The isomorhism $\alpha : \Lambda(V^*) \to \Lambda(V)^*$ is given on a decomposable element $v^* = v_1^* \wedge \cdots \wedge v_k^*$ by $\alpha(v^*)(u) = \det(v_i^*(u_j))$, where $u = u_1 \wedge \cdots \wedge u_k$. The isomorphism $A_k(V) \to \Lambda_k(V)^*$ associates to each alternate k-multilinear $f : V \times \cdots \times V \to \mathbf{R}$ the linear functional $\tilde{f} : \Lambda_k(V) \to \mathbf{R}$ given by $\tilde{f}(u_1 \wedge \cdots \wedge u_k) = f(u_1, \ldots, u_k)$.

Using the above isomorphisms, one gets a graded algebra structure on $A(V)$.

0.0.1 Lemma *If $f \in A_k(V)$ and $g \in A_l(V)$, then $f \wedge g \in A_{k+l}(V)$ is given by*

$$\begin{aligned} f \wedge g\,(u_1, \ldots, u_{k+l}) &= \frac{1}{k!\,l!} \sum_{\sigma \in \mathcal{S}_{k+l}} (sgn\ \sigma) f(u_{\sigma(1)}, \ldots, u_{\sigma(k)})\, g(u_{\sigma(k+1)}, \ldots, u_{\sigma(k+l)}) \\ &= \sum_{\pi \in \mathcal{S}_{k,l}} (sgn\ \pi) f(u_{\pi(1)}, \ldots, u_{\pi(k)})\, g(u_{\pi(k+1)}, \ldots, u_{\pi(k+l)}), \end{aligned}$$

where $\mathcal{S}_{k+l}$ denotes the group of permutations on $k+l$ letters and $\mathcal{S}_{k,l}$ denotes the subgroup of permutations π satisfying $\pi(1) < \ldots \pi(k)$ and $\pi(k+1) < \ldots \pi(k+l)$.

Proof. We prove first the second equality. Every $\sigma \in \mathcal{S}_{k+l}$ can be uniquely decomposed as $\sigma = \rho\tau\pi = \tau\rho\pi$, where $\tau \in \mathcal{S}_k \subset \mathcal{S}_{k+l}$ acts only on the first k letters, $\rho \in \mathcal{S}_l \subset \mathcal{S}_{k+l}$ acts only on the last l letters, and $\pi \in \mathcal{S}_{k,l}$. Then

$$\begin{aligned} &\frac{1}{k!\,l!} \sum_{\sigma \in \mathcal{S}_{k+l}} (\mathrm{sgn}\ \sigma) f(u_{\sigma(1)}, \ldots, u_{\sigma(k)})\, g(u_{\sigma(k+1)}, \ldots, u_{\sigma(k+l)}) \\ &= \frac{1}{k!\,l!} \sum_{\tau,\rho,\pi} (\mathrm{sgn}\ \tau\rho\pi) f(u_{\tau\pi(1)}, \ldots, u_{\tau\pi(k)})\, g(u_{\rho\pi(k+1)}, \ldots, u_{\rho\pi(k+l)}) \\ &= \frac{1}{k!\,l!} \sum_{\pi} (\mathrm{sgn}\ \pi) \sum_{\tau} (\mathrm{sgn}\ \tau) f(u_{\tau\pi(1)}, \ldots, u_{\tau\pi(k)}) \sum_{\rho} (\mathrm{sgn}\ \rho) g(u_{\rho\pi(k+1)}, \ldots, u_{\rho\pi(k+l)}) \\ &= \sum_{\pi} (\mathrm{sgn}\ \pi) f(u_{\pi(1)}, \ldots, u_{\pi(k)})\, g(u_{\pi(k+1)}, \ldots, u_{\pi(k+l)}), \qquad (0.0.2) \end{aligned}$$

as we wished.

Next, we turn to the first equality. By linearity on f and g of both hand sides, it suffices to consider $f = v_1^* \wedge \cdots \wedge v_k^*$ and $g = v_{k+1}^* \wedge \cdots \wedge v_{k+l}^*$. We compute

$$\begin{aligned} f \wedge g(u_1, \ldots, u_{k+l}) &= \det(v_i^*(u_j)) \\ &= \sum_{\sigma \in \mathcal{S}_{k+l}} (\mathrm{sgn}\ \sigma) v_1^*(u_{\sigma(1)}) \cdots v_k^*(u_{\sigma(k)}) v_{k+1}^*(u_{\sigma(k+1)}) \cdots v_{k+l}^*(u_{\sigma(k+l)}) \\ &= \sum_{\tau,\rho,\pi} (\mathrm{sgn}\ \tau)(\mathrm{sgn}\ \rho)(\mathrm{sgn}\ \pi) v_1^*(u_{\tau\pi(1)}) \cdots v_k^*(u_{\tau\pi(k)}) v_{k+1}^*(u_{\rho\pi(k+1)}) \cdots v_{k+l}^*(u_{\rho\pi(k+l)}) \end{aligned}$$

On the other hand, (0.0.2) equals

$$\sum_{\pi} (\mathrm{sgn}\ \pi) \sum_{\tau} (\mathrm{sgn}\ \tau) v_1^*(u_{\tau\pi(1)}) \cdots v_k^*(u_{\tau\pi(k)}) \sum_{\rho} (\mathrm{sgn}\ \rho) v_{k+1}^*(u_{\rho\pi(k+1)}) \cdots v_{k+l}^*(u_{\rho\pi(k+l)}),$$

and this completes the proof. □