) \in G$ and $\pi_i \in \hat{G}$ for $i = 1, 2$.*

Proof. We have $L^2(G_1 \times G_2) = L^2(G_1) \otimes L^2(G_2)$ where $f_1(x_1) \otimes f_2(x_2)$ maps to $f_1(x_1)f_2(x_2)$, and the functions of the latter form span a dense subspace of $L^2(G_1 \times G_2)$. □

7.3.2 Proposition *A compact topological group is Abelian if and only if all of its irreducible representations are one-dimensional.*

Proof. Assume all irreducible representation of a compact topological group G are one-dimensional and let (G, G) be the commutator subgroup of G, that is, the subgroup generated by the elements of the form $aba^{-1}b^{-1}$ for a, $b \in G$. Any (irreducible, finite-dimensional) representation of G, being one-dimensional, is trivial on (G, G). By Corollary 7.1.11, $(G, G) = \{1\}$, that is, G is Abelian.

The converse statement follows from Schur's Lemma. □

Recall that a topological group is said to have no small subgroups if there is a neighborhood of the identity which contains no subgroups other than the trivial one.

7.3.3 Theorem (von Neumann) *Let G be a compact topological group. Then the following assertions are equivalent:*

(a) G does not posses small subgroups.
(b) G admits a faithful representation.
(c) G is a Lie group.

Proof. (a) implies (b): Let U be a neighborhood of 1 which does not contain nontrivial subgroups. There is $f \in C(G)$ such that $f(1) \neq 0$ and whose support is contained in U. By Peter-Weyl, there is a matrix coefficient f' of some $\pi \in \hat{G}$ such that $f'(x) \neq f'(1)$ if $x \notin U$. Clearly f' is constant on $\ker \pi$, necessarily equal to $f'(1)$. Hence $\ker \pi$ is contained in U and thus trivial.

(b) implies (c): G is isomorphic to a closed subgroup of $GL(n, \mathbb{R})$, and any closed subgroup of a Lie group is a Lie group itself, by a theorem of É. Cartan.

(c) implies (a): Let $U = \exp(\frac{1}{2}V)$, where $\exp$ is a diffeomorphism of V onto its image. Then one shows that U does not contain nontrivial subgroups of G. □

Recall that for a compact topological group, the space

$$\mathcal{M}_G = \oplus_{\pi \in \hat{G}} V_\pi \otimes V_\pi^*$$

of all matrix coefficients of unitary representations of G is an algebra of complex functions.

7.3.4 Proposition *Let G be a compact topological group and suppose $\pi : G \to U(V)$ is a faithful representation. Then:*

a. The matrix coefficients of π, together with their complex conjugates, plus the constants, generate $\mathcal{M}_G$ as a complex algebra. In particular, $\mathcal{M}_G$ is finitely generated.

b. Every irreducible representation of G is a subrepresentation of a tensor power of the form

$$\pi^{r,s} := \underbrace{\pi \otimes \cdots \otimes \pi}_{r} \otimes \underbrace{\pi^* \otimes \cdots \otimes \pi^*}_{s}$$

for some r, $s \geq 0$.

c. If H is a closed subgroup of G, then every irreducible representation of H appears as a component of the restriction to H of an irreducible representation of G.

Proof. (a) The subalgebra $\mathcal{A}$ of $\mathcal{M}_G$ generated by the matrix coefficients of π, π^* and the trivial representation is closed under complex conjugation, contains the constants, and separates points (as π is faithful); due to the Stone-Weierstrass Theorem, it is dense in $C(G)$ in the sup-norm. Note that $\mathcal{A}$ is also $G \times G$-invariant. If $\mathcal{A} \subsetneq \mathcal{M}_G$, then there is $\eta \in \hat{G}$ such that $V_\eta \otimes V_{\eta^*} \subsetneq \mathcal{A}$. Since $V_\eta \otimes V_{\eta^*}$ is $G \times G$-irreducible, we must have $\mathcal{A} \cap V_\eta \otimes V_{\eta^*} = \{0\}$. Now we have a direct sum $\mathcal{A} \oplus V_\eta \otimes V_{\eta^*} \subset \mathcal{M}_G$, and we can take $0 \neq f \in V_\eta \otimes V_{\eta^*}$ such that $f \perp \mathcal{A}$. However, this contradicts the denseness of $\mathcal{A}$ in $C(G)$.

(b) Let $\eta \in \hat{G}$. If η is not a subrepresentation of $\pi^{r,s}$ for some r, $s \geq 0$, then the matrix coefficients of η are orthogonal to the matrix coefficients of

$\pi^{r,s}$. The latter comprise monomials of degree r on the matrix coefficients of π and degree s on the matrix coefficients of π^*; therefore, by part (a), the matrix coefficients of $\pi^{r,s}$ for all r, $s \geq 0$ span $\mathcal{M}_G$, and therefore, owing to Peter-Weyl, span a dense subspace of $C(G)$. Hence the matrix coefficients of η cannot be orthogonal to the matrix coefficients of $\pi^{r,s}$ for all r, $s \geq 0$.

(c) Let $\zeta \in \hat{H}$. We can apply part (b) to the faithful representation $\pi|_H$ to deduce that ζ is a subrepresentation of $(\pi|_H)^{r,s}$; but $(\pi|_H)^{r,s} = (\pi^{r,s})|_H$; since ζ is irreducible, it must be a subrepresentation of some component of $(\pi^{r,s})|_H$. □

7.4 Fourier analysis on a compact group

The *Fourier transform* of $f \in L^1(G)$ is given by

$$\hat{f} : \hat{G} \to \prod_{\pi \in \hat{G}} \mathrm{End}(V_\pi), \qquad \hat{f}(\pi) = c_\pi(f),$$

where $c_\pi(f)$ is the *Fourier coefficient* defined by

$$c_\pi(f) = \int_G f(g)\pi(g^{-1})\, dg \in \mathrm{End}(V_\pi) \cong V_\pi \otimes V_{\pi^*}.$$

Note that

$$(f * \chi_\pi)(x) = \int_G f(g) \,\mathrm{trace}\, \pi(x)\pi(g^{-1})\, dg = \mathrm{trace}\left[\pi(x) \int_G f(g)\pi(g^{-1})\, dg\right],$$

so, for $f \in L^2(G)$,

$$\begin{aligned} f(x) &= \sum_{\pi \in \hat{G}} (\dim V_\pi)(f * \bar{\chi}_\pi)(x) \\ &= \sum_{\pi \in \hat{G}} (\dim V_\pi)(f * \chi_\pi)(x) \end{aligned}$$

(Problem 7), and this yields the *Fourier inversion formula* or *Plancherel formula*

$$f(x) = \sum_{\pi \in \hat{G}} (\dim V_\pi) \,\mathrm{trace}\left[c_\pi(f)\pi(x)\right].$$

7.4.1 Proposition (Parseval's formula) *For $f \in L^2(G)$ we have*

$$||f||_2^2 = \sum_{\pi \in \hat{G}} (\dim V_\pi) \,\mathrm{trace}\left(c_\pi(f)c_\pi(f)^*\right). \tag{7.4.2}$$

Proof. See Problem 9. □

As an immediate consequence of Parseval's formula, we have:

7.4.3 Proposition (Riesz-Fischer) *For a given sequence* $(c_\pi)_{\pi\in\hat{G}}$*, where* $c_\pi \in \mathrm{End}(V_\pi)$*, the Fourier series* $\sum_{\pi\in\hat{G}}(\dim V_\pi)\,\mathrm{trace}\,[c_\pi\pi(x)]$ *represents a function* $f \in L^2(G)$ *if and only if the series* $\sum_{\pi\in\hat{G}}(\dim V_\pi)\,\mathrm{trace}(c_\pi c_\pi^*)$ *is convergent.*

7.4.4 Proposition (Riemann-Lebesgue Lemma) *If* $f \in L^1(G)$ *then*

$$\{\pi \in \hat{G} \mid ||c_\pi(f)|| > \epsilon\}$$

if finite for any $\epsilon > 0$*; in other words,* $\hat{f}$ vanishes at ∞.

Proof. Suppose $f \in L^2(G)$. Then Parseval's formula says that $\pi \mapsto \mathrm{trace}\,(c_\pi(f)c_\pi(f)^*)$ vanishes at ∞, but the operator norm

$$||c_\pi(f)|| \leq \mathrm{trace}\,(c_\pi(f)c_\pi(f)^*)^{1/2}\,.$$

Suppose now that $f \in L^1(G)$. Then for $v \in V_\pi$, $||v|| = 1$,

$$||c_\pi(f)v|| = \left\| \int_G f(g)\pi(g^{-1})v\,dg \right\| \leq \int_G |f(g)|\,||\pi(g^{-1})v||\,dg = ||f||_1,$$

so $||c_\pi(f) - c_\pi(h)|| \leq ||f - h||_1$ for f, $h \in L^1(G)$. But $L^2(G)$ is dense in $L^1(G)$. □

The subspace of *central functions* or *class functions* of $L^2(G)$ is the space of Ad-invariant functions

$$L^2(G)^{\mathrm{Ad}} = \{f \in L^2(G) : f(gxg^{-1}) = f(x) \text{ for all } g, x \in G\}.$$

If f is central, then $L_{g^{-1}}f = R_g f$ so (Problem 4) $\hat{f}(\pi)\pi(g) = \pi(g)\hat{f}(\pi)$ for all $\pi \in \hat{G}$. By Schur's lemma, $c_\pi(f) = \hat{f}(\pi)$ is scalar on V_π, $c_\pi(f) = \lambda_\pi \cdot 1$, where $\lambda_\pi \in \mathbb{C}$. Taking the trace, we get

$$\lambda_\pi(\dim V_\pi) = \mathrm{trace}\,c_\pi(f) = \int_G f(g)\chi_\pi(g^{-1})\,dg = (f, \chi_\pi),$$

so the Fourier inversion formula for central functions is

$$f = \sum_{\pi\in\hat{G}}(f, \chi_\pi)\chi_\pi,$$

and there is an isometry of Hilbert spaces

$$L^2(G)^{\mathrm{Ad}} \cong \ell^2(\hat{G}).$$

7.4.5 Examples (i) Let G be the circle group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Then $\hat{G} = \{\pi_n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$ where $\pi_n(z) = z^n$. If we think of $\theta \in \mathbb{R}$ as a parameter for $e^{i\theta} \in S^1$, then $L^2(S^1)$ consists of functions $f(\theta)$ of period

2π and $\{e^{ni\theta} \mid n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(S^1)$. The Plancherel formula says that

$$f(\theta) = \sum_{n\in\mathbf{Z}} c_n e^{ni\theta}, \quad \text{where} \quad c_n = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)e^{-ni\theta}\,d\phi,$$

in the Hilbert space $L^2(G)$.

(ii) Let $G = SU(2)$. Then $\hat{G} = \{\pi_n \mid n = 0, 1, 2\ldots\}$ where $\pi_n$in the representation on the space V_n of complex homogeneous polynomials of degree n in two variables, which has basis $\{z^n, z^{n-1}w, \ldots, w^n\}$ and dimension $n+1$. Since $\pi_n^* = \pi_n$ and $SU(2) = S^3$, we get $L^2(S^3) = \sum_{n=0}^{+\infty} V_n \otimes V_n$. Here the connected isometry group of S^3 is $SO(4)$, doubly covered by $SU(2) \times SU(2)$. Later, we shall identify $V_n \otimes V_n$ with the $SO(4)$-representation on the space of Laplace spherical harmonics of degree n.

The Laplace operator

For a Riemannian manifold M with Levi-Cività connection ∇, the *Laplace operator* or *Laplacian* acting on smooth functions is the differential operator defined by $\Delta f = \mathrm{trace}\,\nabla df$. Hence if $\{E_i\}$ is an orthonormal frame,

$$\Delta f = \sum_i \nabla_{E_i} df(E_i) = E_i(df(E_i)) - df(\nabla_{E_i}E_i) = \sum_i (E_i^2 - \nabla_{E_i}E_i)f.$$

In particular, in a neighborhood of a point $x_0 \in M$ we can take a frame such that $(\nabla_{E_i}E_i)_{x_0} = 0$. Write γ_i for the geodesic with initial speed $E_i(x_0)$. Now

(7.4.6) $$(\Delta f)_{x_0} = \sum_i \frac{d^2}{dt^2}\Big|_{t=0} f(\gamma_i(t)).$$

We make an important remark. Suppose a Lie group G acts by isometries on M. Then the induced action on $C^\infty(M)$, given by $g\cdot f(x) = f(g^{-1}x)$ for $g \in G$, $f \in C^\infty(M)$, $x \in M$, commutes with the Laplacian in the sense that

$$g \cdot \Delta f = \Delta(g \cdot f)$$

(this follows from the fact that isometries preserve the Levi-Cività connection). Because of this, one says that Δ is an *invariant differential operator*.

We can also view Δ as a densely defined, unbounded self-adjoint operator on the Hilbert space $L^2(M)$. In case M is compact, ellipticity of Δ yields that $L^2(M)$ decomposes into the Hilbert space direct sum of the finite-dimensional eigenspaces of Δ. Moreover, the eigenvalues of $-\Delta$ form a countable, discrete subset of $[0, +\infty)$ accumulating at $+\infty$. The corresponding eigenspaces must be invariant under any isometric action on M.

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Endow G with a bi-invariant metric. The geodesics through $g \in G$ are of the form $t \mapsto g \exp tX$ for any left-invariant vector field $X \in \mathfrak{g}$. It follows from (7.4.6) that

$$\Delta f(g) = \sum_i \frac{d^2}{dt^2}\Big|_{t=0} f(g \exp(tE_i))$$

for any orthonormal basis $\{E_i\}$ of $\mathfrak{g}$; note that we are using the right-regular representation of $\mathfrak{g}$ on $L^2(G)$.

The connected isometry group of G is $G \times G$ and we know that the action of Δ on $L^2(G)$ commutes with that of $G \times G$. By Peter-Weyl's theorem, Δ preserves each space of matrix coefficients $V_\pi \otimes V_{\pi^*}$ and in fact by Schur's lemma acts by a scalar there.

7.4.7 Theorem *We have* $-\Delta|V_\pi \otimes V_{\pi^*} = (||\lambda_\pi + \delta||^2 - ||\delta||^2) \cdot 1$, *where* λ_π *is the highest weight of* π *and* δ *is half the sum of positive roots of* G.

Proof. If we forget the action on the left in Peter-Weyl's theorem, we get

$$L^2(G) = \overline{\bigoplus_{\pi \in \hat{G}} V_\pi \otimes V_{\pi^*}} = \overline{\bigoplus_{\pi \in \hat{G}} (\dim V_\pi) V_{\pi^*}} = \overline{\bigoplus_{\pi \in \hat{G}} (\dim V_\pi) V_\pi}.$$

For any matrix coefficient $f_{u,v}$ with $u, v \in V_\pi$, we have

$$\Delta f_{u,v} = f_{u,\omega \cdot v}$$

where

$$\omega = \sum_i E_i^2$$

is the so called *Casimir element*.[2] Let τ denote complex conjugation of $\mathfrak{g}^c$ over $\mathfrak{g}$. Since $\langle X, \tau X\rangle < 0$ for $X \in \mathfrak{g}^c$, we can choose the orthonormal basis of $\mathfrak{g}$ in the form $\{\sqrt{-1}H_j\}_j \cup \{X_\alpha, Y_\alpha \mid \alpha \in \Delta^+\}$, where $\{\sqrt{-1}H_j\}_j$ is an orthonormal basis of a Cartan subalgebra of $\mathfrak{g}$, and $E_\alpha \in (\mathfrak{g}^c)_\alpha$, $F_\alpha = -\tau E_\alpha \in (\mathfrak{g}^c)_{-\alpha}$, $[E_\alpha, F_\alpha] = H_\alpha$, $X_\alpha = \frac{1}{\sqrt{2}}(E_\alpha - F_\alpha)$, $X_\alpha = \frac{\sqrt{-1}}{\sqrt{2}}(E_\alpha + F_\alpha)$. Then

$$\begin{aligned}
\omega &= -\sum_j H_j^2 + \sum_{\alpha>0} X_\alpha^2 + Y_\alpha^2 \\
&= -\sum_j H_j^2 - \sum_{\alpha>0} E_\alpha F_\alpha + F_\alpha E_\alpha \\
&= -\sum_j H_j^2 - \sum_{\alpha>0} H_\alpha + 2F_\alpha E_\alpha.
\end{aligned}$$

[2] Formally speaking, it belongs to the (center of the) universal envelopping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. The algebra $\mathcal{U}(\mathfrak{g})$ is the "smallest" associative algebra containing $\mathfrak{g}$, and its elements can be viewed as the left-invariant differential operators on G.

If $\lambda = \lambda_\pi$ is the highest weight and v is the highest weight vector, then $E_\alpha v = 0$ so

$$\begin{aligned} -\omega \cdot v &= \sum_j \lambda(H_j)^2 v + \sum_{\alpha>0} \lambda(H_\alpha) v \\ &= \langle \lambda, \lambda \rangle v + \sum_{\alpha>0} \langle \lambda, \alpha \rangle v \\ &= \langle \lambda, \lambda + 2\delta \rangle v \\ &= \left(||\lambda + \delta||^2 - ||\delta||^2 \right) v, \end{aligned}$$

as we wished. □

7.4.8 Example (i) Let $G = SU(2)$. Then $\delta = \theta$ and $\lambda_n = n\theta$ where $\theta(\sqrt{-1}H) = 1$ for $H = \mathrm{diag}(1, -1)$. Using $(A, B) \mapsto -\frac{1}{2}\,\mathrm{trace}(AB)$, the multiple of the Killing form such that $SU(2)$ is isometric to the unit sphere S^3, we see that $||\theta|| = 1$. Hence $||\lambda_n + \delta||^2 - ||\delta||^2 = (n+1)^2 - 1 = n(n+2)$. Now

$$L^2(S^3) = \overline{\bigoplus_{n=0}^{+\infty} (n+1) V_n}$$

as unitary left $SU(2)$-spaces and $\Delta_{S^3}|V_n = -n(n+2)$.

(ii) We can also view S^3 as a $SU(2) \cdot SU(2) = SO(4)$-space. Then the Peter-Weyl theorem says that

$$L^2(S^3) = \overline{\bigoplus_{n=0}^{+\infty} V_n \otimes V_n} \tag{7.4.9}$$

as a unitary $SO(4)$-space (note that $V_n = V_n^*$ as a $SU(2)$-space).

Consider the graded algebra $\mathcal{P}(\mathbb{R}^4)$ of real polynomials on x_1, x_2, x_3 x_4, and denote by $\mathcal{P}_n(\mathbb{R}^4)$ its homogeneous component of degree n. The group $SO(4)$ acts on $\mathcal{P}_n(\mathbb{R}^4)$ by substitutions

$$g \cdot p \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = p \left(g^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \right).$$

This representation is orthogonal with respect to the following inner product:

$$\langle p, q \rangle := p(D) q,$$

where

$$p(D) = \sum a_{i_1 i_2 i_3 i_4} \frac{\partial^{i_1+i_2+i_3+i_4}}{\partial x_1^{i_1} \partial x_2^{i_2} \partial x_3^{i_3} \partial x_4^{i_4}} \tag{7.4.10}$$

for $p = \sum a_{i_1 i_2 i_3 i_4} x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}$. The Euclidean Laplace operator

$$\Delta_E = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_4^2}$$

acts on $\mathcal{P}_n := \mathcal{P}_n(\mathbb{R}^4)$, and this action commutes with the action of $SO(4)$. It follows that $\ker \Delta_E \cap \mathcal{P}_n =: \mathcal{H}_n$ is an invariant subspace, the so-called space of *harmonic* polynomials of degree n in $x_1, \ldots, x_4$ (cf. Examples 1.2.1(iv)). It is easy to see from (7.4.10) that the adjoint operator of $\Delta_E : \mathcal{P}_n \to \mathcal{P}_{n-2}$ is the multiplication operator by $||x||^2 = x_1^2 + \cdots + x_4^2$. It follows that there is a $SO(4)$-invariant vector space decomposition

$$\mathcal{P}_n = \mathcal{H}_n \oplus ||x||^2 \mathcal{P}_{n-2}.$$

From this, the dimension of $\mathcal{H}_n$ can be computed as

$$\dim \mathcal{P}_n - \mathcal{P}_{n-2} = \binom{n+3}{n} - \binom{n+1}{n-2} = (n+1)^2.$$

Also, by iteration

$$\mathcal{P}_n = \mathcal{H}_n \oplus ||x||^2 \mathcal{H}_{n-2} \oplus ||x||^4 \mathcal{H}_{n-4} + \cdots .$$

We claim that this decomposition is $SO(4)$-irreducible, and indeed the restriction map $\mathcal{P}_n \to L^2(S^3)$ takes $\mathcal{H}_n$ onto $V_n \otimes V_n$.[3]

In fact the polynomial functions are dense in $L^2(S^3)$ by Stone-Weierstrass. We complexify $\mathcal{P}_n(\mathbb{R}^4) \otimes \mathbb{C} = \mathcal{P}_n(\mathbb{C}^4)$ and note the identifications

$$\mathcal{H}_n^c \oplus ||x||^2 \mathcal{P}_{n-2}(\mathbb{C}^4) = \mathcal{P}_n(\mathbb{C}^4) \cong S^n(\mathbb{C}^{4*}) \cong S^n(\mathbb{C}^4)$$

(the second identification comes from the fact that the algebra of polynomials is the associative and commutative algebra generated by the linear forms $x_1, \ldots, x_4$; the last one follows from Problem 1.4.6 in Chapter 1 and Proposition 5.2.1.)

Let H be the the element of the CSA with $\theta_1(H) = 1$ and $\theta_2(H) = 0$, as in Example 5.4.16(i). The weights of $\mathbb{C}^4$ being $\pm\theta_1$, $\pm\theta_2$, the highest weight of $S^n(\mathbb{C}^{4*})$ is $n\theta_1$ (same as the highest weight of $S^n(\mathbb{C}^4)$), and we claim the polynomial $(x_1 + ix_2)^n$ is a highest weight vector; indeed, if $\{e_1, \ldots, e_4\}$ is the canonical basis of $\mathbb{C}^4$, then $He_1 = ie_2$, $He_2 = -ie_1$, and for the dual basis, $Hx_1 = ix_2$, $Hx_2 = -ix_1$, so

$$H(x_1 + ix_2)^n = n(x_1 + ix_2)^{n-1}(ix_2 + i(-ix_1)) = n(x_1 + ix_2)^n.$$

It is obvious that the the kernel of θ_1 kills $(x_1 + ix_2)^n$; this shows it is a weight vector of weight $n\theta_1$. Now $(x_1 + ix_2)^n$ generates an irreducible

[3]One can also check this claim by relating the Laplacians on $\mathbb{R}^4$ and S^3. The restrictions of harmonic polynomials to the unit sphere are classically called *spherical harmonics*.

component W of $S^n(\mathbb{C}^{4*})$. Note that $(x_1 + ix_2)^n$ is a harmonic (complex) polynomial, and the $SO(4)$ action on $S^n(\mathbb{C}^{4*}) = \mathcal{P}_n(\mathbb{C}^4)$ commutes with the action of the Euclidean Laplacian, so $W \subset \mathcal{H}_n$.

On the other hand, as an $SU(2) \times SU(2)$-representation, we claim $W = V_n \otimes V_n$. In fact, their complexifications coincide, as

$$n\theta_1 = \frac{n}{2}(\theta_1 + \theta_2) + \frac{n}{2}(\theta_1 - \theta_2),$$

where $\pm(\theta_1 + \theta_2)$ are the roots of the first $SU(2)$-factor and $\pm(\theta_1 - \theta_2)$ are the roots of the second $SU(2)$-factor; and, for each $SU(2)$-factor, $\frac{n}{2}$ times the positive root is the highest weight of the $(n+1)$-dimensional complex irreducible representation (Example 6.4.2). Now $\dim W = (\dim V_n)^2 = (n+1)^2 = \dim \mathcal{H}_n$, proving that $\mathcal{H}_n = W$ is irreducible, coincides with $S^n(\mathbb{C}^{4*})$ as a $SO(4)$-representation, and restricts to the eigenspace of the spherical Laplacian with eigenvalue $-n(n+2)$.

The Peter-Weyl Theorem can also be used to prove that every irreducible representation (of arbitrary dimension) of a compact topological group is finite-dimensional. We will not delve into the argument, since we do not want to introduce general infinite-dimensional (unitary) representations.

7.5 Problems

1 Let G be a compact topological group. Prove that $\int_G f = 0$ for every matrix coefficient f of a non-trivial representation.

2 Let π be a (finite-dimensional) complex representation of a compact topological group G. Prove that $\chi_{\Lambda^2\pi}(g) = \frac{1}{2}(\chi_\pi(g)^2 - \chi_\pi(g^2))$ and $\chi_{S^2\pi}(g) = \frac{1}{2}(\chi_\pi(g)^2 + \chi_\pi(g^2))$. Deduce from this that $\otimes^2\pi = S^2\pi \oplus \Lambda^2\pi$.

3 Let G be a compact Lie group, $\pi \in \hat{G}$, and $f_1, f_2, f \in L^2(G)$. Verify that
a. $\pi(f_1 * f_2) = \pi(f_1) \circ \pi(f_2)$;
b. The adjoint of $\pi(f)$ as an operator on V_π is $\pi(\tilde{f})$, where $\tilde{f}(g) = \overline{f(g^{-1})}$.

4 Prove that the Fourier transform has the following properties:
a. $\widehat{(f_1 + f_2)} = \hat{f}_1 + \hat{f}_2$, $\widehat{(af)} = a\hat{f}$ (linearity);
b. $\widehat{(f_1 * f_2)}(\pi) = \hat{f}_2(\pi)\hat{f}_1(\pi)$;
c. $\widehat{(L_g f)}(\pi) = \hat{f}(\pi)\pi(g^{-1})$, $\widehat{(R_g f)}(\pi) = \pi(g)\hat{f}(\pi)$ (equivariance);
for $f_1, f_2 \in L^2(G)$, $a \in \mathbb{R}$, $g \in G$.

5 For $\pi \in \hat{G}$ and $u, v, u', v' \in V_\pi$, compute that $f_{u,v} * f_{u',v'} = \frac{1}{\dim V_\pi}\langle u', v\rangle f_{u,v'}$.

6 Let G be a compact topological group. Prove that $\cap_{\pi\in\hat{G}} \ker \pi = \{1\}$.

7 Verify that the orthogonal projection $P_\pi : L^2(G) \to V_\pi \otimes V_{\pi^*}$ is given by $P_\pi f = d_\pi(f * \bar{\chi}_\pi)$, where $d_\pi = \dim V_\pi$.

8 Let $f \in C(G)$ be a linear combination of matrix coefficients of a complex irreducible representation π of G. If f is Ad-invariant, show that f must be a multiple of the character χ_π.

9 Prove Parseval's formula (7.4.2). (Hint: $||f||_2^2 = \sum_{\pi \in \hat{G}} ||P_\pi f||^2$.)

10 For the inner product defined in Example 7.4.8, check that

$$\langle x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}, x_1^{j_1} x_2^{j_2} x_3^{j_3} x_4^{j_4} \rangle = \begin{cases} i_1! i_2! i_3! i_4! & \text{if } (i_1, i_2, i_3, i_4) = (j_1, j_2, j_3, j_4), \\ 0 & \text{otherwise.} \end{cases}$$

11 Use the Weyl Dimension Formula to verify that the complex irreducible representation of $SO(4)$ with highest weight $n\theta_1$ has degree $(n+1)^2$.

12 Let G be a compact topological group with normalized Haar measure, and let $\pi \in \hat{G}$. Prove that

$$\chi_\pi(x)\chi_\pi(y) = d_\pi \int_G \chi_\pi(xgyg^{-1})\, dg$$

for all x, $y \in G$, where χ_π is the character of π and d_π is its degree.