A.7 Vector fields

Let M be a smooth manifold. A *vector field* on M is a map $X : M \to TM$ such that $X(p) \in T_pM$ for $p \in M$. Sometimes, we also write X_p instead of $X(p)$. As we have seen, TM carries a canonical manifold structure, so it makes sense to call X is a smooth vector field if the map $X : M \to TM$ is smooth. Hence, a smooth vector field on M is a smooth assignment of tangent vectors at the various points of M. From another point of view, recall the natural projection $\pi : TM \to M$; the requirement that $X(p) \in T_pM$ for all p is equivalent to having $\pi \circ X = \mathrm{id}_M$.

More generally, let $f : M \to N$ be a smooth mapping. Then a *vector field along* f is a map $X : M \to TN$ such that $X(p) \in T_{f(p)}N$ for $p \in M$. The most important case is that in which f is a smooth curve $\gamma : [a, b] \to N$. A vector field along γ is a map $X : [a, b] \to TN$ such that $X(t) \in T_{\gamma(t)}N$ for $t \in [a, b]$. A typical example is the tangent vector field $\dot\gamma$.

Let X be a vector field on M. Given a smooth function $f \in C^\infty(U)$ where U is an open subset of M, the directional derivative $X(f) : U \to \mathbf{R}$ is defined to be the function $p \in U \mapsto X_p(f)$. Further, if $(x_1, \ldots, x_n)$ is a coordinate system on U, we have already seen that $\{\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p\}$ is a basis of T_pM for $p \in U$. It follows that there are functions $a_i : U \to \mathbf{R}$ such that

$$X|_U = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}. \tag{A.7.1}$$

A.7.2 Proposition *Let X be a vector field on M. Then the following assertions are equivalent:*

a. X is smooth.

b. For every coordinate system $(U, (x_1, \ldots, x_n))$ of M, the functions a_i defined by (A.7.1) are smooth.

c. For every open set V of M and $f \in C^\infty(V)$, the function $X(f) \in C^\infty(V)$.

Proof. Suppose X is smooth and let $\{\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p\}$ be a coordinate system on U. Then $X|_U$ is smooth and $a_i = dx_i \circ X|_U$ is also smooth.

Next, assume (b) and let $f \in C^\infty(V)$. Take a coordinate system $(U, (x_1, \ldots, x_n))$ with $V \subset U$. Then, by using (b) and the fact that $\frac{\partial f}{\partial x_i}$ is smooth,

$$X(f)|_U = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i} \in C^\infty(U).$$

Since V can be covered by such U, this proves (c).

Finally, assume (c). For every coordinate system $(U, (x_1, \ldots, x_n))$ of M, we have a corresponding coordinate system $(\pi^{-1}(U), x_1 \circ \pi, \ldots, x_n \circ \pi, dx_1, \ldots, dx_n)$ of TM. Then

$$(x_i \circ \pi) \circ X|_U = x_i \quad \text{and} \quad dx_i \circ X|_U = X(x_i)$$

are smooth. This proves that X is smooth. □

In particular, the proposition shows that the coordinate vector fields $\frac{\partial}{\partial x_i}$ associated to a local chart are smooth. The arguments in the proof also show that if X is a vector field on M satisfying $X(f) = 0$ for every $f \in C^\infty(V)$ and every open $V \subset M$, then $X = 0$. This remark forms the basis of our next definition, and is explained by noting that in the local expression (A.7.1) for a coordinate system defined on $U \subset V$, the functions $a_i = dx_i \circ X|_U = X(x_i) = 0$.

Next, let X and Y be smoth vector fields on M. Their *Lie bracket* $[X, Y]$ is defined to be the unique vector field on M that satisfies

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \tag{A.7.3}$$

for every $f \in C^\infty(M)$. By the remark in the previous paragraph, such a vector field is unique if it exists. In order to prove existence, consider a coordinate system $(U, (x_1, \ldots, x_n))$. Then we can write

$$X|_U = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad Y|_U = \sum_{j=1}^n b_j \frac{\partial}{\partial x_j}$$

for a_i, $b_j \in C^\infty(U)$. If $[X,Y]$ exists, we must have

$$[X,Y]|_U = \sum_{i=1}^n \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}, \tag{A.7.4}$$

because the coefficients of $[X,Y]|_U$ in the local frame $\{\frac{\partial}{\partial x_j}\}_{j=1}^n$ must be given by $[X,Y](x_j) = X(Y(x_j)) - Y(X(x_j))$. We can use formula A.7.4 as the definition of a vector field on U; note that such a vector field is smooth and satisfies property (A.7.3) for functions in $C^\infty(U)$. We finally define $[X,Y]$ globally by covering M with domains of local charts: on the overlap of two charts, the different definitions coming from the two charts must agree by the above uniqueness result; it follows that $[X,Y]$ is well defined.

A.7.5 Proposition *Let X, Y and Z be smooth vector fields on M. Then*

a. $[Y,X] = -[X,Y]$.

b. If f, $g \in C^\infty(M)$, then

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X.$$

c. $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$. *(Jacobi identity)*

We omit the proof of this propostion which is simple and only uses (A.7.3). Note that (A.7.3) combined with the commutation of mixed second partial derivatives of a smooth function implies that $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$ for coordinate vector fields associated to a local chart.

Let $f : M \to N$ be a diffeomorphism. For every smooth vector field X on M, the formula $df \circ X \circ f^{-1}$ defines a smooth vector field on N which we denote by f_*X. More generally, if $f : M \to N$ is a smooth map which needs not be a diffeomorphism, smooth vector fields X on M and Y on N are called *f-related* if $df \circ X = Y \circ f$. The proof of the next propostion is an easy application of (A.7.3).

A.7.6 Proposition *Let $f : M \to M'$ be smooth. Let X, Y be smooth vector fields on M, and let X', Y' be smooth vector fields on M'. If X and X' are f-related and Y and Y' are f-related, then also $[X,Y]$ and $[X',Y']$ are f-related.*

Flow of a vector field

Next, we discuss how to "integrate" vector fields. Let X be a smooth vector field on M. An *integral curve* of X is a smooth curve γ in M such that

$$\dot\gamma(t) = X(\gamma(t))$$

for all t in the domain of γ.

In order to study existence and uniqueness questions for integral curves, we consider local coordinates. So suppose $\gamma : (a,b) \to M$ is a smooth curve in M, $0 \in (a,b)$, $\gamma(0) = p$, $(U, \varphi =$

$(x_1, \ldots, x_n))$ is a local chart around p, X is a smooth vector field in M and $X|_U = \sum_i a_i \frac{\partial}{\partial x_i}$ for $a_i \in C^\infty(U)$. Then γ is an integral curve of X on $\gamma^{-1}(U)$ if and only if

$$\text{(A.7.7)} \qquad \frac{d\gamma_i}{dr}\Big|_t = (a_i \circ \varphi^{-1})(\gamma_1(t), \ldots, \gamma_n(t))$$

for $i = 1, \ldots, n$ and $t \in \gamma^{-1}(U)$, where $\gamma_i = x_i \circ \gamma$. Equation (A.7.7) is a system of first order ordinary differential equations for which existence and uniqueness theorems are known. These, translated into manifold terminology yield the following proposition.

A.7.8 Proposition *Let X be a smooth vector field on M. For each $p \in M$, there exists a (possibly infinite) interval $(a(p), b(p)) \subset \mathbf{R}$ and a smooth curve $\gamma_p : (a(p), b(p)) \to M$ such that:*

a. $0 \in (a(p), b(p))$ and $\gamma_p(0) = p$.

b. γ_p is an integral curve of X.

c. γ_p is maximal in the sense that if $\mu : (c, d) \to M$ is a smooth curve satisfying (a) and (b), then $(c, d) \subset (a(p), b(p))$ and $\mu = \gamma_p|_{(c,d)}$.

Let X be a smooth vector field on M. Put

$$\mathcal{D}_t = \{ p \in M \mid t \in (a(p), b(p)) \}$$

and define $X_t : \mathcal{D}_t \to M$ by setting

$$X_t(p) = \gamma_p(t).$$

Note that we have somehow reversed the rôles of p and t with this definition. The collection of X_t for all t is called the *flow* of X.

A.7.9 Example Take $M = \mathbf{R}^2$ and $X = \frac{\partial}{\partial r_1}$. Then $\mathcal{D}_t = \mathbf{R}^2$ for all $t \in \mathbf{R}$ and $X_t(a_1, a_2) = (a_1 + t, a_2)$ for $(a_1, a_2) \in \mathbf{R}^2$. Note that if we replace $\mathbf{R}^2$ by the punctured plane $\mathbf{R}^2 \setminus \{(0, 0)\}$, the sets $\mathcal{D}_t$ become proper subsets of M. ⋆

A.7.10 Theorem *a. For each $p \in M$, there exists an open neighborhood V of p and $\epsilon > 0$ such that the map*

$$(-\epsilon, \epsilon) \times V \to M, \quad (t, p) \mapsto X_t(p)$$

is well defined and smooth.

b. The domain $\mathrm{dom}(X_s \circ X_t) \subset \mathcal{D}_{s+t}$ and $X_{s+t}|_{\mathrm{dom}(X_s \circ X_t)} = X_s \circ X_t$. Further, $\mathrm{dom}(X_s \circ X_t) = X_{s+t}$ if $st > 0$.

c. $\mathcal{D}_t$ is open for all t, $\cup_{t>0} \mathcal{D}_t = M$ and $X_t : \mathcal{D}_t \to \mathcal{D}_{-t}$ is a diffeomorphism with inverse X_{-t}.

Proof. Part (a) is a local result and is simply the smooth dependence of solutions of ordinary differential equations on the intial conditions. We prove part (b). First, we remark the obvious fact that, if $p \in \mathcal{D}_t$, then $s \mapsto \gamma_p(s + t)$ is an integral curve of X with initial condition $\gamma_p(t)$ and maximal domain $(a(p) - t, b(p) - t)$; therefore $(a(p) - t, b(p) - t) = (a(\gamma_p(t)), b(\gamma_p(t)))$. Next, let $p \in \mathrm{dom}(X_s \circ X_t)$. This means that $p \in \mathrm{dom}(X_t) = \mathcal{D}_t$ and $\gamma_p(t) = X_t(p) \in \mathrm{dom}(X_s) = \mathcal{D}_s$. Then $s \in (a(\gamma_p(t)), b(\gamma_p(t)))$, so $s+t \in (a(\gamma_p(t))+t, b(\gamma_p(t))+t) = (a(p), b(p))$, that is $p \in \mathcal{D}_{s+t}$. Further, $X_{s+t}(p) = \gamma_p(s + t) = \gamma_{\gamma_p(t)}(s) = X_s(X_t(p))$ and we have already proved the first two assertions. Next, assume that $s, t > 0$ (the case $s, t \leq 0$ is similar); we need to show that $\mathcal{D}_{s+t} \subset \mathrm{dom}(X_s \circ X_t)$. But this follows from reversing the argument above as $p \in \mathcal{D}_{s+t}$ implies that $s + t \in (a(p), b(p))$, and this implies that $t \in (a(p), b(p))$ and $s = (s + t) - t \in (a(p) - t, b(p) - t) = (a(\gamma_p(t)), b(\gamma_p(t)))$. Finally, we prove part (c). The statement about the union follows from part (a). Note that

$\mathcal{D}_0 = M$. Fix $t > 0$ and $p \in \mathcal{D}_t$; we prove that p is an interior point of $\mathcal{D}_t$ and X_t is smooth on a neighborhood of p (the case $t < 0$ is analogous). Indeed, since $\gamma_p([0,t])$ is compact, part (a) yields an open neighborhood W_0 of this set and $\epsilon > 0$ such that $(s,q) \in (-\epsilon,\epsilon) \times W_0 \mapsto X_s(q) \in M$ is well defined and smooth. Take an integer $n > 0$ such that $t/n < \epsilon$ and put $\alpha_1 = X_{\frac{t}{n}}|_{W_0}$. Then, define inductively $W_i = \alpha_i^{-1}(W_{i-1}) \subset W_{i-1}$ and $\alpha_i = X_{\frac{t}{n}}|_{W_i-1}$ for $i = 2,\dots,n$. It is clear that α_i is smooth and W_i is an open neighborhood of $\gamma_p(\frac{n-i}{n}t)$ for all i. In particular, W_n is an open neighborhood of p in W. Moreover,

$$\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_n|_{W_n} = (X_{\frac{t}{n}})^n|_{W_n} = X_t|_{W_n}$$

by the last assertion of part (b), so X_t is smooth on W_n. Now $\mathcal{D}_t$ is open and X_t is smooth on $\mathcal{D}_t$. It is obvious that the image of X_t is $\mathcal{D}_{-t}$. Since X_{-t} is also smooth on $\mathcal{D}_{-t}$, it follows from part (b) that X_t and X_{-t} are inverses one of the other and this completes the proof of the theorem. □

The Frobenius theorem

Let M be a smooth manifold of dimension n. A *distribution* $\mathcal{D}$ *of rank* (or *dimension*) k is a choice of k-dimensional subspace $\mathcal{D}_p \subset T_pM$ for each $p \in M$. A distribution $\mathcal{D}$ of rank k is called *smooth* if for every $p \in M$ there exists an open neighborhood U of p and k smooth vector fields $X_1,\dots,X_k$ on U such that $\mathcal{D}_q$ equals the span of $X_1(q),\dots,X_k(q)$ for every $q \in U$. A vector field X is said to *belong to* (or *lie in*) the distribution $\mathcal{D}$ (and we write $X \in \mathcal{D}$) if $X(p) \in \mathcal{D}_p$ for $p \in M$. A distribution $\mathcal{D}$ is called *involutive* if X, $Y \in \mathcal{D}$ implies that $[X,Y] \in \mathcal{D}$. A submanifold N of M is called an *integral manifold* of a distribution $\mathcal{D}$ if $T_pN = \mathcal{D}_p$ for $p \in N$.

If X is a nowhere zero smooth vector field on M, then of course the line spanned by X_p in T_pM for $p \in M$ defines a smooth distribution on M. In this special case, Proposition A.7.8 guarantees the existence and uniqueness of maximal integral submanifolds. Our next intent is to generalize this result to arbitrary smooth distributions. A necessary condition is given in the following proposition. The contents of the Frobenius theorem is that the condition is also sufficient.

A.7.11 Proposition *A smooth distribution $\mathcal{D}$ on M admitting integral manifolds through any point of M must be involutive.*

Proof. Given smooth vector fields X, $Y \in \mathcal{D}$ and $p \in M$, we need to show that $[X,Y]_p \in \mathcal{D}_p$. By assumption, there exists an integral manifold N passing thorugh p. By shrinking N, we may further assume that N is embedded. Denote by ι the inclusion of N into M. Then $d\iota_{\iota^{-1}(p)} : T_{\iota^{-1}(p)}N \to T_pM$ is an isomorphism onto $\mathcal{D}_p$. Therefore there exist vector fields $\tilde{X}$ and $\tilde{Y}$ on N which are ι-related to resp. X and Y. Due Theorem A.4.6, $\tilde{X}$ and $\tilde{Y}$ are smooth, so by using Proposition A.7.6 we finally get that $[X,Y]_p = d\iota([\tilde{X},\tilde{Y}]_{\iota^{-1}(p)}) \in \mathcal{D}_p$. □

It is convenient to use the following terminology in the statement of the Frobenius theorem. A coordinate system $(U, \varphi = (x_1,\dots,x_n))$ of a smooth manifold M of dimension m will be called *cubic* if $\varphi(U)$ is an open cube centered at the origin of $\mathbf{R}^m$, and it will be called *centered at a point* $p \in U$ if $\varphi(p) = 0$.

A.7.12 Theorem (Frobenius, local version) *Let $\mathcal{D}$ be a smooth distribution of rank k on a smooth manifold of dimension n. Suppose that $\mathcal{D}$ is involutive. Then, given $p \in M$, there exists an integral manifold of $\mathcal{D}$ containing p. More precisely, there exists a cubic coordinate system $(U, \varphi = (x_1,\dots,x_n))$ centered at p such that the "slices"*

$$x_i = \text{constant} \qquad \textit{for } i = k+1,\dots,n$$

are integral manifolds of $\mathcal{D}$. Further, if $N \subset U$ is a connected *integral manifold of $\mathcal{D}$, then N is an open submanifold of one of these slices.*

Proof. We proceed by induction on k. Suppose first that $k = 1$. Choose a smooth vector field $X \in \mathcal{D}$ defined on a neighborhood of p such that $X_p \neq 0$. It suffices to construct a coordinate system $(U, \varphi = (x_1, \ldots, x_n))$ around p such that $X|_U = \frac{\partial}{\partial x_1}|_U$. Indeed, it is easy to get a coordinate system $(V, \psi = (y_1, \ldots, y_n))$ centered at p such that $\frac{\partial}{\partial y_1} = Y_p$. The map

$$\sigma(t, a_2, \ldots, a_n) = X_t(\psi^{-1}(0, a_2, \ldots, a_n))$$

is well defined and smooth on $(-\epsilon, \epsilon) \times W$ for some $\epsilon > 0$ and some neighborhood W of the origin in $\mathbf{R}^{n-1}$. We immediately see that

$$d\sigma\left(\frac{\partial}{\partial r_1}\Big|_0\right) = X_p = \frac{\partial}{\partial y_1}\Big|_p \quad \text{and} \quad d\sigma\left(\frac{\partial}{\partial r_i}\Big|_0\right) = \frac{\partial}{\partial y_i}\Big|_p \quad \text{for } i = 2, \ldots, n.$$

By the inverse function theorem, σ is a local diffeomorphism at 0, so its local inverse yields the desired local chart φ.

We next assume the theorem is true for distributions of rank $k-1$ and prove it for a given distribution $\mathcal{D}$ of rank k. Choose smooth vector fields $X_1, \ldots, X_n$ spanning $\mathcal{D}$ on a neighborhood $\tilde{V}$ of p. The result in case $k = 1$ yields a coordinate system $(V, y_1, \ldots, y_n)$ centered at p such that $V \subset \tilde{V}$ and $X_1|_V = \frac{\partial}{\partial y_1}|_V$. Define the following smooth vector fields on V:

$$\begin{aligned} Y_1 &= X_1 \\ Y_i &= X_i - X_i(y_1)X_1 \qquad \text{for } i = 2, \ldots, k \end{aligned}$$

Plainly, $Y_1, \ldots, Y_k$ span $\mathcal{D}$ on V. Let $S \subset V$ be the slice $y_1 = 0$ and put

$$Z_i = Y_i|_S \qquad \text{for } i = 2, \ldots, k.$$

Since

$$Y_i(y_1) = X_i(y_1) - X_i(y_1)\underbrace{X_1(y_1)}_{=1} = 0 \qquad \text{for } i = 2, \ldots, k, \tag{A.7.13}$$

we have $Z_i(q) \in T_qS$ for $q \in S$, so $Z_2, \ldots, Z_k$ span a smooth distribution $\mathcal{D}'$ of rank $k-1$ on S; we next check that $\mathcal{D}'$ is involutive. Since Z_i and Y_i are related under the inclusion $S \subset V$, also $[Z_i, Z_j]$ and $[Y_i, Y_j]$ are so related. Eqn. (A.7.13) gives that $[Y_i, Y_j](y_1) = 0$, so, on V

$$[Y_i, Y_j] = \sum_{\ell=1}^{k} c_{ijk} Y_\ell$$

for $c_{ijk} \in C^\infty(V)$. Hence

$$[Z_i, Z_j] = \sum_{\ell=1}^{k} c_{ijk}|_S Z_\ell,$$

as we wished. By the inductive hypothesis, there exists a coordinate system $(w_2, \ldots, w_n)$ on some neighborhood of p in S such that the slices w_i = constant for $i = k+1, \ldots, n$ define integral manifolds of $\mathcal{D}'$.

Let $\pi : V \to S$ be the linear projection relative to $(y_1, \ldots, y_n)$. Set

$$\begin{aligned} x_1 &= y_1, \\ x_i &= w_i \circ \pi \qquad \text{for } i = 2, \ldots, n. \end{aligned}$$

It is clear that there exists an open neighborhood U of p in V such that $(U, \varphi = (x_1, \ldots, x_n))$ is a cubic coordinate system of M centered at p. Now the first assertion in the statement of the theorem follows if we prove that $Y_i(x_j) = 0$ on U for $i = 1, \ldots, k$ and $j = k+1, \ldots, n$, for this will imply that $\frac{\partial}{\partial x_1}|_q, \ldots, \frac{\partial}{\partial x_k}|_q$ is a basis of $\mathcal{D}_q$ for every $q \in U$. In order to do that, note that

$$\frac{\partial x_j}{\partial y_1} = \begin{cases} 1, & \text{if } j = 1, \\ 0, & \text{if } j = 2, \ldots, n \end{cases}$$

on U. Hence

$$Y_1 = X_1 = \frac{\partial}{\partial y_1} = \sum_{j=1}^{n} \frac{\partial x_j}{\partial y_1} \frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_1},$$

so $Y_1(x_j) = 0$ or $j > k$. Next, take $i \leq k$ and $j > k$. Owing to the involutivity of $\mathcal{D}$,

$$[Y_1, Y_i] = \sum_{\ell=1}^{k} c_{i\ell} Y_\ell$$

for some $c_{ik} \in C^\infty(U)$. Therefore

$$\frac{\partial}{\partial x_1}(Y_i(x_j)) = Y_1(Y_i(x_j)) - Y_i(Y_1(x_j)) = Y_1(Y_i(x_j)) = \sum_{\ell=2}^{k} c_{i\ell} Y_\ell(x_j),$$

which, for fixed $x_2, \ldots, x_n$, is a system of $k-1$ homogeneous linear ordinary differential equations in the functions $Y_\ell(x_j)$ of the variable x_1. Of course, the initial condition $x_1 = 0$ corresponds to $S \cap U$ along which we have

$$Y_i(x_j) = Z_i(x_j) = Z_i(w_j) = 0,$$

where the latter equatility follows from the fact that Z_i lies in $\mathcal{D}'$ and $w_j =$ constant for $j > k$ define integral manifolds of $\mathcal{D}'$. By the uniqueness theorem of solutions of ordinary differential equations, $Y_i(x_j) = 0$ on U.

Finally, suppose that $N \subset U$ is a connected integral manifold of $\mathcal{D}$. Let ι denote the inclusion of N into U let and $\pi : \mathbf{R}^n \to \mathbf{R}^k \times \mathbf{R}^{n-k}$ be the projection. Then $d(\pi \circ \varphi \circ \iota)_q(T_q N) = d(\pi \circ \pi)_q(\mathcal{D}_q) = 0$ for $q \in U$. By connectedness of N, $\pi \circ \varphi \circ \iota$ is a constant function on N. Thus N is contained in one of the slices $x_i =$ constant for $i = k+1, \ldots, n$, say S. The inclusion of N into M is continuous (since N is a submanifold of M) and has image contained in S; since S is embedded in M, the inclusion of N into S is continuous and thus smooth by Theorem A.4.6. Since N is a submanifold of M, this inclusion is also an immersion. Of course, $\dim N = \dim S$, so this inclusion is indeed a local diffeomorphism. Hence N is an open submanifold of S. □

A.7.14 Theorem *Integral manifolds of involutive distributions are quasi-embedded submanifolds. More precisely, suppose that $f : M \to N$ is smooth, P is an integral manifold of an involutive distribution $\mathcal{D}$ on M, and $f(M) \subset P$. Consider the induced map $f_0 : M \to P$ that satisfies $\iota \circ f_0 = f$, where $\iota : P \to N$ is the inclusion. Then f_0 is continuous (and hence smooth by Theorem A.4.6).*

Proof. Let U be an open subset of P, $q \in U$ and $p \in f_0^{-1}(q)$. It suffices to prove that p is an interior point of $f_0^{-1}(U)$. By the local version of the Frobenius theorem (A.7.12), there exists a cubic coordinate system $(V, \psi = (x_1, \ldots, x_n))$ of N centered at q such that the slices

$$x_i = \text{constant} \qquad \text{for } i = k+1, \ldots, n \tag{A.7.15}$$

are the integral manifolds of $\mathcal{D}$ in V. Also, we can shrink V so that $V \cap U$ is exactly the slice

$$x_{k+1} = \cdots = x_n = 0. \tag{A.7.16}$$

We have that $f^{-1}(V)$ an open neighborhood of q in M; let W be its connected component containing p. Of course, W is open. It is enough to show that $f_0(W) \subset V \cap U$, or which is the same, $f(W)$ is contained in (A.7.16). Since $f(W)$ is connected, it is contained in a component of $V \cap P$ with respect to the relative topology. Since $f(W)$ meets (A.7.16) at least at the point q, it suffices to show that the components of $V \cap P$ in the relative topology are contained in the slices of the form (A.7.15). Let C be a component of $V \cap P$ with respect to the relative topology; note that C need not be connected in the topology of P, but, by second-countability of P, C is a countable union of connected integral manifolds of $\mathcal{D}$ in V, each of which is contained in a slice of the form (A.7.15). Let $\pi : V \to \mathbf{R}^{n-k}$ be given by $\pi(r) = (x_{k+1}(r), \ldots, x_n(r))$. It follows that $\pi(C)$ is a countable connected subset of $\mathbf{R}^{n-k}$; hence, it is a single point. □

A *maximal integral manifold* of a distribution $\mathcal{D}$ on a manifold M is a connected integral manifold N of $\mathcal{D}$ such that every connected integral manifold of $\mathcal{D}$ which intersects N is an open submanifold of N.

A.7.17 Theorem (Frobenius, global version) *Let $\mathcal{D}$ be a smooth distribution on M. Suppose that $\mathcal{D}$ is involutive. Then through any given point of M there passes a unique integral manifold of $\mathcal{D}$.*

Proof. Let $\dim M = n$ and $\dim \mathcal{D} = k$. Given $p \in M$, define N to be the set of all points of M reachable from p by following piecewise smooth curves whose smooth arcs are everywhere tangent to $\mathcal{D}$. By the local version A.7.12 and the σ-compactness of its topology, M can be covered by countably many cubic coordinate systems $(U_i, x_1^i, \ldots, x_n^i)$ such that the integral manifolds of $\mathcal{D}$ in U_i are exactly the slices

$$x_j^i = \text{constant} \qquad \text{for } j = k+1, \ldots, n. \tag{A.7.18}$$

Note that a slice of the form (A.7.18) that meets N must be contained in N, and that N is covered by such slices. We equip N with the finest topology with respect to which the inclusions of all such slices are continuous. We can also put a differentiable structure on N by declaring that such slices are open submanifolds of N. We claim that this turns N into a connected smooth manifold of dimension k. N is clearly connected since it is path-connected by construction. N is also Hausdorff, because M is Hausdorff and the inclusion of N into M is continuous. It only remains to prove that N is second-countable. It suffices to prove that only countably many slices of U_i meet N. For this, we need to show that a single slice S of U_i can only meet countably many slices of U_j. For this purpose, note that $S \cap U_j$ is an open submanifold of S and therefore consists of at most countably many components, each of which being a connected integral manifold of $\mathcal{D}$ and hence lying entirely in a slice of U_j. Now it is clear that N is an integral manifold of $\mathcal{D}$ through p.

Next, let N' be another connected integral manifold of $\mathcal{D}$ meeting N at a point q. Given $q' \in N'$, there exists a piecewise smooth curve integral curve of $\mathcal{D}$ joining q to q' since connected manifolds are path-connected. Due to $q \in N$, this curve can be juxtaposed to a piecewise smooth curve integral curve of $\mathcal{D}$ joining p to q. We get $q' \in N$ and this proves that $N' \subset N$. Since N' is a submanifold of M, the inclusion of N' into M is continuous. By Theorem A.7.14, the inclusion of N' into N is smooth. Hence N' is an open submanifold of N.

Finally, suppose that N' is in addition to the above a maximal integral manifold. The above argument shows that $N \subset N'$ and N is an open submanifold of N'. It follows that the identity map $N \to N'$ is a diffeomorphism and this proves the uniqueness of N. $\square$