APPENDIX A

Smooth manifolds

A.1 Introduction

The theory of smooth manifolds can be thought of a natural and very useful extension of the differential calculus on $\mathbf{R}^n$ in that its main theorems, which in our opinion are well represented by the inverse (or implicit) function theorem and the existence and uniqueness result for ordinary differential equations, admit generalizations. At the same time, from a differential-geometric viewpoint, manifolds are natural generalizations of the surfaces in three-dimensional Euclidean space.

We will review some basic notions from the theory of smooth manifolds and their smooth mappings.

A.2 Basic notions

A *topological manifold of dimension n* is a Hausdorff, second-countable topological space M which is locally modeled on the Euclidean space $\mathbf{R}^n$. The latter condition refers to the fact that M can be covered by a family of open sets $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ such that each U_α is homeomorphic to an open subset of $\mathbf{R}^n$ via a map $\varphi_\alpha : U_\alpha \to \mathbf{R}^n$. The pairs $(U_\alpha, \varphi_\alpha)$ are then called *local charts* or *coordinate systems*, and the family $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ is called a *topological atlas* for M. A *smooth atlas* for M (smooth in this book always means C^∞) is a topological atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ which also satisfies the following compatibility condition: the map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth for every α, $\beta \in \mathcal{A}$ (of course, this condition is void if α, β are such that $U_\alpha \cap U_\beta = \varnothing$). A *smooth structure* on M is a smooth atlas which is maximal in the sense that one cannot enlarge it by adjoining new local charts of M while having it satisfy the above compatibility condition. (This maximality condition is really a technical one, and one easily sees that every smooth atlas can be enlarged to a unique maximal one.) Formally speaking, a smooth manifold consists of the topological space together with the smooth structure. The basic idea behind these definitions is that one can carry notions and results of differential calculus in $\mathbf{R}^n$ to smooth manifolds by using the local charts, and the compatibility condition between the local charts ensures that what we get for the manifolds is well defined.

A.2.1 Remark Sometimes it happens that we start with a set M with no prescribed topology and attempt to introduce a topology and smooth structure at the same time. This can be done by constructing a family $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ where $\mathcal{A}$ is a countable index set, each $\varphi : U_\alpha \to \mathbf{R}^n$ is a bijection onto an open subset of $\mathbf{R}^n$ and the maps $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are homeomorphisms and smooth maps between open sets of $\mathbf{R}^n$ for all α, $\beta \in \mathcal{A}$ and for some fixed n. It is easy to see then the collection

$$\{\,\varphi_\alpha^{-1}(W) \mid W \text{ open in } \mathbf{R}^n,\ \alpha \in \mathcal{A}\,\}$$

forms a basis for a second-countable, locally Euclidean topology on M. Note that this topology needs not to be automatically Hausdorff, so one has to check that in each particular case and then the family $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ is a smooth atlas for M. Further, instead of considering maximal smooth atlases, one can equivalently define a smooth structure on M to be an equivalence class of smooth atlases, where one defines two smooth atlases to be equivalent if their union is a smooth atlas.

A.2.2 Examples (a) The Euclidean space $\mathbf{R}^n$ itself is a smooth manifold. One simply uses the identity map of $\mathbf{R}^n$ as a coordinate system. Similarly, any n-dimensional real vector space V can be made into a smooth manifold of dimension n simply by using a global coordinate system on V given by a basis of the dual space V^*.

(b) The complex n-space $\mathbf{C}^n$ is a real $2n$-dimensional vector space, so it has a structure of smooth manifold of dimension $2n$.

(c) The *real projective space* $\mathbf{R}P^n$ is the set of all lines through the origin in $\mathbf{R}^{n+1}$. Since every point of $\mathbf{R}^{n+1}\setminus\{0\}$ lies in a unique such line, these lines can obviously be seen as defining equivalence classes in $\mathbf{R}^{n+1}\setminus\{0\}$, so $\mathbf{R}P^n$ is a quotient topological space of $\mathbf{R}^{n+1}\setminus\{0\}$. One sees that the projection $\pi:\mathbf{R}^{n+1}\setminus\{0\}\to\mathbf{R}P^n$ is an open map, so it takes a countable basis of $\mathbf{R}^{n+1}\setminus\{0\}$ to a countable basis of $\mathbf{R}P^n$. It is also easy to see that this topology is Hausdorff. Moreover, $\mathbf{R}P^n$ is compact, since it is the image of the unit sphere of $\mathbf{R}^{n+1}$ under π (every line through the origin contains a point in the unit sphere). We next construct a atlas of $\mathbf{R}P^n$ by defining local charts whose domains cover it all. Suppose $(x_0,\dots,x_n)$ are the standard coordinates in $\mathbf{R}^{n+1}$. Note that the set of all $\ell\in\mathbf{R}P^n$ that are not parallel to the cordinate hyperplane $x_i=0$ form an open subset U_i of $\mathbf{R}P^n$. A line cannot be parallel to all coordinate hyperplanes, so $\bigcup_{i=0}^n U_i=\mathbf{R}P^n$. Now every line $\ell\in U_i$ must meet the hyperplane $x_i=1$ at a unique point; define $\varphi_i:U_i\to\mathbf{R}^n$ by setting $\varphi_i(\ell)$ to be this point and identifying the hyperplane $x_i=1$ with $\mathbf{R}^n$. Of course, the expression of φ_i is

$$\varphi_i(\ell)=\Big(\frac{x_0}{x_i},\dots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\dots,\frac{x_n}{x_i}\Big),$$

where $(x_0,x_1,\dots,x_n)$ are the coordinates of any point $p\in\ell$. This shows that $\varphi_i\circ\pi$ is continuous, so φ_i is continous by the definition of quotient topology. Clearly, φ_i is injective, and one sees that its inverse is also continous. Finally, we check the compatibility between the local charts, namely, that $\varphi_j\circ\varphi_i^{-1}$ is smooth for all i, j. For simplicity of notation, we assume that $i=0$ and $j=1$. We have that

$$\varphi_1\circ\varphi_0^{-1}(x_1,\dots,x_n)=\left(\frac{1}{x_1},\frac{x_2}{x_1},\dots,\frac{x_n}{x_1}\right),$$

so it is smooth since $x_1\neq 0$ on $\varphi_0(U_0\cap U_1)$.

(d) The *complex projective space* $\mathbf{C}P^n$ is the set of all complex lines through the origin in $\mathbf{C}^{n+1}$. One puts s structure of smooth manifold on it in a similar way as it is done for $\mathbf{R}P^n$. Now the local charts map into $\mathbf{C}^n$, so the dimension of $\mathbf{C}P^n$ is $2n$.

(e) If M and N are smooth manifolds, one defines a smooth structure on the product topological space $M\times N$ as follows. Suppose that $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ is an atlas of M and $\{(V_\beta,\varphi_\beta)\}_{\beta\in\mathcal{B}}$ is an atlas of N. Then the family $\{(U_\alpha\times V_\beta,\varphi_\alpha\times\varphi_\beta)\}_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}}$, where

$$\varphi_\alpha\times\varphi_\beta(p,q)=(\varphi_\alpha(p),\varphi_\beta(q)),$$

defines an atlas of $M \times N$. It follows that $\dim M \times N = \dim M + \dim N$.

(f) The *general linear group* $\mathbf{GL}(n, \mathbf{R})$ is the set of all $n \times n$ non-singular real matrices. Since the set of $n \times n$ real matrices can be identified with a $\mathbf{R}^{n^2}$ and as such the determinant becomes a continuous function, $\mathbf{GL}(n, \mathbf{R})$ can be viewed as the open subset of $\mathbf{R}^{n^2}$ where the determinant does not vanish and hence acquires the structure of a smooth manifold of dimension n^2.

(g) Similarly as above, the *complex general linear group* $\mathbf{GL}(n, \mathbf{C})$, which is the set of all $n \times n$ non-singular complex matrices, can be viewed as an open subset of $\mathbf{C}^{2n^2}$ and hence admits the structure of a smooth manifold of dimension $2n^2$. ⋆

Before giving more examples of smooth manifolds, we introduce a new definition. Let N be a smooth manifold of dimension $n + k$. A subset M of N is called an *embedded submanifold* of N of dimension n if M has the topology induced from N and, for every $p \in M$, there exists a local chart (U, φ) of N such that $\varphi(U \cap M) = \varphi(U) \cap \mathbf{R}^n$, where we view $\mathbf{R}^n$ as a subspace of $\mathbf{R}^{n+k}$ in the standard way. We say that (U, φ) is a local chart of M *adapted* to N. Note that an embedded submanifold M of N is a smooth manifold in its own right in that an atlas of M is furnished by the restrictions of the local charts of N to M. Namely, if $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ is an atlas of N, then $\{(U_\alpha \cap M, \varphi_\alpha|_{U_\alpha \cap M})\}_{\alpha \in \mathcal{A}}$ becomes an atlas of M. Note that the compatibility condition for the local charts of M is automatic.

A.2.3 Examples (a) Suppose M is a smooth manifold and $\{(U_\alpha, \varphi_\alpha)\}$ is an atlas of M. Any open subset U of M carries a natural structure of embedded submanifold of M of the same dimension as M simply by restricting the local charts of M to U, namely, one takes the atlas $\{(U_\alpha \cap U, \varphi_\alpha|_{U_\alpha \cap U})\}$.

(b) Let $f : U \to \mathbf{R}^m$ be a smooth mapping, where U is an open subset of $\mathbf{R}^n$. Then the graph of f is a smooth submanifold M of $\mathbf{R}^{n+m}$ of dimension n. In fact, an adapted local chart is given by $\varphi : U \times \mathbf{R}^m \to U \times \mathbf{R}^m$, $\varphi(p, q) = (p, q - f(p))$, where $p \in \mathbf{R}^m$ and $q \in \mathbf{R}^n$. More generally, if a subset M of $\mathbf{R}^{m+n}$ can be covered by open sets each of which is the graph of a smooth mapping from an open subset of $\mathbf{R}^n$ into $\mathbf{R}^m$, then M is an embedded submanifold of $\mathbf{R}^{n+m}$.

(c) The *n-sphere*

$$S^n = \{ (x_1, \ldots, x_{n+1}) \mid x_1^2 + \cdots x_{n+1}^2 = 1 \}$$

is an n-dimensional embedded submanifold of $\mathbf{R}^{n+1}$ since each open hemisphere given by an equation of type $x_i > 0$ or $x_i < 0$ is the graph of a smooth mapping $\mathbf{R}^n \to \mathbf{R}$.

(d) The product of n-copies of the circle S^1 is a n-dimensional manifold called the *n-torus* and it is denoted by T^n. ⋆

A smooth mapping between two smooth manifolds is defined to be a continuous mapping whose local representations with respect to charts on both manifolds is smooth. Namely, let M and N be two smooth manifolds and let $\Omega \subset M$ be open. A continuous map $f : \Omega \to N$ is called *smooth* if and only if

$$\psi \circ f \circ \varphi^{-1} : \varphi(\Omega \cap U) \to \psi(V)$$

is smooth as a map between open sets of Euclidean spaces, for every local charts (U, φ) of M and (V, ψ) of N. Clearly, the composition of two smooth maps is again smooth. Also, a map $f : M \to N$ is smooth if and only if M can be covered by open sets such that the restriction of f to each of which is smooth. We denote the space of smooth functions from M to N by $C^\infty(M, N)$. If $N = \mathbf{R}$, we also write $C^\infty(M, \mathbf{R}) = C^\infty(M)$.

A smooth map $f : M \to N$ between smooth manifolds is called a *diffeomorphism* if it is invertible and the inverse $f^{-1} : N \to M$ is also smooth. Also, $f : M \to N$ is called a *local diffeomorphism* if every $p \in M$ admits an open neighborhood U such that $f(U)$ is open and f defines a diffeomorphism from U onto $f(U)$.

A.2.4 Example Let $\mathbf{R}$ denote as usual the real line with it standard topology and standard smooth structure. We denote by M the manifold whose underlying topological space is $\mathbf{R}$ and whose smooth structure is defined as follows. Let $f(x) = \sqrt[3]{x}$. Then f defines a homeomorphism of $\mathbf{R}$, so we can use it to define a global chart, and we set the smooth sructure of M to be given by the maximal atlas containing this chart. Note that $\mathbf{R}$ and M are different as smooth manifolds, since f viewed as a function $\mathbf{R} \to \mathbf{R}$ is not smooth, but f viewed as a function $M \to \mathbf{R}$ is smooth (since the composition $f \circ f^{-1} : \mathbf{R} \to \mathbf{R}$ is smooth). On the other hand, M is diffeomorphic to $\mathbf{R}$. Indeed, $f : M \to \mathbf{R}$ is a diffeomorphism, since it is smooth, bijective, and its inverse $f^{-1} : \mathbf{R} \to M$ which is given by $f^{-1}(x) = x^3$ is also smooth (since $f \circ f^{-1} : \mathbf{R} \to \mathbf{R}$ is smooth). ⋆

A.3 The tangent space

We next set the task of defining the tangent space to a smooth manifold at a given point. Recall that for a surface S in $\mathbf{R}^3$, the tangent space T_pS is defined to be the subspace of $\mathbf{R}^3$ consisting of all the tangent vectors to the smooth curves in S through p. Here a curve in S is called smooth if it is smooth viewed as a curve in $\mathbf{R}^3$, and its tangent vector at a point is obtained by differentiating the curve as such. In the case of a general smooth manifold M, in the absence of a circumventing ambient space, we construct the tangent space T_pM using the only thing at our disposal, namely, the local charts. The idea is to think that T_pM is the abstract vector space whose elements are represented by vectors of $\mathbf{R}^n$ with respect to a given local chart around p, and using a different local chart gives another representation, so we need to identify all those representations via local charts by using an equivalence relation.

Let M be a smooth manifold of dimension n, and let $p \in M$. Suppose that $\mathcal{F}$ is the maximal atlas defining the smooth structure of M. The *tangent space* of M at p is the set T_pM of all pairs (a, φ) — where $a \in \mathbf{R}^n$ and $(U, \varphi) \in \mathcal{F}$ is a local chart around p — quotiented by the equivalence relation

$$(a, \varphi) \sim (b, \psi) \quad \text{if and only if} \quad d(\psi \circ \varphi^{-1})_{\varphi(p)}(a) = b.$$

The fact that this is indeed an equivalence relation follows from the chain rule in $\mathbf{R}^n$. Denote the equivalence class of (a, φ) be $[a, \varphi]$. Each such equivalence class is called a *tangent vector* at p. Note that for a fixed local chart (U, φ) around p, the map

$$a \in \mathbf{R}^n \mapsto [a, \varphi] \in T_pM \tag{A.3.1}$$

is a bijection. It follows from the linearity of $d(\psi \circ \varphi^{-1})_{\varphi(p)}(a)$ that the equivalence relation $\sim$ is compatible with the vector space structure of $\mathbf{R}^n$ in the sense that if $(a, \varphi) \sim (b, \psi)$, $(a', \varphi) \sim (b', \psi)$ and $\lambda \in \mathbf{R}$, then $(\lambda a + a', \varphi) \sim (\lambda b + b', \psi)$. The bottom line is that we can use the bijection (A.3.1) to define a structure of a vector space on T_pM by declaring it to be an isomorphism. The preceding remark implies that this structure does not depend on the choice of local chart around p. Note that $\dim T_pM = \dim M$.

Let $(U, \varphi = (x_1, \ldots, x_n))$ be a local chart of M, and denote by $\{e_1, \ldots, e_n\}$ the canonical basis of $\mathbf{R}^n$. The *coordinate vectors* at p are defined to be

$$\left.\frac{\partial}{\partial x_i}\right|_p = [e_i, \varphi].$$

Note that

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p \right\} \tag{A.3.2}$$

is a basis of T_pM.

In the case of $\mathbf{R}^n$, for each $p \in \mathbf{R}^n$ there is a canonical isomorphism $\mathbf{R}^n \to T_p\mathbf{R}^n$ given by

$$a \mapsto [a, \mathrm{id}], \tag{A.3.3}$$

where id is the identity map of $\mathbf{R}^n$. Usually we will make this identification without further comment. If we write $\mathrm{id} = (r_1, \ldots, r_n)$ as we will henceforth do, then this means that $\frac{\partial}{\partial r_i}\big|_p = e_i$.

In particular, $T_p\mathbf{R}^n$ and $T_q\mathbf{R}^n$ are canonically isomorphic for every p, $q \in \mathbf{R}^n$. In the case of a general smooth manifold M, obviously there are no such canonical isomorphisms.

Tangent vectors as directional derivatives

Let M be a smooth manifold, and fix a point $p \in M$. For each tangent vector $v \in T_pM$ of the form $v = [a, \varphi]$, where $a \in \mathbf{R}^n$ and (U, φ) is a local chart of M, and for each $f \in C^\infty(U)$, we define the *directional derivative of f in the direction of v* to be the real number

$$\begin{aligned} v(f) &= \frac{d}{dt}\Big|_{t=0} (f \circ \varphi^{-1})(\varphi(p) + ta) \\ &= d(f \circ \varphi^{-1})(a). \end{aligned}$$

It is a simple consequence of the chain rule that this definition does not depend on the choice of representative of v.

In the case of $\mathbf{R}^n$, $\frac{\partial}{\partial r_i}\big|_p f$ is simply the partial derivative in the direction e_i, the ith vector in the canonical basis of $\mathbf{R}^n$. In general, if $\varphi = (x_1, \ldots, x_n)$, then $x_i \circ \varphi^{-1} = r_i$, so

$$v(x_i) = d(r_i)_{\varphi(p)}(a) = a_i,$$

where $a = \sum_{i=1}^n a_i e_i$. Since $v = [a, \varphi] = \sum_{i=1}^n a_i [e_i, \varphi]$, it follows that

$$v = \sum_{i=1}^n v(x_i) \frac{\partial}{\partial x_i}\Big|_p. \tag{A.3.4}$$

If v is a coordinate vector $\frac{\partial}{\partial x_i}$ and $f \in C^\infty(U)$, we also write

$$\frac{\partial}{\partial x_i}\Big|_p f = \frac{\partial f}{\partial x_i}\Big|_p.$$

As a particular case of (A.3.4), take now v to be a coordinate vector of another local chart $(V, \psi = (y_1, \ldots, y_n))$ around p. Then

$$\frac{\partial}{\partial y_j}\Big|_p = \sum_{i=1}^n \frac{\partial x_i}{\partial y_j}\Big|_p \frac{\partial}{\partial x_i}\Big|_p.$$

Note that the preceding formula shows that even if $x_1 = y_1$ we do not need to have $\frac{\partial}{\partial x_1} = \frac{\partial}{\partial y_1}$.

The differential

Let $f : M \to N$ be a smooth map between smooth manifolds. Fix a point $p \in M$, and local charts (U, φ) of M around p, and (V, ψ) of N around $q = f(p)$. The *differential of f at p* is the linear map

$$df_p : T_pM \to T_qN$$

given by

$$[a, \varphi] \mapsto [d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}(a), \psi].$$

It is easy to check that this definition does not depend on the choices of local charts. Using the identification (A.3.3) , one checks that $d\varphi_p : T_pM \to \mathbf{R}^n$ and $d\psi_q : T_pM \to \mathbf{R}^n$ are linear isomorphisms and

$$df_p = (d\psi_q)^{-1} \circ d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)} \circ d\varphi_p.$$

It is also a simple exercise to prove the following important proposition.

A.3.5 Proposition (Chain rule) *Let M, N, P be smooth manifolds. If $f \in C^\infty(M,N)$ and $g \in C^\infty(N,P)$, then $g \circ f \in C^\infty(M,P)$ and*

$$d(g \circ f)_p = dg_{f(p)} \circ df_p$$

for $p \in M$.

If $f \in C^\infty(M,N)$, $g \in C^\infty(N)$ and $v \in T_pM$, then it is a simple matter of unravelling the definitions to check that

$$df_p(v)(g) = v(g \circ f).$$

Now (A.3.4) together with this equation gives that

$$\begin{aligned} df_p\left(\frac{\partial}{\partial x_j}\Big|_p\right) &= \sum_{i=1}^n df_p\left(\frac{\partial}{\partial x_j}\Big|_p\right)(y_i)\frac{\partial}{\partial y_i}\Big|_p \\ &= \sum_{i=1}^n \frac{\partial(y_i \circ f)}{\partial x_j}\Big|_p \frac{\partial}{\partial y_i}\Big|_p. \end{aligned}$$

The matrix

$$\left(\frac{\partial(y_i \circ f)}{\partial x_j}\Big|_p\right)$$

is called the *Jacobian matrix of f at p* relative to the given coordinate systems. Observe that the chain rule (Proposition A.3.5) is equivalent to saying that the Jacobian matrix of $g \circ f$ at a point is the product of the Jacobian matrices of g and f at the appropriate points.

Consider now the case in which $N = \mathbf{R}$ and $f \in C^\infty(M)$. Then $df_p : T_pM \to T_{f(p)}\mathbf{R}$, and upon the identification between $T_{f(p)}\mathbf{R}$ and $\mathbf{R}$, we easily see that $df_p(v) = v(f)$. Applying this to $f = x_i$, where $(U, \varphi = (x_1, \ldots, x_n))$ is a local chart around p, and using again (A.3.4) shows that

$$\{dx_1|_p, \ldots, dx_n|_p\}$$

is the basis of T_pM^* dual of the basis (A.3.2), and hence

$$df_p = \sum_{i=1}^n df_p\left(\frac{\partial}{\partial x_i}\Big|_p\right) dx_i|_p = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\, dx_i|_p.$$

Finally, we discuss smooth curves on M. A *smooth curve* in M is simply a smooth map $\gamma : (a,b) \to M$ where (a,b) is an interval of $\mathbf{R}$. One can also consider smooth curves γ in M defined on a closed interval $[a,b]$. This simply means that γ admits a smooth extension to an open interval $(a-\epsilon, b+\epsilon)$ for some $\epsilon > 0$.

If $\gamma : (a,b) \to M$ is a smooth curve, the *tangent vector* to γ at $t \in (a,b)$ is

$$\dot{\gamma}(t) = d\gamma_t \left(\frac{\partial}{\partial r}\Big|_t \right) \in T_{\gamma(t)}M,$$

where r is the canonical coordinate of $\mathbf{R}$. Note that an arbitrary vector $v \in T_pM$ can be considered to be the tangent vector at 0 to the curve $\gamma(t) = \varphi^{-1}(t, 0, \dots, 0)$, where (U, φ) is a local chart around p with $\varphi(p) = 0$ and $d\varphi_p(v) = \frac{\partial}{\partial r_1}|_0$.

In the case in which $M = \mathbf{R}^n$, upon identifying $T_{\gamma(t)}\mathbf{R}^n$ and $\mathbf{R}^n$, it is easily seen that

$$\dot{\gamma}(t) = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}.$$

The tangent bundle

For a smooth manifold M, there is a canonical way of assembling together all of its tangent spaces at its various points. The resulting object turns out to admit a natural structure of smooth manifold and even the structure of a vector bundle which we will discuss later in **??**.

Let M be a smooth manifold and consider the disjoint union

$$TM = \dot{\bigcup_{p \in M}} T_pM.$$

We can view the elements of TM as equivalence classes of triples (p, a, φ), where $p \in M$, $a \in \mathbf{R}^n$ and (U, φ) is a local chart of M such that $p \in U$, and

$$(p, a, \varphi) \sim (q, b, \psi) \quad \text{if and only if } p = q \text{ and } d(\psi \circ \varphi^{-1})_{\varphi(p)}(a) = b.$$

There is a natural projection $\pi : TM \to M$ given by $\pi[p, a, \varphi] = p$, and then $\pi^{-1}(p) = T_pM$. Next, we use Remark A.2.1 to show that TM inherits from M a structure of smooth manifold of dimension $2 \dim M$. Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ be a smooth atlas for M. For each $\alpha \in \mathcal{A}$, $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a diffeomorphism and, for each $p \in U_\alpha$, $d(\varphi_\alpha)_p : T_pU_\alpha = T_pM \to \mathbf{R}^n$ is the isomorphism mapping $[p, a, \varphi]$ to a. Set

$$\tilde{\varphi}_\alpha : \pi^{-1}(U_\alpha) \to \varphi_\alpha(U_\alpha) \times \mathbf{R}^n, \qquad [p, a, \varphi] \to (\varphi_\alpha(p), a).$$

Then $\tilde{\varphi}_\alpha$ is a bijection and $\varphi_\alpha(U_\alpha)$ is an open subset of $\mathbf{R}^{2n}$. Moreover, the maps

$$\tilde{\varphi}_\beta \circ \tilde{\varphi}_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbf{R}^n \to \varphi_\beta(U_\alpha \cap U_\beta) \times \mathbf{R}^n$$

are given by

$$(x, a) \mapsto (\varphi_\beta \circ \varphi_\alpha^{-1}(x)\, , \, d(\varphi_\beta \circ \varphi_\alpha^{-1})_x(a)).$$

Since $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a smooth diffeomorphism, we have that $d(\varphi_\beta \circ \varphi_\alpha^{-1})_x$ is a linear isomorphism and $d(\varphi_\beta \circ \varphi_\alpha^{-1})_x(a)$ is also smooth on x. It follows that $\{(\pi^{-1}(U_\alpha), \tilde{\varphi}_\alpha)\}_{\alpha \in \mathcal{A}}$ defines a topology and a smooth atlas for M so that it becomes a smooth manifold of dimension $2n$.

If $f \in C^\infty(M, N)$, then we define the *differential of* f to be the map

$$df : TM \to TN$$

that restricts to $df_p : T_pM \to T_{f(p)}N$ for each $p \in M$. Using the above atlases for TM and TN, we immediately see that $df \in C^\infty(TM, TN)$.

The inverse function theorem

We have now come to state the version for smooth manifolds of the first theorem mentioned in the introduction.

A.3.6 Theorem (Inverse function theorem) *Let $f : M \to N$ be a smooth function between two smooth manifolds M, N, and let $p \in M$ and $q = f(p)$. If $df_p : T_pM \to T_qN$ is an isomorphism, then there exists an open neighborhood W of p such that $f(W)$ is an open neighborhood of q and f restricts to a diffeomorphism from W onto $f(W)$.*

Proof. The proof is really a transposition of the inverse function theorem for smooth mappings between Euclidean spaces to manifolds using local charts. Note that M and N have the same dimension, say, n. Take local charts (U, φ) of M around p and (V, ψ) of N around q such that $f(U) \subset V$. Set $\alpha = \psi \circ f \circ \varphi^{-1}$. Then $d\alpha_{\varphi(p)} : \mathbf{R}^n \to \mathbf{R}^n$ is an isomorphism. By the inverse function theorem for smooth mappings of $\mathbf{R}^n$, there exists an open subset $\tilde{W} \subset \varphi(U)$ with $\varphi(p) \in \tilde{W}$ such that $\alpha(\tilde{W})$ is an open neighborhood of $\psi(q)$ and α restricts to a diffeomorphism from $\tilde{W}$ onto $\alpha(\tilde{W})$. It follows that $f = \psi^{-1} \circ \alpha \circ \varphi$ is a diffeomorphism from the open neighborhood $W = \varphi^{-1}(\tilde{W})$ of p onto the open neighborhood $\psi^{-1}(\alpha(\tilde{W}))$ of q. ☐

A.3.7 Corollary *Let $f : M \to N$ be a smooth function between two smooth manifolds M, N, and let $p \in M$ and $q = f(p)$. Then f is a local diffeomorphism at p if and only if $df_p : T_pM \to T_qN$ is an isomorphism.*

Proof. Half of the statement is just a rephrasing of the theorem. The other half is the easy part, and follows from the chain rule. ☐

If M is a smooth manifold of dimension n with smooth structure $\mathcal{F}$, then a map $\tau : W \to \mathbf{R}^n$, where W is an open subset of M, is a diffeomorphism onto its image if and only if $(W, \tau) \in \mathcal{F}$ by the maximality of the smooth atlas $\mathcal{F}$. It follows from this remark and the inverse function theorem that if $f : M \to N$ is a local diffeomorpshim at $p \in M$, then there exist local charts (U, φ) of M around p and (V, ψ) of N around $f(p)$ such that the local representation $\psi \circ f \circ \varphi^{-1}$ of f is the identity.

A.4 Immersions and submanifolds

The concept of embedded submanifold that was introduced in section A.2 is too strong for some purposes. There are other, weaker notions of submanifolds one of which we discuss now. We first give the following definition. A smooth map $f : M \to N$ between smooth manifolds is called an *immersion* at $p \in M$ if $df_p : T_pM \to T_{f(p)}N$ is an injective map, and f is called simply an *immersion* if it is an immersion at every point of its domain.

Let M and N be smooth manifolds such that M is a subset of N. We say that M is an *immersed submanifold of* N or simply a *submanifold of* N if the inclusion map of M into N is an immersion. Note that embedded submanifolds are automatically immersed submanifolds, but the main point behind this definition is that the topology of M can be finer than the induced topology from N. Note also that it immediately follows from this definition that if P is a smooth manifold and $f : P \to N$ is an injective immersion, then the image $f(P)$ is a submanifold of N.

A.4.1 Example Take the 2-torus $T^2 = S^1 \times S^1$ viewed as a submanifold of $\mathbf{R}^2 \times \mathbf{R}^2 = \mathbf{R}^4$ and consider the map

$$f : \mathbf{R} \to T^2, \qquad f(t) = (\cos at, \sin at, \cos bt, \sin bt),$$

where a, b are non-zero real numbers. Since $f'(t)$ never vanishes, this map is an immersion and its image a submanifold of T^2. We claim that if b/a is an irrational number, then $M = f(\mathbf{R})$ is *not* an embedded submanifold of T^2. In fact, the assumption on b/a implies that M is a dense subset of T^2, but an embedded submanifold of some other manifold is always locally closed. ⋆

A.4.2 Theorem (Local form of an immersion) *Let M and N be smooth manifolds of dimensions n and $n+k$, respectively, and suppose that $f : M \to N$ is an immersion at $p \in M$. Then there exist local charts of M and N such that the local expression of f at p is the standard inclusion of $\mathbf{R}^n$ into $\mathbf{R}^{n+k}$.*

Proof. Let (U, φ) and (V, ψ) be local charts of M and N around p and $q = f(p)$, respectively, such that $f(U) \subset V$, and set$\alpha = \psi \circ f \circ \varphi^{-1}$. Then $d\alpha_{\varphi(p)} : \mathbf{R}^n \to \mathbf{R}^{n+k}$ is injective, so, up to rearranging indices, we can assume that $d(\pi_1 \circ \alpha)_{\varphi(p)} = \pi_1 \circ d\alpha_{\varphi(p)} : \mathbf{R}^n \to \mathbf{R}^n$ is an isomorphism, where $\pi_1 : \mathbf{R}^{n+k} = \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^n$ is the projection onto the first factor. By the inverse function theorem, by shrinking U, we can assume that $\pi_1 \circ \alpha$ is a diffeomorphism from $U_0 = \varphi(U)$ onto its image V_0; let $\beta : V_0 \to U_0$ be its smooth inverse. Now we can describe $\alpha(U_0)$ as being the graph of the smooth map $\gamma = \pi_2 \circ \alpha \circ \beta : V_0 \subset \mathbf{R}^n \to \mathbf{R}^k$, where $\pi_2 : \mathbf{R}^{n+k} = \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^k$ is the projection onto the second factor. By Example A.2.3, $\alpha(U_0)$ is an embedded submanifold of $\mathbf{R}^{n+k}$ and the map $\tau : V_0 \times \mathbf{R}^k \to V_0 \times \mathbf{R}^k$ given by $\tau(x, y) = (x, y - \gamma(x))$ is a diffeomorphism such that $\tau(\alpha(U_0)) = V_0 \times \{0\}$. Finally, we put $\tilde{\varphi} = \pi_1 \circ \alpha \circ \varphi$ and $\tilde{\psi} = \tau \circ \psi$. Then $(U, \tilde{\varphi})$ and $(V, \tilde{\psi})$ are local charts, and for $x \in \tilde{\varphi}(U) = V_0$ we have that

$$\tilde{\psi} \circ f \circ \tilde{\varphi}(x) = \tau \circ \psi \circ f \circ \varphi^{-1} \circ \beta(x) = \tau \circ \alpha \circ \beta(x) = (x, 0).$$

□

A.4.3 Scholium *If $f : M \to N$ is an immersion at $p \in M$, then there exists an open neighborhood U of p in M such that $f|_U$ is injective and $f(U)$ is an embedded submanifold of N.*

Proof. The local injectivity of f at p is an immediate consequence of the fact that some local expression of f at p is the standard inclusion of $\mathbf{R}^n$ into $\mathbf{R}^{n+k}$, hence, injective. Moreover, in the proof of the theorem, we have seen that $\alpha(U_0)$ is an embedded submanifold of $\mathbf{R}^{n+k}$. Since $\psi(f(U)) = \alpha(U_0)$ and ψ is a diffeomorphism, it follows that $f(U)$ is an embedded submanifold of N. □

The preceding result is particularly useful in geometry when dealing with local properties of an isometric immersion.

A smooth map $f : M \to N$ between manifolds is called an *embedding* if it is an injective immersion which is also a homeomorphism into $f(M)$ with the relative topology.

A.4.4 Scholium *If $f : M \to N$ is an embedding, then the image $f(M)$ is an embedded submanifold of N.*

Proof. In the proof of the theorem, we have seen that $\tilde{\psi}(f(U)) = V_0 \times \{0\}$. Since f is an open map into $f(M)$ with the relative topology, we can find an open subset W of N contained in V such that $W \cap f(M) = f(U)$. The result follows. □

Recall that a continuous map between locally compact, Hausdorff topological spaces is called *proper* if the inverse image of a compact subset of the counter-domain is a compact subset of the domain. It is known that proper maps are closed. Also, it is clear that if the domain is compact, then every continuous map is automatically proper. An embedded submanifold M of a smooth manifold N is called *properly embedded* if the inclusion map is proper. Now the following proposition is a simple remark.

A.4.5 Proposition *If $f : M \to N$ is an injective immersion which is also a proper map, then the image $f(M)$ is a properly embedded submanifold of N.*

If $f : M \to N$ is a smooth map between manifolds whose image lies in a submanifold P of N and P does not carry the relative topology, it may happen f viewed as a map into P is discontinuous.

A.4.6 Theorem *Suppose that $f : M \to N$ is smooth and P is an immersed submanifold of N such that $f(M) \subset P$. Consider the induced map $f_0 : M \to P$ that satisfies $i \circ f_0 = f$, where $i : P \to N$ is the inclusion.*
a. If P is an embedded submanifold of N, then f_0 is continuous.
b. If f_0 is continuous, then it is smooth.

Proof. (a) If $V \subset P$ is open, then $V = W \cap P$ for some open subset $W \subset N$. By continuity of f, we have that $f_0^{-1}(V) = f^{-1}(W)$ is open in M, hence also f_0 is continuous.

(b) Let $p \in M$ and $q = f(p) \in P$. Take a local chart $\psi : V \to \mathbf{R}^n$ of N around q. By the local form of an immersion, there exists a projection from $\mathbf{R}^n$ onto a subspace obtained by setting certain coordinates equal to 0 such that $\tau = \pi \circ \psi \circ i$ is a local chart of P defined on a neighborhood U of q. Note that $f_0^{-1}(U)$ is a neighborhood of p in M. Now

$$\tau \circ f_0|_{f_0^{-1}(U)} = \pi \circ \psi \circ i \circ f_0|_{f_0^{-1}(U)} = \pi \circ \psi \circ f|_{f_0^{-1}(U)},$$

and the latter is smooth. ☐

A submanifold P of N with the property that given any smooth map $f : M \to N$ with image lying in P, the induced map into P is also smooth will be called a *quasi-embedded submanifold*.

A.5 Submersions and inverse images

Submanifolds can also be defined by equations. In order to explain this point, we introduce the following definition. A smooth map $f : M \to N$ between manifolds is called a *submersion* at $p \in M$ if $df_p : T_pM \to T_{f(p)}N$ is a surjective map, and f is called simply a *submersion* if it is a submersion at every point of its domain.

A.5.1 Theorem (Local form of a submersion) *Let M an N be smooth manifolds of dimensions $n+k$ and n, respectively, and suppose that $f : M \to N$ is a submersion at $p \in M$. Then there exist local charts of M and N such that the local expression of f at p is the standard projection of $\mathbf{R}^{n+k}$ onto $\mathbf{R}^n$.*

Proof. Let (U, φ) and (V, ψ) be local charts of M and N around p and $q = f(p)$, respectively, and set $\alpha = \psi \circ f \circ \varphi^{-1}$. Then $d\alpha_{\varphi(p)} : \mathbf{R}^{n+k} \to \mathbf{R}^n$ is surjective, so, up to rearranging indices, we can assume that $d(\alpha \circ \iota_1)_{\varphi(p)} = d\alpha_{\varphi(p)} \circ \iota_1 : \mathbf{R}^n \to \mathbf{R}^n$ is an isomorphism, where $\iota_1 : \mathbf{R}^n \to \mathbf{R}^{n+k} = \mathbf{R}^n \times \mathbf{R}^k$ is the standard inclusion. Define $\tilde\alpha : \varphi(U) \subset \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^n \times \mathbf{R}^k$ by $\tilde\alpha(x, y) = (\alpha(x, y), y)$. Since $d\alpha_{\varphi(p)} \circ \iota_1$ is an isomorphism, it is clear that $d\tilde\alpha_{\varphi(p)} : \mathbf{R}^n \oplus \mathbf{R}^k \to \mathbf{R}^n \oplus \mathbf{R}^k$ is an isomorphism. By the inverse function theorem, there exists an open neighborhood U_0 of $\varphi(p)$ contained in $\varphi(U)$ such that $\tilde\alpha$ is a diffeomorphism from U_0 onto its image V_0; let $\tilde\beta : V_0 \to U_0$ be its smooth inverse. We put $\tilde\varphi = \tilde\alpha \circ \varphi$. Then $(\varphi^{-1}(U_0), \tilde\varphi)$ is a local chart of M around p and

$$\psi \circ f \circ \tilde\varphi^{-1}(x, y) = \psi \circ f \circ \varphi^{-1} \circ \tilde\beta(x, y) = \alpha \circ \tilde\beta(x, y) = x.$$

☐

A.5.2 Corollary *Let $f : M \to N$ be a smooth map, and let $q \in N$ be such that $f^{-1}(q) \neq \varnothing$. If f is a submersion at all points of $P = f^{-1}(q)$, then P admits the structure of an embedded submanifold of M of dimension* $\dim M - \dim N$.

Proof. It is enough to construct local charts of M that are adapted to P and whose domains cover P. So suppose $\dim M = n + k$, $\dim N = n$, let $p \in P$ and consider local charts (U, φ) and (V, ψ) as in Theorem A.5.1 such that $p \in U$ and $q \in V$. We can assume that $\psi(q) = 0$. Now it is obvious that $\varphi(U \cap P) = \varphi(U) \cap \mathbf{R}^n$, so φ is an adapted chart around p. □

A.5.3 Examples (a) Let A be a non-degenerate real symmetric matrix of order $n+1$ and define $f : \mathbf{R}^{n+1} \to \mathbf{R}$ by $f(p) = \langle Ap, p\rangle$ where $\langle , \rangle$ is the standard Euclidean inner product. Then $df_p : \mathbf{R}^{n+1} \to \mathbf{R}$ is given by $df_p(v) = 2\langle Ap, v\rangle$, so it is surjective if $p \neq 0$. It follows that f is a submersion on $\mathbf{R}^{n+1} \setminus \{0\}$ and $f^{-1}(r)$ for $r \in \mathbf{R}$ is an embedded submanifold of $\mathbf{R}^{n+1}$ of dimension n if it is nonempty. In particular, by taking A to be the identity matrix we get a manifold structure for S^n which coincides with the one previously constructed.

(b) Denote by V the vector space of real symmetric matrices of order n, and define $f : GL(n, \mathbf{R}) \to V$ by $f(A) = AA^t$. We first claim that f is a submersion at the identity matrix I. One easily computes that

$$df_I(B) = \lim_{h \to 0} \frac{f(I + hB) - f(I)}{h} = B + B^t,$$

where $B \in T_I GL(n, \mathbf{R}) = M(n, \mathbf{R})$. Now, given $C \in V$, df_I maps $\frac{1}{2}C$ to C, so this checks the claim. We next check that f is a submersion at any $D \in f^{-1}(I)$. Note that $DD^t = I$ implies that $f(AD) = f(A)$. This means that $f = f \circ R_D$, where $R_D : GL(n, \mathbf{R}) \to GL(n, \mathbf{R})$ is the map that multiplies on the right by D. We have that R_D is a diffeomorphism of $GL(n, \mathbf{R})$ whose inverse is plainly given by $R_{D^{-1}}$. Therefore $d(R_D)_I$ is an isomorphism, so the chain rule $df_I = df_D \circ d(R_D)_I$ yields that df_D is surjective, as desired. Now $f^{-1}(I) = \{ A \in GL(n, \mathbf{R}) \mid AA^t = I \}$ is an embedded submanifold of $GL(n, \mathbf{R})$ of dimension

$$\dim GL(n, \mathbf{R}) - \dim V = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Note that $f^{-1}(I)$ is a group with respect to the multiplication of matrices; it is called the *orthogonal group* of order n and is usually denoted by $O(n)$. ⋆

A.6 Partitions of unity

In general, a locally compact, Hausdorff, second-countable topological space is paracompact (every open covering of the space admits an open locally finite refinement) and σ-compact (it is a countable union of compact subsets). The σ-compactness immediately implies that every open covering of the space admits a countable open refinement. Paracompactness can used to prove that the existence of smooth partitions of unity on smooth manifolds, an extremely useful tool in the theory. Partitions of unity are used to piece together locally defined smooth objects on the manifold to construct a global one, and conversely to represent global objects by locally finite sums of locally defined ones. Recall that a *partition of unity* subordinate to an open covering $\{U_i\}_{i \in I}$ of a smooth manifold M is a collection $\{\lambda_i\}_{i \in I}$ of nonnegative smooth functions on M such that the family of supports $\{\operatorname{supp} \lambda_i\}$ is locally finite (this means that every point of M admits an open neighborhood intersecting only finitely many members of the family), $\operatorname{supp} \lambda_i \subset U_i$ for every $i \in I$, and $\sum_{i \in I} \lambda_i = 1$.

The starting point of the construction of smooth partitions of unity is the remark that the function

$$f(t) = \begin{cases} e^{-1/t}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$$

is smooth everywhere. Therefore the function

$$g(t) = \frac{f(t)}{f(t) + f(1-t)}$$

is smooth, flat and equal to 0 on $(-\infty, 0]$, and flat and equal to 1 on $[1, +\infty)$. Finally,

$$h(t) = g(t+2)g(2-t)$$

is smooth, flat and equal to 1 on $[-1, 1]$ and its support lies in $(-2, 2)$. We refer to [War83, Theorem 1.11] for the proof of the existence of smooth partitions of unity subordinate to an arbitrary open covering of a smooth manifold. In the following, we will do some applications.

A.6.1 Examples (a) Suppose $\{U_i\}_{i\in I}$ is an open covering of M and for each $i \in I$ we are given $f_i \in C^\infty(U_i)$. Take a partition of unity $\{\lambda_i\}_{i\in I}$ subordinate to that covering. Then the formula

$$f = \sum_{i\in I} \lambda_i f_i \tag{A.6.2}$$

defines a smooth function on M. In fact, given $p \in M$, for each $i \in I$ it is true that either $p \in U_i$ and then f_i is defined at p, or $p \notin U_i$ and then $\lambda_i(p) = 0$. Moreover, since $\{\operatorname{supp} \lambda_i\}$ is locally finite, there exists an open neighborhood of p on which all but finitely many terms in the sum in (A.6.2) vanish, and this shows that f is well defined and smooth.

(b) Let C be closed in M and let U be open in M with $C \subset U$. Then there exists a smooth function $\lambda \in C^\infty(M)$ such that $0 \leq \lambda \leq 1$, $\lambda|_C = 1$ and $\operatorname{supp} \lambda \subset U$. Indeed, it suffices to consider a partition of unity subordinate to the open covering $\{U, M \setminus C\}$. ⋆

The following result is a related application. We note that the full Whitney embedding theorem does not require compactness of the manifold and it also provides an estimate on the dimension of the Euclidean space.

A.6.3 Theorem (Weak form of the Whitney embedding theorem) *Let M be a compact smooth manifold. Then there exists an embedding of M into $\mathbf{R}^n$ for n suffciently big.*

Proof. Since M is compact, there exists an open covering $\{(V_i, \varphi_i)\}_{i=1}^a$ such that for each i, $\bar{V}_i \subset U_i$ where (U_i, φ_i) is a local chart of M. For each i, we can find $\lambda_i \in C^\infty(M)$ such that $0 \leq \lambda_i \leq 1$, $\lambda_i|_{\bar{V}_i} = 1$ and $\operatorname{supp} \lambda_i \subset U_i$. Put

$$f_i(x) = \begin{cases} \lambda_i(x)\varphi_i(x), & \text{if } x \in U_i, \\ 0, & \text{if } x \in M \setminus U_i. \end{cases}$$

Then $f_i : M \to \mathbf{R}^m$ is smooth, where $m = \dim M$. Define also smooth functions

$$g_i = (f_i, \lambda_i) : M \to \mathbf{R}^{m+1} \quad \text{and} \quad g = (g_1, \ldots, g_a) : M \to \mathbf{R}^{a(m+1)}.$$

It is enough to check that g is an injective immersion. In fact, on the open set V_i, we have that $g_i = (\varphi_i, 1)$ is an immersion, so g is an immersion. Further, if $g(x) = g(y)$ for x, $y \in M$, then $\lambda_i(x) = \lambda_i(y)$ and $f_i(x) = f_i(y)$ for all i. Take an index j such that $\lambda_j(x) = \lambda_j(y) \neq 0$. Then x, $y \in U_j$ and $\varphi_j(x) = \varphi_j(y)$. Due to the injectivity of φ_j, we must have $x = y$. Hence g is injective. □